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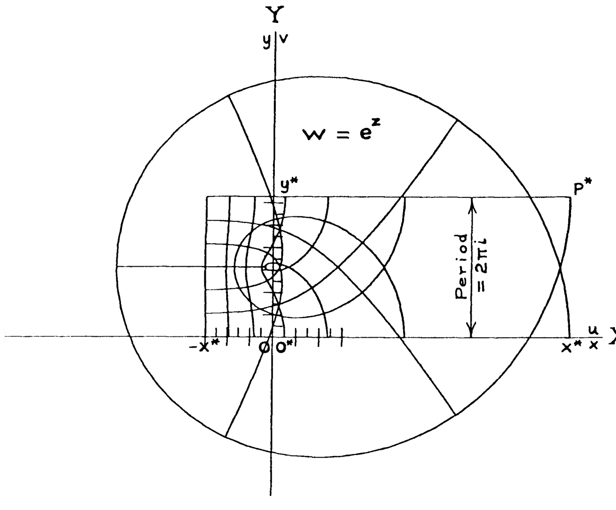

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LOGARITHMS!

N. DAVID MERMIN

Introduction. Among other things, this is indeed a paper about Logarithms! I hope the difference between Logarithms! and logarithms will be clear at the end. (It is rather like the difference between Physics! and physics. If you know what I mean, please skip to the next section.) Evidently Logarithms! are the nicer kind, but if that is not enough to entice you to read on, let me at once answer three questions:

(1) Whom am I writing for?

(2) What about?

(3) Why?

(1) I am writing for physicists who teach the one-year course in "liberal arts physics" for students with absolutely no trace of professional interest in the subject. Although the paper itself simply presents the contents of the lectures on Logarithms! that I have given such students for several years, the presentation here is aimed at a more sophisticated audience of colleagues, to reveal to them briefly and efficiently how the techniques can be used. The paper might be of some interest to the students themselves, but if they can follow the argument without help they probably don't belong in the course for which it is intended.

(2) I am writing about three different things:

(a) An efficient and entertaining way to review arithmetic skills in manipulating powers, which, for typical students in the "liberal arts" physics course, may have lain unused for as many as four years. Without restoring such skills to them it is virtually impossible to teach any physics whatever.

(b) The attitude of physicists toward numbers. I believe this is one of the most important points that can be put across in a physics course for liberal arts students. A number unaccounted for cries out for explanation, and the act of explaining is an act of conquest. The number explained is a trophy to be treasured. Needless to say, nothing this blatant is said explicitly, either in my own course or in the text that follows, but the attitude is made plain enough to most students in the course of the discussion.

(c) An introduction, by example, to the art of estimating and a demonstration of its remarkable power. Many students have never been taught that you can estimate anything without losing dignity. In my physics course we estimate things all day long, and the confidence gained by starting the estimating on known numbers is important. By furnishing us all with portable computers, the electronics industry has deprived us of many great sources of insight and pleasure.

(3) I am writing about these matters for this Journal, because they bear directly on the interests and needs of college and university physics teachers and students, at the level of the "liberal arts physics" course. If calculus is the mathematical language of the introductory course for physics majors, and algebra is the language of the introductory course for premedical students, then arithmetic is the language of liberal arts physics. Students quite impervious to the

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"Logarithms!" originally appeared in the *American Journal of Physics*, 46 (1978) 101-105, and is reprinted by permission of the author and the American Association of Physics Teachers. It is reprinted here partly for its intrinsic charm and partly because much of it is also relevant to the teaching of mathematics.—*Editors*

usual algebraic treatment of the relativity of simultaneity can be carried remarkably far if only the case $v/c=3/5$ is studied, for much of the distracting analytic baggage underlying the derivation of the Lorentz transformation vanishes away when letters are replaced by numbers. In a similar way, many students become frightened or bored, or simply refuse to listen, when "reminded" that $a^{b/c}$ is the cth root of a product of b a's. Wrestling with log 2 for a week convinces many of them that it is, after all, simple, useful, and even fun.

It would be an act of downright silliness on my part to try to enumerate the areas where a knowledge of even liberal arts physics is hampered by an almost phobic refusal to focus on $a^{b/c}$. At least two distinguished physics texts devote several pages to the computation of base 10 logarithms, for much the same pedagogic reasons as I have outlined **[1]**, **[2]**. The approach I shall describe succeeds in being much less arithmetically cumbersome than these, by replacing a systematic but somewhat turgid approach by an *ad hoc* blend of artistry and good fortune. But it belongs to the same tradition of physics pedagogy and should find similar applications.

Logarithms! The numerical explorations to be described below were developed in the course of reviewing the elementary properties of positive, negative, and fractional exponents that inevitably arise, even in physics courses for nonscientists. Students with astonishingly slight capacities for algebraic abstraction have found these constructions entertaining and instructive, those with a little taste for numbers have found them intriguing, and, indeed, I myself have found them a recurring obsession, which this article is written in the hope of curing.

If you consult a ten-place table of base 10 logarithms, or the kind of pocket calculator the nonscientist is unlikely to have access to, you will find that

$$\log 2 = 0.301\,029\,995\,6\ldots. \tag{1}$$

Everybody knows (or at least knew, before they all had pocket calculators) that with much information of this kind one can reduce the odious task of multiplying two long numbers to the merely distasteful one of adding two others, and a little hunting through tables. By having many sunny hours of our adolescence filled with such pursuits, some of us have acquired a deep hatred for logarithms, and even for the elementary properties of positive, negative, and fractional exponents from which they derive.

To overcome this antipathy, while illustrating and applying the elementary properties, I suggest that one should inquire not into the uses of Eq. (1), but into its very origins. Why should anyone believe that to eight place accuracy the logarithm of 2 is 0.301 03?

We approach the question in stages. It does not take long to figure out that

$$2^{10} = 1024, \tag{2}$$

a number which, by sheer chance, is very little different from

$$10^3 = 1000. \tag{3}$$

Ignoring the difference, we then have

$$2^{10} \approx 10^3, \tag{4}$$

so that

$$2 \approx 10^{3/10} = 10^{0.3} \tag{5}$$

which tells us that

$$\log 2 \approx 0.3, \tag{6}$$

a pretty good guess. Since the left side of Eq. (4) is actually a little bigger than the right side, we know that log 2 is somewhat larger than 0.3, and Eq. (1) begins to make a little sense.

But how much bigger than 0.3? How much of that long tail can one come to terms with? There are two ways to proceed: by systematic analysis, or by hoping for more lucky accidents of the kind that gave us so good an estimate for log 2 (missing, after all, by only a third of a percent). The second method is the more entertaining, and we pursue it first.

Equations (2) and (3) give about as good a coincidence as you can hope for, if you want a simple approximate relation involving only powers of 2 and 10 and accept the ground rules that everything should be calculable with a pencil and one small sheet of paper. But if we let a few more integers into the game, relations at least as good as (4) (which makes about a 2.5% error) lie all about. The only integers whose logarithms need concern us are the prime numbers, since the logarithms of composites are the sums of those of their prime factors. We therefore inquire into the logarithms of 3, 7, and 11, as well as 2. (Since $\log 5 = 1 - \log 2$, its determination is not of separate concern.) The data we wish to account for are:

$$\log 3 = 0.477\,121\,254\,7\ldots, \tag{7}$$

$$\log 7 = 0.845\,098\,040\,0\ldots, \tag{8}$$

$$\log 11 = 1.041\,392\,685\ldots. \tag{9}$$

Let us first estimate these by finding some 1% or 2% coincidences like the one that gave us $3/10$ for log 2. By making use of that first estimate, the search for coincidences is not a hard one. Thus

$$3^4 = 81, \quad 2^3 \times 10 = 80, \tag{10}$$

so with little more than a one percent error,

$$3^4 \approx 2^3 \times 10, \tag{11}$$

or

$$\log 3 \approx (3\log 2 + 1)/4 \approx 19/40 = 0.475, \tag{12}$$

which misses (7) by only $1/2\%$. In a similar vein, since

$$7^2 = 49, \quad 10^2/2 = 50, \tag{13}$$

with the estimate of $3/10$ for log 2, we find

$$\log 7 \approx 17/20 = 0.85, \tag{14}$$

which is less than a percent away from Eq. (8). Using the estimates Eqs. (12) and (6), and noting that

$$11^2 = 121, \quad 2^2 \times 3 \times 10 = 120, \tag{15}$$

we find

$$\log 11 \approx 83/80 = 1.0375, \tag{16}$$

which is less than $\frac{1}{2}\%$ off Eq. (9).

Those whom the schools have taught to hate numbers will probably want to stop after this preliminary warmup, but the real fun is only beginning. For if we look a little longer at numbers whose prime factors are only powers of 2, 3, 5, 7, and 11, we uncover some truly splendid coincidences. I begin with my prize specimen:

$$3^4 \times 11^2 = 9801, \quad 2 \times 7^2 \times 10^2 = 9800. \tag{17}$$

I must confess to having made this discovery while leafing through a table of prime factors up to 10 000. Once found, of course, it is easily verified, but I felt the search that brought it to light to be offensively at odds with the back of the envelope spirit of the whole enterprise. My mind was set at rest when I noticed, without anybody's help, that 98, 99, and 100 are *three* consecutive numbers composed only of primes less than or equal to 11, and Eq. (17) is merely the expression of the fact that $100 \times 98 = 99^2 - 1$.

Equation (17) gives us an 0.01% coincidence, and therefore a much improved linear relation between the logarithms of 2, 3, 7, and 11. Since this is one equation in four unknowns, we need three more independent coincidences of comparable accuracy to solve for the logarithms. Counting up from 100 until boredom set in, I came upon no additional trios of consecutive numbers composed only of the primes 2, 3, 5, 7, and 11. Counting downwards, however, one

encounters the groups [54,55,56], [48,49,50], and [20,21,22]. These yield

$$10^2\times 11^2/2^2=3025,\quad 2^4\times 3^3\times 7=3024 \tag{18}$$

$$7^4=2401,\quad 2^3\times 3\times 10^2=2400 \tag{19}$$

$$3^2\times 7^2=441,\quad 2^2\times 10\times 11=440. \tag{20}$$

Equations (18) and (19), while not up to the standard set by Eq. (17), still give 0.03% and 0.04% coincidences. The estimates they furnish are more than an order of magnitude better than those of our first round. Equation (20), however, while still substantially better than the best of the first round coincidences, is unquestionably the weak link in the chain. I would be delighted to learn of something simple and better. But it is not helpful to note that $2\times 3^7=4374$, while $7\times 10^4/2^4=4375$. The resulting equation for the logarithms is not linearly independent of those coming from Eqs. (17) and (18). Equally useless, for similar reasons, is the fact that $2^2\times 3^6\times 7^3=1\ 000\ 188$. I mention this last one not merely to show off, but to indicate that I did struggle a bit before settling for Eq. (20).

If one ignores the slight differences in the pairs of numbers appearing in Eqs. (17)-(20), the four simple linear equations for the four logarithms are easily solved and give:

$$\log 2\approx\frac{72}{239}=0.301\ 25\ldots\quad (\text{exact: } 0.301\ 02\ldots), \tag{21}$$

$$\log 3\approx\frac{114}{239}=0.476\ 98\ldots\quad (\text{exact: } 0.477\ 12\ldots), \tag{22}$$

$$\log 7\approx\frac{202}{239}=0.845\ 18\ldots\quad (\text{exact: } 0.845\ 09\ldots), \tag{23}$$

$$\log 11\approx\frac{249}{239}=1.041\ 84\ldots\quad (\text{exact: } 1.041\ 39\ldots). \tag{24}$$

Off by no more than 2 in the 4th significant place!

About half the class may feel some joy at this. The rest will be bored no matter what happens next, so you may as well go on to point out that one can, with only a little more effort, do better. Much better!

The search for bigger and better coincidences is about over if one wishes to pursue it by amateur means. I offer with a slight blush

$$2^{53}=9.007\ 199\ldots\times 10^{15} \tag{25}$$

and

$$2\times 3^{35}=1.000\ 630\ 9\ldots\times 10^{17}, \tag{26}$$

which I admit to having discovered with the aid of a calculator. Not only does this method violate the rules for amateurs. It considerably undermines one's stand against students who might wonder sullenly why you do not ask a calculator to tell you the wretched logarithms in the first place. Most students crass enough to feel this way, however, have probably never been exposed to calculators that handle logarithms, but only to those that multiply, so the danger is not great. However a more honorable defense can be constructed around the fact that these relations are not as hard to verify by hand as one might think. Thus

$$2^{53}=8\times(1024)^5=8\times 10^{15}\times(1+24/1000)^5, \tag{27}$$

from which an application of the binomial theorem (known to many amateurs) easily extracts Eq. (25) to the required accuracy. The other relation (26) requires a little more effort, but fits with little trouble onto a well-organized page. The resulting approximations,

$$2^{53}\approx 3^2\times 10^{15},\quad 2\times 3^{35}\approx 10^{17}, \tag{28}$$

give two linear equations which yield:

$$\log 2 \approx \frac{559}{1857} = 0.301\,023\,1\ldots \quad (\text{exact: } 0.301\,029\,9\ldots), \tag{29}$$

$$\log 3 \approx \frac{886}{1857} = 0.477\,113\,6\ldots \quad (\text{exact: } 0.477\,121\,2\ldots). \tag{30}$$

The disagreement is down to less than one in the 5th significant place.

To improve on this we must invoke a small consequence of systematic analysis:

When X is a number (positive or negative) small in size compared with one, then the logarithm of $1+X$ is very accurately given by X itself, multiplied by the number 0.434 29...:

$$\log(1+X) \approx 0.4343X, \quad \text{for} |X| \ll 1. \tag{31}$$

The smaller the size of X, the more accurate the approximation. The number 0.434 29...(which professionals will recognize as the base 10 logarithm of e) can be truncated to 0.4343 with two gratuitous benefits: the accuracy is virtually to five places (rather than the apparent four), and tender minds need only remember the number 43 (and then again 43) rather than a full four-digit number.

Persuading the amateur that there is some truth in Eq. (31) can be arduous. There is something to be said for simply producing it as one might a laser: a technological spin-off of higher research, with spectacular capabilities, now to be shown. For the more thoughtful students, however, one can first observe that since the logarithm of one is zero, when X is a very small number the logarithm of $1+X$ can also be expected to be a very small number. The question is why that second very small number should be simply proportional to the first. One can next point out that when Y and Z are both very small, then the product $(1+Y)(1+Z)$ is equal to $1+(Y+Z)$ except for a correction that is *very*, very small, and therefore the logarithm of $1+(Y+Z)$ must be *very*, very close to the sum of the logarithms of $(1+Y)$ and $(1+Z)$. Anyone can see that log $(1+X) \approx AX$, A a constant, has this necessary property, and with somewhat more squirming one can even be fairly persuasive on the point that nothing else does.

This leaves only the sticky question of why the constant A should be 0.434 29.... There was a time in the evolution of this pedagogical adventure when I found it necessary to break the ground rules by producing a list of successive square roots of 10, and noting that the values of

$$A_n = \frac{1}{2^n[10^{1/2^n} - 1]} \tag{32}$$

settled down to 0.4343 with gratifying efficiency. I lived with this departure from the back of the envelope by noting to myself that the calculator purchased by the amateur at the corner drugstore often gives square roots, but rarely logarithms. Then one day it occurred to me that such inferior instruments would be more than likely to yield round-off errors which would do the amateur in before that second 43 had a chance to be established. With this, my conscience rebelled, and out of another appeal to the binomial theorem and no more than the usual amount of abstract mumbo-jumbo, I constructed a computation in the purest amateur spirit, which readers who persevere to the end of this essay are invited to examine in the Appendix.

Armed with the approximation (31), we can improve our efforts at all stages. At the crudest level we replace the estimate Eq. (4) by the exact identity:

$$2^{10} = 10^3 \times (1 + 24/1000). \tag{33}$$

Since 24/1000 is a small number, we can use Eq. (31) to estimate with some accuracy the size of the logarithm of the correction factor, and with no more additional effort than that required to multiply 24 by 0.4343 we arrive at the improved value:

$$\log 2 \approx 0.301\,042\ldots \quad (\text{exact: } 0.301\,029\ldots). \tag{34}$$

This is about as good as the result (29) of the most extreme extension of the coincidence method, and will convince those students who still have a taste for the game of the marvelous power of the 0.4343 method.

If one similarly improves the rough estimates furnished by Eqs. (10), (13), and (15), using the values $1/80$, $-1/50$, and $1/120$ for X, all of which the amateur can multiply by 0.4343 through an effortless exercise in division (*short* division, assuming they still teach the 12-table in school), then one finds with remarkable ease:

$$\log 3 \approx 0.477\,139 \quad (\text{exact: } 0.477\,121\ldots), \tag{35}$$

$$\log 7 \approx 0.845\,136 \quad (\text{exact: } 0.845\,098\ldots), \tag{36}$$

$$\log 11 \approx 1.041\,421 \quad (\text{exact: } 1.041\,392\ldots). \tag{37}$$

Using (31) to improve the second level of approximation (Eqs. (21)-(24)) requires a little more effort, but it can still be done by hand in a page or two. The best students will be eager to try, and will be rewarded by

$$\log 2 \approx 0.301\,030\,15\ldots \quad (\text{exact: } 0.301\,029\,99\ldots), \tag{38}$$

$$\log 3 \approx 0.477\,121\,15\ldots \quad (\text{exact: } 0.477\,121\,25\ldots), \tag{39}$$

$$\log 7 \approx 0.845\,097\,90\ldots \quad (\text{exact: } 0.845\,098\,04\ldots), \tag{40}$$

$$\log 11 \approx 1.041\,393\,07\ldots \quad (\text{exact: } 1.041\,392\,68\ldots). \tag{41}$$

As a special treat for fanatics, the improvememt can be applied to Eqs. (25) and (26). In fact, less effort is required to do this by hand than to find the results (38)-(41). The only real burden is carrying to many places the long division by 1857, but for the pure of heart, the ineffable joy of seeing one after another of that amazing stretch of 9's come marching out of the deep interior of log 2 is more than enough reward:

$$\log 2 \approx 0.301\,029\,998\,7\ldots \quad (\text{exact: } 0.301\,029\,995\,6\ldots), \tag{42}$$

$$\log 3 \approx 0.477\,121\,246\,0\ldots \quad (\text{exact: } 0.477\,121\,254\,7\ldots). \tag{43}$$

After this it is almost too gross to observe that those who have fought through to these triumphant conclusions will be eager to be introduced to e itself, as $10^{0.434\,29\cdots}$; or that still finer results can be retained by introducing the quadratic term into Eq. (31); or that higher primes, at the very worst,* can be got at successively through $p^2 \approx (p-1)(p+1)$, since all prime factors of the right side must be less than p. Better, surely, to rest with the extraction of that glorious...999....

Acknowledgments. This paper was written while recuperating from surgery performed at Tompkins County Hospital. I would like to thank an anesthetist who seemed to say that his name was Dr. Eeyore, for inducing the appropriate postoperative state of mind. I am also grateful to those many people whose names I do not know, who remembered to close the door when they left the room and to N. W. Ashcroft, for a vase of lilies-of-the-valley.

Appendix

By taking the very small number X in

$$\log(1+X) \approx AX, \quad |X| \ll 1, \tag{44}$$

to be the inverse of a very large integer n, one finds that the constant A should be given by

$$A \approx \log(1+1/n)^n, n \gg 1. \tag{45}$$

With the aid of the binomial theorem and a little rearranging, one has

$$\left(1+\frac{1}{n}\right)^n = 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\frac{1}{4!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)+\cdots. \tag{46}$$

*In any given case, one can almost always do better. One can, for example, get the logarithm of 13 through $7\times 11\times 13 = 1001$, or, if one is really serious, through $2^4\times 3^4\times 7^3\times 11^3\times 13^2 = 99\,991\,683\,792$. Aficionados should also note that if one plays the game on the primes through 13 one essentially gets 37 free of charge, since $3^3\times 7\times 11\times 13\times 37 = 999\,999$.

From a few concrete examples one can persuade the amateur that when n is a *very* large number, then the first dozen or so terms in Eq. (42) will hardly depend on n at all. By the time one gets to terms that *do* depend appreciably on n, they are so small (because of the inverse factorials) as to make negligible contributions to the sum. This not only provides a plausible basis for the original assertion that an approximation of the form Eq. (44) exists, but it also gives the numerical value of the constant:

$$A=\log(2+1/2!+1/3!+1/4!+\cdots) \tag{47}$$

as any professional knew all along.

The short computation of A starts from Eq. (47). If we need four place accuracy, it surely suffices to retain only terms through $1/7!$ in the sum. So truncated, the sum is most efficiently evaluated by giving each term the common denominator $7!=5040$, which gives the result in the form $137\,00/5040$. This fraction is more suggestively written as $2740/1008$, not because it thereby bears a closer resemblance to the more familiar 2.718 28..., but because, as the seasoned hunter of coincidences cannot fail to notice, it is thereby revealed on the 1% level of accuracy to be nothing but $3^3/10$. With the crudest estimate (Eq. (12)) for log 3, this gives

$$A\approx 3\log 3-1\approx 17/40=0.425. \tag{48}$$

We can improve on this as follows: More accurately,

$$A\approx\log\frac{2740}{1008}\approx 3\log 3-1+\log\left(1+\frac{4}{270}\right)-\log\left(1+\frac{8}{1000}\right). \tag{49}$$

But by the nature of A (Eq. (44)) we can express the last two logarithms in terms of A again, to find

$$A\left(1+\frac{8}{1000}-\frac{4}{270}\right)\approx 3\log 3-1. \tag{50}$$

To finish, we need merely repeat whichever of our calculations of log 3 strikes us as sufficiently accurate. If we take one of the estimates that exploited the correction formula (31), however, we must let the (momentarily unknown) symbol A replace the value 0.4343. Thus the procedure that gave Eqs. (34) and (35) now gives

$$10\log 2\approx 3+(24/1000)A, \tag{51}$$

$$4\log 3\approx 3\log 2+1+(1/80)A, \tag{52}$$

and therefore

$$\log 3\approx\frac{19}{40}+\left(\frac{1}{320}+\frac{9}{5000}\right)A. \tag{53}$$

Using this to eliminate log 3 from Eq. (50) we find that

$$A\left(1+\frac{13}{5000}-\frac{3}{320}-\frac{2}{135}\right)\approx\frac{17}{40}, \tag{54}$$

which a little scribbling shows to give

$$A\approx 0.434\,37\ldots\quad(\text{exact}: 0.434\,29\ldots). \tag{55}$$

(Those whom it offends slightly to overshoot the second 43 can repeat the procedure with one of the better methods for getting at log 3, but for all the uses we have put it to, 0.4344 works just about as well.)

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ON A HYPERANALYTIC GEOMETRY FOR COMPLEX FUNCTIONS

MALCOLM MURRILL

Summary. This paper deals with certain aspects of complex geometry. Our field of study, which we have called **hyperanalytic geometry**, is concerned with two problems: *the geometrical representation of complex functions of a single complex variable*, and *the geometric properties of these representations*.

In Section 1, a graphical basis is constructed for a complex hyperanalytic geometry of four dimensions. This analytic geometry is used in Section 2 to obtain graphs of some elementary complex functions. The complex line and the complex circle are examined in detail. In Section 3, the hyperanalytic model is applied to resolve some present paradoxes in complex geometry.

1. Hyperanalytic Geometry. For our frame of reference in the *complex hyperanalytic geometry of four dimensions*, we postulate and construct four coordinate axes, x, y, u, and v, mutually perpendicular *by definition* at a common origin O (Fig. 1). We then have six mutually perpendicular coordinate planes: xy, xu, xv, yu, yv, uv; and four mutually perpendicular coordinate hyperplanes: xyu, xyv, xuv, yuv. The four hyperplanes partition hyperspace into sixteen four-dimensional cells, which we call **hexadekants**.

For convenience in working in the z- and w-planes of complex variables, which is one of our objectives here, we use *equal scales* on all four axes, and preserve *the right angles and the usual orientation of the paired axes* in the two complex planes. Although this deliberate distortion will attenuate our figures projected from 4- onto 3-space, we will see later it is a serendipitous choice which leads to a *conformal projection* of our graphs *from 4-space to 2-space*.

To avoid crowded figures and the confusion of coinciding lines, we use the particular asymmetrical arrangement of axes shown in Figure 1, in which the viewpoint is such that the projections of the positive x- and u-axes are 150° apart. Note the *apparent* intersection of the xu- and yv-planes, which actually have only the origin in common. Such "spurious" intersections also appear in the four-dimensional graphs of catastrophe surfaces [**45**, p. 68].

Points are plotted in this four-dimensional coordinate system, as indicated in Figure 1, by starting at the origin and moving successively, in any order, the given coordinate distance parallel to each axis. We define the vector $\vec{t}$, drawn from the origin to the point $P(x,y,u,v)$, to be the vector sum of $\vec{x},\vec{y},\vec{u},\vec{v}$; it may be represented by the notation $\vec{t}=(x,y,u,v)$. When a relation exists among the variables x,y,u,v, we call $\vec{t}$ a **transformation vector**. We make the following

DEFINITION. A **graph** in n-dimensions $x_1,x_2,\ldots,x_n$ is the locus of the terminal point of the transformation vector $\vec{t}=(x_1,x_2,\ldots,x_n)$.

The graphing process, then, becomes basically that of vector addition. When $n \leqslant 3$, this definition gives us the graphs obtained by the ordinary methods of plotting. The *dimensionality* of the graph, i.e., whether it is a point, curve, surface, solid, or hypersolid, is given by $n-k$, where k is the number of constraints on the available n degrees of freedom. The *configuration* of the graph displays the mathematical relationship of these constraints.

We recall that a function of a complex variable $w=f(z)$, where $w=u+iv$ and $z=x+iy$, yields the two equations $u=u(x,y)$ and $v=v(x,y)$. Since these constitute two constraints among four variables, we see that *geometrically*, with two degrees of freedom remaining, we are dealing

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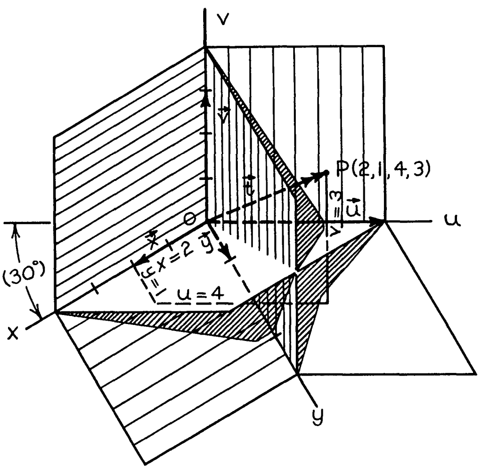
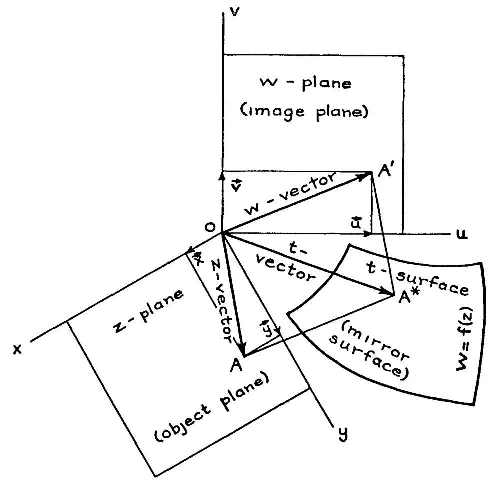


First Hexadekant of Four-Dimensional Space

FIG. 1



Four-Dimensional Graph of a Complex Function

FIG. 2

with a **skew surface** or two-dimensional locus in 4-space. This is the locus traced by the transformation vector $\vec{t}$, and is **the four-dimensional graph of the complex function $w=f(z)$** (Fig. 2).

We call this locus a **transformation surface** or ***t*-surface**, it being the locus of all points A^* (the tip of the t-vector) performing the transformation of points A, representing the independent variable z, into points A', representing the corresponding dependent variable w. Since such a surface is interposed between and performs the transformation from an "object" in the z-plane to its "image" in the w-plane, we also refer to this transformation surface as a **mirror surface**.

With $\vec{z}=(x,y)=\overrightarrow{OA}$ and $\vec{w}=(u,v)=\overrightarrow{OA'}$, we have $\vec{t}=(x,y,u,v)=(z,w)=(OA,OA')=\overrightarrow{OA^*}$. Thus, A^* is the fourth vertex of the vector addition parallelogram (actually a **position rectangle** in 4-space) having $\overrightarrow{OA}$ and $\overrightarrow{OA'}$ as sides.

It follows that this mirror surface in four dimensions not only gives us an accurate pictorial representation or graph of the complex function itself, but—in the ultimate test of a complete geometric-graphic analogue—*it will also perform the actual complex transformations from the z-plane to the w-plane graphically*. We will need, of course, just as we do for correct interpretation of an ordinary graph of a three-dimensional surface, some system, such as mirror curves of object plane grid lines, to *correlate* the points of the mirror surface with those of the object plane. Each such *pair of correlated points* in the two dimensions of the graph paper supply the four constraints necessary to uniquely specify the position rectangle of a four-dimensional mirror point (compare [37]).

Now consider Figure 2 as simply *a two-dimensional diagram*. Superimpose X,Y axes on the u,v axes, and let θ be the counterclockwise angle from the u-axis to the x-axis. Then the transformation equations for $A^*(X,Y)$ are $X=u+x\cos\theta-y\sin\theta, Y=v+x\sin\theta+y\cos\theta$. We compare the angle between two arbitrary curves through A with the corresponding angle between their projected mirror curves through A^* for some analytic function $w=f(z)$. Using parametric and partial differentiation to obtain the slopes dY/dX of the curves at A^*, we can show that, under the Cauchy-Riemann conditions, the angles at A and A^* are equal for all non-exceptional object points and object curves, for all θ, and for all analytic $f(z)$. Similar results hold for image curves through A'. We have thus proved the following *two-dimensional*

THEOREM. *Under the plotting methods of this paper, the two-dimensional projections of mappings onto analytic mirror surfaces are conformal.*

For example, in Figure 2, if the t-surface represents an analytic function over a rectangular region of the z-plane (or, inversely, the w-plane), then the angles *in the figure itself* at the four corners of the t-surface will be right angles.

Now let us reconsider the original skew surface in four dimensions (Fig. 2). If we examine the perpendicular vectors $z=1$ and $z=i$ in the object plane, we find that for an analytic function the mirrors of these vectors, $z^*=1^*$ and $z^*=i^*$ lying on the four-dimensional mirror surface, are also perpendicular. (Consider the inner product of the direction numbers of these mirror vectors: $(1,0,\partial u/\partial x,\partial v/\partial x)$ and $(0,1,\partial u/\partial y,\partial v/\partial y)$, under the Cauchy-Riemann conditions.) We thus have a partial verification of the *four-dimensional*

CONJECTURE. Mappings onto analytic mirror surfaces are conformal.

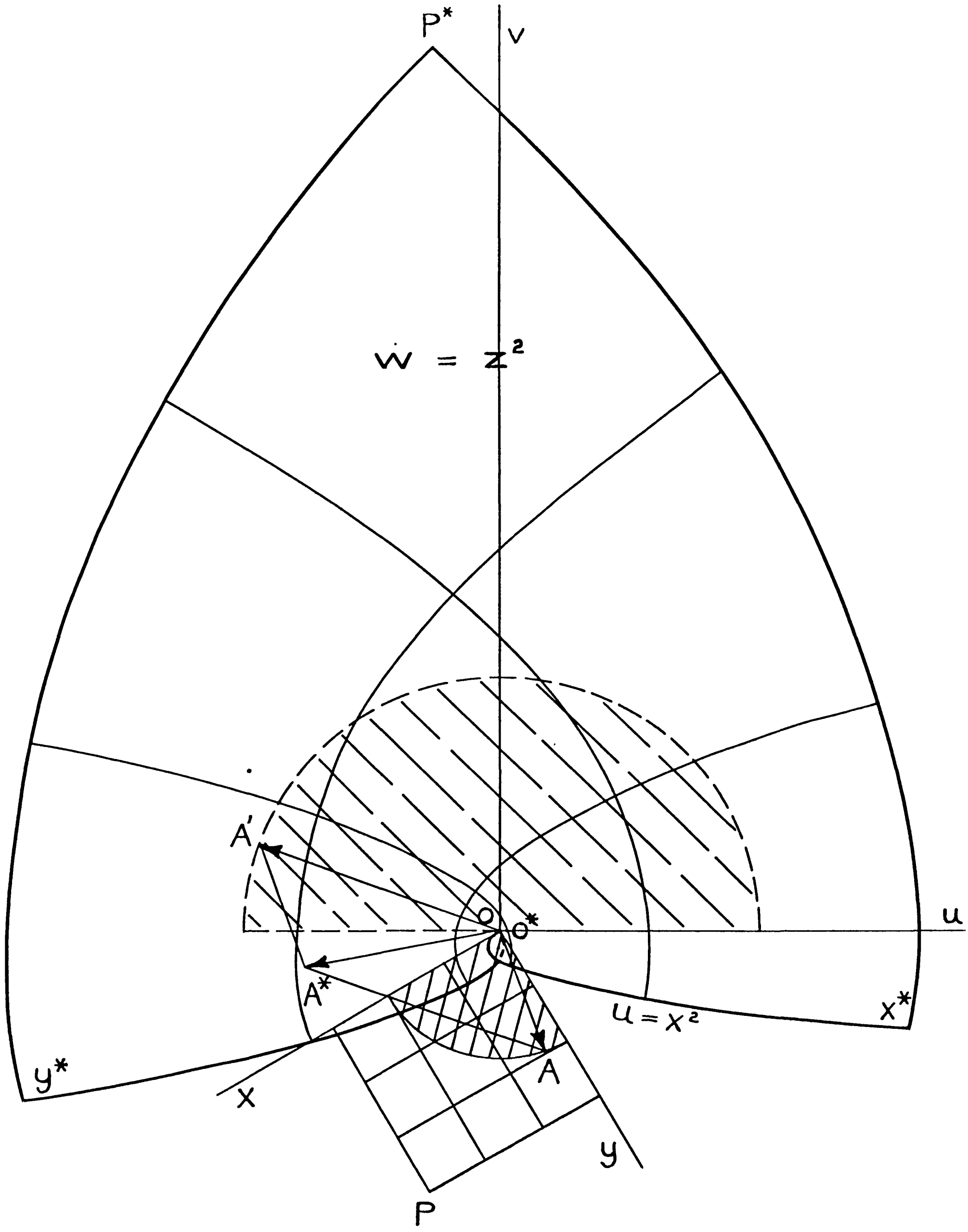
That is, we believe that our conformal mirror projections in 2-space are reflecting an intrinsic conformality of analytic mirror surfaces in 4-space.

It is shown in an earlier paper by the author **[29**, Chaps. 2, 3] that *these four-dimensional graphs are valid, complete, and comprehensible geometrical representations for functions of a single variable in the fields of real and complex numbers*. They therefore include, as subsets, the graphs of real functions, and also of *semi-complex functions* **[39]** in which one variable is real and the other complex. Thus we find that *the mirrors of the real axes represent the three-dimensional graphs of semi-complex functions*, such as described by Kempner **[22]**, **[23]**, Ward **[44]**, Karst **[21]**, and others. (The interested reader will find a variety of graphical constructions for portraying complex quantities in the preceding four articles and in **[6]**, **[9]**, **[10]**, **[11]**, **[15]**, **[16]**, **[19]**, **[24]**, **[25]**, **[27]**, **[28]**, **[32]**, **[34]**, **[35]**, **[38]**. However, all of these interpretations have been confined to three or fewer dimensions, and thus none provides a *complete* graphical representation for general complex functions.)

We have chosen in this paper to use **asymmetric axes** for most of our graphs (Figs. 1–4,7,8), since they show all six coordinate planes and thus give more information and more of a "four-dimensional feel" to the figures. Of course, other viewpoints, involving different angles and scales for the axes, are also possible and instructive. In certain arrangements (by accepting further distortions in the vector sums), we can reduce the graphs to *two-dimensional plotting*. With **symmetric axes** (Fig. 5), in which the complex planes are rotated so that the y- and u-axes coincide, we plot the two-dimensional points (X,Y) after first computing $X=x-sv$ and $Y=y+su$, where s is an arbitrary scale factor, usually set equal to 1. An advantage of symmetric axes is that they can display the corresponding *graph of the real function* (the trace on the xu-plane) *in true size and shape*. By *superimposing* the z- and w-planes, we obtain the two-dimensional plotting of **coincident axes** (Fig. 6), in which $X=x+su$ and $Y=y+sv$. Finally, coincident axes can be separated into three **parallel axes** (not shown), where the usual Gauss z-plane and w-plane pair of corresponding graphs are drawn, but *with extra space between them in which the mapping t-surface is exhibited*, related through sums of vector pairs parallel to corresponding object and image vectors.

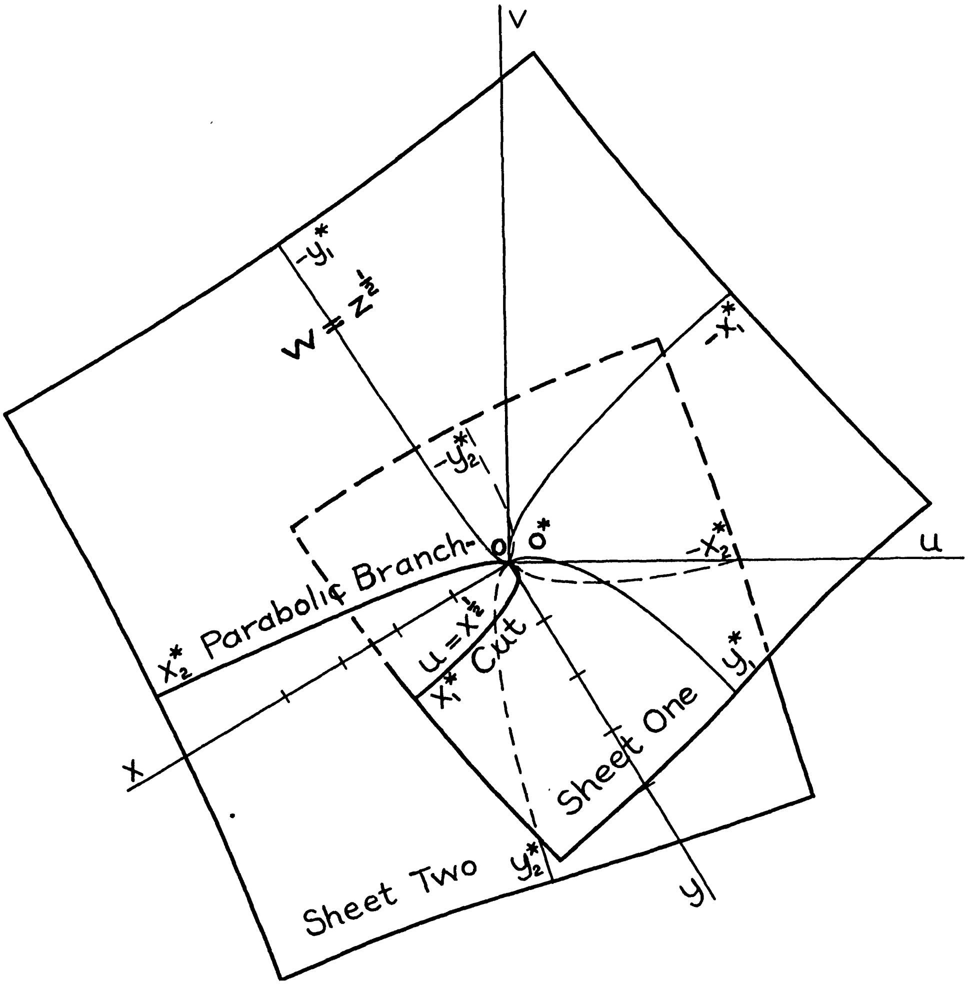
All of these viewpoints provide valid graphs of complex functions, since all perform the *graphical transformation* of an object in the z-plane into its image in the w-plane, in the role of a "complex nomograph." A computer-generated *Dictionary of Complex Transformation Surfaces*, containing symmetric or coincident axes graphs of commonly encountered complex functions, should be a useful mathematical resource.

2. Graphs of Complex Functions. The graphs of a few basic analytic functions, along with typical transformations effected by some of them, are shown here. Included are the complex parabola, or **parabolex**, $w=z^2$, and its inverse, $w=z^{\frac{1}{2}}$ (Figs. 3 and 4); the complex sine function, or **sinex**, $w=\sin z$ (Fig. 5); and the complex exponential function, or **exponex**, $w=e^z$ (Fig. 6). (For a different approach to the graph of this last function, see **[4]**.) Two graphs will be discussed



The Parabolex $w=z^2$ $(u=x^2-y^2,\ v=2xy)$
Scan: $0 \leq x \leq 3$, $0 \leq y \leq 3$, $\Delta x = \Delta y = 1$

FIG. 3



The Inverse Parabolex $\quad w=z^{\frac{1}{2}}$

$$\left(u=\pm\sqrt{\tfrac{1}{2}\left(\sqrt{x^2+y^2}\,+x\right)}\;,v=\pm\sqrt{\tfrac{1}{2}\left(\sqrt{x^2+y^2}\,-x\right)}\;\right).$$

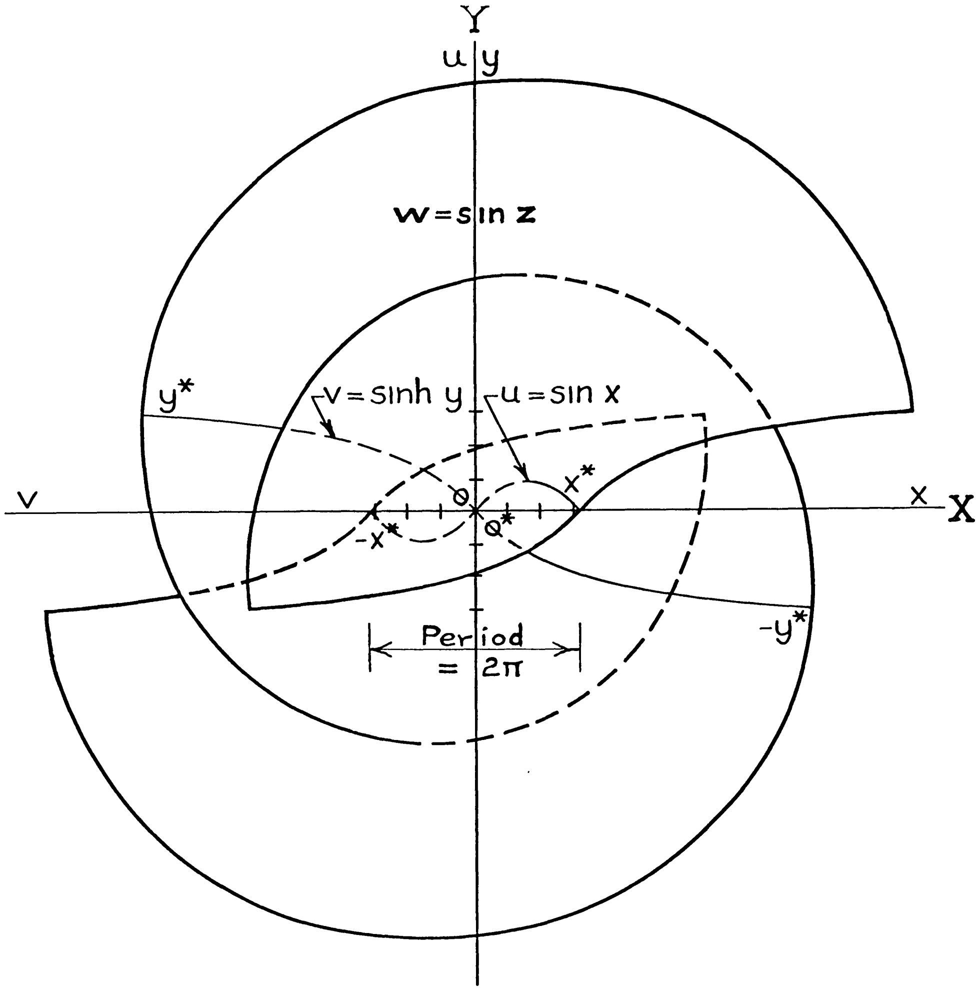
Scan: $-4\leqslant x\leqslant 4,\ -4\leqslant y\leqslant 4$

FIG. 4

in detail: that of the complex line, or **linex**, $w=z$ (Fig. 7), and that of the complex circle (or hyperbola), the **circlex-hyperbolex**, $w=1/z$ (Fig. 8).

We cannot expect, of course, to plot the entire graph of a complex function any more than we can plot an entire real line or parabola. We can, however, construct and examine the mirror surface for any portion, or **scan**, of the object plane we wish, such as a region near the origin or about a pole, and, just as with real variables, become familiar with its characteristic shape.

Also, from such a surface, we can gain an insight into the kind of transformation of any particular curve or region it will effect. To "read" this information from the graph (see Fig. 2), it



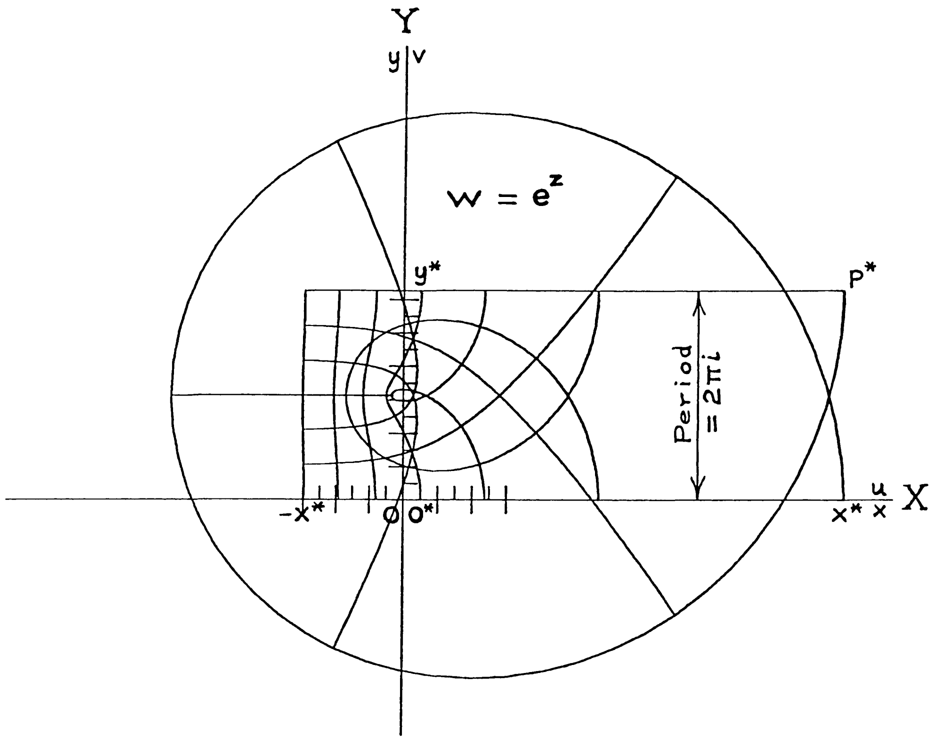
The Sinex $w = \sin z$ ($u = \sin x \cosh y$, $v = \cos x \sinh y$)
Symmetric axes, with $s = 1$
Scan: $-\pi \leqslant x \leqslant \pi$, $-3 \leqslant y \leqslant 3$

FIG. 5

is only necessary to keep in mind that if the vector from each object point (A) in the z-plane to its correlated mirror point (A^*) on the transformation surface be moved to the origin, it maps the image (A') of that point in the w-plane.

Visualization of the four-dimensional surfaces of complex functions is sometimes difficult. *Computer graphics* is our greatest resource here [**4**], [**12**], [**45**]. Also, perception is greatly aided by examining projections in three dimensions instead of two, using *stereoscopic* techniques [**12**, Fig. 4], [**29**, Figs. 18–21], [**45**, pp. 132ff].

Today's computers are fast enough to make machine study of these complex surfaces feasible. A program written by the author in the 1960's required at that time *18 minutes of actual CPU time* to produce one four-dimensional graph. The *same program*, recently dug out of the files and submitted to our present computer, ran in *3.49 seconds*!



The Exponex $w=e^z$ ($u=e^x\cos y$, $v=e^x\sin y$)
Coincident axes, with $s=\frac{1}{2}$
Scan: $-3<x<3$, $0<y<2\pi$, $\Delta x=1$, $\Delta y=\pi/3$

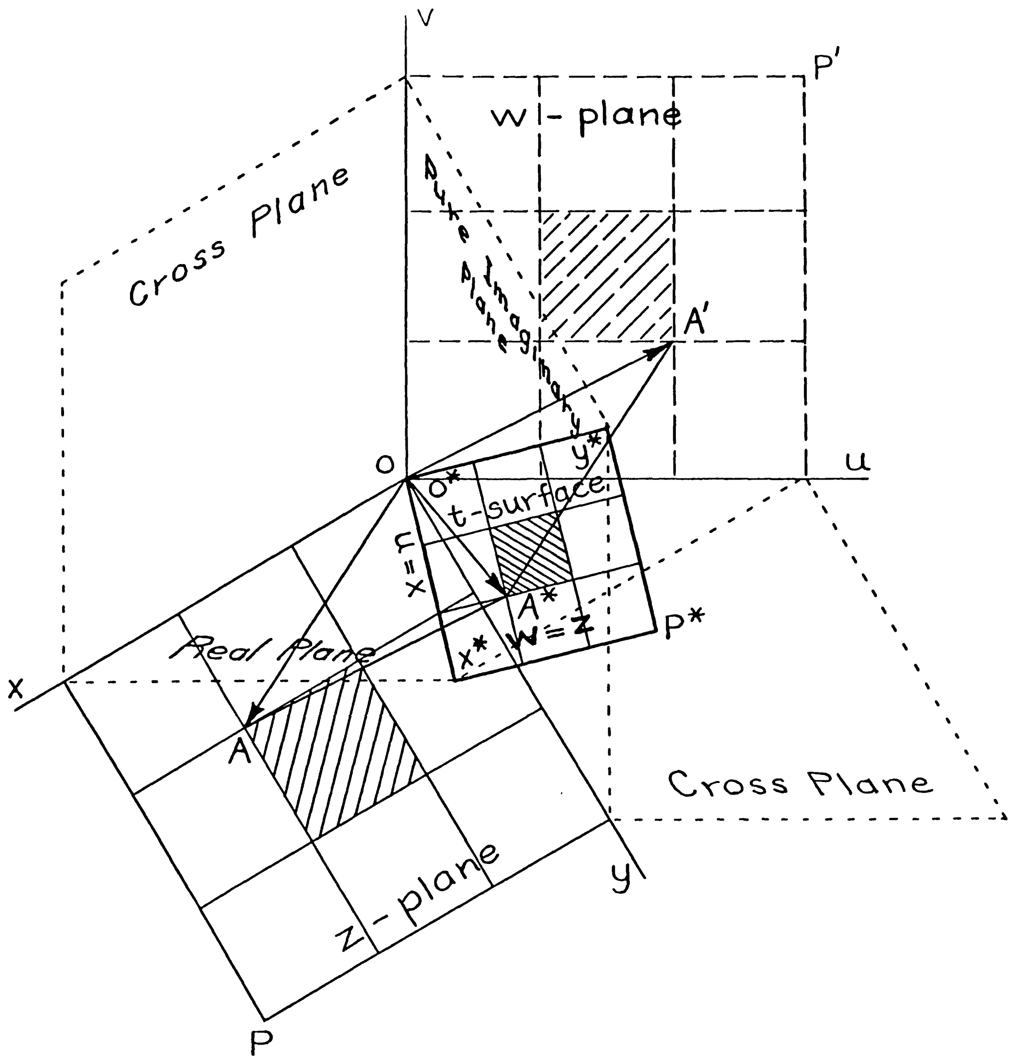
FIG. 6

With the employment of *interactive* computer graphics, in which the observer has real-time control over the display, these four-dimensional configurations (viewed either in three dimensions or two) can be manipulated dynamically, so that they are rotated, enlarged, "walked through," varied parametrically, projected onto various hyperplanes, or sectioned in any desired sequence **[18]**, **[30]**, **[41]**. The clearest insights into the geometry of these objects will no doubt come from this direction. Banchoff and Strauss **[3]**, **[4]** have used this technique to produce a number of computer-generated films of complex functions graphed in 4-space, which have been shown at several national meetings of the MAA and the AMS.

More prosaically, the *manual* computation and sketching of rotations and of traces on various planes and hyperplanes is highly informative, and is a standard procedure for "visualizing" 4-D objects **[5]**, **[26**, p. 18].

(A different but notable computer approach, not involving transformation surfaces, has been developed by McKeeman and Rousseau **[14**, p. 458] and by Piele, Firebaugh and Manulik **[33]**. In this, *real-time conformal mapping* is generated at the interactive graphics console. A full description is given by the latter team of investigators in the November 1977 issue of this MONTHLY. It is interesting that this method for illuminating complex functions is a computer-age realization of the *kinematic* interpretation of complex transformations described by Cole **[9]** in 1890.)

Often, for effective visualization of a transformation surface, we need only the four-dimensional graph (plotted either by computer or by hand) and the following three readily derived rules.

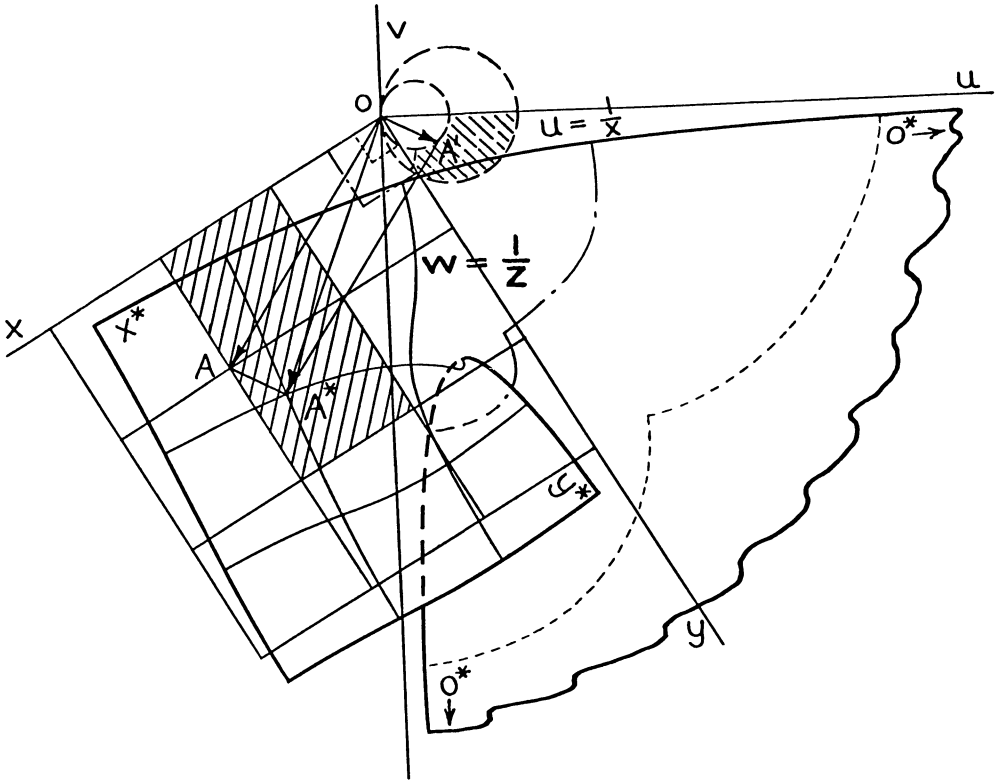


The Linex $w=z\ (u=x,\ v=y)$
Scan: $0 \leqslant x \leqslant 3$, $0 \leqslant y \leqslant 3$, $\Delta x = \Delta y = 1$

FIG. 7

First, keeping in mind that in our particular asymmetric oblique projection, all u and v values are plotted in planes parallel to the w-plane (the plane of the paper) from points in the z-plane (which extends toward the observer), we see that the "relative nearness" or "order of proximity" of any point of the transformation surface corresponds to the proximity of its object point in the z-plane to the observer stationed at the lower edge of the paper. *Second*, if a mirror point is farther than its object point from the bottom edge of the paper, it lies above the z-plane; if nearer, it lies below. *Third*, if v is negative, the mirror points are below both the real xu-plane and the z-plane. Similar rules can be applied for symmetric and coincident axes.

Thus, while computer displays, stereoscopic views, films, models, rotations, and traces are a considerable help with the more complicated figures, for many of the simpler loci the three rules



The Circlex-Hyperbolex $w=\frac{1}{z}\left(u=\frac{x}{x^2+y^2}, v=\frac{-y}{x^2+y^2}\right)$

Scan: $0\leq x\leq 3$, $0\leq y\leq 3$, $z\neq 0$, $\Delta x=\Delta y=1$

FIG. 8

above are sufficient to enable us to visualize the surface in three dimensions (and possibly in four?).

It is surprising, actually, how much insight can be gained from the construction and contemplation of a single four-dimensional graph of a complex function. For example, *the continuous single-surface character of a Riemann surface is clearly exhibited* in Figure 4 by the mirror surface of two sheets for the inverse complex parabola [**8**, p. 199]. *Embedded graphs of semi-complex functions* are also illustrated here, and in Figure 3. The close relationship through complex numbers of the circular and hyperbolic functions is plainly evident in Figure 5, where *the zeros of the trigonometric sine are hyperbolic sines* (also see [**21**]). The meaning of *periodicity* in the complex sine function and in the complex exponential function is readily apparent in Figures 5 and 6. And so on.

The linex. Figure 7 shows a portion of a *linex*, or four-dimensional graph of a *linear complex function*. The t-surface seen here is that for $w=z$ plotted over a 3×3 region in the first quadrant of the z-plane, starting at the origin. Although, in four dimensions, this mirror plane is square and twice the area of the z region, the three-dimensional projection is actually diamond-shaped, extending out from point O^* at the origin to point P^* nearest the observer, as our first rule above, or a stereoscopic projection, will readily verify. The two-dimensional graph happens to appear square again due to the particular type of projection we have employed (a *cavalier*

projection); however, the "apparent size" of the mirror surface is now considerably smaller than that of the four-dimensional original. On the other hand, if we plot a graph of $w = -z$, we find, due to our different *viewpoint* (plus the distortion of our cavalier projection), that the mirror plane appears much larger. This change in apparent size of an object with change in viewpoint is one of the more startling effects of a two-dimensional projection of a four-dimensional object. It is due to the resulting **double-foreshortening** or change of projected distance in two perpendicular directions, which thus produces a change in projected apparent area and, consequently, in our impression of the "size" of the object. Thus, with the change in our viewpoint as we either "move around" or "move past" a four-dimensional object, or as the object itself is rotated or translated in a fourth dimension, we will see it apparently change in size (perhaps even diminishing to zero), since a four-dimensional curve, surface, or solid is a *changing* three-dimensional curve, surface, or solid in this fourth direction [**5**], [**26**, p. 18].

Human experience has produced paired names for only the three directions of ordinary space, e.g., "up, down," "right, left," "toward, away"; we have no corresponding pair for this fourth direction. Since we need for this two words denoting opposite directions, but unattached to any of the three ordinary dimensions, we shall call this fourth direction **zig** and **zag**. Thus, in Figure 7, we consider the positive directions along the v, u, y, and x axes, respectively, as: "up," "right," "toward," and "zig."

If, now, we shift our viewpoint so that the z-plane appears rotated 30° clockwise and the four directions coincide with the four points of a compass, we find that the apparent size of the mirror plane for $w = z$ has reduced to zero. That is, we are seeing this plane "end-on"; thus points of our four-dimensional graphs *may* represent "end-views" of planes as well as of lines, or merely single points.

It is interesting to see the geometrical meaning of the Cauchy-Riemann equations in Figure 7 as the slope $\partial u/\partial x$ of the trace O^*X^* in the horizontal real plane equal to the slope $\partial v/\partial y$ of the trace O^*Y^* in the vertical pure imaginary plane, with $\partial u/\partial y = -\partial v/\partial x = 0$ at the origin. Equivalently, we note that the slope of the t-surface is oriented such that it fulfills the Cauchy-Riemann condition in its basic form: $\partial w/\partial x = \partial w/\partial iy$ [**2**].

The mapping of a typical region (cross-hatched) of the object plane onto the image plane by the mirror surface is indicated by the vectors.

Let us consider now the *general linear complex function*, $w = Mz + B$, M and B complex. The mirror surface of this linear function is always a four-dimensional plane, or "linex." When $M = B = 0$, then $w = 0$, and the mirror plane coincides with the object plane. Similarly, $z = 0$ is the w-plane. If $M = 0$, then $w = B$, a complex constant, and the mirror plane is now parallel to the z-plane, but translated, with the mirror origin now **hinged** at point B in the w-plane; that is, it intersects at its mirror origin the image plane in the single point B. This is the complex counterpart of the translation of a line to a parallel position by an additive constant. The mapping is correspondingly translated by this shift [**8**, p. 51].

(We noted in Figure 1 that two planes in four dimensions *may* have only a single point in common [**13**, p. 418]. We see this illustrated again in Figure 7, e.g., the origin, for the **complex planes** (the object and image or z- and w-planes), also for the **component planes** (the real and pure imaginary or r- and i-planes), and for the **cross planes** (the xv-cross and yu-cross or c- and k-planes). If, as in these three instances, the two planes are also perpendicular, they are said to be "absolutely perpendicular," and *every line* of one plane passing through the common point of intersection is perpendicular to *every line* of the other through this point [**26**, pp. 80–81]. The traces of a four-dimensional mirror surface $f(z,w) = 0$ on the *complex planes* will consist (unless there is complete coincidence) of only a finite number of isolated points at the most. Its traces on the *component and cross planes*, however, may be either null, isolated points, or plane curves, but not coincidence. The *cross plane traces* are, in fact, *Poncelet's supplementaries* [**10**, Chap. 3], [**11**, pp. 220–221], [**34**] (see Figs. 3 and 4). And since the simultaneous solution of two distinct functions of a complex variable always yields a disjoint set of complex number pairs, we see that

two distinct complex mirror surfaces are always "absolute" with respect to each other, i.e., they will intersect only in isolated complex points, never in lines or curves.)

When $B=0$ in the general linear equation, then $w=Mz$, and the mirror plane is "doubly rotated" [**26**, p. 145] about the mirror origin and expanded (or contracted) in size. It is interesting to note that when M is real and positive, although the mirror is rotated, any image map it "reflects" is not, since the rotation angle in the w-plane is then $\arg M=0$. However, mirror and map are both expanded. With negative or complex values of M, both mirror and map are rotated and expanded [**8**, p. 51].

Finally, if $M\neq 0$ and $B\neq 0$, then $w=Mz+B$ is represented hyperanalytically by a linex which has been rotated, expanded, and translated, and whose resultant mappings are correspondingly rotated, expanded, and translated. Also, we see that $z/A+w/B=1$ is a linex intersecting the z- and w-planes in the complex points A and B, respectively; and so on. The similarity to results in plane analytic geometry is obvious.

Using inner products and a generalized Euclidean distance formula (see Section 3), we can prove the following Pythagorean variation. (Consider the corresponding object, image, and mirror areas of an arbitrary rectangular element in the object plane bounded on two sides by orthogonal vectors):

THEOREM. *Under the general linear transformation, $w=Mz+B$, M and B complex, the area of any region of the mirror plane is the sum of the areas of the corresponding regions of the object and image planes.*

Specifically, the area of the *image* region is the area of the *object* region magnified by a factor of $|M|^2$, while the area of the *mirror* region is that of the object region magnified by $(1+|M|^2)$.

We note in the graphs of *analytic functions* (Figs. 3–8) that *the intersecting curves* defining the projected mirror grid intersections *are invariably perpendicular*. The *conformality theorem* of Section 1 explains the reason for this, and also accounts for the square appearance of the linex t-surface in Figure 7. In contrast, we discover that *projected mirror grids of non-analytic functions*, such as $\Re(z)$, $\Im(z)$, $\bar{z}$, $|z|$, and $\arg z$, *do not in general meet at right angles*. This significant difference under the same projection methods reinforces our earlier *conjecture* by again suggesting that conformality must be an intrinsic geometric property of the 4-space analytic mirror surfaces themselves.

The circlex-hyperbolex. Figure 8 shows the *complex hyperbola*, or **hyperbolex**, $w=1/z$ for a 3×3 scan of the first quadrant (origin deleted). The positive branch of the real equilateral hyperbola, $u=1/x$, is seen here as the trace O^*X^*, the mirror of the positive x-axis, lying in the xu-plane and curving from the right to the zig of us. The positive mirror y-axis, O^*Y^*, curving upward and toward us in the pure imaginary yv-plane, is also a branch of an equilateral hyperbola, $v=-1/y$. Since v is zero or negative throughout the first quadrant, the surface shown here, except for the real hyperbola trace, lies below both the xu- and the xy-planes, although this may be difficult to visualize.

This surface has a *singular point*, the origin. As the z values approach the origin from the first quadrant (indicated by the nested squares), the surface twists into a vortex away from the observer, then spreads out indefinitely downward and to the right (as shown by the mirrors of the nested squares), approaching asymptotically the fourth quadrant of the w-plane. Similar and complementary surfaces are obtained from the other three quadrants, the four regions being smoothly connected along the real and pure imaginary hyperbolic traces into a single surface, which spuriously passes through itself as it mushrooms out in all directions toward the singular mirror point of the origin at infinity.

In the complex analogy to the real hyperbola, the surface is symmetrical with respect to the origin and is asymptotic to both the z and w coordinate planes, changing, as z decreases, from a surface which is asymptotic to the first, second, third, and fourth quadrants of the z-plane into one asymptotic, respectively, to the fourth, third, second, and first quadrants of the w-plane.

A rectangular strip in the first quadrant of the z-plane parallel to the y-axis is mapped by the surface, as shown, into a fourth quadrant portion of the area between two circles in the w-plane passing through the origin and with centers on the u-axis.

There is a **prime circle** of this surface, the Poncelet supplementary of the real trace, with center at the origin and radius $\sqrt{2}$, which passes through the four vertices of the real and pure imaginary hyperbolas. (In Fig. 8, this circle, if shown, would appear *edge-on*.)

The three-dimensional traces of this hyperbolex are suggestive of intersecting hyperboloids of one sheet lying askew on their sides. When rotated to the "upright" position by the linear transformation $z=(z'+iw')/\sqrt{2}$, $w=(z'-iw')/\sqrt{2}$, the surface becomes that of the **circlex** $z'^2+w'^2=2$, the prime circle now being the real circle $x'^2+u'^2=2$ passing through the vertices of the absolutely perpendicular hyperbolic traces (Poncelet supplementaries) in the xv- and yu-cross planes. The complex asymptotic planes now make paired 45° traces on these cross planes.

As one might expect from the dual "real-imaginary" nature of complex numbers, the graph of a *general* complex function exhibits a *duality* with respect to the absolutely perpendicular real and pure imaginary planes. Thus, the *general circlex* $(z-H)^2+(w-K)^2=R^2$, where $H=h_1+ih_2$, $K=k_1+ik_2$, $R=r_1+ir_2$, boasts *two* prime circles, a real one and a pure imaginary one, about the *center of the translated circlex*: (h_1,h_2,k_1,k_2). The real circular trace, with center at (h_1,k_1) and radius r_1, lies in a plane parallel to the real plane and hinged at (h_2,k_2) in the pure imaginary plane. The pure imaginary circular trace, with center at (h_2,k_2) and radius r_2, lies in a plane parallel to the pure imaginary plane and hinged at (h_1,k_1) in the real plane.

The general *linear fractional* or *Möbius transformation* $w=(Az+B)/(Cz+D)$, A,B,C,D complex, with $C\neq 0$ and $AD-BC\neq 0$, can be written in the form of the successive transformations:

$$\tilde{w}=\frac{1}{\tilde{z}},\ \text{where}\ \tilde{w}=w-\frac{A}{C},\quad \text{and}\ \tilde{z}=\frac{-C(Cz+D)}{AD-BC}.$$

Since the first of these equations is the circlex-hyperbolex, and the other two merely translate and expand the coordinate planes, we see that the mirror surface of a Möbius transformation is that of a translated **ellipsex-hyperbolex**, having a singular point, asymptotic planes, elliptic and hyperbolic traces, and so on.

In 1813, while a prisoner of war in Russia, Poncelet made the interesting discovery that all circles, regardless of their size or their position in the (real) plane, pass through the same two imaginary "circular points at infinity" $(1,\pm i,0)$ [**42**, pp. 132-133]. We generalize this to include *all (complex) circles whose prime circles lie in, or in planes parallel to, the real and/or the pure imaginary planes*:

THEOREM (Poncelet). *All circlexes* $(z-H)^2+(w-K)^2=R^2$, H,K,R *complex, intersect the linex at infinity in the two imaginary circular points at infinity* $(1,\pm i,0)$.

We will examine the hyperanalytic meaning of this theorem among items in the next section.

3. The Hyperanalytic Model in Complex Geometry.

Modern geometers have long sought an adequate geometric model for a general theory of the imaginary [**1**, pp. 121-122], [**13**, p. 80]. Although Coolidge [**10**, pp. 103-104] disparages the four-dimensional approach, much of the present confusion in the complex geometry of C^2 can be readily cleared up by the following **two-point program**:

1. *Use complex forms for all unspecified constants and variables, whenever complex quantities are in any way involved. That is, replace literal constants, such as a, with their complex forms*: $A=a_1+ia_2$, *and replace the real variables x and y with the complex variables*: $z=x+iy$ *and* $w=u+iv$. (*The real variables are now x and u.*)

2. *Use the four-dimensional hyperanalytic model presented here to clarify the meaning of such things as complex curves and intersections, and to motivate proper definitions for the* ***geometric properties*** (*distance, angle, perpendicularity, etc.*) *of complex quantities.*

Such a program will accomplish several immediate results. It will rectify the previous situation where *most* points, "curves," and intersections could not be plotted or even visualized [7, Chap. 6], [**31**, p. 4]. It will disclose and disperse the fallacies underlying various "paradoxes" which have long been associated with complex geometry. And it will extend the concepts of distance and the Pythagorean theorem into the realm of complex quantities.

We have already seen in the preceding sections that use of the hyperanalytic model makes possible the plotting of complex curves as four-dimensional graphs, projected to 2-space, which are valid, complete, workable, graphically transforming, and even conformal. And we have glimpsed some of the myriad insights which flow naturally from such isomorphic representations.

In this section we will touch on other facets and implications of the hyperanalytic model in complex geometry. For example, the graphical meaning of the *complex roots of an equation* is clearly seen in these four-dimensional figures. Let us look at the translated parabolex $w-k=(z-h)^2$, h and k real and positive. The trace on the real xu-plane is the parabola $u-k=(x-h)^2$, which does not intersect the x-axis. That is, setting $u=0$ yields the quadratic equation $x^2-2xh+h^2+k=0$, which has the complex roots $x=h\pm i\sqrt{k}$.

However, our "two-point program" immediately exhibits these complex roots. Since x takes on complex values, we replace it with $z=x+iy$. Thus, we are working with the semi-complex function $u-k=(x+iy-h)^2$ in the xyu-hyperplane $v=0$. We see from this equation that if $u\geqslant k$, then $y=0$, and the graph of this portion of the function lies in the real xu-plane; while if $u\leqslant k$, then $x-h=0$, and the graph lies in the perpendicular plane $x=h$ parallel to the yu-cross plane. Using asymmetric axes, we sketch, in the horizontal xu-plane, the real parabolic trace $u-k=(x-h)^2$ through the vertex (h,k) and opening to the right $(u\geqslant k)$. In the perpendicular plane $x=h$, we sketch the "Poncelet supplementary" parabola $u-k=-y^2$ through the same vertex and opening to the left $(u\leqslant k)$. The intersection of this latter parabola with the xy-plane $u=0$ in the points $(h,\pm\sqrt{k}\,)$ depicts the complex roots of the quadratic equation above. These roots represent, of course, the intersection of the parent parabolex $w-k=(z-h)^2$ with the z-plane $w=0$, in the two complex points $z=h\pm i\sqrt{k}$. (See [**21**] for additional discussion.)

And the following "classic problem" [**1**, pp. 68, 121-122], [**13**, p. 80], [**31**, pp. 3-4] can now be resolved. We see, from a brief analysis and a quick freehand sketch, that the intersection of the circle $x^2+y^2=1$ (x and y real axes) with the line $x=2$ in the imaginary points $(2,\pm\sqrt{3}\,i)$ is simply the intersection in the xv-cross plane of the hyperbolic trace $x^2-v^2=1$ of the circlex $z^2+w^2=1$ and the linear trace $x=2$ of the linex $z=2$, the two traces crossing at the points $(2,0,0,\pm\sqrt{3}\,)$.

Further, we can easily answer questions such as these:

Question: How do we plot the imaginary circle $x^2+y^2=-r^2$?

Answer: Draw a centered circle of radius r in the pure imaginary plane. (The parent circlex is now "lying on its side.")

Question: The transverse axis connects the two vertices of the real hyperbola $x^2-y^2=1$; what points does the conjugate axis connect?

Answer: As a four-dimensional linex, it connects the vertices of the hyperbola $v^2-y^2=1$ (x and u are now the real axes) which is the pure imaginary trace of $z^2-w^2=1$, the parent hyperbolex of both hyperbolas. (Imaginary loci and intersections, in general, however, are not confined to the six coordinate planes.)

In addition, the geometrical properties of imaginary points, lines, and curves, such as given by Campbell [**7**, Chap. 6], become intuitively apparent from the corresponding hyperanalytic models.

Now let us examine some of the long-standing paradoxes of complex geometry: the "imaginary circular points at infinity," "minimal" or "isotropic lines," etc. As Klein [**24**, p. 119] points out, the imaginary circular points are expected to lie both a finite distance away (on the circle) and at infinity. While Veblen and Young [**43**, pp. 120ff], Graustein [**17**, Chap. 8], Campbell [**7**, pp. 181-183], Eves [**13**, 10.3], and others show that a so-called "minimal" line (a

line with slope i or $-i$) not only has the peculiar property for which it was named: that *any* two points lying on it are always *zero* distance apart, but also that the tangent of the angle it makes with *any* other (non-minimal) line is always the same (whence the name "isotropic"), and furthermore (except Graustein), it is "perpendicular to itself." Graustein [**17**, pp. 119-120] further "proves" that *every* point in a plane, other than the center of a given circle, lies on the circumference of that circle.

The basic error underlying all of these anomalies has been pointed out by some writers, notably by Coolidge [**10**, Preface and pp. 200-201], and briefly by Graustein [**17**, p. 120], Kempner [**22**, pp. 466-467], Campbell [**7**, p. 182], Tuller [**42**, p. 133], and others. It can be simply stated: *The real variable formulas involving* ***geometrical*** *concepts, such as distance, perpendicularity, etc., are not, in general, valid for complex quantities.* That is, we need to discover **the general complex geometrical laws** from which the real variable formulas proceed as *special cases*. As Coolidge [**10**, p. 6] expresses it: "There are more things in Heaven and Earth than are dreamt of in our philosophy of reals."

Thus, for example, from examination of our hyperanalytic model, we readily see that the (real) distance between two *complex* points is *not* given by the usual relationship

$$D=\sqrt{(z_2-z_1)^2+(w_2-w_1)^2}\,,$$

which involves a *circlex*, but instead should be written

$$D=\sqrt{|z_2-z_1|^2+|w_2-w_1|^2}$$
$$=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(u_2-u_1)^2+(v_2-v_1)^2}\,,$$

which is based on a *hypersphere*. In the special case when z and w are real, *both* of these definitions for D reduce to the real distance formula

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\,;$$

that is, these two surfaces pass through the same real plane circle. So, while reduction to the real variables formula is a *necessary* condition, it is *not sufficient*, and *generalization* here from the real to the complex case is *not uniquely defined.*

Further, if

$$M=m+in=\frac{w_2-w_1}{z_2-z_1}=\frac{dw}{dz}$$

is the **complex slope** of the linex $w=Mz+B$, we find that $M_1=-1/M_2$ is *not* the requirement for perpendicularity here. Instead, recalling that two real lines are perpendicular when the inner product of their direction numbers is zero, which in turn requires that their slopes be negative reciprocals, we note that *the vectors representing these slopes* are necessarily *collinear* (along the real axis) *but opposite in sense.*

In like fashion, for two complex lines to be perpendicular, a study of the hyperanalytic model shows us that a necessary condition for the inner product to be zero is that *the vectors representing their complex slopes be collinear but opposite in sense.* But on examining the model's Argand diagram basis, we find that this condition obtains only when one of the slopes and the *conjugate* of the other are negative reciprocals. Thus the correct complex formula for orthogonality should read $M_1=-1/\overline{M}_2$, from which we obtain $m_1=-m_2/(m_2^2+n_2^2)$ and $n_1=-n_2/(m_2^2+n_2^2)$. If either M_1 or M_2 is real, the other is also, since then $n_1=n_2=0$; only then do we have, as a special case, the familiar real variables formula: $m_1=-1/m_2$.

We call this complex orthogonality requirement, $M_1\overline{M}_2+1=0$, **the fundamental theorem of hyperanalytic geometry**, since it generates the basic difference between hyperanalytic and the other 4-D geometries. As indicated earlier, it has long been customary in complex geometry to extend the real variables formulas, such as $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ and $m_1m_2+1=0$, *without*

change to the realm of complex numbers [**13**, Def. 10.3.13, p. 85; Def. 10.3.19, p. 87], assumptions which produce the so-called "minimal" or "isotropic" lines with their unusual properties mentioned above. Here, we are proceeding instead from a generalized complex relationship, "the fundamental theorem," which reduces to the real variables formulas only as a special case.

The validity of these generalized distance and perpendicularity formulas for complex functions, *here obtained directly from the geometrical properties of our four-dimensional model as "perceived theorems"* [**12**], can be readily verified:

THE FUNDAMENTAL THEOREM OF HYPERANALYTIC GEOMETRY. *Two complex lines are perpendicular if and only if the slope of one and the conjugate of the slope of the other are negative reciprocals.*

Proof. Analytically, the two linexes $w=M_1z$ and $w=M_2z$, where $M_1=m_1+in_1$ and $M_2=m_2+in_2$, have direction numbers $(x_1,y_1,m_1x_1-n_1y_1,n_1x_1+m_1y_1)$ and $(x_2,y_2,m_2x_2-n_2y_2,n_2x_2+m_2y_2)$, respectively. If their slopes satisfy the fundamental theorem $M_1\overline{M}_2+1=0$, which can be written in the form $m_2n_1-n_2m_1=0$ (i.e., the vectors are collinear) and $m_1m_2+n_1n_2+1=0$ (i.e., the vectors are opposite), then the inner product of their direction numbers is identically zero, and therefore, analytically, the linexes are perpendicular. Q.E.D.

Geometrically, in the hyperanalytic model, the traces of $w=M_1z$ and $w=M_2z$, under the condition $M_1\overline{M}_2+1=0$, are two perpendicular lines in the real plane, as well as in the pure imaginary plane, and in each of the two cross planes. (The traces in the object and image planes are coincident points at the origin.) Thus, since we have two perpendicular lines in each of two absolutely perpendicular planes (the real and pure imaginary planes, also in the cross planes), we have (in both cases) four mutually perpendicular lines, which thus determine two absolutely perpendicular planes [**26**, pp. 80-81, 87]. But these lines lie respectively in the two linexes, $w=M_1z$ and $w=M_2z$. Therefore, geometrically also, these linexes are (absolutely) perpendicular. Q.E.D.

From the fundamental theorem, using $\mathfrak{R}(M_1\overline{M}_2+1)=0$, we obtain the hyperanalytic real distance formula $D=\sqrt{|z_2-z_1|^2+|w_2-w_1|^2}$ discussed above, and the *real Pythagorean theorem* for (real) distance. These involve the *absolute value* of z_k-z_j [and of w_k-w_j], or the *real distance* from z_j to z_k, which can be written as $\sqrt{(z_k-z_j)\overline{(z_k-z_j)}}$.

Similarly, using $\mathfrak{I}(M_1\overline{M}_2+1)=0$, we obtain a **pure imaginary Pythagorean theorem** for "paradistance," involving the (complex) **paravalue** of z_k-z_j [and w_k-w_j], which we define as $\sqrt{(z_k-z_j)\overline{(z_k+z_j)}}$. This **paradistance from** z_j **to** z_k is intimately related to several geometrical properties of the diagonals and the area of what we term the "**position parellelogram** of vector $\overrightarrow{z_k-z_j}$," that is, the parallelogram having position vectors $\vec{z}_j$ and $\vec{z}_k$ as two of its sides. However, the paradistance is not Euclidean metric and is not invariant under translation.

We note the desirable result that our orthogonality and real distance definitions are equivalent to the inner product and Hermitian norm established for a unitary space of complex vectors [**20**, pp. 190-195]. Alternatively, then, we could describe the hyperanalytic model as **a graphical representation of unitary 2-space.**

Now, with these generalized complex definitions and the hyperanalytic model in mind, let us first examine the last paradox listed, that of Graustein's circle. The error in the "proof" becomes obvious: Imaginary intersections with circles do not lie on the circumference of the circle. They lie, instead, on the parent circlex, not at a distance r from the center, but at a distance D given by the formula above involving *absolute values* of the complex quantities. Only the points on the real circular trace of a circlex are at the (minimum) distance of the radius from the center—recall the hyperboloid-like shape of a circlex. Again, the basic distinction here is that, even though the trace of each in the real plane is a circle, a circlex is not the same surface as the locus of the generalized distance formula, a hypersphere.

Next we examine the improperly named "minimal lines." From the definition of *complex* perpendicularity above, we see immediately that $w = iz + B$ is no longer required to be "perpendicular to itself." In fact, consider the pair of lines $w = \pm iz$: our definition tells us that instead of each being "self-perpendicular," these minimal lines are really *mutually* perpendicular, and represent a degenerate circlex-hyperbolex, $z^2 + w^2 = 0$, in the same manner that the perpendicular asymptotic lines $y = \pm x$ can be considered a degenerate hyperbola. A pair of minimal lines through a point thus consists of the two absolutely perpendicular four-dimensional planes (linexes) asymptotic to all circlexes centered at that point. Furthermore, the revised distance formula now gives a proper *non-zero* value for the distance between two given distinct points on a minimal line (linex). It is conjectured that with the development of a valid expression for the *relative slope* between two linexes, it can be shown that the name "isotropic" is also inappropriate for these lines; that in fact they possess no embarrassing peculiarities, but instead are just as ordinary as any other complex lines we might examine. Their special role in the formulation of complex geometric theory derives not from any difference in intrinsic geometrical properties, but in their connection with the circular points at infinity, which are invariant under similarity transformations [**1**, pp. 145-148].

(The relation between *slope* and *angle* in complex hyperanalytic geometry needs further investigation. A **complex angle** represents the *double rotation* [**26**, pp. 142-145] in 4-space of a linex through the origin as it unfolds from the object plane toward the image plane. It can be treated as the resultant of two orthogonal component rotations about a pair of absolutely perpendicular planes as *axis-planes*, such as the *complex, component*, or *cross planes*. The complete hyperanalytical relationships remain to be worked out.)

Now for a look at the "imaginary circular points at infinity." First let us resolve Klein's dilemma. These points are to lie at infinity, but they are *not*, as Klein supposed, also required to be at the finite distance of the radius away, since they are only required to be on the circlex, not on the real circle. The circlex, we have seen, extends to infinity, so there is no contradiction here, and the distance formula, as extended above to complex quantities, *no longer yields an indeterminate expression*, as espoused by Klein. At infinity, we consider that the circlex intersects its two asymptotic linexes, the "minimal lines," each in a single infinite point, the surfaces being relatively absolute. These are the two imaginary *circular points at infinity* [**36**].

Since all "parallel" circlexes centered about a *given* point in the real plane, regardless of their size, meet the common pair of complex asymptotes through that point in the same two circular points at infinity, and since all such pairs of asymptotes, or "minimal lines," one pair through each point of the real plane, form two orthogonal pencils of absolutely parallel complex lines, and all parallel linexes in a pencil meet in the same infinite point, we see that all (complex) circles, regardless of the size or position of their (circular) trace in the real plane, pass through these two imaginary circular points at infinity: $(1, \pm i, 0)$ on the infinite linex L_∞. The argument can be extended to all circular traces in planes parallel to the real or to the pure imaginary plane.

Perhaps the reader can now find an explanation for the paradoxical result given by the Cayley-Klein definition [**1**, p. 162]: that *the projective distance from the center to the circumference of a circle is infinite.*

Of course, in applying the hyperanalytic model to resolve some of the paradoxes of complex geometry, we may have opened a Pandora's box, since we then need to re-examine *all* extensions of real variable *geometrical* concepts to complex quantities. Thus, Klein's Erlanger Programm [**13**, Chap. 11] may bear re-study, since some *complex geometric properties* may not prove to be invariant under certain transformations which leave the corresponding real properties unchanged.

If, further, we negate the similar claims to pecularity of the isotropics in the other metrical geometries, we give up, in special relativity theory, the geometrical explanation for the constant velocity of light in a Minkowski 4-space [**31**, pp. 293-295], [**40**, pp. 7-11].

However, we must be careful not to tear down or neglect the elegant structures of the established geometries, so painstakingly built up over the years [**11**]. Even if based on the inadequate notion that, except for order, complex numbers should behave *geometrically* in all respects as real numbers, these traditional geometries will continue to serve as rich and useful models in many fields. But, on the other hand, this should not deter us from the exciting prospect of building a new and interestingly different model for complex geometry, starting with a hyperanalytic geometry of four dimensions.

Note added in proof: Regarding **paradistance**, it can be shown that (1) *the absolute value of the square of the paradistance is the product of the diagonal lengths of the related position parallelogram*, and (2) *the amplitude of the square is the angle between these diagonals*; (3) *the paravalue "rectifies" the position parallelogram, its vector being the diagonal of a rectangle with real and imaginary sides whose area equals that of the parallelogram*, and (4) *this area is also equal to one-half the imaginary component of the square of the paradistance*; (5) *if the position parallelogram is rotated so that its diagonal extending from the origin lies along the positive real axis, the original* (*and the rotated*) *paradistance is the* ***geometric mean vector*** *of the rotated diagonal vectors*.

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A "NATURAL" ENUMERATION OF NON-NEGATIVE RATIONAL NUMBERS—AN INFORMAL DISCUSSION

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Among the infinitely many definable enumerations of the non-negative rational numbers, the following seems to be a particularly "natural" one. It is also quite "primitive" and very rich in interesting mathematical consequences. Let

$$\gamma_0=0, \qquad \gamma_{2k}=\gamma_k+1 \quad \text{for } k>0, \qquad \gamma_{2k+1}=1/(\gamma_k+1).$$

An equally "natural" enumeration of the rational numbers between 0 and 1, including 1, is given by $\rho_k=\gamma_{2k+1}$.

Alternatively, one may also define (the function) ρ thus: $\rho_0=1$, $\rho_{2k}=\rho_k/(\rho_k+1)$ for $k>0$, $\rho_{2k+1}=1/(\rho_k+1)$. Then one may define (the function) γ in terms of ρ: $\gamma_0=0$, $\gamma_{2k}=1/\rho_k$ for $k>0$, $\gamma_{2k+1}=\rho_k$.

A fundamental relation between γ and ρ is

$$\gamma_k+1=1/\rho_k. \tag{A}$$

This makes possible alternative definitions of γ and ρ in terms of each other.

As regards order of magnitude, if $\gamma_k<\gamma_l$, then $\rho_k>\rho_l$. For example, corresponding to

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$$\gamma_0<\gamma_5<\gamma_3<\gamma_7<\gamma_1<\gamma_6<\gamma_2<\gamma_4$$

we have

$$\rho_0>\rho_5>\rho_3>\rho_7>\rho_1>\rho_6>\rho_2>\rho_4.$$

Numerically,

$$0<\frac{1}{3}<\frac{1}{2}<\frac{2}{3}<1<\frac{3}{2}<2<3,$$

$$1>\frac{3}{4}>\frac{2}{3}>\frac{3}{5}>\frac{1}{2}>\frac{2}{5}>\frac{1}{3}>\frac{1}{4}.$$

Since in our first definition of γ the relation $\gamma_{2k}=\gamma_k+1$ is true for all positive k and not true for $k=0$, it may be thought that the enumerations γ and ρ are not "perfectly natural." However, the following considerations will convince us that they are.

We effect an enumeration of all finite sets of natural numbers in the obvious way:

$$M_0=\text{the null set},$$

$$M_{2^k}=M_0+\{k\},M_{2^k+1}=M_1+\{k\},\ldots,M^1_{2^{k+1}-1}=M_{2^k-1}+\{k\}.$$

Observe that when $k=0$, all the statements after the first become a single statement since $2^{0+1}-1=2^0$. Similarly when $k=1$, they become two statements. And so on. Then we define an enumeration S of all finite sequences of positive integers in such a way that if

$$S_k=(a_0,a_1,\ldots,a_{l-1}),$$

then

$$M_k=\{-1+a_0,-1+a_0+a_1,\ldots,-1+a_0+a_1+\cdots+a_{l-1}\}.$$

Thus if S_k has l terms, M_k has l members.

Now the following relation holds without exception: If

$$S_k=(a_0,a_1,\ldots,a_{l-1}),$$

then

$$\rho_k=\cfrac{1}{a_0+\cfrac{1}{a_1+\cfrac{}{\ddots+\cfrac{1}{a_{l-1}+1}}}}.$$

As we see, it results naturally from this, without special agreement, that $\rho_0=1$ corresponds to the null sequence S_0 (which in turn corresponds to the null set M_0). Also, no separate formulas of correspondence are needed for odd and even subscripts. Thus ρ is seen to be "perfectly natural," and it follows from (A) that γ is "perfectly natural" too.

We have proposed very natural one-to-one correspondences between any two of the following totalities:

(1) all finite sets of natural numbers,

(2) all finite sequences of positive integers,

(3) all positive rational numbers not exceeding unity,

(4) all non-negative rational numbers.

By a similar method we can also establish very natural one-to-one correspondences between any two of the following non-denumerable totalities:

(1′) all infinite sets of natural numbers,

(2′) all ω-sequences of positive integers,

(3′) all positive irrational numbers less than unity,

(4′) all positive irrational numbers.

Thus to the ω-sequence of positive integers $(a_0, a_1, a_2, \ldots)$ correspond the infinite set of natural numbers $\{-1+a_0, -1+a_0+a_1, -1+a_0+a_1+a_2, \ldots\}$, the positive irrational number less than unity

$$\cfrac{1}{a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\ddots}}}$$

and by a relation analogous to (A), the positive irrational number

$$-1+a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\ddots}}$$

As an interesting example, let us note that to the ω-sequence $(1,1,1,\ldots)$ correspond the set of all natural numbers $\{0,1,2,\ldots\}$, the positive irrational number less than unity $\frac{1}{2}(-1+\sqrt{5})$, and the positive irrational number

$$\frac{2}{-1+\sqrt{5}}-1=\frac{-1+\sqrt{5}}{2}.$$

In this particular example the corresponding elements of (3′) and (4′) coincide.

It is a curious fact that there does not seem to exist an enumeration of all rational numbers as natural as the enumeration we have given of *non-negative* rational numbers. Any enumeration of all rational numbers seems bound to be artificial. We shall first give one such enumeration γ^*, which, though rather "unnatural," establishes a loose analogy between the passage from ρ to γ and that from γ to γ^*. Here γ_0^* is left undefined. We define:

$$\gamma_{2k}^* = \gamma_k \quad \text{for} \quad k>0, \qquad \gamma_{2k+1}^* = -\gamma_k.$$

Fundamental relations between γ and γ^* are:

$$1+\gamma_{2k}^* = \gamma_{2k} \quad \text{for} \quad k>0; \qquad 1-\gamma_{2k+1}^* = \frac{1}{\gamma_{2k+1}}. \tag{B}$$

This makes possible an alternative definition of γ^* in terms of γ.

γ_0^*, which we left undefined, may conveniently be set equal to an entity called absolute infinity, $1/0=\infty$.

If we read ∞ as $-\infty$, we obtain the following result as regards order of magnitude: If $\gamma_k<\gamma_l$, then $\gamma_k^*<\gamma_l^*$. For example, corresponding to

$$\gamma_0<\gamma_5<\gamma_3<\gamma_7<\gamma_1<\gamma_6<\gamma_2<\gamma_4$$

we have

$$\gamma_0^*<\gamma_5^*<\gamma_3^*<\gamma_7^*<\gamma_1^*<\gamma_6^*<\gamma_2^*<\gamma_4^*;$$

numerically,

$$-\infty<-2<-1<-\frac{1}{2}<0<\frac{1}{2}<1<2.$$

We have defined one-to-one correspondences not only between any two of the totalities (1), (2), (3), (4), but also, by (B), between any one of these and the totality

(5) all rational numbers, including ∞ or $1/0$ among the rational numbers.

Likewise, we have established one-to-one correspondences not only between any two of the non-denumerable totalities (1′), (2′), (3′), (4′), but also, by relations analogous to (B), between any one of these and the non-denumerable totality

(5′) all irrational numbers.

The relation analogous to the first or the second formula of (B) is used according as the element of (4′) is greater or less than unity.

To complete our previous example, the element of (5′) which corresponds to the set of all natural numbers (an element of (1′)) is $\frac{1}{2}(1-\sqrt{5})$, while $\frac{1}{2}(-1+\sqrt{5})$, which as an element of (3′) or (4′) corresponds to the set of all natural numbers, as an element of (5′) corresponds now to the set of all positive integers (also as an element of (1′)).

A further interesting step: Let us abolish absolute infinity or γ_0^*, and equate γ_{k+1}^* with M_k. Then, curiously enough, every natural number n is now equated with the set $\{0,1,\ldots,n-1\}$, as is usually the convention in set theory; 0 is of course equated with the null set.

It is desirable to have a similar shift in the infinite case. But we shall not dwell on this now.

A more "natural" way to extend the method by which we defined γ_k and ρ_k to all rational numbers is as follows: Let n be any integer and k any non-negative integer (i.e., natural number). We assume that γ_k has been defined as before. Let

$$\gamma_k^0=\gamma_k,$$

and let

$$\gamma_k^{n+1}=\frac{1}{\gamma_k^n+1}. \tag{C}$$

It follows that

$$\gamma_k^{n-1}=\frac{1}{\gamma_k^n}-1,$$

and this formula may be needed in proceeding in the negative direction. From the first definition of γ_k we have

$$\gamma_{2k+1}^0=\frac{1}{\gamma_k^0+1}.$$

From this, together with the above statements, it can be proved in general that

$$\gamma_{2k+1}^{n} = \frac{1}{\gamma_k^n + 1}.$$

Applying (C), we have

$$\gamma_k^{n+1} = \gamma_{2k+1}^{n}. \tag{D}$$

The first definition of ρ_k yields $\gamma_k^1 = \rho_k$. We define:

$$v_k^{n+1} = \gamma_{2k}^{n}. \tag{E}$$

Now it can be proved that every rational number can be written in the form v_k^n, where n and k are uniquely determined; also that if n and k are not both 0, v_k^n is always a rational number. We set $v_0^0 = \infty$. Applying (E), we have $\gamma_0^{-1} = \infty$; applying (D), we have

$$\gamma_1^{-2} = \infty, \quad \gamma_3^{-3} = \infty, \quad \ldots.$$

In general, $\gamma_{2^k-1}^{-k-1} = \infty$. The expression γ_k^0, as stated above, ranges through all non-negative rational numbers as k varies; but there exists no n such that γ_k^n (for fixed n) embraces all rational numbers as values. Finally we define:

$$w_k^1 = \gamma_k^1, \tag{F}$$

$$w_k^{n+1} = w_k^n + 1. \tag{G}$$

It follows that $w_k^{n-1} = w_k^n - 1$. One easily shows that every rational number (not including ∞) can be written in the form w_k^n, where n and k are uniquely determined; also that w_k^n is always a rational number. It is interesting to note that

$$w_k^0 = v_k^0 \quad \text{for } k > 0. \tag{H}$$

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THE EIGHTH U.S.A. MATHEMATICAL OLYMPIAD

SAMUEL L. GREITZER

The Eighth U.S.A. Mathematical Olympiad (USAMO) took place on May 1, 1979. Participation is by invitation only and is based almost entirely on the student's performance on the Annual High School Mathematics Examination (HSME). In 1979, the HSME was given on March 6, and all participants who had scored 118 points or better were invited to take the Olympiad. Of the 101 students invited, 3 did not take the Olympiad for various reasons, and this article concerns itself with the 98 who did participate.

The Olympiad is also the basis for the selection of a team of eight students to represent the United States at the International Mathematical Olympiad (IMO), held in England this time. With at most three weeks to go between the Olympiad and the selection of a team and of students for a three-week training program, time was indeed at a premium. We allowed ten days for the completed examinations to reach us, and the papers were graded on May 11.

Our heartfelt gratitude is due those Rutgers faculty members who voluntarily gave of their

time and effort and graded the papers. The committee consisted of Professors Michael Aissen, John Bender, Richard Bumby, Solomon Leader, Benjamin Muckenhoupt, and Barbara Osofsky. Professor Murray Klamkin came to Rutgers from the University of Alberta to oversee. He also reread more than half of the papers on Friday night and Saturday morning. After a few hours on the telephone, we had notified our team members and also selected the students for the training session. This year, the U.S. Military Academy at West Point was our host.

In previous reports, I have tried my hand at statistical procedures, but since these actually have little validity, they will be omitted this time. However, I think some data are sufficiently significant to be presented. The problems themselves will be found at the end of this article. The tables should be examined in conjunction with the problems.

Problem	Average Score
1	8.2
2	2.8
3	4.2
4	3.1
5	3.7

Table 1

Problem	Score		
	0	1–10	11–20
1	34	26	38
2	81	3	14
3	52	35	11
4	71	17	10
5	31	59	8

Table 2

Score	No
80–99	2
60–79	5
40–59	13
20–39	26
0–19	52

Table 3

It is no surprise to find that Problems 2 and 4, being geometric in content, were found the most difficult. What is surprising is that students failed to make use of the substitutes for geometry they had received in the classroom. Problem 2 could have been done using transformation geometry, which is a popular substitute for classical geometry. Problem 4 could even have been done using calculus, and reaching calculus in the lower and intermediate grades is the goal of many educators.

Table 2 lists the number of students who scored zero in any problem. We refrain from moralizing on the results, and hope only that secondary school teachers will draw a lesson from this table and from Table 3.

The students did best on Problem 1, both because it was the first and because it required very little modular arithmetic. They did poorly on the last problem because it was last and they were pressed for time.

We have been told that the mathematical content in the USAMO is not "relevant." Without going into a discussion of the nature of relevance, we can point out that the content is certainly relevant as far as the IMO is concerned, that at least twenty nations participate, that these nations also have educators, and that these educators consider the content relevant. I merely call this to the attention of our own mathematics educators.

Addressing ourselves to the question of whether the Olympiad is doing what it was developed to do—namely, to discover and encourage students gifted in mathematics—we present the following data from the William Lowell Putnam Competition held on December 2, 1978:

a. The Putnam Fellows—the five highest-ranking individuals—were all members of U.S.A. teams taking part in previous International Olympiads.
b. Three of the next five highest-ranking individuals have been team members in previous International Olympiads.
c. Two of the three members of the team in first place—Case Western Reserve University —are former Olympiad team members. The third also took the U.S.A. Olympiad.
d. The lists of Honorable Mentions, both individual and team, are liberally peppered with names of students who either were Olympiad team members or took previous Olympiads.

I believe that the Olympiad's objectives are being attained.

On Monday, June 4, the eight finalists and their parents met at the new quarters of the MAA for an informal collation and to receive awards for their efforts. The Association presented books and a membership to each student, NCTM gave a briefcase and books, and there were books donated by Springer-Verlag and by Wiley, as well as by Mu Alpha Theta. IBM presented each student with a silver tray, and Hewlett-Packard gave each one an HP-67 calculator. On Tuesday, June 5, all met at the National Academy of Sciences, where we heard an excellent lecture from Professor Charles Fefferman, and proceeded thence to the State Department for a formal dinner, thanks to the generosity of IBM. On June 6, the students made their way to West Point to begin three weeks of preparation for the International Mathematical Olympiad. A report on that event is in preparation.

EIGHTH U.S.A. MATHEMATICAL OLYMPIAD—May 1, 1979

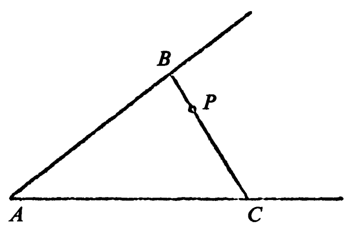
1. Determine all non-negative integral solutions $(n_1, n_2, \ldots, n_{14})$, if any, apart from permutations, of the Diophantine equation

$$n_1^4 + n_2^4 + \cdots + n_{14}^4 = 1{,}599.$$

2. A *great circle* $\mathfrak{S}$ on a sphere is one whose center is the center O of the sphere. A *pole* P of the great circle $\mathfrak{S}$ is a point on the sphere such that OP is perpendicular to the plane of $\mathfrak{S}$. On any great circle through P, two points, A and B, are chosen equidistant from P. For any *spherical triangle* ABC (the sides are great circle arcs) where C is on $\mathfrak{S}$, prove that the great circle arc CP is the angle bisector of angle C.

3. Given three identical n-faced dice whose corresponding faces are identically numbered with arbitrary integers, prove that if they are tossed at random the probability that the sum of the top three face numbers is divisible by three is greater or equal to $1/4$.

4. Show how to construct a chord BPC of a given angle A through a given point P within the angle A such that $1/BP + 1/PC$ is a maximum.



5. A certain organization has n members ($n \geqslant 5$) and it has $n+1$ three-member committees, no two of which have identical membership. Prove that there are two committees which share *exactly one* member.

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CORRECTION AND ADDENDUM TO

"Evolution of the Topological Concept of 'Connected'"

(THIS MONTHLY, 85 (1978) 720–726)

R. L. WILDER

On page 724 of my article "Evolution of the Topological Concept of 'Connected'" delete the second paragraph (beginning "If Riesz..."). That the identification mentioned in this paragraph

was indeed made by Riesz (see [**16**, p. 321]) has been called to my attention by Professor W. J. Thron.

Also, Professor C. E. Aull has informed me that Professor Thron mentioned Riesz's contribution to the theory of connectedness in his book *Topological Structures*, Holt, Rinehart & Winston, New York, 1966, pages 29–30.

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MATHEMATICAL NOTES

EDITED BY DEBORAH TEPPER HAIMO AND FRANKLIN TEPPER HAIMO

Material for this department should be sent to Professor Deborah Tepper Haimo, Department of Mathematical Sciences, University of Missouri, St. Louis, MO 63121.

Advice to prospective authors: The editors have recently been receiving about **ten times** as many Mathematical Notes as can be used. It will simplify our work if authors will submit only especially interesting manuscripts. Mathematical Notes should be short papers of one to four printed pages which give new insights, new and improved proofs of old theorems, brief bits of mathematical folklore that have not found a home in the literature, or (occasionally!) new results that are not too technical. The topics should be of wide current interest. Papers that have already been rejected by a research journal are only very rarely suitable as Mathematical Notes.

R.P.B.

A PROBLEM OF THE POMPEIU TYPE

K. W. THOMPSON AND T. SCHONBEK

In this article we study the following problem posed by L. Zalcman. Let S be the closed unit square. For each z in the interior S^0 of S, let $S(z)$ be the largest closed square in S with centroid z. Let $0<\alpha\leqslant 1$. For each z in S^0 let $S_\alpha(z)$ be the square homothetic to $S(z)$ with linear ratio α. If f is a continuous function on S such that the integral of f over $S_\alpha(z)$ vanishes for all z in S^0, is $f\equiv 0$?

Zalcman observed that the answer is yes in the cases $\alpha=1$ and $\alpha=\frac{1}{3}$.

We were drawn to this problem by an expository article by C. A. Berenstein on the Pompeiu problem ([**1**]). (For a discussion of the Pompeiu problem and an exhaustive bibliography, see [**3**].) In this article, Berenstein wonders about the answer to Zalcman's question if $\alpha=\frac{1}{2}$.

We show that the question has an affirmative answer whenever $\alpha=n/(n+2)$, n a positive integer; and in particular if $\alpha=\frac{1}{3}$ or $\alpha=\frac{1}{2}$.

The case $\alpha=1$ appeared as Problem A-6 in the 1977 Putnam Competition. Since our methods are a generalization of a method which works in this case, we begin by presenting a solution to the Putnam problem.

Let $(a,b)\in S$, and consider the rectangle of vertices (0,0), $(a,0)$, (a,b), $(0,b)$.

Figure 1 indicates how this rectangle can be paved by squares of the type $S(z)$. What one has at first, and what possibly remains unpaved after each "tile" is placed, is a rectangle having (0,0) as a vertex and two sides contained in the boundary of S. The smallest of the remaining sides is then used as the side for the next tile.

It is easy to see that all points of the original rectangle are eventually covered (in fact, if a/b is rational, the number of "tiles" is finite); hence we conclude that $\int_0^a\int_0^b f(x,y)\,dx\,dy=0$ for all

$a,b\in S$. Since $\int_\beta^b\int_\alpha^a=\int_0^b\int_0^a+\int_0^\beta\int_0^\alpha-\int_0^b\int_0^\alpha-\int_0^\beta\int_0^a$, it follows that the integral of f vanishes over each rectangle $[\alpha,a]\times[\beta,b]\subseteq S$. If $f(x,y)\neq 0$ for some $(x,y)\in S$, say $f(x,y)>0$, then there exist α,β,a,b such that $\alpha<a,\beta<b,(x,y)\in[\alpha,a]\times[\beta,b]\subseteq S$ and $f(x',y')>0$ for $(x',y')\in[\alpha,a]\times[\beta,b]$, contradicting $\int_\alpha^a\int_\beta^b f(x,y)\,dx\,dy=0$. This proves $f\equiv 0$.

Let now $\alpha=n/(n+2)$, n a positive integer, and assume f is a continuous function on S having zero integral over $S_\alpha(z)$ for all $z\in S^0$.

To prove $f\equiv 0$, we begin by observing that, if $0<b\leqslant a\leqslant\frac{1}{2}$, then (a,b) is the upper right-hand vertex of $S_\alpha(z)$, where

$$z=\left(a-\frac{n}{2(n+1)}b,\frac{n+2}{2(n+1)}b\right).$$

Set $S^1=S_\alpha(z)$. Partitioning the lower side of S^1 into $n+1$ intervals of equal length, we observe that each one of these intervals is the upper side of a square of type $S_\alpha(z)$; let us denote by $S_1^2,\dots,S_{n+1}^2$ these squares. Proceeding in this fashion we can pave the rectangle of vertices (a,b), $(a-nb/(n+1),b)$, $(a-nb/(n+1),0)$, $(a,0)$ (i.e., the rectangle with one side in the x-axis, and one side coinciding with the upper side of S^1) completely by squares of the type $S_\alpha(z)$. At the kth step we are adding $(n+1)^k$ new squares, and what remains unpaved is a rectangle of vertical side of length $b/(n+1)^k$. See Figure 2 where this process is illustrated in case $n=2$ (and $\alpha=\frac{1}{2}$).

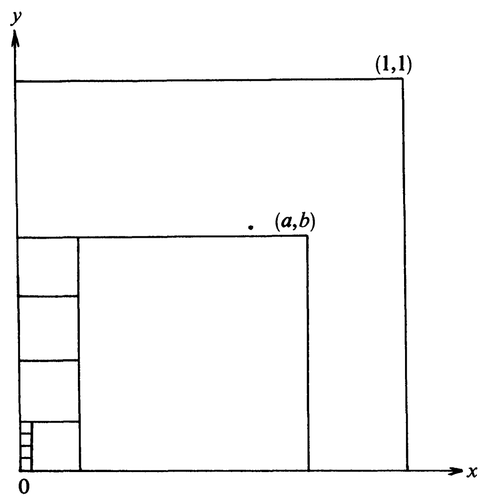

FIG. 1

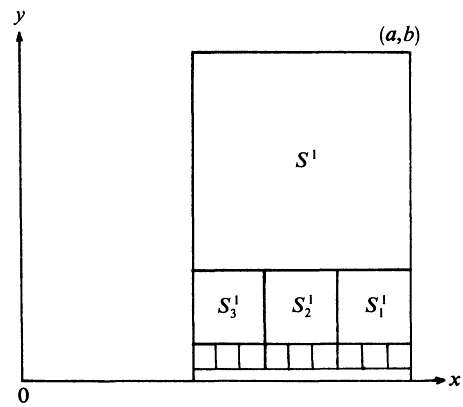

FIG. 2

This proves

$$\int_{a-\frac{nb}{n+1}}^{a}\int_0^b f(x,y)\,dx\,dy=0\quad\text{if}\quad 0<b\leqslant a\leqslant\frac{1}{2}. \tag{1}$$

For reasons of symmetry, we also have

$$\int_0^c\int_{d-\frac{nc}{n+1}}^{d} f(x,y)\,dx\,dy=0\quad\text{if}\quad 0<c\leqslant d\leqslant\frac{1}{2}. \tag{2}$$

If $0<b\leqslant a\leqslant\frac{1}{2}$, we denote by $R(a,b)$ the rectangle of vertices (a,b), $(a-nb/(n+1),b)$, $(a-nb/(n+1),0)$, $(a,0)$. If $0<c\leqslant d\leqslant\frac{1}{2}$, let $Q(c,d)$ be the rectangle of vertices (c,d), $(0,d)$, $(0,d-nc/(n+1))$, $(c,d-nc/(n+1))$.

Let $(a,b)\in S, a\leqslant\frac{1}{2}, b\leqslant\frac{1}{2}$. Figure 3 shows how the rectangle of vertices (a,b), $(0,b)$, $(0,0)$, $(a,0)$

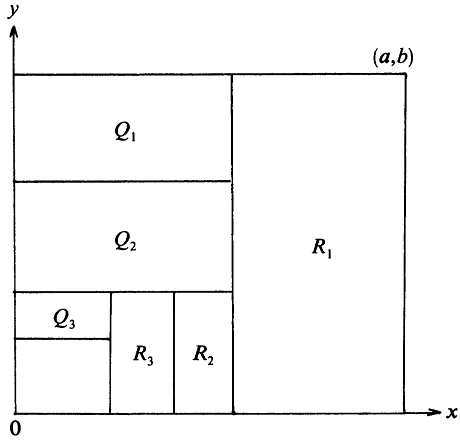

FIG. 3

can be paved by rectangles of the type R and Q. If $b \leqslant a$, take $R(a,b)$ for the first "tile"; otherwise take $Q(a,b)$.

At each step what remains unpaved is a rectangle of vertices (0,0), $(x,0)$, (x,y), $(0,y)$ with $0 \leqslant x,y \leqslant \frac{1}{2}$. If $y \leqslant x$, the next tile is $R(x,y)$; otherwise it is $Q(x,y)$.

Furthermore, the unpaved area converges to 0. For example, if $0<b \leqslant a$, we start with tiles of type R, switch to type Q after using k tiles of type R, where k is the first integer such that $a-nkb/(n+1)<b$. At that stage, the upper right-hand vertex of the unpaved region is $(a-nkb/(n+1),b)$ giving an area of

$$A=ab-\frac{nk}{n+1}b^2.$$

Now $k \geqslant 1$; hence we have

$$a<b+\frac{nk}{n+1}b \leqslant \frac{2nk}{n+1}b+\frac{nk}{n+1}b=\frac{3nk}{n+1}b.$$

Thus

$$3A=3ab-\frac{3nk}{n+1}b^2<3ab-ab=2ab; \text{ i.e., } A \leqslant \frac{2}{3}ab.$$

In this way one shows that, every time one switches from tiles of type R to tiles of type Q and vice-versa, the unpaved area has been reduced by a factor of $\frac{2}{3}$.

It follows that

$$\int_0^a \int_0^b f(x,y)\,dxdy=0 \quad \text{if} \quad 0 \leqslant a,b \leqslant \frac{1}{2}. \tag{3}$$

As in the case of the Putnam problem, this proves $f \equiv 0$ in $[0,\frac{1}{2}]\times[0,\frac{1}{2}]$.

If a or b is larger than $\frac{1}{2}$, the same argument (*mutatis mutandis*) proves that the integral of f vanishes over the smallest rectangle having (a,b) as one vertex, another vertex coinciding with a vertex of S and sides parallel to the sides of S.

We thus obtain

$$\int_a^1 \int_0^b f(x,y)\,dxdy=0 \quad \text{if} \quad \frac{1}{2} \leqslant a<1, 0 \leqslant b \leqslant \frac{1}{2}; \tag{4}$$

$$\int_0^a \int_b^1 f(x,y)\,dxdy=0 \quad \text{if} \quad 0<a \leqslant \frac{1}{2}, \frac{1}{2} \leqslant b<1; \tag{5}$$

$$\int_a^1 \int_0^1 f(x,y)\,dx\,dy = 0 \quad \text{if} \quad \frac{1}{2} \leqslant a,b < 1. \tag{6}$$

It follows that $f \equiv 0$ also in $[\frac{1}{2},1]\times[0,\frac{1}{2}]$, $[0,\frac{1}{2}]\times[\frac{1}{2},1]$ and $[\frac{1}{2},1]\times[\frac{1}{2},1]$. For example, if $[\alpha,a]\times[\beta,b]$ is a rectangle contained in $[\frac{1}{2},1]\times[0,\frac{1}{2}]$, we have

$$\int_\alpha^a \int_\beta^b = \int_\alpha^1 \int_0^b - \int_a^1 \int_0^b - \int_\alpha^a \int_0^\beta .$$

This proves $f \equiv 0$ in S.

The continuity of f is not essential in these considerations. If we assume only that f is integrable over S, the arguments leading to equations (1)–(6) remain valid; hence $\int \int_R f(x,y)\,dx\,dy = 0$ if R is a rectangle of sides parallel to the sides of S contained in the interior of one of the subsquares obtained by partitioning S by the lines $x=\frac{1}{2}, y=\frac{1}{2}$. The conclusion now is $f=0$ a.e. in each one of the subsquares (cf. [2, Theorem 8.8]) and hence $f=0$ a.e. in S.

If $0<\alpha<1$, the question also makes sense for f locally integrable in S^0. In fact, $S_\alpha(z)$ is a relatively compact subset of S^0 for all $z \in S^0$. But in this case our proofs of (1), (2), and (3) are no longer valid. In all these proofs we used implicitly the following fact, which is an immediate consequence of the Lebesgue dominated-convergence theorem: Let $f \in L^1(\mu)$, $(\overline{X},\mu)$ a measure space. Let $A, A_1, A_2, \ldots$ be measurable sets, such that $\mu(A_i \cap A_j)=0$ if $i \neq j$ and $A = \cup_{i=1}^\infty A_i$. Then $\int_A f\,d\mu = \sum_{i=1}^\infty \int_{A_i} f\,d\mu$.

To see that the integrability condition of f is really essential we will show that for each α, $0<\alpha<1$, there exists a continuous function f on S^0 having zero integral on $S_\alpha(z)$ for all z in S^0 and such that $f \not\equiv 0$.

The basic result is

LEMMA. *Let $0<\alpha<1$. There exists a differentiable function F on $(0,1)$ such that*
(i) $F' \not\equiv 0$
(ii) $F(1-x) = -F(x)$ *for all* $x \in (0,1)$
(iii) $F((1+\alpha)x) = F((1-\alpha)x)$ *for all* $x \in (0,\frac{1}{2}]$.

Proof. Let F_1 be differentiable in $((1-\alpha)/2,(1+\alpha)/2)$, $F_1' \not\equiv 0$, of compact support, and such that $F_1(1-x) = -F_1(x)$. Define F_2 in $((1-\alpha)^2/2, 1-(1-\alpha)^2/2)$ by setting $F_2 = F_1$ in $((1-\alpha)/2,(1+\alpha)/2)$;

$$F_2(x) = F_1\left(\frac{1+\alpha}{1-\alpha}x\right) \text{ if } \frac{(1-\alpha)^2}{2} < x \leqslant \frac{1-\alpha}{2}$$

and then

$$F_2(x) = -F_2(1-x) \text{ if } \frac{1+\alpha}{2} \leqslant x < 1 - \frac{(1-\alpha)^2}{2}.$$

Continuing in this fashion we get a sequence of functions $F_1, F_2, \ldots$, where F_n is defined on $I_n = ((1-\alpha)^n/2, 1-(1-\alpha)^n/2)$, differentiable, of compact support and satisfying $F_n(1-x) = -F_n(x)$ for all $x \in I_n$; $F_n((1-\alpha)x) = F_n((1+\alpha)x)$ if $x \in (0,\frac{1}{2}]$ and $(1-\alpha)x \in I_n$. Also $F_{n+1}|I_n = F_n$. Setting $F = \lim_{n\to\infty} F_n$, we are done.

We now let F be as in the Lemma and define f on S^0 by

$$f(x,y) = F'(x)F'(y).$$

Then one verifies easily that $\int\int_{S_\alpha(z)} f(x,y)\,dx\,dy = 0$ for all $z \in (0,1)\times(0,1)$ while clearly $f \not\equiv 0$.

We conclude by remarking that an analogue of Zalcman's problem can be stated for the unit interval, the unit cube, etc. A method similar to the one used in solving the Putnam problem proves that the problem has an affirmative solution if $\alpha=1$ in all cases, while it is easy to see that it has an affirmative solution for all α, $0<\alpha\leqslant 1$ in the case of the unit interval.

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THE ALTERNATING SERIES DEFINED BY AN INCREASING FUNCTION

RICHARD JOHNSONBAUGH

In an introductory calculus course, the usual hypotheses imposed during the discussion of the alternating series $\Sigma(-1)^{k+1}a_k$ are that $\{a_k\}$ should decrease and have limit zero (see [**9**, page 587]). Under these conditions, $\Sigma(-1)^{k+1}a_k$ converges. In this note we shall investigate the alternating series for $\{a_k\}$ increasing.

We should restrict our attention to those increasing sequences $\{a_k\}$ for which $\lim_{k\to\infty}a_k=\infty$. For if $\{a_k\}$ is bounded above, $\{a_k\}$ converges, say to L. Then $\{L-a_k\}$ decreases to zero and we may study the alternating series $\Sigma(-1)^{k+1}(L-a_k)$ with the traditional hypotheses.

Let f be a real-valued function on $[1,\infty)$ with continuous third derivative and let n be an odd positive integer. We will use the equation

$$\sum_{k=1}^{n}(-1)^{k+1}f(k)=[f(1)+f(n)]/2+[f'(n)-f'(1)]/4+\int_1^n f'''(t)X(t)dt \tag{1}$$

where $X(t)=-(t^2/4)+(3t/4)-1/2$ for $1\leqslant t\leqslant 2$, $X(t)=-X(t-1)$ for $2\leqslant t\leqslant 3$, and $X(t+2)=X(t)$ for $t\geqslant 1$. (See Fig. 1, which illustrates the graph of $y=X(t)$.) Equation (1) may be easily verified by applying the integration-by-parts formula to the integral. The motivation for obtaining this equation is given in [**6**].

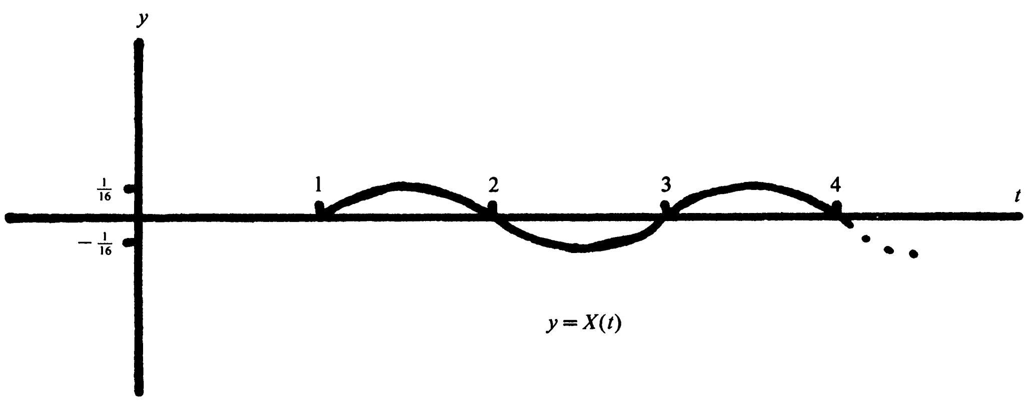

FIG. 1

As an example of the use of equation (1) let $f(t)=\ln(t+1)$. Then $f(t)$ tends monotonically to ∞. Equation (1) becomes

$$\sum_{k=1}^{n}(-1)^{k+1}\ln(k+1)=[\ln 2+\ln(n+1)]/2+[1/(n+1)-1/2]/4+\int_1^n f'''(t)X(t)dt.$$

We first evaluate the integral $\int_1^\infty f'''(t)X(t)dt$. Setting

$$\alpha_n = \sum_{k=1}^{n} (-1)^{k+1} f(k) - f(n)/2$$
$$= \sum_{k=1}^{n} (-1)^{k+1} \ln(k+1) - \ln(n+1)/2,$$

we have

$$\alpha_n = (\ln 2)/2 + [1/(n+1) - 1/2]/4 + \int_1^n f'''(t)X(t)\,dt. \tag{2}$$

Now

$$\alpha_n = \ln \frac{2\cdot 4\cdot 6\cdot \cdots \cdot (n+1)}{1\cdot 3\cdot 5\cdot \cdots \cdot n\sqrt{n+1}},$$

which is one-half the logarithm of Wallis's product (see [**1**, p. 299]). Hence

$$\lim \alpha_n = (1/2)\ln(\pi/2).$$

Taking the limit in (2) we obtain

$$(1/2)\ln(\pi/2) = (\ln 2)/2 - 1/8 + \int_1^\infty f'''(t)X(t)\,dt.$$

We therefore find that

$$\int_1^\infty f'''(t)X(t)\,dt = 4.217762364\times 10^{-3}.$$

We may estimate

$$\sum_{k=1}^{n} (-1)^{k+1}\ln(k+1) \doteq [\ln 2 + \ln(n+1)]/2 + [1/(n+1) - 1/2]/4 + \int_1^\infty f'''(t)X(t)\,dt. \tag{3}$$

The absolute error $|\int_n^\infty f'''(t)X(t)\,dt|$ is at most

$$\left|\int_n^{n+1} f'''(n)X(t)\,dt\right| = f'''(n)/24 \tag{4}$$

as we can see by referring to the graph of $X(t)$ and noting that $f'''(t)$ is decreasing.

Suppose we ask for the least N for which $\sum_{k=1}^{N}(-1)^{k+1}\ln(k+1)$ surpasses 10. Using equation (3) we see that we must solve

$$10 = [\ln 2 + \ln(N+1)]/2 - 1/8 + \int_1^\infty f'''(t)X(t)\,dt.$$

We have dropped the term $1/(N+1)$ since it will be so small as not to affect the initial computation. We find

$$N = 308{,}865{,}755.26.$$

Using (3) and (4) we find that if $s(n) = \sum_{k=1}^{n}(-1)^{k+1}\ln(k+1)$,

$$s(308865753) = 9.9999999995\ldots$$

$$s(308865755) = 10.000000002\ldots,$$

with error less than 10^{-26}. Thus the minimum number of terms of $\sum_{k=1}^{\infty}(-1)^{k+1}\ln(k+1)$ needed to surpass 10 is exactly 308,865,755. Similar computations show that to surpass 100 one would need approximately 4.6×10^{86} terms.

The method illustrated above, or a modification thereof, will apply to any function f for which there exists K such that for $k \geqslant K$, $f^{(k)}f^{(k+1)} < 0$ and $\lim_{t\to\infty} f^{(k)}(t) = 0$. For the function $f(t) = \ln(t+1)$ discussed above, we have $K=1$. If $K>1$ one needs a more refined version of equation (1), which can be obtained by successive applications of the integration-by-parts formula. (See [**6**] and especially equation (2.7).) A function f for which $f^{(k)}f^{(k+1)} < 0$ for $k = 0, 1, 2, \ldots$ is called *completely monotone* (see [**10**]).

For an arbitrary function $f(t)$, one might not be so lucky as to find an explicit formula for $\lim \alpha_n$, which hereafter we call α. This complicates the calculation of the integral $\int_1^\infty f'''(t)X(t)dt$. The number α is analogous to the number γ which occurs when one uses the Euler-Maclaurin formula to analyze a divergent series (see [4] or [7]).

Assume that f' and f''' decrease and that

$$\lim_{t\to\infty} f'(t) = 0 = \lim_{t\to\infty} f'''(t).$$

In this case an argument like that used to establish (4) assures us that $\lim \alpha_n$ exists. We may write

$$\begin{aligned}\alpha = \lim \alpha_n &= \lim\left[\sum_{k=1}^{n}(-1)^{k+1}f(k) - f(n)/2\right]\\ &= f(1)/2 - f'(1)/4 + \int_1^\infty f'''(t)X(t)dt\\ &= \sum_{k=1}^{n}(-1)^{k+1}f(k) - f(n)/2 - f'(n)/4 + \int_n^\infty f'''(t)X(t)dt.\end{aligned}$$

Thus we may estimate

$$\alpha \doteq \sum_{k=1}^{n}(-1)^{k+1}f(k) - f(n)/2 - f'(n)/4$$

by taking n large enough so that the absolute error $|\int_n^\infty f'''(t)X(t)dt|$ is sufficiently small. We may employ estimate (4) to assure the desired accuracy.

For example, if $f(t) \doteq \sqrt{t}$, we obtain

$$\begin{aligned}\alpha &\doteq \sum_{k=1}^{101}(-1)^{k+1}\sqrt{k} - \sqrt{101}/2 - 1/(8\sqrt{101})\\ &= .3801047364,\end{aligned}$$

which is correct to six places since the absolute error is at most

$$\frac{3}{8\cdot 24\cdot 101^{5/2}} = 1.524110979\times 10^{-7}.$$

We now find that $\int_1^\infty f'''(t)X(t)dt = .380105$, and hence we can obtain information about the rate of growth of the partial sums $s_n = \sum_{k=1}^{n}(-1)^{k+1}\sqrt{k}$. We find, for example, that the minimum values for N for which s_N surpasses 100, 1,000, and 10,000 are, respectively, 39,697, 3,996,961, and 399,969,593.

Finally, we call attention to references [2], [3], [4], [5], [6], and [8], which are concerned with sharp remainder estimates and rates of growth of partial sums of series.

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A COMMUTATIVE BANACH ALGEBRA OF FUNCTIONS OF BOUNDED VARIATION

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Let BV$[a,b]$ denote the linear space of functions of bounded variation on the closed interval $[a,b]$, and let

$$\mathrm{BV}^*[a,b]=\{f; f\in \mathrm{BV}[a,b], f(a)=0\}.$$

Functions f which have the property that $f(a)=0$ are often said to be anchored at a.

We shall denote the total variation of f on $[a,b]$ by $V(f;a,b)$, or just by $V(f)$ when no confusion can arise. It is a well-known result that BV$^*[a,b]$ is a Banach space under the variation norm $\|\cdot\|$, where $\|f\|=V(f)$. If pointwise multiplication is employed, we show that $\|fg\|\leqslant\|f\|\|g\|$ for all f and g in BV$^*[a,b]$, which makes BV$^*[a,b]$ a Banach algebra under the variation norm, $\|f\|=V(f)$. It is in fact commutative and possesses a unit element. That BV$^*[a,b]$ is a Banach algebra under the variation norm does not appear to be well known in the literature, although, for example, the reader is led to the result in problem (17.35) of **[2]**. The result is also an easy consequence of Lemma 2 of **[1]**.

We offer an elegant and easy proof that BV$^*[a,b]$ is a Banach algebra under the variation norm. For expedience we assume that BV$^*[a,b]$ is a Banach space. Under pointwise operations it is obvious that BV$^*[a,b]$ is a commutative algebra, possessing a unit element e defined by $e(a)=0, e(x)=1, a<x\leqslant b$. Consequently, our main result is to show that $V(fg)\leqslant V(f)V(g)$ for all f and g in BV$^*[a,b]$. Accordingly, we present the following

THEOREM 1. *If f and g belong to* BV$^*[a,b]$, *then so does fg, and*

$$V(fg)\leqslant V(f)V(g). \tag{1}$$

Proof. Let $a=x_0,x_1,\ldots,x_n=b$ be any subdivision of $[a,b]$ such that $a=x_0<x_1<\cdots<x_n=b$. Denote the difference $f(x_{i+1})-f(x_i)$ by $\Delta f(x_i)$. Then

$$\begin{aligned}
\sum_{i=0}^{n-1}|\Delta f(x_i)g(x_i)| &= \sum_{i=0}^{n-1}|f(x_{i+1})g(x_{i+1})-f(x_i)g(x_i)|\\
&= \sum_{i=0}^{n-1}|f(x_{i+1})\Delta g(x_i)+(\Delta f(x_i))g(x_i)|\\
&\leqslant \sum_{i=0}^{n-1}|f(x_{i+1})\Delta g(x_i)|+\sum_{i=0}^{n-1}|(\Delta f(x_i))g(x_i)|\\
&= \sum_{i=0}^{n-1}\Big|\sum_{j=0}^{i}\Delta f(x_j)\Delta g(x_i)\Big|+\sum_{i=0}^{n-1}\Big|\sum_{j=0}^{i-1}\Delta f(x_i)\Delta g(x_j)\Big|\\
&\leqslant \sum_{i=0}^{n-1}\sum_{j=0}^{i}|\Delta f(x_j)\Delta g(x_i)|+\sum_{i=0}^{n-1}\sum_{j=0}^{i-1}|\Delta f(x_i)\Delta g(x_j)|\\
&= \sum_{j=0}^{n-1}\sum_{i=j}^{n-1}|\Delta f(x_j)\Delta g(x_i)|+\sum_{i=0}^{n-1}\sum_{j=0}^{i-1}|\Delta f(x_i)\Delta g(x_j)|\\
&= \sum_{i=0}^{n-1}\sum_{j=i}^{n-1}|\Delta f(x_i)\Delta g(x_j)|+\sum_{i=0}^{n-1}\sum_{j=0}^{i-1}|\Delta f(x_i)\Delta g(x_j)|\\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}|\Delta f(x_i)\Delta g(x_j)|
\end{aligned}$$

$$=\left(\sum_{i=0}^{n-1}|\Delta f(x_i)|\right)\sum_{j=0}^{n-1}|\Delta g(x_j)|\leqslant V(f)V(g),$$

from which the required inequality follows. Consequently, $\|fg\|\leqslant\|f\|\,\|g\|$, as required.

An Application. If f is an absolutely continuous function on $[a,b]$, then it is shown in §18 of [2] that

$$V(f)=\int_a^b|f'(t)|\,dt.$$

Consequently, when f and g are absolutely continuous functions anchored at a, then we obtain from (1) the integral inequality

$$\int_a^b|(f(x)g(x))'|\,dx\leqslant\left(\int_a^b|f'(x)|\,dx\right)\left(\int_a^b|g'(x)|\,dx\right).$$

The restriction that f and g be anchored at a can, of course, be easily removed, in which case the inequality becomes

$$\int_a^b|[(f(x)-f(a))(g(x)-g(a))]'|\,dx\leqslant\left(\int_a^b|f'(x)|\,dx\right)\left(\int_a^b|g'(x)|\,dx\right).$$

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TEMPERED SCALES AND CONTINUED FRACTIONS

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1. Scales and Arithmetic. From ancient times on, the construction of musical scales has been described in arithmetic terms. We begin this note by outlining this numerical form of music theory.

It will be assumed that the reader is familiar with the concept of the *pitch* of a sound. A pitched sound corresponds in physical terms to a sound wave which is periodic with respect to time and so has a *frequency*. It is an empirical fact that pitch is determined by frequency and as pitch goes up so does frequency. The *interval* between two pitches may be thought of as the distance between them. It turns out that the interval between two pitches is determined by the *ratio* of their frequencies. Since the interval between two pitches is measured by the ear, we can say that the ear hears a frequency ratio.

Two very common intervals which play an important role in scale construction are the *octave* and the *perfect fifth*. (We usually omit the word "perfect".) The octave, which is the natural interval between the male voice and the female voice, can be demonstrated by playing on a keyboard instrument a C and the next C above it, or a D and the next D above it, etc. An example of a fifth is the interval between two consecutive strings on a violin. The interval between a C on the piano and the G above it is very nearly a fifth. The frequency ratios corresponding to an octave and a fifth are 2 : 1 and 3 : 2, respectively.

The octave and fifth are two intervals which are given names. A few other intervals have names, but it is desirable to have an interval name for any frequency ratio. A common way of

doing this is to describe an interval in terms of *cents*. A cent is 1/1200th of an octave. If we give the obvious definition to a rational multiple of an interval we find that m/n cents corresponds to a frequency ratio of $2^{m/1200n}$. We extend this by saying that for an arbitrary frequency ratio r the corresponding number of cents is $1200 \log r/\log 2$.

It is surprising that with the octave and the fifth alone we can construct a scale (i.e., a string of pitches) which is close to the scale obtained by playing the white keys of a piano. To describe this construction we first adopt the convention that we will not distinguish between two pitches an octave apart. With this convention we can construct a *Pythagorean scale* as follows: pick any pitch and ascend six fifths. The resulting seven pitches form our scale. If we reduce these pitches to a one-octave range with our initial pitch the lowest and arrange them in increasing order, we get the following string of relative frequencies: 1, 9/8, 81/64, 729/512, 3/2, 27/32, 243/128. This scale, arranged in this order, is often called the "Lydian mode" and corresponds closely to the pitches F, G, A, B, C, D, E on the piano.

2. Tempering. From the fundamental theorem of arithmetic it follows that no positive integral power of 3/2 is an integral power of 2 so that no positive integral number of fifths is an integral number of octaves. From this it follows that if we want to extend our Pythagorean scale so that the extended scale contains a Pythagorean scale starting on any one of its pitches we must have infinitely many pitches in our extended scale. The ability to start a Pythagorean scale on any pitch in our extended scale is of importance to the musician, but a scale with infinitely many pitches presents a formidable problem to the construction of a keyboard instrument or a stringed instrument with fixed frets. This problem is solved by "tempering" the fifth; if k fifths equal approximately l octaves, the fifth is modified so that k of these modified or "tempered" fifths equal l octaves. The extended scale has k distinct pitches and when they are reduced to a one-octave range and arranged in increasing order they divide the octave into k equal parts.

If we use as our fifth the tempered fifth described above, we may say that k fifths make up l octaves so that we have a "cycle of fifths" where we should have a "spiral of fifths." The *gap* in the spiral of fifths is the interval between k fifths and l octaves. This interval in cents is $1200|k \log (3/2)/\log 2 - l|$. The *error per fifth* is the difference between a true fifth and a tempered fifth and is $1200|l/k - \log (3/2)/\log 2|$ cents. The gap in the spiral of fifths and the error per fifth are both measures of the error in a particular tempering (i.e., a particular choice of l and k).

3. Continued Fractions. We now show how the theory of continued fractions can be used to find values of k and l which are in a certain sense best for tempering. (The connection between continued fractions and tempering has been noted by Jeans in [3].) It is clear that k fifths equal approximately l octaves if $(3/2)^k$ is approximately 2^l or if l/k is approximately $\log (3/2)/\log 2$. We will call this last ratio ρ for the remainder of this note. A rational approximation to ρ, which is what l/k is, may be found by expanding ρ in a simple continued fraction. Let the expansion be

$$a_1 + \frac{1}{a_2 +} \frac{1}{a_3 +} \cdots$$

and let the nth convergent be, in reduced terms, p_n/q_n. Values of a_n, p_n, and q_n for $n = 1$ through 10 are given below. We also give a decimal expansion of $p_n/q_n - \rho$.

Each convergent may be given a meaning in musical terms. q_n is the number of fifths which approximates p_n octaves. If C is the pitch on which we started, then the second, third, and fourth convergents give the approximations G, D, and B, respectively, to C. The fifth convergent corresponds to the tempering actually used today. The sixth and seventh, according to A. L. L. Silver [5], were approximations known to the ancient Chinese. Helmholtz [2] also discusses dividing the octave into 53 parts. Jeans gives the eighth convergent, and Silver all through the ninth.

n	a_n	p_n	q_n	$p_n/q_n-\rho$
1	0	0	1	-5.850×10^{-1}
2	1	1	1	4.150×10^{-1}
3	1	1	2	-8.496×10^{-2}
4	2	3	5	1.504×10^{-2}
5	2	7	12	-1.629×10^{-3}
6	3	24	41	4.034×10^{-4}
7	1	31	53	-5.684×10^{-5}
8	5	179	306	4.820×10^{-6}
9	2	389	665	-9.470×10^{-8}
10	23	9126	15601	1.683×10^{-9}

The rational number l/k is called a *best approximation of the second kind* to ρ if

$$\frac{l'}{k'}\neq\frac{l}{k} \quad\text{and}\quad |l'-k'\rho|\leqslant|l-k\rho|\Rightarrow k'>k.$$

Looking at our formula for the gap in the spiral of fifths, we see that l/k is a best approximation of the second kind to ρ if and only if a decrease in k results in an increase in the gap in the spiral of fifths. According to theorems 16 and 17 of [4] k/l is a best approximation of the second kind to ρ if and only if it is a convergent in the expansion of ρ as a simple continued fraction. This may be rephrased as follows: Suppose we divide the octave into k equal parts. Then, unless $k=q_n$ for some n, we can get a gap in the spiral of fifths just as small or smaller by subdividing the octave into fewer than k equal parts. This is the sense in which the numbers 12, 41, 53,... are optimal.

Now let us examine the error per fifth. The error per fifth, we see from our formula, is essentially the error in approximating ρ by l/k, and the appropriate tool for studying this error is the sequence of intermediate convergents to our continued fraction (see [5, Chap. 32, §12]). With the aid of the intermediate convergents we can answer the following kind of question: Suppose that we are given $\varepsilon>0$ and we specify whether we want our tempered fifth larger than or smaller than a true fifth. Into how many parts must we divide the octave to get an error per fifth less than ε cents? As an example we take $\varepsilon=1.955$, which is the error per fifth corresponding to the standard division of the octave into 12 parts. First we look for a tempered fifth which is too small, as is the case with the standard tempering. Since there are no intermediate convergents between the fifth and the seventh, we conclude that we must go to a division of the octave into 53 parts to achieve this result. If we are willing to accept a tempered fifth which is too large, we start with the fourth convergent and we get the following sequence of intermediate convergents:

$$3/5,\ 10/17,\ 17/29,\ 24/41$$

If we compute the error per fifth in each intermediate convergent, we see that the first time we get an error per fifth less than ε is at 17/29. A division of the octave into 29 equal parts gives a fifth which is too large by 1.493 cents and no division of the octave into fewer than 29 parts gives any improvement over the standard fifth. Further improvement requires at least 41 parts.

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RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada T2N 1N4.

THE CONGRUENT NUMBER PROBLEM

RONALD ALTER

The positive rational integer a is called **congruent** if and only if there exist rational nonzero integers x, y, z, and t such that

$$x^2 + ay^2 = z^2 \quad \text{and} \quad x^2 - ay^2 = t^2. \tag{1}$$

It is easy to see that cn^2 is a congruent number if and only if c is a congruent number. According to Dickson [4] the problem of congruent numbers goes back to an anonymous Arab manuscript written before 972.

Many articles have appeared on congruent numbers. Among the authors who have made contributions to the problem of whether or not a given square-free number is congruent are: Leonardo Pisano (Fibonacci), G. Gosselin, Fermat, L. Euler, A. Genocchi, F. Woepcke, B. Boncompagni, L. Pisanus, J. Hartley, P. Barlow, J. Cunliffe, M. Collins, G. Destournelles, E. Lucas, S. Günther, S. Roberts, A. Desboves, G. Heppel, M. Jenkins, G. B. Mathews, R. Aiyar, G. Bisconcini, R. D. Carmichael, H. B. Mathieu, L. Aubry, G. Métrod, A. Cunningham, A. Gérardin, L. Bastien, T. Ono, and G. Candido. References to their works, the related works of others, and more information on the history and origin of congruent numbers can be found in the encyclopedic treatise of Dickson [4].

More recent references to congruent numbers can be found in the texts by Uspensky and Heaslet [12], Sierpiński [10], and Mordell [8]. In his text, Sierpiński [10, pp. 63–67] requires that $y = 1$ in Equation (1).

Roberts [9] proved that every congruent number a satisfying Equation (1) must also satisfy (2)

$$uv(u^2 - v^2) = aw^2. \tag{2}$$

In Alter, Curtz, and Kubota [2] other results on congruent numbers can be found. (A list of known results on noncongruent numbers is also provided.)

In 1915 Bastien listed all square-free congruent numbers less than 100 and Gérardin listed 62 square-free congruent numbers less than 1000. In 1972, Alter, Curtz, and Kubota [2], using new results and a computer search based on Equation (2), determined 333 of the 608 square-free numbers less than 1000 to be either congruent (198 numbers) or noncongruent (135 numbers). In 1974, Alter and Curtz [1], using a computer search based on other forms a congruent number may take, found 18 more congruent numbers less than 1000. Godwin [6] introduced a method which enabled 19 primes of the form $8k+7$ to be added to the list of congruent numbers less than 1000. Hunter [7] found, in addition to other numbers, 133 and 183 to be congruent and 115, 123, and 187 to be noncongruent.

The following conjecture, which first appeared in Alter, Curtz, and Kubota [2], is still open.

CONJECTURE 1. *If* $n \equiv 5$, 6 *or* 7 (mod 8), *then* n *is a congruent number.*

TABLE 1

Some Congruent Numbers

157	269	373	503	607	661	743	829	911
173	277	389	541	613	677	757	853	941
197	293	397	543	631	701	773	863	967
229	317	421	557	647	727	797	877	983
239	367	461	599	653	733	823	887	997

TABLE 2

Some Square-Free Numbers

105	259	341	415	493	570	627	703	767	822	881	946
113	266	345	418	494	573	635	705	770	826	885	953
139	267	354	427	497	577	638	706	771	830	893	955
155	273	355	430	498	579	642	707	779	831	894	965
185	282	365	435	501	581	654	713	781	834	895	969
195	295	374	437	506	589	655	714	782	835	898	970
203	301	377	447	515	590	667	715	785	843	899	973
213	303	381	451	519	591	678	717	786	851	902	977
217	305	385	453	521	593	679	723	789	854	906	978
230	309	402	469	534	595	682	730	790	857	917	979
235	322	403	471	545	597	685	742	795	858	923	986
237	327	406	474	551	598	687	745	802	861	930	989
238	329	407	481	553	606	690	749	803	865	933	994
247	335	409	482	555	611	695	755	807	869	938	
253	337	411	483	559	618	697	762	809	871	939	
258	339	413	485	569	623	699	763	815	874	942	

TABLE 3

Known Noncongruent Numbers < 1000

1	2	3	10	11	17	19	26	33	35
42	43	51	57	58	59	66	67	73	74
82	83	89	91	97	106	107	114	115	122
123	129	130	131	139	146	163	170	177	178
179	186	187	193	201	202	209	211	218	227
233	241	249	251	274	281	283	290	298	307
314	321	331	346	347	362	370	379	393	394
401	417	419	433	443	449	458	466	467	473
489	491	499	523	530	537	538	547	554	562
563	571	586	587	601	610	617	619	626	633
634	641	643	649	659	673	681	683	691	698
737	739	746	753	754	769	778	787	794	811
817	818	827	842	849	859	883	907	913	914
921	922	929	937	947	962	971	993		

Stephens [11] proved that Conjecture 1 is a special case of the Selmer conjecture for elliptic curves. From this he established the following theorem.

THEOREM 1. *If n is a prime congruent to* 5 *or* 7 (mod 8), *or* $n=2p$ *where* p *is a prime congruent to* 3 (mod 4), *then* n *is a congruent number.*

Using Theorem 1 one finds 45 new congruent numbers. These numbers, for which no explicit representation has yet been found, are exhibited in Table 1. This leaves only 189 of the 608 square-free numbers less than 1000 for which it is still not known whether or not they are congruent. These numbers are exhibited in Table 2. Table 3 contains all known noncongruent numbers.

Note that in Table 1, every number is congruent to 5 (29 numbers) or 7 (16 numbers) modulo 8. Also, the numbers 213, 222, and 247 are the smallest numbers congruent to 5, 6, and 7 (mod 8), respectively, for which Conjecture 1 is still unsettled.

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CLASSROOM NOTES

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Advice to prospective authors: The editors have recently been receiving about **ten times** as many Classroom Notes as can be used. It will simplify our work if authors will submit only especially interesting manuscripts.

R.P.B.

CONCURRENCES FOR A TRIANGLE

D. KIRBY

The main purpose of this note is to give publicity to the following little-known theorem of plane Euclidean geometry.

THEOREM 1. *If similar isosceles triangles BCA', CAB', ABC' are described on the sides of a triangle ABC, then the lines AA', BB', CC' are concurrent.*

With special values for the common base angle θ of the isosceles triangles the concurrence is well known. Thus $\theta=\pi/2$ gives the concurrence of the altitudes; $\theta=0$ that of the median lines; $\theta=\pi/3$ appears in Casey [1, p. 49, Example 7], for example, and has connections with Fermat's problem of finding a point P such that $PA+PB+PC$ is a minimum (see Coxeter [2, p. 21]); also $\theta=\pm\pi/3$ appear in Eves [3, p. 84, Examples 17, 18], for example. The only reference I have found to the general result (and that thanks to Dr. J. A. Tyrrell of King's College, London) is the oblique one in Casey [1, p. 288, Examples 58, 59].

In this note we investigate the general problem of when a concurrence occurs by describing triangles on the sides of ABC and joining opposite vertices. Each such concurrence gives rise to three others, and Theorem 1 is one of four infinite families for which concurrence occurs. Our approach is algebraic; no doubt the reader will find more elegant arguments for himself.

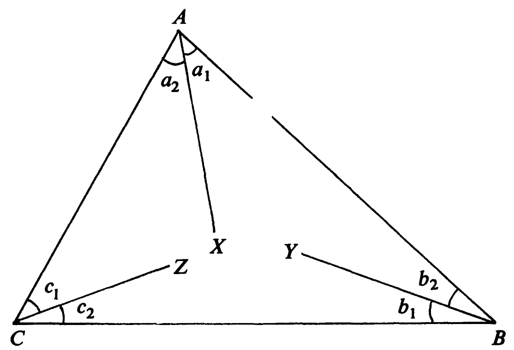

FIG. 1

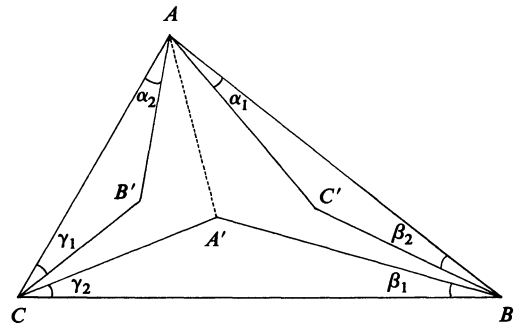

FIG. 2

Either with an argument on the same lines as that used to establish Ceva's theorem or by translating Ceva's theorem in terms of angles, we have in the notation of Figure 1:

PROPOSITION 2. *AX, BY, CZ are concurrent if, and only if,*

$$\sin a_1 \sin b_1 \sin c_1 = \sin a_2 \sin b_2 \sin c_2.$$

By applying Proposition 2 four times to the situation depicted in Figure 2 we have our main result.

THEOREM 3. *AA', BB', CC' are concurrent if, and only if,*

$$\frac{\sin\alpha_1 \sin\beta_1 \sin\gamma_1}{\sin(A-\alpha_1)\sin(B-\beta_1)\sin(C-\gamma_1)} = \frac{\sin\alpha_2 \sin\beta_2 \sin\gamma_2}{\sin(A-\alpha_2)\sin(B-\beta_2)\sin(C-\gamma_2)}.$$

This criterion for concurrence remains unchanged under each of the following changes of angles.

(i) $\alpha_i \to A-\alpha_i$, $\beta_i \to B-\beta_i$, $\gamma_i \to C-\gamma_i$ $(i=1,2)$.

(ii) $\alpha_1 \leftrightarrow \alpha_2$, $\beta_1 \leftrightarrow \beta_2$, $\gamma_1 \leftrightarrow \gamma_2$.

(iii) $\alpha_1 \to A-\alpha_2$, $\alpha_2 \to A-\alpha_1$, $\beta_1 \to B-\beta_2$, $\beta_2 \to B-\beta_1$, $\gamma_1 \to C-\gamma_2$, $\gamma_2 \to C-\gamma_1$.

Thus concurrence in one case implies concurrence in the other three cases. In geometric terms (iii) amounts to interchanging the rôles played by pairs of lines such as AB', AC' issuing from

each vertex: (i) amounts to replacing each line issuing from a vertex by its isogonal conjugate, i.e., its reflection in the internal angle bisector; (ii) is thus the product of (i) and (iii) in either order.

It is immediate from Theorem 3 that a triply infinite family of concurrences occurs by taking $\alpha_1=\alpha_2$, $\beta_1=\beta_2$, and $\gamma_1=\gamma_2$. In particular, when AB',AC'; BC',BA'; CA',CB' are the angle trisectors we have a connection with the configuration of Morley's Theorem which tells us the $A'B'C'$ is equilateral (see Coxeter [2, p. 23]).

Two particular, simply infinite families are worthy of mention.

I. *If* $\alpha_1=\alpha_2=\beta_1=\beta_2=\gamma_1=\gamma_2=\theta$, *then* AA',BB',CC' *are concurrent. This establishes Theorem* 1.

II. *If* $\alpha_1=\alpha_2=A-\theta$, $\beta_1=\beta_2=B-\theta$, $\gamma_1=\gamma_2=C-\theta$, *then* AA',BB',CC' *are concurrent. This is obtained from* I *by applying either procedure* (i) *or* (iii) *to the angles.*

Two further infinite families of concurrences connected in a similar way are:

III. *If* $\beta_2=\gamma_1=\theta-A$, $\gamma_2=\alpha_1=\theta-B$, $\alpha_2=\beta_1=\theta-C$, *then* AA',BB',CC' *are concurrent. This family is transformed to itself by* (iii).

IV. *If* $\beta_1=\gamma_2=\theta-A$, $\gamma_1=\alpha_2=\theta-B$, $\alpha_1=\beta_2=\theta-C$, *then* AA',BB',CC' *are concurrent. This arises from* III *by* (ii).

We end by finding the locus of the point of concurrence of AA',BB',CC' as θ varies in each of the four cases I, II, III, and IV.

For any point on AA' the ratio of its perpendicular distances from AC and AB is

$$\frac{\sin C\hat{A}A'}{\sin B\hat{A}A'}=\frac{\sin\beta_1\sin(C-\gamma_2)}{\sin(B-\beta_1)\sin\gamma_2}$$

by Proposition 1. Thus in trilinear coordinates (x,y,z) the equation of AA' is

$$y\frac{\sin(B-\beta_1)}{\sin\beta_1}=z\frac{\sin(C-\gamma_2)}{\sin\gamma_2},$$

and similarly for BB' and CC'. From this it is a simple matter to calculate that in cases I and III the trilinear coordinates of the point of concurrence are $(x,y,z)=(\operatorname{cosec}(A-\theta), \operatorname{cosec}(B-\theta), \operatorname{cosec}(C-\theta))$. If we eliminate θ the equation of locus as θ varies is

$$yz\sin(B-C)+zx\sin(C-A)+xy\sin(A-B)=0$$

which, unless $A=B=C$, is the equation of the conic through points A,B,C, the orthocenter $(\theta=\pi/2)$ and the centroid $(\theta=0)$. The locus is therefore a rectangular hyperbola. Similarly, in cases II and IV the point of concurrence has coordinates

$$(\sin(A-\theta),\ \sin(B-\theta),\ \sin(C-\theta))\quad\text{and}\quad(\sin(A+\theta),\ \sin(B+\theta),\ \sin(C+\theta)),$$

respectively, and in each case the locus has equation

$$x\sin(B-C)+y\sin(C-A)+z\sin(A-B)=0.$$

This is the line joining the circumcenter $(\theta=\pi/2)$ to the symmedian point $(\theta=0)$.

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Note that $c(t)$ as given by Equation (9) is (a) a non-increasing function of t; (b) always less than 90°; and (c) $\lim_{t\to\infty} c(t)=0$. Theoretical considerations dictate that $c(t)$ must have each of these properties.

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ON THE ABSOLUTE CONVERGENCE OF A MULTIPLE INTEGRAL OF SIN x/x.

HARRY DYM

Let $f(y)=\min[1,|y|^{-1}]$. In a recent issue of this MONTHLY P. Ungar [1] proved that the integral

$$I=\int\cdots\int f(y_1)f(y_2)\cdots f(y_n)f(y_1+\cdots+y_n)\,dy_1\cdots dy_n$$

is finite and used that to deduce the absolute convergence of

$$\int\cdots\int\frac{\sin y_1}{y_1}\frac{\sin y_2}{y_2}\cdots\frac{\sin y_n}{y_n}\frac{\sin(y_1+\cdots+y_n)}{(y_1+\cdots+y_n)}\,dy_1\cdots dy_n.$$

(The limits of all integrals are $\pm\infty$ unless designated otherwise.) This resolved a problem posed in this MONTHLY some 10 years earlier by D. R. Brillinger [2]. Ungar's solution depends upon the clever use of an inequality drawn from the treasure chest of Hardy, Littlewood, and Pólya.

The purpose of this note is to present an elementary self-contained proof of the finiteness of the integral I. At the same time the ideas used seem sufficiently instructive to warrant their inclusion in analysis courses at either the graduate or the advanced undergraduate level.

The proof rests upon the observation that

$$I=\int\cdots\int f(0-x_n)f(x_n-x_{n-1})\cdots f(x_2-x_1)f(x_1)\,dx_1\cdots dx_n$$

is the n-fold convolution of f evaluated at 0. Hence I is, or ought to be, equal to

$$(2\pi)^n\int e^{ixt}[f^\vee(t)]^{n+1}\,dt,\qquad \text{evaluated at } x=0,$$

in which

$$f^\vee(t)=\frac{1}{2\pi}\int e^{-ixt}f(x)\,dx$$

designates the inverse Fourier transform of f. This line of reasoning is not directly applicable since neither f nor $[f^\vee]^{n+1}$ is known to be summable. Accordingly define

$$f_\delta(y)=[f(y)]^{1+\delta}$$

for $0<\delta\leqslant 1$, set

$$I_\delta=\int\cdots\int f_\delta(y_1)f_\delta(y_2)\cdots f_\delta(y_n)f_\delta(y_1+\cdots+y_n)\,dy_1\cdots dy_n$$

and notice that, since $0 \leqslant f_\delta \uparrow f$ as $\delta \downarrow 0$, the monotone convergence theorem guarantees that

$$I = \lim_{\delta \downarrow 0} I_\delta,$$

both sides being finite or infinite together. Now (unlike f) f_δ is summable and, since f_δ is even,

$$\begin{aligned} f_\delta^\vee(t) &= \frac{1}{\pi}\int_0^1 \cos tx\,dx + \frac{1}{\pi}\int_1^\infty \frac{\cos tx}{x^{1+\delta}}\,dx \\ &= \frac{\sin t}{\pi t} + \frac{t^\delta}{\pi}\int_t^\infty \frac{\cos x}{x^{1+\delta}}\,dx \\ &= \frac{\sin t}{\pi t} + \frac{t^\delta}{\pi}\left\{ \frac{\sin x}{x^{1+\delta}}\Big|_t^\infty + (1+\delta)\int_t^\infty \frac{\sin x}{x^{2+\delta}}\,dx \right\} \\ &= \frac{(1+\delta)t^\delta}{\pi}\int_t^\infty \frac{\sin x}{x^{2+\delta}}\,dx, \end{aligned}$$

as follows by an elementary change of variables $[tx \to x]$ in the integral of line 2 and integration by parts in line 3. The bound

$$|f_\delta^\vee(t)| < (1+\delta)\frac{t^\delta}{\pi}\int_t^\infty \frac{1}{x^{2+\delta}}\,dx = \frac{1}{\pi t}$$

is immediate for $t > 0$ and will be sufficient for large t. A better bound will be needed for small t. In particular, for $0 < t < 1/e$,

$$\begin{aligned} |f_\delta^\vee(t)| &\leqslant (1+\delta)\frac{t^\delta}{\pi}\int_t^1 \left|\frac{\sin x}{x}\right| \frac{dx}{x^{1+\delta}} + (1+\delta)\frac{t^\delta}{\pi}\int_1^\infty \frac{1}{x^{2+\delta}}\,dx \\ &\leqslant \frac{(1+\delta)}{\pi}\int_t^1 x^\delta \frac{dx}{x^{1+\delta}} + \frac{t^\delta}{\pi} \\ &\doteq \frac{(1+\delta)}{\pi}\ln\frac{1}{t} + \frac{t^\delta}{\pi} \\ &\leqslant \ln\frac{1}{t}. \end{aligned}$$

Thus, $[f_\delta^\vee(t)]^{n+1}$ is summable and

$$\begin{aligned} I_\delta &= (2\pi)^n \int [f_\delta^\vee(t)]^{n+1}\,dt \\ &\leqslant 2\cdot(2\pi)^n \int_0^{1/e}\left(\ln\frac{1}{t}\right)^{n+1} dt + 2\cdot(2\pi)^n \int_{1/e}^\infty \left(\frac{1}{\pi t}\right)^{n+1} dt \\ &< \infty, \end{aligned}$$

independently of δ. The proof is complete.

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MATHEMATICAL EDUCATION

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A CASE-STUDY COURSE IN APPLIED MATHEMATICS USING REGIONAL INDUSTRIES

J. L. AGNEW AND M. S. KEENER

Introduction. During the past decade the mathematical community has been forced to reconsider its curriculum in light of the anxiety-reaction of society toward mathematics on the one hand and the challenges of the computer age on the other [**2**], [**5**], [**6**], [**9**]. One aspect of this problem is the preparation of students who expect to use their mathematical training in areas other than the traditional academic ones. Mathematical faculties have developed and experimented with a variety of approaches to this complicated and important topic. See, for example, [**1**], [**3**], [**4**], and [**8**].

At Oklahoma State University we were challenged to act by the responses of our graduates to the question, "Were you academically prepared for your professional life?" Those graduates in academic employment gave affirmative and complimentary answers, but among those in nonacademic activities we did not fare so well. Typical of their responses is: "I regret not having my upper-level mathematics brought into focus in courses dealing with real-world problems and phenomena." All expressed the need for an appreciation of the relationship between the academic discipline and its application in an industrial setting. (We use the word "industry" throughout to include corporations, businesses, firms, and government agencies.) This article describes our effort to bridge the gap between the academic and the nonacademic environments.

Specific Goals. We sought the advice of people in regional industry who act as supervisors (for example, project leaders) and also of people who had been employed in industry for a relatively short time (two or three years) and thus could still recall the adjustment period. These discussions led us to identify four skills which mathematics majors lack: (a) modeling a problem and interpreting its solution in the actual setting in which the problem arose; (b) integrating the mathematics learned in a variety of courses and extracting from the mathematical literature the new techniques needed in working a problem; (c) communicating with people who are involved with *using* rather than teaching mathematics; (d) communicating in writing about a problem, its solution, and the interpretation of the solution.

We provide these four experiences in a customized applied mathematics course at the senior level. Ordinarily, seniors will have had differential equations, linear algebra, some working knowledge of a programming language, and the mathematical maturity derived from other electives. We have found that courses in numerical analysis and in basic statistics are also quite helpful. Students with different backgrounds share their knowledge with each other.

Ingredients of the Course. We chose a case-study format in order to make the course as much a simulation of an actual employment experience as is possible. Instead of selecting problems from textbooks or preparing materials ourselves, we directly involved representatives of regional industries. Thus we needed (1) individuals in the nonacademic community who are able and willing to discuss their mathematical needs and interests; (2) mathematical problems encountered by these individuals which are of sufficient difficulty to involve college-level mathematics but are still accessible to a well-prepared undergraduate; (3) a mechanism for interaction between the student and these people.

The identification of appropriate individuals has proved to be easier than we anticipated. Initial contacts were made through professional associates of faculty members, graduates of the school of mathematical sciences, and friends. These people in turn suggested others. The people contacted were most cooperative about taking time to visit with us. Many were pleased to have some input into the educational process. In no case did the management of the industry decline to allow its personnel to participate. The regional nature of these contacts, besides making them reasonably accessible, ensures that the students will be involved with people from industries with which they have some acquaintance.

A mathematical problem is considered "suitable" if it can be presented to a senior mathematics major with the expectation that the student can understand the setting of the problem well enough to take some part in the development of the mathematical model; already knows or has the background to learn the techniques needed to approach the problem; and can make substantial progress toward solving the problem in about a month, assuming an average of three hours a week of class time and six hours a week outside class.

Negotiations for suitable problems are done in personal visits to the offices of the industrial resource people. Here we discuss the kind of problem with which they are primarily involved, and, when possible, the kind on which new employees might work. We restrict our attention to problems for which at least a working solution has already been found. The initial reaction of the industrial resource people is often that the problems are too easy to interest us. After discussing some possible problems and recalling the knowledge and preparation required in solving them, they are inclined to think the problems are too involved. The discussions usually converge to one or more problems which promise to be both interesting and usable. These we take home with us to discuss and study in more detail.

A semester of the case-study course is based on three problems chosen from those that appeared suitable. The criteria used in selecting the problems for a given semester are balance of the mathematical content, variety of settings, appropriate length and level of difficulty, "non-textbook" nature, and opportunity for the student to do original work.

With the cooperation of the industrial resource people, we make a detailed plan for the presentation of each problem. They are reminded of the students' level of understanding in the area of the problem and are asked to suggest references for background reading, as well as to prepare lists of vocabulary with which the students should be familiar. Our role is to identify the areas of mathematics which the student might find useful.

An Example. A description of a problem presented by the Boeing Wichita Company is illuminating. The cooperation extended to us by the administration and personnel of this company was very much appreciated and was typical of that received from many companies. On our first visit we spent a day discussing possible problems with people in a number of groups in design and analysis. These problems ranged from simple ones involving only spherical trigonometry and geometry to very complicated ones requiring numerical solutions of partial differential equations. We were provided with outlines of some problems, as well as reports and publications. Afterwards we studied these along with our notes of conversations. The most suitable problem was presented by a structural engineer, Kenneth L. Roger, and concerns the determination of the "gust design envelope of interacting loads" [7].

When airframe stress is a function of n loads L_i, the stress x is given by

$$x = x_{\text{mean}} + \Delta x = \sum_{i=1}^{n} c_i(L_{i(\text{mean})} + \Delta L_i).$$

The gust-loading region consists of all n-tuples $(\Delta L_1, \Delta L_2, \ldots, \Delta L_n)$ which satisfy a certain design criterion for all sets of constants c_i, $i = 1, 2, \ldots, n$. The particular criterion considered in our problem is $(\Delta x)^2 \leqslant U_\sigma^2 \sigma_x^2$, where σ_x^2 is the variance of x and U_σ is a given constant. In matrix notation the criterion is

$$\left([c_i]^T[\Delta L_i]\right)^2 - U_\sigma^2 [c_i]^T[\sigma_{ij}][c_i] \leqslant 0, \quad \text{where} \quad \sigma_{ij} = \operatorname{cov}(\Delta L_i, \Delta L_j).$$

The design loads envelope is the boundary of the gust-loading region. It includes all of the potentially critical loading combinations which must be investigated by the stress analyst. The problem is to devise a way of determining and describing the design loads envelope, given the covariance matrix for the loads. We shall return to this example after discussing the structure of the course.

Structure of the Course. A unique feature of the case study as we have used it is the direct involvement of the industrial resource person with the students. The resource person visits the campus and describes the problem to the students. Before the visit the students are given the references for background reading suggested by the resource person, and they have an opportunity to look up some basic vocabulary. During the visit the resource person talks about the background of the problem, how the problem arose, and why its solution is important. If the problem is complicated or very technical, suggestions for developing a suitable mathematical model may be given. The students are free to ask questions, and they do so, although at this presentation their questions are not very penetrating. They do, however, have a chance to ask about parts of the background reading which are not completely clear to them. Because they lack experience in communicating with nonacademic professionals, this visit is important.

After the first visit, the students need a few days to review what has been said and to bring the model and problem into focus. They are given the responsibility for deciding what directions they will take in determining and refining the model and in solving the problem. The instructor plays more the role of a critic than that of a leader or guide. Thus, the students seek out (under some guidance from the instructor) reference material and suggest possible approaches to solutions. These activities are usually group efforts, although some problems can be approached individually. Most of the actual work is done outside of class, with the class time given to discussing and comparing various aspects of the problem and methods of solution.

After a couple of weeks it becomes apparent which of the suggested approaches have merit and which are inappropriate. The students then choose a particular method of solution and solve the problem. If more than one method of attack seems appropriate, the students are divided into groups.

The final task is for each student to write a report. This includes an introduction to the problem, a precise statement of what was to be accomplished, a precise statement of the interpretation of the student's results, and a discussion of the method used. Tables and computer programs are also included.

The industrial resource person returns to campus to discuss the solutions arrived at by the class and to compare them with the industry's solution. The resource person also discusses with the class any further considerations which would be included in a more complete analysis. The students are now ready to ask questions more freely and with much more insight. Some of the resource people graded the student reports and made helpful individual comments.

An Example Concluded. Mr. Roger aroused student interest in his problem by presenting a brief history of the development of the B52 bomber and of the difficulties caused by irregular gusts when this plane was used at low altitudes. The students were first given a covariance matrix for the case $n=2$ and asked to find the design loads envelope for this particular case. They were then to devise a technique for the general case. The students were encouraged to prepare for the problem by reviewing some linear algebra, especially quadratic forms. They were also asked to find the mathematical meaning of the word "envelope"—a topic which was once a favorite in differential equations texts. The students were then able to identify two basic approaches to the problem. One of these involves considering the boundary of the gust-loading region as the surface enveloped by the hyperplanes $(\Delta x)^2 - U_\sigma^2\sigma_x^2 = 0$. Since this was the way in which Mr. Roger had himself formulated the problem, the students were led to think of this approach for the case $n=2$, in which the envelope is an ellipse determined by a family of straight lines.

Some students had trouble generalizing the notion of an envelope. They chose a more direct

approach based on the fact that the design criterion requires that the quadratic form $(\Delta x)^2 - U_\sigma^2\sigma_x^2$ in the variables $c_1, c_2, \ldots, c_n$ be negative definite. This requirement yields, as the set of boundary points, the points determined by $\det\{[\Delta L_i][\Delta L_i]^T - U_\sigma^2[\sigma_{ij}]\} = 0$. This is itself a quadratic equation in the variables $\Delta L_1, \Delta L_2, \ldots, \Delta L_n$; the surface that it represents can be identified by orthogonal diagonalization. Both approaches lead to the same quadratic equation.

Reactions to the Course. A feeling for the immediate impact of the course can be obtained from the reactions of the participants. At the end of the year we asked the students to list what they felt they had learned from the course. Among the things mentioned were: an awareness of the ways in which mathematics can be applied; the difference in point of view between attacking problems in theoretical courses and in practical applications; a good deal of mathematics not previously studied; creativity and initiative; the ability to read mathematics, to write accurately about a problem, to work in groups, and to think and persevere. The general opinion was that the workload in the course was about average, although it tended to be concentrated at certain times, mostly because of procrastination. Here are two typical comments: "I learned in this course that one can never learn enough mathematics to handle every problem he or she may be involved in. There are always new techniques to be found in books and articles. I also gained a vast amount of confidence in my abilities to do work of this type." "I think that many students have a stalemated view of their occupation; they often fail to realize that their job experience will be a perpetual learning encounter. As these last two semesters have transpired, I have become quite aware of the dynamic state of one's work position."

The industrial resource people were also asked to comment about their experience. All felt it had been very useful and were willing to do it again. Three have approached us with new problems they wish to present. Why are people willing to take on this extra assignment in spite of the demands of full-time employment in industry? Two primary reasons were suggested: a desire for closer contact between industry and education, and the feeling that they are able to make a unique contribution to the students' learning. Participation in this course helped the industrial people become more aware of the potential of mathematics students.

Certainly no claim is made that a course of this nature should replace the traditional courses in applied mathematics—partial differential equations, Fourier series, optimization, numerical methods, etc. The course does not emphasize specific techniques in mathematics. Rather, it is intended to help the students see that they can learn mathematics on their own as well as relate the mathematics they have learned to the solution of "real world" problems. Nevertheless, the students did learn new mathematics in several areas, for example, Fourier series, ordinary differential equations, partial differential equations, statistics, and numerical analysis. In the specimen problem discussed, the students had taken a course in linear algebra which mentioned quadratic forms only briefly. With this background they could study the topic independently.

Conclusion. This method of instruction requires more effort than a traditional approach and more out-of-class time for the teacher and student alike. Identification of resource people and problems requires the most time. Moreover, since the students are required to be more forward and involved, shy students may feel at a disadvantage. We have found that the students become much more able to discuss mathematical concepts, to defend and write about their ideas, and to search the literature. They also gain confidence in building and analyzing mathematical models. They learn a significant amount of new mathematical material. However, it is difficult to predict the areas in which this material will lie, since we use different problems each semester and the students choose the methods of solution.

As we enter our third year of teaching this course, we continue to find the entire enterprise challenging, satisfying, rewarding, and—most of all—fun.

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An expanded version of this article was presented to a colloquium at Virginia Polytechnic Institute in May, 1978.

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PROBLEMS AND SOLUTIONS

EDITED BY VLADIMIR DROBOT

The editors receive more problems than they can use, but they are always on the lookout for interesting and novel problems. The task of selection will be much easier if proposers will keep the following guidelines in mind:

Send all proposed problems, in duplicate if possible, to Vladimir Drobot, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053. Please include solutions and any information that will help the editors, including reasons why the problem is interesting. Problems in well-known textbooks and results in generally accessible sources are not acceptable.

Solutions should be sent to the addresses given at the head of each problem set.

An asterisk (*) indicates that the proposer did not supply a solution. If you submit a problem without a solution, you should tell the editors whether you know (or somebody else knows) how to solve the problem. If you are uncertain, the problem might be more suitable for the Research Problems section; see the instructions given there.

Proposers are asked to aim for the same audience as for the rest of the MONTHLY; *a rule of thumb is to think of people who have had at least a year of graduate work in Mathematics, very likely some years ago. Proposals that can be both understood and solved by high school students, or by those in the first two years of college, are usually more appropriate for the Mathematics Magazine or the Two-Year College Mathematics Journal. No precise line can be drawn between elementary and advanced problems, but in general an elementary problem will involve notions no more difficult than those encountered in the first year of graduate work and should be solvable by methods common at the same level. An advanced problem may deal with more advanced concepts or require more sophisticated techniques for its solution.*

A problem that takes half a page or more to state, or that involves several ad hoc definitions, is usually unsuitable. The statement of a problem should avoid specialized jargon and symbols whenever the English language is more readily understood: for example, "f is a continuous function" is preferable to "$f \in C$."

Before submitting a problem, be sure that it says exactly what you intend it to say. Solvers are not telepathic and will usually solve what was proposed, not what you had in mind.

PROBLEMS DEDICATED TO EMORY P. STARKE

Solutions of these problems dedicated to E. P. Starke should be mailed to Prof. A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, N.M. 87131, by May 31, 1980. To facilitate consideration, type with double spacing in the format used below. If acknowledgment is desired, include a self-addressed card, label, or envelope.

S 24. *Proposed by Dragomir Ž. Djoković and Jerry Malzan, University of Waterloo, Waterloo, Ontario, Canada.*

If u is a unit column vector in C^n, let $R(u)$ be the reflection matrix $I_n - 2uu^*$. Let $a_1, \ldots, a_n$ be a basis for C^n consisting of unit vectors and $R_i = R(a_i)$. Define $A = R_1 R_2 \cdots R_n$ and $H = (A + I_n)(A - I_n)^{-1}$. [Note that $A - I_n$ is invertible.] Show that $a_i^* H a_i = 0$ for $i = 1, \ldots, n$.

S 25. *Proposed by J. Mycielski, University of Colorado, Boulder.*

Let S be the spiral $\{(\cos t, \sin t, 1/t): t > 0\}$. Is the loop $f(t) = (0, 1 + \cos t, 1 + \sin t)$, $-\pi \leqslant t \leqslant \pi$, contractible to a constant in the space $\mathbf{R}^3 \setminus S$?

ELEMENTARY PROBLEMS

Solutions of these Elementary Problems should be mailed in duplicate to Dr. J. L. Brenner, 10 Phillips Road, Palo Alto, CA 94303 (U.S.A.), by May 31, 1980. Please type with double spacing and place the solver's name and mailing address on each sheet. If acknowledgment is desired, include a self-addressed card or label.

E 2809. *Proposed by Leo J. Alex, State University College, Oneonta, N.Y.*

Let n be a positive integer greater than 1.
(a) Show that if $2^n + n^2$ is a prime, then $n \equiv 3 \pmod 6$.
(b) Investigate the converse.

E 2810. *Proposed by C. Notari, Mégrine-Coteaux, Tunisia.*

Let $T(z)$ be a polynomial with integral coefficients, having a common root with $P(z) = z^n - 1$. Supposing that for each root u_i of $P(z)$ we have $|T(u_i)| \leqslant 1$; prove that $T(z)$ is divisible by $z^n - 1$.

E 2811. *Proposed by Ç. Notari, Mégrine-Coteaux, Tunisia.*

Let $T(z)$ be a non-constant polynomial with integral coefficients and $T(0) \neq 0$. If for each root u_i of $P(z)$ we have $|T(u_i)| \leqslant 1$, prove the existence of a unique integer k $(0 \leqslant k < n)$ such that $T(z) + z^k$ or $T(z) - z^k$ is divisible by $z^n - 1$. Here $P(z) = z^n - 1$.

E 2812. *Proposed by Barry J. Powell, Kirkland, Wash.*

Prove that for every odd prime p, there exists an infinite set of pairwise relatively prime integers n such that the equation $x^{np} + y^{np} = z^{np}$ has no solution in positive integers x, y, z with $xyz \not\equiv 0 \bmod p$).

[This is an extension of the proposer's theorem in "Proof of a Special Case of Fermat's Last Theorem," vol. 85, pp. 750–751, this MONTHLY, November 1978.]

E 2813. *Proposed by A. D. Sands, University of Dundee, Scotland.*

For $0 \leqslant x < 1$, show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{[2^n x]}}{2^n} = 1 - 2x,$$

where $[u]$ is the greatest integer in u. Also, find the sum of the series for $x \geqslant 1$.

E 2814. *Proposed by Walter Leighton, University of Missouri–Columbia.*

Consider the system

$$\dot{x} = ax + by, \qquad \dot{y} = cx + dy, \tag{1}$$

where a, b, c, d are real constants with $(a-d)^2 + 4bc < 0$. It is well known that there exist linear affine transformations

$$x = pu + qv, y = ru + sv \ (p, q, r, s \text{ real constants})$$

of (1) such that

$$\dot{u} = hu - kv, \qquad \dot{v} = ku + hv. \tag{2}$$

Find one such set of values p, q, r, s.

SOLUTIONS OF ELEMENTARY PROBLEMS

Sequence of Integral Parts of $na + b$

E 2726* [1978, 593]. *Proposed by Roy Streit, Naval Underwater Systems Center, New London, Connecticut.*

For $(a,b) \in \mathbf{R}^2$, let $F(a,b)$ be the sequence $(c_0, c_1, c_2, \ldots)$ where $c_n = [na + b]$. Which (a,b) have the property that

$$F(x,y) = F(a,b) \quad \text{implies} \quad (x,y) = (a,b)?$$

Solution by M. J. Knight, University of Texas, El Paso, Texas. Any (a,b) with a irrational has the required property. Since $a = \lim c_n/n$ $(n \to \infty)$, $F(x,y) = F(a,b)$ implies $x = a$. Suppose a is rational and b arbitrary, then $\{na + b\}$ takes on only a finite number of values (here $\{\ \}$ denotes fractional part). Thus, for an $\epsilon > 0$ sufficiently small $F(a, b + \epsilon) = F(a,b)$.

If we have a irrational and $b_1 < b_2$, then since $\{an\}$ is dense on $(0,1)$, there exists an n such that $an + b_1 < m < an + b_2$ for m an integer. (See Niven, Ivan, *Irrational Numbers*, MAA, 1956, p. 75.) It follows that $F(a, b_1) \neq F(a, b_2)$, as claimed.

Also solved by R. Breusch, J. M. Cohen, J. Dou (Spain), M. W. Ecker, L. L. Foster, R. Gilmer, G. Gripenberg (Finland), W. Habakkuk, D. C. Hanna, C. Hurd, E. L. Isaacson, T. Jager, A. Jansson (Germany), E. Johnson, D. J. Kaplan, L. Keener, L. Kuipers (Switzerland), J. Levy, P. W. Lindstrom, O. P. Lossers (Netherlands), J. G. Mauldon, D. Mick, A. Nijenhuis, V. Pambuccian (Rumania), R. Pinch (England), Santa Clara Problem Solving Ring, K. R. P. Singh (India), J. C. Smith, C. R. Wall, P. C. Washburn, V. K. W. Wei, and P. Zwier.

Permuting the Digits of a Real Number

E 2738 [1978, 764]. *Proposed by Michael W. Ecker, City University of New York.*

Let σ be a permutation of the digits $0, 1, \ldots, 9$. Let f: $[0,1] \to [0,1]$ be the "extension" of σ, i.e., $f(x)$ is obtained from x by applying σ to each digit in the decimal expansion of x. (For uniqueness of decimal expansions, we do not allow expansions with all but finitely many digits equal to 9.)

(1) Find the points where f is continuous (or differentiable).

(2) Show that f is Riemann integrable and compute the integral.

Solution by Alberto Guzman, City College of CUNY. We show the following:

(i) f is always continuous, except possibly from the left at a number $a = .a_1a_2\ldots a_n00\ldots$, $a_n \neq 0$. At such a number, f is continuous iff σ fixes 0 and 9 and $\sigma(a_n - 1) = \sigma(a_n) - 1$, or else σ transposes 0 and 9 and $\sigma(a_n - 1) = \sigma(a_n) + 1$.

(ii) f is differentiable everywhere only if $\sigma = $ identity or $\sigma(i) = 9 - i$ for all i; f is nowhere differentiable in all other cases.

(iii) $\int_0^1 f(x)dx = \frac{1}{2}$.

To show (i) let $a = .a_1a_2\ldots$. For any m and all x with $.a_1a_2\ldots a_m \leqslant x < .a_1\ldots a_m99\ldots$, $f(x)$ matches $f(a)$ in the first m digits and so $|f(x) - f(a)| \leqslant 10^{-m}$; therefore f is always continuous from the right, and the same inference is valid from the left, unless $a = .a_1a_2\ldots a_n$. For continuity at such a point, it is necessary and sufficient that the sequence $x_p = .(\sigma(a_1))\ldots(\sigma(a_n - 1))$ $(\sigma(9))(\sigma(9))\ldots(\sigma(9))$ (with p digits of $\sigma(9)$) should converge to $.(\sigma(a_1))\,(\sigma(a_2))\ldots(\sigma(a_n))\,(\sigma(0))\ldots$. Such convergence occurs iff either σ fixes 9 and 0, in which case $\sigma(a_n - 1)$ must be $\sigma(a_n) - 1$, or σ

transposes 0 and 9, forcing $\sigma(a_n - 1) = \sigma(a_n) + 1$.

To show (ii), assume that f is differentiable at some $a = .a_1a_2\ldots$. Let d be any digit that appears infinitely many times in this expansion. Let $x_{i,n}$ be the number obtained by replacing the nth appearance of d with i; $x_{i,n} \to a$ as $n \to \infty$, and the quotient $(f(x_{i,n}) - f(a))/(x_{i,n} - a)$ is constant, equal to $(\sigma(i) - \sigma(d))/(i - d)$. The limits of the quotients must be the same for all i. Yet among the quotients one has denominator ± 1, another ± 5 and the numerators are between -9 and $+9$. Hence all quotients must be 1 or all of them must be -1, which forces $\sigma(i) = i$ or $\sigma(i) = 9 - i$, giving $f(x) = x$ or $f(x) = 1 - x$.

To prove (iii) we note that $\int f(x)$ exists since $f(x)$ is bounded and continuous almost everywhere. Partition the interval into 10^n equal parts and use the values of f at left endpoints. The effect of f on these numbers is to rearrange the leading n-digit sequences and to append a string of $\sigma(0)$'s. Therefore we note that this Riemann sum is a reordered sum for $\int_0^1 x\,dx$. We conclude that $\int_0^1 f(x)\,dx = \frac{1}{2}$.

Comment by the Santa Clara Problem Solving Ring: f should be defined on $[0,1)$ rather than on $[0,1]$ since $f(1.00\ldots) = \sigma(1).\sigma(0)\sigma(0)\ldots$ is not in $[0,1]$, unless $\sigma(1) = 0$, or $\sigma(1) = 1$ and $\sigma(0) = 0$. Also f assumes all values in $[0,1]$ except the number $.\sigma(9)\sigma(9)\ldots$.

Also solved completely by CWRU Problem Solving Team, E. L. Isaacson, J. Navratil (Czechoslovakia), V. Pambuccian (Rumania), K. Rusnak, and Santa Clara Problem Solving Ring. Almost complete solutions were received from F. S. Cater, M. Eisner, B. M. O'Connor, A. R. Solomon and the proposer. Almost complete solutions did not mention the continuity from the left for certain σ's at the special points indicated in (i).

Trigonometric Inequality

E 2739 [1978, 765]. *Proposed by Marvin C. Papenfuss, Loras College, Iowa.*

Prove that

$$x\sec^2 x - \tan x \leqslant \frac{8\pi^2 x^3}{(\pi^2 - 4x^2)^2} \qquad (0 \leqslant x < \pi/2).$$

Solution by Günter Bach, Universität Hohenheim, Stuttgart, Germany. From the expansion [1]: $\tan x = \sum_{n=0}^{\infty} 8x/D_n$, $(D_n = (2n+1)^2\pi^2 - 4x^2)$, we have $\sec^2 x = (\tan x)' = \sum_{n=0}^{\infty} 8/D_n + \sum_{n=0}^{\infty} 64x^2/D_n^2$, so that $x\sec^2 x - \tan x = \sum 64x^3/D_n^2$. Now $D_n^2 \geqslant (2n+1)^4E^2$ $(E = \pi^2 - 4x^2)$, so that $x\sec^2 x - \tan x \leqslant \sum 64x^3/[(2n+1)^4E^2] = [64x^3/E^2]\sum(2n+1)^{-4} = 2\pi^4x^3/(3E^2)$. (See [2].) Since $2\pi^4/3 < 8\pi^2$, the proposed inequality is improved.

References

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2. M. Abramowitz and I. Stegun, National Bureau of Standards Applied Mathematics Series no. 59, Handbook of Mathematical Tables.

Also solved by D. M. Bloom, James Smith, George Trytten, Case Western Reserve University Problem Group, G. M. Ortner, V. N. Murty, J. M. Stark, John Miltech, R. C. Carson, Peter Ehlers, Eli L. Isaacson, Thomas Jager, Otto G. Ruehr, Robert E. Shafer, St. Olaf College Problems Group, and the proposer.

The improvement was also obtained by Ruehr and Shafer; the latter called attention to another inequality, $x\sec^2 x - \tan x < 2\pi^2(\tan x - x)/E$, which he obtained by noting that $D_n^2 > (2n+1)^2ED_n$.

Similarity and the Diagonal of a Matrix

E 2741 [1978, 765]. *Proposed by H. S. Witsenhausen, Bell Laboratories, Murray Hill, New Jersey.*

Given a complex square matrix A, show that there exists a unitary matrix U such that U^*AU has all diagonal entries equal. If A is real, U can be taken real orthogonal.

Solution I. See Faddeev-Sominskii, Problems in Higher Algebra, W. H. Freeman, 1965, page 497. It is proved there that if C has trace 0, then a matrix W exists such that $W^{-1}CW$ has 0 diagonal. In fact, W can be taken to be unitary (and real orthogonal if C is real).

M. Marcus refers to P. A. Fillmore's article, On similarity and the diagonal of a matrix, this MONTHLY, 1969, pp. 167–169, where the explicit assertion is again established, but for a matrix with complex elements. The argument there is more advanced; it uses the fact that the field of values $W(A)$ of the matrix A is convex, and hence includes tr A. First suppose tr $A=0$. Then $z^*Az=0$ for some nonzero vector z. If U is a unitary matrix with first row z, then U^*AU has 0 in the $(1,1)$ position. By induction then, V^*AV has 0 diagonal for some unitary matrix V. Next, set $A=B-n^{-1}(\text{tr } B)I$. Clearly the diagonal elements of V^*BV are equal.

If A is hermitian or real, the same argument produces a real orthogonal matrix V, as soon as a real vector w can be found such that $w^*Aw=0$. To find w, use the fact that $(x+iy)^*A(x+iy)=0$, from which it follows easily that for some t, $-1\leqslant t\leqslant 1$, the vector $w=x+ty$ satisfies the requirements. Indeed, $(x-y)^*A(x-y)$, $(x+y)^*A(x+y)$ have opposite signs, or else they are both 0.

Solution II by Eli L. Isaacson, New York University. There is a numerically implementable algorithm for carrying out the transformation. The assertion is clear for a 2×2 matrix B. Using (generalized) plane rotations (which are unitary matrices) successively transform $B=[b_{ij}]$ so that $b_{11}=b_{22}$. Then transform the new matrix $B^{(1)}=[b_{ij}^{(1)}]$ so that $b_{22}^{(1)}=b_{33}^{(1)}$, and so on down the diagonal of each transformed matrix; finally proceed back up the diagonal. This series of transformations constitutes one pass. The net effect of one pass is to average the [vector of] diagonal elements, that is, to multiply this vector by a nontrivial doubly stochastic matrix. Attention to details shows that, after several passes, the diagonal will be as close to constant as desired. To prove the assertion of the problem, it is necessary to remark that the succession of (products of) unitary matrices involved does have a convergent subsequence.

Also solved by C. Bierlânt (Belgium), Å. Björck, Thomas Jager, Donald W. Robinson, and the proposer.

Å. Björck showed how to solve the problem in one pass, by writing $B=B_1+iB_2$, where $B_1=\frac{1}{2}(B+B^*)$, $B_2=\frac{1}{2i}(B-B^*)$ are both hermitian.

T. L. Markham located the problem also in M. L. Mehta's book, Elements of Matrix Theory, Delhi, 1977, theorems 11.7.1, 11.7.2.

Rarely Commuting Matrices

E 2742 [1978, 765]. *Proposed by P. M. Gibson, University of Alabama at Huntsville.*

In a ring with 1, find two matrices such that only the scalar matrices commute with both.

Solution by Duane M. Broline, Auburn University; Joe Flowers, Northeast Missouri State University; Thomas Jager, Calvin College; Michael Josephy, Universidad de Costa Rica; N. H. McCoy, Smith College; Donald W. Robinson, Brigham Young University; Gerald Thompson, Augusta College; Paul A. Vojta, Harvard University; Gregory P. Wene, University of Texas at San Antonio; and E. T. Wong, Oberlin College. For one of the matrices, take A, the matrix with $n-1$ 1's in the first superdiagonal, and zeros in the other n^2-n+1 positions. Let $X=(x_{ij})$ be arbitrary. The last row of AX is 0, so that if $AX=XA$, the last row of X is $(0,0,\ldots,0,x_{nn})$. A short induction, using powers of A in place of A, establishes that X is triangular with equal diagonal elements, so that A, A^* form a pair that satisfy the requirements.

Also solved by Theodore S. Bolis, F. S. Cater, Edmond D. Dixon, Thomas E. Elsner, Lorraine L. Foster, Robert Gilmer, Yasuhiko Ikeda, Eli L. Isaacson, Thomas Jager, Victor Pambuccian (Rumania), Santa Clara Problem Solving Ring, A. Shuchat, F. B. Strauss, Edward T. Wong, J. Zelmanowitz, Paul J. Zwier, and the proposer.

N. H. McCoy notes that if all the elements of S are in the center of the ring then $S^{-1}AS$, $S^{-1}A^*S$ can replace A, A^*. The editor asks for a characterization of all pairs A, A^* with the required property.

Circles for a Convex Polygon

E 2746* [1978, 824]. *Proposed by George F. Shum, Ohio State University.*

Let $A_1,\ldots,A_n$ be distinct non-collinear points in the plane. A circle with center P and radius r is called *minimal* if $A_kP \leqslant r$ for all k and equality holds for at least three values of k.

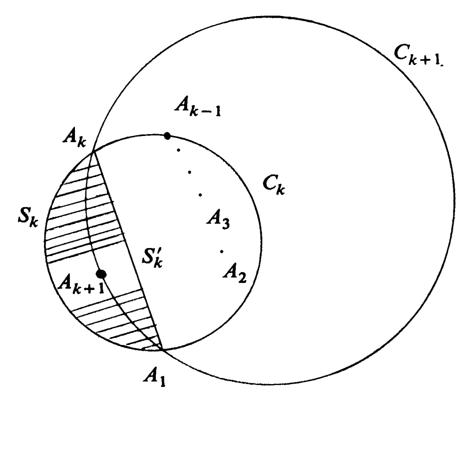
If $A_1,\ldots,A_n$ vary (n being fixed) what is the maximum number of minimal circles?

Solution by E. G. Strauss, U.C.L.A.. The answer is $n-2$. First observe that every A_i which appears on a minimal circle must be on the boundary of the convex hull of $\{A_i\}$. We may therefore restrict attention to n-tuples which are vertices of a convex n-gon, P_n.

If A_i and A_j lie on a minimal circle C_1 and A_k and A_h lie on a minimal circle C_2, and if the segments A_iA_j and A_kA_h have an interior point in common, then $C_1=C_2$. To see this, consider the convex quadrilateral $Q=A_iA_kA_jA_h$. From the fact that Q lies inside C_1 it follows that $\angle A_kA_jA_h+\angle A_hA_iA_k \leqslant 180^0$, and from the fact that Q lies inside C_2 it follows that $\angle A_iA_kA_j+\angle A_jA_hA_i \leqslant 180^0$; since the sum of all four angles is 360^0 it follows that equality holds and Q is circular.

Thus the sides of triangles which determine different circles have no interior points in common. Since P_n can be triangulated into no more than $n-2$ such triangles, it follows that there are no more than $n-2$ minimal circles.

To show that the bound $n-2$ is attained, we construct the n-tuple $\{A_k\}$ by induction so that the minimal circles are the circumcircles C_k of A_1, A_{k-1}, A_k for $k=3,\ldots,n$. Choose A_1, A_2, A_3 any non-collinear triple. Now assume that $A_1,\ldots,A_k$ ($k \geqslant 3$) have been chosen with minimal circles $C_3,\ldots,C_k$, and pick any point A_{k+1} in the interior of that segment S_k of C_k which is bounded by the chord A_1A_k and does not contain A_{k-1}. Then the circle C_{k+1} contains the segment S_k' of C_k complementary to S_k and hence encloses all points $A_1,\ldots,A_{k+1}$. On the other hand, we have $S_3 \supset S_4 \supset \cdots \supset S_k$, so that A_k lies inside all circles C_i, $i=1,\ldots,k$.



Actually it is easy to see that for any strictly convex n-gon with no four vertices concyclic there exist exactly $n-2$ minimal circles.

Also solved by Aage Bondesen (Denmark) and Paul A. Vojta, graduate student, Harvard University. Vojta's demonstration that the bound $n-2$ is attained was similar to the above. His proof that $n-2$ is in fact a bound was somewhat different. He established that there always exists a point Y in $\{A_i\}$ which lies on precisely one minimal circle. (He found such a Y on the largest minimal circle.) Since all minimal circles for $\{A_i\}$, except possibly the one that contains Y, are also minimal circles for $\{A_i\}-\{Y\}$, it follows that the number of minimal circles for $\{A_i\}$ does not exceed one more than the number of minimal circles for $\{A_i\}-\{Y\}$. By induction we can descend to a set containing three points, for which there is clearly only one minimal circle.

ADVANCED PROBLEMS

Solutions of these Advanced Problems should be sent to Professor Roger C. Lyndon, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, by May 31, 1980. To facilitate their consideration, please type (in duplicate, with double spacing) and place the solver's mailing address on the solution sheets. If acknowledgment is desired, include a self-addressed card.

6285*. *Proposed by Michael Brozinsky, Queensborough Community College, N.Y.*

Define the operation $*$ on the positive integers Z^+ by $a*b=ab+2$, and call a subset S of Z^+ *anticlosed* under $*$ if and only if $a,b\in S$ implies $a*b\notin S$.

(a) Prove that there exists an infinite set T of primes such that T is either closed under $*$ or anticlosed under $*$.

(b*) Find such a set T.

6286. *Proposed by F. Gerrish, The Polytechnic, Kingston-upon-Thames, England.*

Let R be a *-ring, i.e., a ring (not necessarily with unity element) together with a conjugation $*: R\to R$ such that $(x+y)^*=x^*+y^*$, $(xy)^*=y^*x^*$, and $x^{**}=x$ (for all x,y in R). An element x in R is called *projective* if x is idempotent and self-conjugate, i.e., $xx=x=x^*$. Let P be the set of all projective elements in R. Then the relation $\leqslant$ on P defined by "$x\leqslant y$ if and only if $x=xy$" is a partial order on P. Give an example, preferably simple, of a *-ring R for which $(P,\leqslant)$ is *not* a lattice.

(This problem arose from a remark, near the foot of page 49, in the article, "The Current Interest in Orthomodular Lattices," by S. S. Holland, in *Trends in Lattice Theory*, by J. C. Abbott, Van Nostrand, 1970.)

6287. *Proposed by Brook Taylor, Phoenix, Arizona.*

What is the smallest integer n with the following property? There exist a partition of $\mathbf{R}$ into n sets D_i and n real analytic functions f_i, each f_i defined on some open set containing D_i, such that, if we define a function f: $\mathbf{R}\to\mathbf{R}$ by setting

$$f(x)=f_i(x) \quad \text{for } x \text{ in } D_i,$$

then $f(f(x))=-x$ for all x in $\mathbf{R}$.

SOLUTIONS OF ADVANCED PROBLEMS

Rational Triangles

5499 [1967, 599]. *Proposed by D. E. Daykin, University of Malaya, Kuala Lampur, and J. S. Biggerstaff, Portland, Oregon.*

(1) Are there rational numbers a,b such that no rational numbers c,k exist for which there is a triangle having sides a, b, c and area k?

(2) Prove or disprove the conjecture: for every positive rational number k there exist positive rational a, b, c such that a triangle with sides a, b, c has area k (e.g., triangle with sides $3/2$, $5/3$, $17/6$ has area 1).

Solution by D. E. Daykin, Reading University, England. Part (1) has been dealt with [1968, 1019]. For part (2) if $k>2$ then the three rationals

$$\frac{5k^2-4k+4}{k^2-4} \quad \text{and} \quad \frac{k(k^2-4k+20)}{2(k^2-4)} \quad \text{and} \quad \frac{k+2}{2}$$

are positive and form the sides of a triangle with area k. The case $0<k\leqslant 2$ follows by a change of scale.

This solution was derived by P. and S. Chowla from N. J. Fine's unpublished solution. S. L. Segal has pointed out that it is a special case of Brahmagupta's triangle of the 7th century.

Sequences of Independent Random Variables

5884 [1972, 1140]. *Proposed by Gérard Letac, University of Clermont, France.*

Let $(X_j^{(i)})_{j=1}^{\infty}, i=1,2,\dots,d$ be sequences of independent random variables with positive integer values and having distributions not depending on j. Denote

$$S_n^{(i)}=\sum_{j=1}^{\infty} X_j^{(i)}$$

and

$$S=\inf\{s\colon \text{there exist } n_1,\dots,n_d \text{ such that } s=S_{n_i}^{(1)}=\cdots=S_{n_d}^{(d)}\}.$$

Prove that $E(X_1^{(i)})<\infty$ for all $i=1,2,\dots,d$ implies $S<\infty$ almost surely and $E(S)=E(X_1^{(1)})\cdots E(X_1^{(d)})$.

Solution by L. E. Clarke, University of East Anglia, Norwich, England. There are three corrections to be made. First, the upper limit ∞ in $\Sigma_{j=1}^{\infty}$ should be replaced by n. Also, in the second display the subscript i should be replaced by 1 in the subscript n_i to $S^{(1)}$. Third, the final equation should read

$$E(S)=\frac{h}{g_1\cdots g_d}E(X_1^{(1)})\cdots E(X_1^{(d)}), \tag{1}$$

where g_i is the period of $X_1^{(i)}$, i.e., the greatest common divisor (g.c.d.) of those n for which $P(X_1^{(i)}=n)>0 (i=1,\dots,d)$; and h is the least common multiple of $g_1,\dots,g_d$.

The problem should also state that all the random variables $X_j^{(i)}$ $(i=1,\dots,d\colon j=1,2,\dots)$ are independent, not merely that the random variables in each of the d sequences are independent (as it seems to imply).

We need the following results.

Suppose that $f_1,f_2,\dots$ are non-negative numbers for which $0<\Sigma_1^{\infty}f_n\leqslant D$. Let

$$u_1=f_1,$$

$$u_n=f_n+u_1f_{n-1}+u_2f_{n-2}+\cdots+u_{n-1}f_1 \quad (n>1),$$

$$F(x)=\Sigma_1^{\infty}f_nx^n \quad\text{and}\quad U(x)=\Sigma_1^{\infty}u_nx^n. \tag{2}$$

Then:

(a) both power series have radius of convergence $\geqslant 1$;
(b) $U(x)=F(x)+U(x)F(x)$ $(|x|<1)$;
(c) the g.c.d. of those n for which $f_n>0$ equals the g.c.d. of those n for which $u_n>0$;
(d) $u_{ng}>0$ for all sufficiently large n, where g is the g.c.d. in (c);
(e) $u_{ng}\to g/m$ as $n\to\infty$ if $\Sigma_1^{\infty}f_n=1$, where $m=\Sigma_1^{\infty}nf_n$.

These results may either be found in W. Feller, An Introduction to Probability Theory and Its Applications, vol.1, 3rd. ed., Chap. 13, or are implicit in the work of that chapter. (a)–(d) are reasonably straightforward: (e) is a consequence of Feller's "basic limit theorem" (see Feller pp. 335–338 for the basic limit theorem, and p. 313 for (e)). In Feller's terminology, u_n is the probability that a recurrent event occurs at time n, and f_n is the probability that it occurs for the first time at time n.

For $i=1,\dots,d$, let $F_i(x)$ be the probability generating function of $X_1^{(i)}$, and let $u_n^{(i)}$ be the

probability that $S_k^{(i)}=n$ for some k $(n=1,2\ldots)$. Then

$$u_n^{(i)}=\Sigma_{k=1}^{\infty}P(S_k^{(i)}=n),$$

because the events $\{S_k^{(i)}=n\}$ $(k=1,2,\ldots)$ are mutually exclusive, and so $u_n^{(i)}$ is the coefficient of x^n in the power series expansion of

$$\Sigma_{k=1}^{\infty}\{F_i(x)\}^k=F_i(x)/\{1-F_i(x)\}.$$

Therefore $U_i(x)=F_i(x)+U_i(x)F_i(x)$ $(|x|<1)$, where $U_i(x)=\Sigma_{n=1}^{\infty}u_n^{(i)}x^n$, and so, by (e),

$$u_{ng_i}^{(i)}\to g_i/m_i \quad \text{as } n\to\infty, \tag{3}$$

where $m_i=E(X_1^{(i)})$.

Now let $\mathfrak{S}$ denote the event

$$S_{k_1}^{(1)}=\cdots=S_{k_d}^{(d)} \text{ for some } k_1,\ldots,k_d,$$

and let $\mathfrak{S}$ be said to occur at time n $(\geqslant 1)$ if

$$S_{k_1}^{(1)}=\cdots=S_{k_d}^{(d)}=n \text{ for some } k_1,\ldots,k_d.$$

Then $\mathfrak{S}$ is a recurrent event (see Feller, p. 308). For $n\geqslant 1$, let u_n be the probability that $\mathfrak{S}$ occurs at time n, and let f_n be the probability that $\mathfrak{S}$ occurs for the first time at time n, i.e., $f_n=P(S=n)$. Then

$$u_1=f_1$$

and

$$u_n=f_1u_{n-1}+\cdots+f_{n-1}u_1+f_n \qquad (n>1)$$

(see Feller p. 311).

By the independence condition

$$u_n=u_n^{(1)}\cdots u_n^{(d)}.$$

Since the g.c.d. of those n for which $u_n^{(i)}>0$ is g_i (by (c)), and $u_{ng_i}^{(i)}>0$ for all sufficiently large n (by (d)), it follows that the g.c.d. of those n for which $u_n>0$ is the least common multiple h of $g_1,\ldots,g_d$. By (3)

$$u_{nh}^{(i)}\to g_i/m_i \text{ as } n\to\infty,$$

and so

$$u_{nh}\to\frac{g_1\cdots g_d}{m_1\cdots m_d} \text{ as } n\to\infty. \tag{4}$$

Since the limit is positive, $\Sigma_1^{\infty}u_n=\infty$.

Let $F(x)$ and $U(x)$ be given by (2). Then

$$F(x)=\frac{U(x)}{1+U(x)} \text{ (by (b))}$$
$$\to 1 \text{ as } x\to 1-.$$

Therefore $\Sigma_1^{\infty}f_n=1$, i.e., $P(S<\infty)=1$.

Finally, by (e),

$$u_{nh}\to\frac{h}{E(S)} \text{ as } n\to\infty, \tag{5}$$

and (1) follows from (4) and (5).

Uniform Distribution

6024 [1975, 409]. *Proposed by L. Kuipers, Southern Illinois University.*

If α is rational and different from 0, and β is irrational, then the sequence $([n\alpha]n\beta)$, $n=1,2,\ldots$ is uniformly distributed mod 1.

Solution by H. Niederreiter, University of the West Indies, Kingston. Write $\alpha=r/s$ with integers r and s satisfying $r\neq 0$, $s\geqslant 1$. Partition the sequence (x_n), $n=1,2,\ldots$, with $x_n=[n\alpha]n\beta$ into s subsequences, by considering for $t=1,2,\ldots,s$ the sequences (x_{ks+t}), $k=0,1,\ldots$. For fixed t we have

$$x_{ks+t}=\left[(ks+t)\frac{r}{s}\right](ks+t)\beta=\left(kr+\left[\frac{rt}{s}\right]\right)(ks+t)\beta=f_t(k)$$

for $k=0,1,\ldots$, where f_t is the polynomial $f_t(x)=rs\beta x^2+(rt+s[rt/s])\beta x+t[rt/s]\beta$. Since f_t has an irrational leading coefficient, it follows from a classical result on uniform distribution (see Theorem 3.2 on p. 27 in L. Kuipers and H. Niederreiter, Uniform Distribution of Sequences, Wiley, New York, 1974) that the sequence (x_{ks+t}), $k=0,1,\ldots$, is uniformly distributed mod 1. Since a superposition of finitely many uniformly distributed sequences mod 1 is easily shown to be uniformly distributed mod 1 (see also Exercise 1.18 on p. 180 of L. Kuipers and H. Niederreiter, op. cit.), it follows that the original sequence (x_n), $n=1,2,\ldots$, is uniformly distributed mod 1.

Also solved by the proposer.

Torsion Group with Two Generators

6052* [1975, 857]. *Proposed by J. R. Gard, University of South Florida.*

If G is a torsion group such that there exists an element $x\in G$ with the property that x and y generate G whenever $y\in G$ is not a power of x, is G finite? And what other properties does G have?

Comment by the editors. Ol'shanskiĭ [Izvestiya, 1979–1980] and, independently, E. Rips, have found infinite groups that respond to the question. These are infinite torsion groups, all of whose maximal subgroups are cyclic.

Ranges in Banach Spaces

6203 [1978, 203]. *Proposed by Albert Wilansky, Lehigh University.*

Let X, Y, Z be Banach spaces and $T:X\to Z$, $S:Y\to Z$ continuous and linear. Show that the (equivalent) conditions

(i) $\overline{TD_1}\supset SD_\varepsilon$ for some $\varepsilon>0$ (D_ε is the disk of radius ε),

(ii) $\|S'(f)\|\leqslant k\|T'(f)\|$ for all $f\in Z'$ for some $k>0$,

do not imply that $TX\supset SY$. (See M. Embry: Proc. A. M. S. 38(1973), 587–589.)

Solution by Gustaf Gripenberg, Helsinki University of Technology. Let $X=C([0,1])$ (with sup norm), $Y=L^\infty(0,1)$ and $Z=L^1(0,1)$ and let T: $X\to Z$ and $S:Y\to Z$ be the natural injections. Clearly T and S are continuous and linear and $SD_1\subset\overline{TD_1}$. But it is also obvious that $SY\not\subset TX$.

The same solution was given by Dave Joyner & Fred Roush, and by Bertram Walsh. Also solved by J. Borwein and the proposer. Borwein shows that (i) implies $TX\supset SY$ for all S iff T is almost range closed (Wilansky, Topological Vector Spaces, McGraw-Hill, 1978, Problems 11-4-114, 11-4-120), and conjectures that this holds for all T iff X is reflexive.

REVIEWS

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We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

Dynamic Topology. By Gordon Whyburn and Edwin Duda. Springer-Verlag, New York, 1979. xi + 152 pp. $12.00. (Telegraphic Review, August-September 1979.)

From the Preface: "This book was prepared from a set of notes used by the late Professor Gordon T. Whyburn in an introductory course in topology. The intention of the book is to lead the student, through his own efforts, rather quickly to some important theorems concerning mappings on topological spaces, in particular mappings on continua or generalized continua.... The book, for the most part, is set up in the form of definitions and notations followed by exercises for the student to attempt and then solutions to the exercises."

The book has a Foreword by John L. Kelley. From the Foreword: "It was not a lecture course. He gave definitions, drew pictures, gave examples, and stated theorems. Our task was to prove the theorems and to present the proofs in class. The ground rules were simple: we were each to work alone, and we were not to consult references until after the results in question had been proven in class." Kelley took a forerunner of the course presented in this book in the academic year 1937-1938. It was my privilege to be a member of the class the first time Whyburn offered this course at Charlottesville, his initial year there, 1934-1935. At that time we were permitted to work together, but "hints for the solutions" were never given—and I don't mean "hardly ever." I remember quite vividly that we worked for five or six weeks on what was then called "Theorem 6," but which is now Exercise 10, page 7, of this new text. We could make no progress so we went, as a group, to ask G. T. for assistance. We got none, and were so angry that, almost immediately, we worked out several correct proofs, by ourselves. The impact of this course, and the scholarly atmosphere that was built up when G. T. would greet us in the morning with "What's the news?" before he said "Good morning!" is almost beyond belief. We found out that we could, so we did. Guilford Spencer and I wrote a text on *Topology,* and Kelley's book *General Topology* has become a classic. So much for background. The point is that I was so thoroughly trained in this course, and in my subsequent work with Whyburn, that a great deal of it remains with me after almost half a century.

The text is divided into sections, with the first part of each section consisting of definitions, examples, and statements of theorems. This is followed by a set of exercises. Complete solutions of all the exercises are then given. Thus the "Solutions" occupy a great deal more than half the text. They provide a splendid reference for those of us who wish to refresh our memories or discover how very nicely these notes have been modernized since those very early days. But these very solutions mean that I can never consider using this book as a text for my undergraduate course. Imagine that you, a student, have come to Section XIX, "Simple Arcs and Simple Closed Curves," page 70. Near the bottom of this left-hand page is:

Exercise 1. A nondegenerate metric continuum T is a simple arc "from a to b" (with end points a and b) if and only if each point of T other than a and b separates a and b in T.

You look at the facing page (page 71) headed by the word "Solutions." Unless you are blind in your right eye, it is clear to you that the solution to Exercise 1 requires at least a full page of print. Will you struggle for several weeks to devise your own independent proof, or will you peek at page 71, and, perhaps, turn the page?

DICK WICK HALL, State University of New York, Binghamton

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print.*

GENERAL, S***, L***. *The Chess Mysteries of Sherlock Holmes.* Raymond Smullyan. Alfred A. Knopf, 1979, xiii + 171 pp, $10. [ISBN: 0-394-50488-7] Fifty captivating chess problems of the "retrograde" type, where the object is to deduce certain events in the game's past, each set up and integrated into a whole by a stimulating Holmes-Watson dialogue. Along with Watson, the intrepid Holmes uses his remarkable abilities of deduction to awe and delight, and to demonstrate precisely what must have happened, move by move, at the "scene of the crime." If you like intellectual puzzles of any kind, this book should be in your collection. LCL

GENERAL, P, L. *Mathematics and the Real World.* Ed: Bernhelm Booss, Mogens Niss. Birkhäuser, 1979, 136 pp, $22 (P). [ISBN: 3-7643-1079-0] Largely unedited proceedings (talks, discussion, notes) of an International Workshop held at Roskilde, Denmark ("close in space and time" to the 1978 meetings of the International Congress of Mathematicians) to massage problems (scientific, psychological, ethical, economic) caused by the changing relation between theoretical mathematics and the needs of society. Includes an extensive bibliography of books about mathematics. LAS

GENERAL, P. *Lecture Notes in Mathematics-710: Séminaire Bourbaki Vol. 1977/78 Exposés 507-524.* Springer-Verlag, 1979, iv + 328 pp, $16 (P). [ISBN: 0-387-09243-9]

PRECALCULUS, S. *Barron's How to Prepare for the College Board Achievement Test in Mathematics Level 2.* Howard P. Dodge. Barron's Educ Ser, 1979, v + 144 pp, $5.95 (P). [ISBN: 0-8120-0325-X] Includes a diagnostic test, review of topics of advanced algebra and trigonometric functions, exercises, and five model examinations with explanations of all answers. LLK

HISTORY, P. *Mathesis en Mystiek.* G. Mannoury. Bohn, Scheltema & Holkema, 1978, xii + 116 pp, (P). [ISBN: 90-313-0344-5] A republication of the 1924 philosophical treatise in Dutch. JAS

HISTORY, S(16-18), P, L. *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert, Reprint.* Felix Klein. Springer-Verlag, 1979, xxiii + 593 pp, $19.80 (P). [ISBN: 0-387-09235-8] A reprint of both original (1926) volumes in one relatively inexpensive paperback. JAS

HISTORY, P, L. *Saunders Mac Lane, Selected Papers.* Ed: I. Kaplansky. Springer-Verlag, 1979, xiii + 556 pp, $29.80. [ISBN: 0-387-90394-1] 21 papers, supplemented by a biographical note by Alfred Putnam, personal tribute by Roger Lyndon, technical reviews of Mac Lane's work by Irving Kaplansky, Samuel Eilenberg and Max Kelly, a complete Mac Lane bibliography and list of his Ph.D. students. LAS

FOUNDATIONS, S(17-18), P. *Was Ist Und Was Soll Die Mathematische Biologie?* Detlef D. Spalt. Wissenschaftliche Buchgesellschaft, 1979, xxi + 161 pp, (P). [ISBN: 3-534-08016-5] A philosophical discussion, in the form of several dialogues with footnotes, of the nature of mathematical--more accurately, what the author calls "algebraic" (i.e., not analytic or topological)--biology. Argues, among other things, that most of what passes for biomathematics is really biophysics. JD-B

COMBINATORICS, S(16-17), P, L. *Combinatorial Problems and Exercises.* L. Lovász. North-Holland, 1979, 551 pp, $26.75 (P). [ISBN: 0-444-85219-0] This book consists of three parts. The first consists of over 600 challenging exercises from combinatorics with an emphasis on graph theory. The second part gives hints to the exercises, and the third gives detailed solutions. Includes a glossary of terms.CEC

COMBINATORICS, T(17: 2), S, L. *Graph Theory, An Introductory Course.* Béla Bollobás. Grad. Texts in Math., V. 63. Springer-Verlag, 1979, x + 180 pp, $16.80. [ISBN: 0-387-90399-2] This modest-sized book is not at all modest in content. In addition to the usual topics dealing with the structure of graphs, it includes chapters on graphs and groups, electrical networks, extremal problems, coloring and much more. Most attractive are notes (appended to each chapter) that provided an excellent guide to the literature. Many interesting non-routine exercises. SS

LINEAR ALGEBRA, T(15-16: 1, 2). *Fundamentals of Linear Algebra, Second Edition.* Katsumi Nomizu. Chelsea, 1979, x + 325 pp, $14.95. [ISBN: 0-8284-0276-0] First edition published 1966. Only changes in this one are correction of typographical errors and provision of answers to more problems. JD-B

LINEAR ALGEBRA, T(15: 1). *Elements of the Theory of Generalized Inverses for Matrices.* Randall E. Cline. EDC/UMAP, 1979, vi + 87 pp, (P). A concise introduction to the theory of generalized inverses for matrices. Covers the Moore-Penrose inverse, Drazin inverse, and includes a short chapter on inverses that are not unique. LLK

LINEAR ALGEBRA, S**(15-16), P**, L**. *Circulant Matrices.* Philip J. Davis. Wiley, 1979, xv + 250 pp, $18.95. [ISBN: 0-471-05771-1] Artfully woven blend of elementary geometry (much in the complex plane) and matrix algebra. The complete theory of circulants follows their introduction via nested polygons. Fascinating further applications of circulants in geometry. Generalizations. A reference on circulants and a source of alternate or supplemental material for courses in matrix algebra. Bibliography with about 140 entries. Over 200 problems at various levels of difficulty. A gem of a book. JK

LINEAR ALGEBRA, T, L. *Matrices and Linear Programming with Applications.* Toshinori Munakata. Holden-Day, 1979, ix + 469 pp, $18. [ISBN: 0-8162-6166-0] Intended as a text for a course in matrix algebra (not linear algebra, e.g., linear independence is mentioned only in passing) with applications to business, game theory, Markov chains and particularly, linear programming (including duality and sensitivity analysis). The low theoretical level is compensated for with a multitude of examples and exercises. TAV

ALGEBRA, T(18: 1), S, P. *Lecture Notes in Mathematics-723: Commutative Rings whose Finitely Generated Modules Decompose.* Willy Brandal. Springer-Verlag, 1979, 116 pp, $9 (P). [ISBN: 0-387-09507-1] A commutative ring R with identity is an FGC ring if every finitely generated unitary R-module is a direct sum of cyclic submodules. The author's intent is to "characterize the FGC rings and give as many examples as possible" at a level of exposition accessible to a graduate student who has had a "good" first year of algebra. In this he is relatively successful, obtaining a characterization in terms of valuation rings, Bezout domains, and torch rings. Bibliography, index. JS

ALGEBRA, T(18: 1), S, P. *The Burnside Problem and Identities in Groups.* S.I. Adian. Ergebnisse der Math., B. 95. Springer-Verlag, 1979, x + 311 pp, $39. [ISBN: 0-387-08728-1] An English translation from the 1975 Russian version, presenting a simplified version of the Novikov-Adian negative solution to the Burnside conjecture on finitely generated groups of finite exponent. Last chapter includes further applications of techniques developed in the proof. Bibliography, subject and notational indexes. JS

ALGEBRA, S(16-17), P. *Théorie des Treillis, En Vue Des Applications.* A. Kaufmann, G. Boulaye. Masson (US Distr: SMPF, 14 E. 60th St., NY 10022), 1978, 146 pp. [ISBN: 2-225-48862-2] A rigorous presentation of lattice theory from an algebraic viewpoint. The final chapter studies Boolean algebras. Many typos to confuse and dismay. TLS

FINITE MATHEMATICS, T(13: 2). *Finite Mathematics and Calculus with Applications.* Thomas Koshy. Goodyear, 1979, xiv + 578 pp, $16.95. [ISBN: 0-87620-321-7] A text in which concepts are carefully reinforced with examples and applications. An excellent approach for a general audience. LLK

NUMERICAL ANALYSIS, P. *Lecture Notes in Mathematics-704: Computing Methods in Applied Sciences and Engineering, 1977, I.* Ed: R. Glowinski, J.L. Lions. Springer-Verlag, 1979, vi + 391 pp, $17.80 (P). [ISBN: 0-387-09123-8]; *Lecture Notes in Physics-91: Computing Methods in Applied Sciences and Engineering, 1977, II,* vi + 359 pp, $17.80 (P). These two volumes together contain the proceedings of the symposium which was held December 5-9, 1977 at IRIA LABORIA in Rocquencourt, France. JAS

FUNCTIONAL ANALYSIS, T(17-18: 1), S. *Nichtlineare Funktionalanalysis.* Hansgeorg Jeggle. B.G. Teubner Stuttgart, 1979, 255 pp, (P). [ISBN: 3-519-02057-2] An introduction to existence theory for solutions of nonlinear problems, with applications to differential and integral equations. Assumes knowledge of the elements of functional analysis but not of homology. Some problems, many comments on sources of results and directions for further study. JD-B

FUNCTIONAL ANALYSIS, P. *The Classification and Structure of C*-algebra Bundles.* Maurice J. Dupré. Memoirs No. 222. AMS, 1979, ix + 77 pp, $6.40 (P). [ISBN: 0-8218-2222-5] The author contends that a C*-algebra bundle is the appropriate vehicle to generalize to the non-commutative case the relationship between a commutative Banach algebra and its maximal ideal space. He deals mainly with C*-bundles having finite-dimensional fibres and obtains a classification for second order bundles stated in terms of homotopy classes. Quite technical and demanding a sophisticated reader. Bibliography, no index. JS

OPTIMIZATION, T*, P*, L. *Multiobjective Programming and Planning.* Jared L. Cohon. Math. in Sci. and Eng., V. 140. Acad Pr, 1978, xiv + 333 pp, $23. [ISBN: 0-12-178350-2] In the introductory chapters, the author gives a compelling argument for considering many optimization problems as multiobjective problems, particularly in the area of public planning. After a discussion of the mathematical prerequisites, he discusses the formulation of solution techniques for such problems in a variety of settings. A chapter on methods applicable to a multiple-decision scenario precedes some very powerful case studies. The book contains an extensive and very useful bibliography. A welcome addition to the operations research literature, this would easily serve as a text for a one-semester course. TAV

OPTIMIZATION. *Lecture Notes in Control and Information Sciences-15: Semi-Infinite Programming.* Ed: R. Hettich. Springer-Verlag, 1979, x + 178 pp, $9.80 (P). [ISBN: 0-387-09479-2] Proceedings of workshop at Bad Honnef (Germany), August 1978. Contains a dozen papers, 3 in each category: theory, linear methods, nonlinear methods, and applications. TAV

ANALYSIS, S(16-18), P. *Approximation with Rational Functions.* D.J. Newman. CBMS Reg. Conf. in Math., No. 41. AMS, 1979, iii + 52 pp, $8.80 (P). [ISBN: 0-8218-1691-8] Reproduction of ten delightful expository lectures on approximation with rational functions given at the CBMS Regional Conference at the University of Rhode Island in June, 1978. Peppered with remarks reflecting keen discernment. Humorous asides. Light in tone and delivered with aplomb. Many surprises with peeks at a master at work. From approximation to x^k and e^x, Müntz rational functions and quadrature in Hardy spaces to open questions. References. JK

ANALYSIS, T(15-17: 1), S, L. *Digital Spectral Analysis.* C.K. Yuen, D. Fraser. Fearon-Pitman, 1979, viii + 156 pp, $12 (P). [ISBN: 0-273-08439-9] Expanded notes from a 1975 course at the Australian National University, designed not for engineers, but for workers in any field; presumes only advanced calculus. Covers Fourier and fast Fourier transforms, spectrum estimation, digital filtering, and diverse applications. LAS

ALGEBRAIC GEOMETRY, P. *Lectures on Introduction to Moduli Problems and Orbit Spaces.* P.E. Newstead. Tata Inst, 1978, vi + 183 pp, $9.90 (P). A monograph based on the author's lectures at the Tata Institute of Fundamental Research in 1975 and aimed at presenting a more accessible, simplified

version of parts of D. Mumford's book *Geometric Invariant Theory*. Topics include the concept of moduli, vector space endomorphisms, quotients, and vector bundles over a curve. TRS

DIFFERENTIAL GEOMETRY, T(16-18), S, P, L. *Singular Points of Smooth Mappings*. C.G. Gibson. Research Notes in Math., No. 25. Fearon-Pitman, 1979, 239 pp, $16.50 (P). [ISBN: 0-273-08410-0] An introduction to the "less technical" aspects of the theory of singularities, providing a foundation for both differential geometry and catastrophe theory. Concentrates on elementary cases rather than general theory. LAS

DIFFERENTIAL GEOMETRY, S(17-18), P. *Surfaces of Nonpositive Curvature*. Patrick Eberlein. Memoirs No. 218. AMS, 1979, x + 90 pp, $6.40 (P). [ISBN: 0-8218-2218-7] Topologically noncompact surfaces are Riemann surfaces with a finite number of points removed. Geometrically one can study the isometry group of the surface by classifying the behavior "near" these removed points. Like most classification problems, this one brings much good geometry to bear in a clever entertaining way. TLS

GEOMETRY, S, P, L**. *Selected Papers on Geometry*. Ed: Ann K. Stehney, *et al.* Brink Selected Papers, V. 4. MAA, 1979, x + 338 pp, $20 [ISBN: 0-88385-204-7] 37 papers reprinted from the *Monthly* and *Mathematics Magazine*, arranged in chronological order "to exhibit, as much as possible, the changing emphasis within the geometry curriculum." Concludes with a selection of problems (with separate solutions), a supplementary bibliography, and author and keyword indexes. LAS

PROBABILITY, P. *Multidimensional Diffusion Processes*. Daniel W. Stroock, S.R. Srinivasa Varadhan. Grund. der math. Wissenschaften, B. 233. Springer-Verlag, 1979, xii + 338 pp, $34.80. [ISBN: 0-387-90353-4] In order to elucidate the martingale approach to the theory of Markov processes, the authors develop carefully the area of diffusion theory in R^d. The book provides a well written and complete treatment. Contains an extensive bibliography. TAV

PROBABILITY, P. *Markov Chain Models--Rarity and Exponentiality*. Julian Keilson. Appl. Math. Sci., V. 28. Springer-Verlag, 1979, xiii + 184 pp, $12 (P). [ISBN: 0-387-90405-0] Presumes a familiarity with Feller's Volume I and parts of Volume II. The object of the book is the development of the tools needed to quantify the ergodic and transient behaviour of systems of many degrees of freedom. An underlying theme is reversibility. References, index. TAV

PROBABILITY, T(17-18: 1), S, P. *Bedienungsprozesse*. Gennadi P. Klimow. Math. Reihe, B. 68. Birkhäuser, 1979, xi + 244 pp, $38. [ISBN: 3-7643-1049-9] A systematic development of queueing theory by analytic rather than graph-theoretic methods, intended for probabilists as well as engineers and other users. Problems. JD-B

COMPUTER PROGRAMMING. *Tiny Assembler 6800: Design and Implementation of a Microprocessor Self Assembler*. Jack Emmerichs. BYTE Pub, 1978, v + 74 pp, $9 (P). [ISBN: 0-931718-08-2] Contains articles on the design and implementation of the Tiny Assembler. Also contains its User's Guide, appendices with source code listings for versions 3.0 and 3.1, and bar code representation of the object code for version 3.1. RJA

COMPUTER PROGRAMMING, S, P. *How to Program Your Programmable Calculator*. Stephen L. Snover, Mark A. Spikell. P-H, 1979, xiv + 271 pp, $6.95 (P); $7.95. [ISBN: 0-13-429357-6; 0-13-429365-7] Designed specifically to teach programming on the TI-57, EC-4000 and HP-33. Goes through the use of all the functions along with decision making and loops. Meant to be used as a self-study manual with many examples which are to be followed with a calculator while you read. Lots of exercises. CEC

COMPUTER PROGRAMMING, S, P, L. *FORTRAN 77*. Harry Katzan, Jr. D. Van Nostrand, 1978, xvi + 207 pp, $10.95. [ISBN: 0-442-24278-6] Presents the new Fortran Standard which replaces the previous one defined in 1966. Delineates the extensions in the areas of I/O facilities, data declaration facilities, subprogram facilities, integer valued expressions, and various structured programming enhancements. References. Appendices. Index. RJA

COMPUTER PROGRAMMING, S(13), L. *Computer Games for Businesses, Schools, and Homes*. J. Victor Nahigian, William S. Hodges. Winthrop Pub, 1979, xiii + 157 pp, $10.95 (P).[ISBN: 0-87626-166-7] Written on the premise that playing computer games "helps you learn the computer language and also gives you more confidence in the use of a computer terminal," this book contains a description and common use list (in Basic) for 27 games from Tic-Tac-Toe to the familiar Star Trek. No analyses of how the programs work are given--unfortunately. TAV

COMPUTER PROGRAMMING, T(13: 2), S, L. *Introduction to Computer Programming*. Walter S. Brainerd, Charles H. Goldberg, Jonathan L. Gross. Har-Row, 1979, xxi + 534 pp, $16.95. [ISBN: 0-06-042396-X] A highly readable and detailed introduction to programming via the language BPL (Beginner's Programming Language). Discusses loops, lists, subroutines, string variables, data processing and simulation. Includes many interesting examples and exercises. CEC

COMPUTER PROGRAMMING, T*(13: 1), S, L. *Introduction to BASIC Programming: A Structured Approach*. Peter B. Worland. HM, 1979, xii + 328 pp, $10.50 (P). [ISBN: 0-395-26775-7]; *Solutions Manual*, x + 68 pp, $.60 (P). [ISBN: 0-395-26776-5] A highly detailed, well-written introduction to Basic, stressing programming style. Many good examples and exercises. CEC

COMPUTER SCIENCE, T(13: 1), S. *Understanding Computers*. Paul M. Chirlian. Dilithium Pr, 1978, x + 193 pp, $8.95 (P). [ISBN: 0-918398-15-0] A very readable introduction to how computers work. In addition to describing the parts of a computer, the author discusses elementary machine language programming, assembly languages and higher level languages, with the intent not to teach specific languages but to tell what languages are and how they operate. Exercises. TRS

COMPUTER SCIENCE, S(13-18). *Programming Techniques, Volume I: Program Design.* Ed: Blaise W. Liffick. BYTE Pub, 1978, 102 pp, $6 (P). [ISBN: 0-931718-12-0] A collection of papers submitted to BYTE Publications concerning the design of efficient, effective, maintainable programs for use on home computers. Includes structured program design, modular programming techniques, program logic design, common errors, hash tables, and binary tree processing. RJA

COMPUTER SCIENCE, T(15-18: 1, 2), S, P, L. *Information Retrieval: Computational and Theoretical Aspects.* H.S. Heaps. Acad Pr, 1978, xii + 344 pp, $19.50. [ISBN: 0-12-335750-0] Commences with a survey of main ideas and proceeds to chapters on document data bases, question logic, data structures, search programs, vocabulary characteristics, information theory, coding and compression of data bases, document indexing, term associations, automatic question modification and document classification. Text attempts to close the gap between relevant computer science concepts and information science. Chapter problems. Subject index. RJA

COMPUTER SCIENCE, L*, P*. *Machines Who Think: A Personal Inquiry into the History and Prospects of Artificial Intelligence.* Pamela McCorduck. Freeman, 1979, xiv + 375 pp, $14.95. [ISBN: 0-7167-1072-2] A lively, personal history of artificial intelligence, featuring the words, the hopes and the dreams of those who "were present when this art was transformed into a science." Written by an experienced novelist, *Machines Who Think* sets artificial intelligence research in both a scientific and literary context, tracing its roots, not just in science, but also in philosophy, literature and religion. A superb bridge between the two cultures. LAS

COMPUTER SCIENCE, T*(13: 1), L. *Problem Solving and Basic, A Modular Approach.* Frances G. Gustavson, Marian V. Sackson. SRA, 1979, x + 251 pp, $9.95 (P). [ISBN: 0-574-21240-X] An excellent introductory computer science text, which successfully integrates algorithm development, language, fundamentals and computer concepts. An appealing text for majors and nonmajors alike. Lots of exercises. TRS

COMPUTER SCIENCE, T*(13-18: 1), S, L. *Programming Language Structures.* Elliott I. Organick, Alexandra I. Forsythe, Robert P. Plummer. Acad Pr, 1978, xviii + 659 pp, $19.95. [ISBN: 0-12-528260-5] Consists of three parts: (1) basic concepts and models for understanding syntax and semantics; (2) comparative semantics and syntax of Algol, Fortran, Lisp, and Snobol; (3) recent language design directions--asynchronous tasks, coroutines, data structure and management issues. Emphasizes semantics throughout with the use of contour diagrams that contain the control structures and data structures of each language situation represented. Special feature is the extensive chapter devoted to recursion. Chapter references and exercises. Appendix. Bibliography. Answers to selected exercises. Index. RJA

SYSTEMS THEORY, P. *Control and Dynamic Systems, V. 15: Advances in Theory and Application.* Ed: C. T. Leondes. Acad Pr, 1979, xviii + 360 pp, $19.50. [ISBN: 012-012715-6] A diverse collection of mathematical models for complex systems: prescription of clinical drugs, chemical engineering, fish behavior, control of chemical reaction processes, water resource systems, sensitivity analysis, and adaptive filtering. LAS

APPLICATIONS (ENGINEERING), S(16-17), P. *Applied Mathematics: An Intellectual Orientation.* Francis J. Murray. Math. Concepts and Methods in Sci. and Eng., V. 12. Plenum Pr, 1978, xiv + 225 pp, $29.50. [ISBN: 0-306-39252-6] The author analyzes the characteristics and structure of the computer-activated simulations used in modern applied mathematics. The justification for the use of these simulations is that they incorporate scientific and technical understanding in a mathematically formulated model. On the basis of a historical discussion, the author argues that mathematical developments structured scientific understanding, but that the intensive effort to establish what is logically acceptable has isolated mathematics from the other sciences. He also discusses philosophical questions associated with mathematically formulated understanding. RPB

APPLICATIONS (MODELLING), P, L. *New Trends in Mathematical Modelling.* Ed: A. Straszak, J.W. Owsiński. PWN, 1978, 368 pp, (P). Proceedings of the Second International Seminar on Trends in Mathematical Modelling, Jabłonna, Poland, December 1974. Papers on a wide variety of modelling techniques and experiences. Most models considered are large scale, computer based. TAV

APPLICATIONS (PHYSICS), P. *Convexity in the Theory of Lattice Gases.* Robert B. Israel. Princeton U Pr, 1979, lxxxv + 167 pp, $16.50; $6.95 (P). Classical and quantum lattice systems are examined in terms of equilibrium statistical mechanics, with emphasis on the characterization of translation-invariant equilibrium states by a variational principle and on the use of convexity in studying these states. A motivational and historical introduction is written by A.S. Wightman. TRS

APPLICATIONS (SOCIAL SCIENCE), S. *Voting Made Easy: A Mathematical Theory of Election Procedures.* Lewis Lum, David C. Kurtz. Journal of Undergraduate Mathematics (Dept. of Math., Guilford Coll., Greensboro, NC 27410), 18 pp, $3.50 (P). Brief résumé of Arrow's theorem and related results, with some history, some proofs, exercises and a minimal bibliography. Suitable as the starting point for a senior seminar. LAS

Reviewers Whose Initials Appear Above

Richard J. Allen, St. Olaf; Ralph P. Boas, Northwestern University; Clifton E. Corzatt, St. Olaf; John Dyer-Bennet, Carleton; Lorraine L. Keller, St. Olaf; Joseph Konhauser, Macalester; Loren C. Larson, St. Olaf; Thomas R. Savage, St. Olaf; John Schue, Macalester; Seymour Schuster, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; Timothy L. Strotman, St. Olaf; T.A. Vessey, St. Olaf.

NEWS AND NOTICES

EDITED BY PAUL A. HAEDER, University of Nebraska at Omaha

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036

PERSONAL ITEMS

CORRECTION: An item that appeared in the October, 1979, *Monthly* (page 718) was worded incorrectly. It should have read: Professor Gail S. Young, University of Rochester, has been appointed Professor of Mathematics and Chairman of the Department of Mathematics and Statistics at Case Western Reserve University. He was President of the Mathematical Association of America in 1969-70.

John Carroll University, Cleveland, Ohio: Dr. James F. Smith, former Dean of LeMoyne College, is Visiting Professor for the academic year 1979-80. Associate Professors Leo J. Schneider and Robert J. Kolesar have been promoted to Professors.

Vassar College: John H. McCleary, Bates College, and Frank Purcell, Columbia University, have been appointed Assistant Professors.

Adrian College, Adrian, Michigan: Dr. James O. Watson has been promoted to Associate Professor. Associate Professor Margaret O. Marchand has been appointed Chairperson of the Mathematics Department.

Ripon College, Ripon, Wisconsin: Associate Professor C. Wayne Larson has been promoted to Professor. Assistant Professor Karl A. Beres has been promoted to Associate Professor.

McNeese State University, Lake Charles, Louisiana: Assistant Professor Frank W. Carter has been promoted to Associate Professor. Assistant Professor Colleen O'Neal has retired.

Rose-Hulman Institute of Technology, Terre Haute, Indiana: Dr. Lo-Yung Su, formerly of Texas A. & M. University, has been appointed Assistant Professor. Assistant Professor Roger G. Lautzenheiser has been promoted to Associate Professor.

Goucher College, Baltimore, Maryland: Associate Professor Elaine Koppelman has been promoted to Professor and has been named Chairman of the Mathematics Department. Assistant Professor Robert Lewand has been promoted to Associate Professor. Professor Dorothy L. Bernstein has retired with the title of Professor Emeritus. (Professor Bernstein is President of the Mathematical Association of America).

Delta State University, Cleveland, Mississippi: Mrs. Rose E. Strahan has been appointed Acting Chairman of the Department of Mathematics. Professor Eleanor Walters has retired with the title of Professor Emeritus.

University of Texas at San Antonio: Dr. Patricia Semmes, formerly at Montgomery College, has been appointed Assistant Professor. Dr. Gregory Wene, University of Iowa, has been appointed Assistant Professor.

Professor Kenneth P. Goldberg, New York University, has been awarded tenure as of September 1979.

Dr. Jonathan Skinner, formerly at the University of Pennsylvania and St. John's College, Annapolis, Maryland, has been appointed Associate Professor of Mathematics and Statistics at St. Mary's College, Winona, Minnesota.

Professor James H. Stoddard, Montclair State College, is on sabbatical leave at Bell Laboratories, Murray Hill, New Jersey, from September 1, 1979, to May 30, 1980.

Professor Ernest Heighton, Dalhousie University, Halifax, Nova Scotia, has retired.

Association Professor and Chairman, R. Vic Morgan, Sul Ross State University, Alpine, Texas, has been appointed Director of the Science Division.

Associate Professor, Ray O. Hamel, Eastern Washington University at Cheney, has been promoted to Professor.

Dr. Delia Klingbell, Senior Analyst, Sonalysts, Inc., is Visiting Assistant Professor at Connecticut College in New London.

Assistant Professor, Kathleen A. Hall, Southern Technical Institute, Marietta, Georgia, has been promoted to Associate Professor.

Professor B. D. Arendt, University of Missouri-Columbia, has resigned to enter private business.

Associate Professor Stavros Busenberg, Harvey Mudd College, Claremont, California, has been promoted to Professor.

Assistant Professor Kent Wooldridge, California State College, Stanislaus, has been promoted to Associate Professor.

Associate Professor David F. Addis, Texas Christian University, assumed the Chairmanship of the Department of Mathematics on June 1, 1979.

Mr. Alan Avery, University of Illinois, has been appointed Instructor at East Texas State University, Commerce.

Associate Professor Robert S. Fisk, Pacific Lutheran University, has accepted a position at Colorado School of Mines, Golden.

Professor Stanley Gudder, University of Denver, has been named Chairman of the Department of Mathematics and Computer Science.

Dr. Theodore S. Erickson, formerly at Hunter College, has been appointed Assistant Professor at Bethany College, Bethany, New York.

Professor Nathan Simms, Winston-Salem State University, North Carolina, has been named Division Director of Liberal Arts and Sciences.

Dr. Wulf T. Rossmann, formerly of McMaster University, has been appointed Assistant Professor at the University of Ottawa, Ontario.

Mr. Charles Babb has been appointed Instructor II and Head of the Mathematics Department at Michigan Christian College, Rochester, Michigan.

Dr. Curtis D. Herink, formerly at Allegheny College, has been appointed Assistant Professor of Mathematics at Augustana College, Rock Island, Illinois.

H. Leon Harter has retired from Federal service after more than twenty six years as a Mathematical Statistician at Wright-Patterson Air Force Base. He has accepted an appointment as a part-time Research Professor of Mathematics at Wright State University.

Professor S. J. Farber, Mt. Vernon, New York, died on May 7, 1979, at the age of 39. He was a member of the Association for thirteen years.

Professor N. J. Hicks, University of Michigan, died in September, 1979, at the age of 50. He was a member of the Association for five years.

Mr. C. H. Lewis, Costa Mesa, California, died in October, 1979, at the age of 58. He was a member of the Association for twenty five years.

Professor Zarem N. Pirenian, Gainesville, Florida, died in October, 1979, at the age of 77. He was a member of the Association for fifty one years.

Professor M. E. Taylor, Mary Baldwin College, died in September, 1979, at the age of 81. He was a member of the Association for fifty five years.

Dr. Ernest Rubin, Arlington, Virginia, died in November, 1978. He was a member of the Association for nineteen years.

Professor M. L. Browne, Durham, North Carolina, died in September, 1979. He was a member of the Association for twenty eight years.

Professor H. F. Simmons, Cal State Polytechnic University, died in 1979 at the age of 52. He was a member of the Association for twenty seven years.

NORTHEASTERN SECTION 1980 SHORT COURSE

Professor William F. Lucas of Cornell University will be the principal lecturer at the 1980 Short Course. The course will be held at the University of Maine in Orono the week of June 16-20. Professor Lucas will present ten lectures under the title of *Selected Mathematical Models for Applications in the Managerial Sciences*. It is expected that an additional ten lectures will be presented by participants in the course.

SHORT COURSE ON BOUNDARY INTEGRAL EQUATION METHODS

The short course is presented by the University of Arizona College of Engineering and the Cornell University Civil and Environmental Engineering Department. It will be held March 17-22, 1980, at the University of Arizona Computer Center, Tucson, Arizona. The object of this course is to introduce the Boundary Integral Equation Method (BIEM) as an efficient numerical tool for the solution of various types of ground-water problems. The course is designed to provide a working knowledge of the BIEM so that the participants will be able to use and modify the existing computer programs and to develop their own programs for their specific problems. Several such programs will be made available to participants.

For further information contact:

Professor James A. Liggett or Professor Phillip L.-F. Liu
School of Civil and Environmental Engineering
Cornell University
Hollister Hall
Ithaca, New York 14853

NEW PUBLICATION: HISTORY AND PHILOSOPHY OF LOGIC

The first volume of this new publication will appear in mid-1980. It will be edited by Dr. I. Grattan-Guinness, Middlesex Polytechnic of Enfield, England. Publication will be as a volume, which, it is hoped, will be published annually. 'Logic' will be understood to mean that body of knowledge which was regarded as logic at the time in question. 'History' refers to ancient times and also to work in this century, but not to very recent work on a topic. 'Philosophy' refers to broad and general questions, but not to specialist articles which are now classed as 'philosphical logic'. Articles will be considered on the relationship between logic and other branches of knowledge, but the component of logic must be substantial.

Further information may be obtained from the editor.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

REPORT OF THE TREASURER FOR THE YEAR 1978

Herewith is a summary of the report of the Treasurer on the operating funds of the Association. The Association also handles several grant funds as well as the funds for the High School Contests.

The full report has been approved by the Finance Committee and accepted by a vote of the Board of Governors. Any member of the Association who wishes to have a copy of the full report may obtain one by writing to the Washington Office of the Association.

ASSETS	December 31, 1978
Cash	$ 12 163
Accounts receivable	98 405
Publication inventory	121 112
Prepaid expenses	6 105
Due from other funds	251 285
Investments (at cost)	388 462
Furniture and equipment (at cost)	61 298
Less: Accumulated depreciation	(34 586)
Deferred publication costs	102 109
Total Assets	$1 006 353

LIABILITIES AND FUND BALANCES	
Accounts payable	$ 62 700
Accrued liabilities	11 195
Unearned dues and subscriptions	384 768
Bank note	60 000
Unearned advertising and other	12 980
Fund balances	474 710
Total Liabilities and Fund Balances	$1 006 353

INCOME	
Dues	$ 443 249
Publications	408 155
Investments	41 580
Contributions	21 885
Registration fees	21 197
Miscellaneous	4 473
Total Income	$940 539

EXPENSES	
Salaries	$ 343 267
Administrative expenses	142 002
Publications (excluding salary & administrative exp.)	269 711
Travel and meeting expenses	67 369
Dues and Contributions	20 859
Professional services	10 938
Taxes, Interest, Fees	53 052
Awards and Grants	3 440
Office maintenance	20 698
Miscellaneous	1 284
Total Expenses	$ 932 620
Income over (under) expenses	$ 7 919

In addition, the Association solicited gifts for the building fund and (in 1978) the membership contributed $176 889 to this fund.

LEONARD GILLMAN, *Treasurer*

OCTOBER MEETING OF THE SEAWAY SECTION

The Seaway Section of M.A.A. held its fall meeting at the State University of New York Center at Albany on October 26 and 27, 1979. There was a registered attendance of 79 individuals. Section Chairman Howard Bell presided.

On Friday the Executive Committee and Committee Chairmen met to consider future activities of the Section. This was followed by a banquet and later in the evening a talk entitled, *Sex Differences in Math Aptitudes* by Malcolm J. Sherman of SUNY at Albany.

The Saturday Sessions were devoted to the presentation of the following papers:
The Spectral Theorem and the Jordan Form for Applied Linear Algebra, Malcolm F. Smiley, SUNY at Albany
A Characterization of the Semigroup of Matrix Units, Bernard R. Gelbaum, SUNY at Buffalo
A First Course in Computing, Robert Ellison, Hamilton College
Crux Mathematicorum, F. G. B. Maskell, Algonquin College
Graphing Analytic Functions (A Geometric Point of View), Edwin T. Hoefer, Rochester Institute of Technology
Modules Used in Calculus with Computers Course, Sarah Brooks, Mohawk Valley Community College
Partitions, Partial Fractions and Characteristic Polynomials, Lindsay N. Childs, SUNY at Albany

D. W. TRASHER, *Secretary-Treasurer*

FALL MEETING OF THE NORTH CENTRAL SECTION

The Fall meeting of the North Central Section was held on October 26 and 27, 1979, at the University of North Dakota, Grand Forks, North Dakota. Fifty five persons attended the meetings.

An Invited Address, *Independence Numbers in Complementary Graphs* was presented on Friday evening by Professor Sy Schuster of Carleton College. On Saturday, Professor J. Marshall Osborn of the University of Wisconsin gave an Invited Address entitled, *Rings with Involution.* A presentation on the UMAP Project was given by Professor Richard Allen of St. Olaf College.

President Hubert Walczak chaired the business meeting. Contributed papers were as follows:
Covering of Sets of Lattice Points by Clifton Corzatt (Presenter) and Richard Allen of St. Olaf College
The UMAP Project: Authoring, Reviewing and Classroom Use by Richard Allen, St. Olaf College
Anesthetic Uptake and Action: A sample UMAP Module by Loren Larson, St. Olaf College
Continuous Images of Compact Hausdorff Spaces by James Hatzenbuhler, Moorhead State University
Nonstandard Dice by Jeremy Teitelbaum, Student at Carleton College
Mathematician at the Bus Company by Wayne Roberts, Macalester College
Concrete Examples in Abstract Algebra by Joe Gallian, University of Minnesota, Duluth
A Solution to 1064-Mathematics Magazine by Gerald E. Bergum, South Dakota State University
Stieltjes Integrals and the Theory of Interest by John M. Holte, Gustavus Adolphus College
The Use of Some Second Order Partial Differential Equations in Picture Processing by Paul Rowe, North Dakota State University
Solzhenitsyn the Mathematician by David J. Uherka, University of North Dakota.

CHARLES V. HEUER, *Secretary-Treasurer*

FALL MEETING OF THE NEW JERSEY SECTION

The New Jersey Section of the MAA held its annual Fall meeting on Saturday November 3, 1979, at Essex County College, Newark, New Jersey. Fifty members were registered for the meeting, which was presided over by our section chairman, B. Melvin Kiernan.

Two papers were presented during the morning session. They were: *Local Homogeneous Graphs* presented by Pavol Hell of Rutgers University, and *Shape and Mathematics — an Invitation to the Geometric Calculus of Variations* presented by Fred Almgren of Princeton University.

After the luncheon a short business meeting was held. The following officers were elected unanimously: Vice chair for the HS Contest — Francis Masat, Glassboro State College; Vice chair for Innovations — Dorothy Wolff, Caldwell College; Vice chair for Two Year Colleges — Margaret Piedem, Somerset County College. An additional action was the movement of Jean Lane from the office of Chair-elect to the office of Chair, while B. Melvin Kiernan moved from Chair to Past Chair.

After the business meeting there was a short presentation of the benefits of the Undergraduate Mathematics Applications Project (UMAP). The final speaker of the day was Marcia Sward of Trinity College and the US Department of Transportation, who spoke on *The MAA Placement Testing Program.*

JAMES MAGLIANO, *Secretary*

CALENDAR OF FUTURE MEETINGS

Sixty-third Annual Meeting, San Antonio, Texas, January 5–7, 1980.

Sixtieth Summer Meeting, University of Michigan, Ann Arbor, August 18–20, 1980.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, West Virginia Wesleyan College, Buckhannon, April 25–26, 1980.

EASTERN PENNSYLVANIA AND DELAWARE, Saturday before Thanksgiving.

FLORIDA, Jacksonville University, Jacksonville, March 7–8, 1980.

ILLINOIS, John A. Logan College, Carterville, April 25–26, 1980.

INDIANA

INTERMOUNTAIN, Utah State University, Logan, late April or early May 1980.

IOWA, Simpson College, Indianola, April 18–19, 1980.

KANSAS, Kansas State University, Manhattan, spring 1980.

KENTUCKY, Western Kentucky University, Bowling Green, April 11–12, 1980.

LOUISIANA–MISSISSIPPI, Louisiana Tech University, Ruston, February 15–16, 1980.

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, University of Richmond, Richmond, Virginia, April 12, 1980.

METROPOLITAN NEW YORK, Mercy College, Dobbs Ferry, May 4, 1980.

MICHIGAN, Hope College, Holland, May 2–3, 1980.

MISSOURI, late March/early April. Deadline for papers January 31.

NEBRASKA, April.

NEW JERSEY, Hyatt House, Cherry Hill, March 15, 1980.

NORTH CENTRAL, end of October and April. Deadline for papers October 1 and April 1.

NORTHEASTERN, Saturday after Thanksgiving, and third week in June in odd-numbered years.

NORTHERN CALIFORNIA, Naval Postgraduate School, Monterey, February 23, 1980.

OHIO, Wittenberg University, Springfield, April 25–26, 1980.

OKLAHOMA–ARKANSAS, Westark Community College, Fort Smith, Arkansas, March 28–29, 1980.

PACIFIC NORTHWEST, Central Washington University, Ellensburg, Washington, June 20–21, 1980.

ROCKY MOUNTAIN, University of Colorado, Boulder, March 28–29, 1980.

SEAWAY, first Saturday in November and Saturday in late April. Deadline for papers 6 weeks before meeting.

SOUTHEASTERN, Appalachian State University, Boone, North Carolina, April 11–12, 1980.

SOUTHERN CALIFORNIA, California State University, Northridge, March 8, 1980.

SOUTHWESTERN, Northern Arizona University, Flagstaff, spring 1980.

TEXAS, East Texas State University, Commerce, April 4–5, 1980.

WISCONSIN, University of Wisconsin, Milwaukee, March 28–29, 1980.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, San Francisco, January 3–8, 1980.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES

AMERICAN MATHEMATICAL SOCIETY, San Antonio, Texas, January 3–6, 1980.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Amherst, Massachusetts, June 23–26, 1980.

ASSOCIATION FOR COMPUTING MACHINERY, Kansas City, Missouri, February 12–14, 1980.

ASSOCIATION FOR SYMBOLIC LOGIC

ASSOCIATION FOR WOMEN IN MATHEMATICS, San Antonio, Texas, January 3–7, 1980.

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHÉMATIQUES

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS, University of Michigan, Ann Arbor, August 18–21, 1980.

MU ALPHA THETA

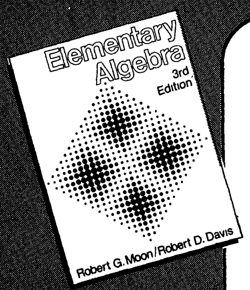
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Seattle, Washington, April 16–19, 1980.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Shoreham Hotel, Washington, D.C., May 5–7, 1980.

PI MU EPSILON

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS



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Larry Baggett and Watson Fulks
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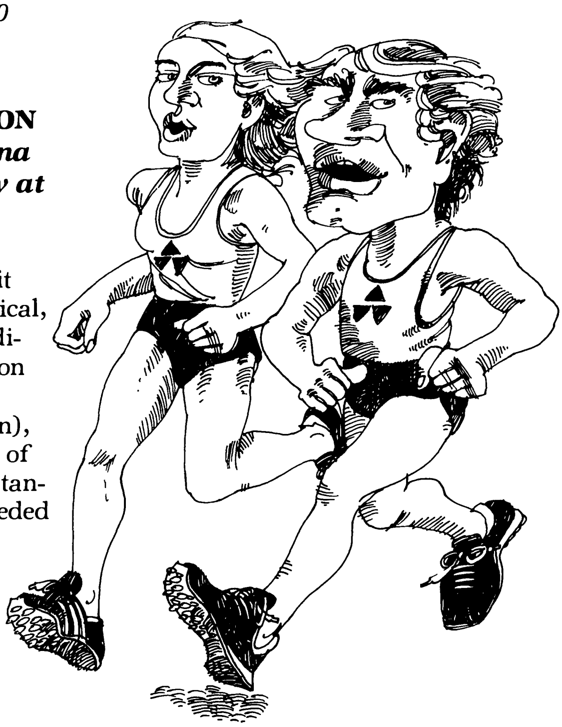

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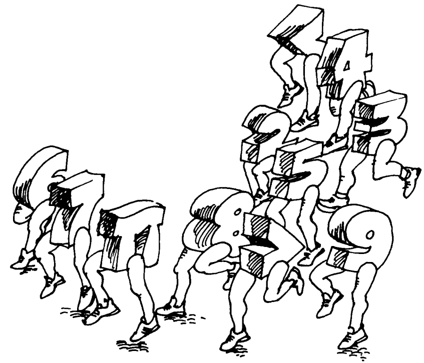

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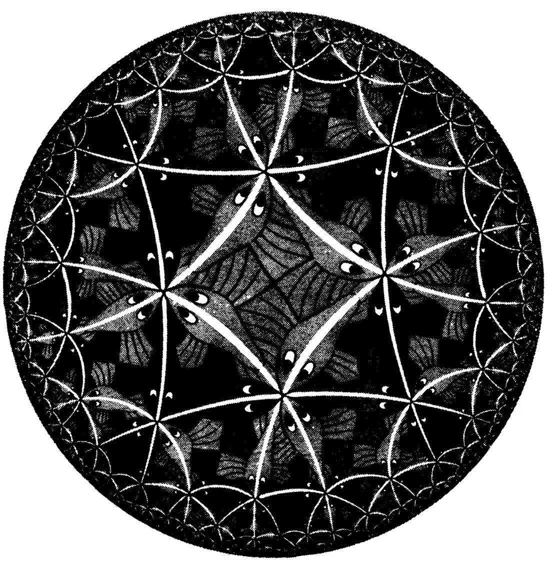

M. C. Escher's Circle Limit III, reproduced with the permission of the Escher Foundation, Gemeentemuseum, The Hague.

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THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

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THE MATHEMATICAL ASSOCIATION OF AMERICA

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AWARD FOR DISTINGUISHED SERVICE TO PROFESSOR HENRY L. ALDER

This year's recipient of the Award for Distinguished Service to Mathematics is a most familiar and well known member of our community. He has had tremendous and profound influence on all of mathematics and, more than any other person, he has charted the course of the Mathematical Association of America for the past twenty years. There are mathematicians who have added greatly to mathematics through their research activities. Still others have contributed through their teaching. Finally, some mathematicians are willing and energetic workers on behalf of the professional organizations. But there are few among us who fill all three of these roles. Thus, it is appropriate that we today honor Professor Henry L. Alder, an able researcher, an honored teacher, and a man who has made unparalleled contributions of time and energy to the professional activities of mathematicians.

Born in Duisburg, Germany, in 1922, Henry Alder moved with his family to Zürich, Switzerland, in 1933. After graduating from the Kantonschule in Zürich in 1940, he studied chemistry for a semester at the Eidgenössische Technische Hochschule. When his family moved to the United States in 1941 he enrolled immediately in the University of California at Berkeley, where he received his A.B. degree in 1942 and, after service in the U.S. Air Force, his Ph.D. in mathematics in 1947. After receiving his Ph.D., he was appointed to the faculty at Berkeley. In 1948 he joined the faculty of the University of California at Davis where he rose through the ranks, becoming Professor of Mathematics in 1965. Joining the MAA in 1950, he quickly became actively involved in the affairs of the Northern California Section and in 1957 became the first president of Mu Alpha Theta. In 1965, Mu Alpha Theta awarded him its International Distinguished Service Award. Also in 1957 he became Vice-Chairman of the Board of Governors of the *Pacific Journal of Mathematics,* and he remained active in the leadership of that journal, presently serving as Vice-Chairman of the Board of Governors and as Chairman of its Investment Committee.

Upon the occasion of Henry's retirement as MAA Secretary in 1975, Dr. Alfred B. Willcox, MAA Executive Director, wrote an article which appeared in the February 1975 issue of the MONTHLY, pages 110–112. Hence, in true mathematical fashion, I will cite a few of those activities which Henry Alder has been engaged in since 1975.

Henry Alder served as President of the Association in 1977-78 and has recently been a member of the Joint MAA-NCTM Committee on the Reported Decline in the Preparation of High School Students for Collegiate Mathematics Courses, the Committee on the Teaching of Undergraduate Mathematics, a member of CUPM, Chairman of the Joint AMS-MAA Fact Finding Committee on Archives, and Chairman of the Newsletter Subcommittee of the Committee on Publications. He is also currently a member of the CBMS Council, the United States Commission on Mathematical Instruction, a member of the Program Committee for the International Congress on Mathematical Education IV, and Chairman of the Council of Scientific Society Presidents and a member of that organization's Executive Board and Chairman of its Committee on Membership.

Henry serves the University of California, Davis, as Chairman of the Committee on Admissions and Enrollment. In October 1976 he received the Davis campus' Citation for Distinguished Teaching, an award given to at most six persons annually. For the University of California, Henry serves as member of the university-wide Board of Admissions and Enrollment and as Chairman of its Subcommittee on Evaluation of CLEP Examinations.

In January of 1975, Henry was awarded NCTM's Certificate of Merit and was made by the Association an honorary life member.

Henry is a number theorist and has been particularly interested in identities of the Rogers-Ramanujan type. A generalization of these identities which Henry published in 1954 embodies a polynomial now commonly known as the Alder polynomial.

The Mathematical Association of America extends to Henry Alder its Award for Dis-

HENRY L. ALDER

tinguished Service to Mathematics in recognition of his many teaching, research, and service activities. We are indebted to Henry for his good work in all areas of professional concern and thank him for being a willing helper and a good friend.

David P. Roselle, *Secretary*

AWARD OF THE CHAUVENET PRIZE TO PROFESSOR HEINZ BAUER

The Board of Governors of the Mathematical Association of America voted to award the 1980 Chauvenet Prize to Professor Heinz Bauer for his paper "Approximation and Abstract Boundaries," which appeared in this MONTHLY, 85 (1978) 632–647.

A certificate and monetary award in the amount of five hundred dollars were presented to Professor Bauer at the Business Meeting of the Association on January 6, 1980.

The Chauvenet Prize is awarded for a noteworthy paper of an expository or survey nature published in English that comes within the range of profitable reading for members of the Association. The purpose of the prize is to stimulate the writing of expository and survey articles. The 1980 prize, awarded for a paper published in the three-year period 1976–78, is the twenty-eighth award of the Chauvenet Prize since its institution by the Association in 1924. For the list of names of the previous winners, see this MONTHLY, 71 (1964) 589; 72 (1965) 2–3; 74 (1967) 3; 75 (1968) 3–4; 77 (1970) 117–118; 78 (1971) 112–113; 79 (1972) 112–113; 80 (1973) 120; 81 (1974) 113–114; 82 (1975) 108–109; 83 (1976) 84–85; 84 (1977) 417; 85 (1978) 74–75; and 86 (1979) 79. The award-winning papers are now available in the two-volume collection *The Chauvenet Papers*, published by the MAA.

Professor Bauer was born January 31, 1928, at Nürnburg, Germany. Between 1948 and 1953, he studied at the University of Erlangen and the Université de Nancy, France. He was awarded the Ph.D. (summa cum laude) by the University of Erlangen in 1953 and, until 1956, served as Assistant Professor there. He has held Associate and Full Professorships at the University of Hamburg and, since 1965, has served as Full Professor at the University of Erlangen–Nürnberg. Professor Bauer has also held visiting positions at the University of Munich, the University of Washington, the Sorbonne, the California Institute of Technology, New Mexico State University, and Aarhus University. He has also been a Research Fellow at both the Centre National de la Recherche Scientifique, Paris, and Institut des Hautes Etudes, Bures-sur-Yvette, France.

Professor Bauer is involved in research in integration theory, functional analysis (convexity and approximation theory), potential theory, and Markov processes. He is the author of more than fifty papers and three books: *Wahrscheinlichkeitstheorie und Grundzüge der Masstheorie*, Walter de Gruyter and Co., Berlin, 1968 (3rd ed., 1978), *Probability Theory and Elements of Measure Theory*, Holt, Rinehart and Winston, New York, 1972, and *Mehrdimensionale Integration*, Sammlung Göschen, Walter de Gruyter & Co., Berlin, 1976.

Professor Bauer has served as Editor of both *Inventiones Mathematicae* (1966–79) and *Math. Annalen* (1970–). He has served as Dean of the Faculty of Sciences at the University of Hamburg (1964–65), as member of the Board of the Mathematical Research Institute of Oberwolfach (1966–), as President of the German Mathematical Society (1976–77), and as Regular Member of the Bavarian Academy of Sciences (1975–).

The paper for which Professor Bauer received the Chauvenet Prize discusses three famous theorems of P. P. Korovkin that concern uniform approximation of functions. These theorems are presented in a well-chosen setting and are illustrated and illuminated superbly with a collection of examples and applications. The paper is accessible to graduate students who have learned about the Lebesgue integral.

David P. Roselle, *Secretary*

THE UNREASONABLE EFFECTIVENESS OF MATHEMATICS

R. W. HAMMING

Prologue. It is evident from the title that this is a philosophical discussion. I shall not apologize for the philosophy, though I am well aware that most scientists, engineers, and mathematicians have little regard for it; instead, I shall give this short prologue to justify the approach.

Man, so far as we know, has always wondered about himself, the world around him, and what life is all about. We have many myths from the past that tell how and why God, or the gods, made man and the universe. These I shall call *theological explanations.* They have one principal characteristic in common—there is little point in asking why things are the way they are, since we are given mainly a description of the creation as the gods chose to do it.

Philosophy started when man began to wonder about the world outside of this theological framework. An early example is the description by the philosophers that the world is made of earth, fire, water, and air. No doubt they were told at the time that the gods made things that way and to stop worrying about it.

From these early attempts to explain things slowly came philosophy as well as our present science. Not that science explains "why" things are as they are—gravitation does not explain why things fall—but science gives so many details of "how" that we have the feeling we understand "why." Let us be clear about this point; it is by the sea of interrelated details that science seems to say "why" the universe is as it is.

Our main tool for carrying out the long chains of tight reasoning required by science is mathematics. Indeed, mathematics might be defined as being the mental tool designed for this purpose. Many people through the ages have asked the question I am effectively asking in the title, "Why is mathematics so unreasonably effective?" In asking this we are merely looking more at the logical side and less at the material side of what the universe is and how it works.

Mathematicians working in the foundations of mathematics are concerned mainly with the self-consistency and limitations of the system. They seem not to concern themselves with why the world apparently admits of a logical explanation. In a sense I am in the position of the early Greek philosophers who wondered about the material side, and my answers on the logical side are probably not much better than theirs were in their time. But we must begin somewhere and sometime to explain the phenomenon that the world seems to be organized in a logical pattern that parallels much of mathematics, that mathematics is the language of science and engineering.

Once I had organized the main outline, I had then to consider how best to communicate my ideas and opinions to others. Experience shows that I am not always successful in this matter. It finally occurred to me that the following preliminary remarks would help.

In some respects this discussion is highly theoretical. I have to mention, at least slightly, various theories of the general activity called mathematics, as well as touch on selected parts of it. Furthermore, there are various theories of applications. Thus, to some extent, this leads to a theory of theories. What may surprise you is that I shall take the experimentalist's approach in discussing things. Never mind what the theories are supposed to be, or what you think they should be, or even what the experts in the field assert they are; let us take the scientific attitude and look at what they are. I am well aware that much of what I say, especially about the nature

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Richard W. Hamming received his Ph.D. from the University of Illinois in 1942 and has had a distinguished career in computing, mainly at the Bell Telephone Laboratories. He has also done part-time teaching. Since 1976 he has been on the faculty of the Naval Postgraduate School, Monterey, California, in the Computer Science Department. He is a past president of the Association for Computing Machinery and received its Turing prize. He has published six books and many articles. His research interests are in numerical methods, software, simulation, and system performance and measurement. This article is the text of a talk given at the meeting of the Northern California Section of the MAA, February 24, 1979.—*Editors*

of mathematics, will annoy many mathematicians. My experimental approach is quite foreign to their mentality and preconceived beliefs. So be it!

The inspiration for this article came from the similarly entitled article, "The Unreasonable Effectiveness of Mathematics in the Natural Sciences" [1], by E. P. Wigner. It will be noticed that I have left out part of the title, and by those who have already read it that I do not duplicate much of his material (I do not feel I can improve on his presentation). On the other hand, I shall spend relatively more time trying to explain the implied question of the title. But when all my explanations are over, the residue is still so large as to leave the question essentially unanswered.

The Effectiveness of Mathematics. In his paper, Wigner gives a large number of examples of the effectiveness of mathematics in the physical sciences. Let me, therefore, draw on my own experiences that are closer to engineering. My first real experience in the use of mathematics to predict things in the real world was in connection with the design of atomic bombs during the Second World War. How was it that the numbers we so patiently computed on the primitive relay computers agreed so well with what happened on the first test shot at Almagordo? There were, and could be, no small-scale experiments to check the computations directly. Later experience with guided missiles showed me that this was not an isolated phenomenon—constantly what we predict from the manipulation of mathematical symbols is realized in the real world. Naturally, working as I did for the Bell System, I did many telephone computations and other mathematical work on such varied things as traveling wave tubes, the equalization of television lines, the stability of complex communication systems, the blocking of calls through a telephone central office, to name but a few. For glamour, I can cite transistor research, space flight, and computer design, but almost all of science and engineering has used extensive mathematical manipulations with remarkable successes.

Many of you know the story of Maxwell's equations, how to some extent for reasons of symmetry he put in a certain term, and in time the radio waves that the theory predicted were found by Hertz. Many other examples of successfully predicting unknown physical effects from a mathematical formulation are well known and need not be repeated here.

The fundamental role of *invariance* is stressed by Wigner. It is basic to much of mathematics as well as to science. It was the lack of invariance of Newton's equations (the need for an absolute frame of reference for velocities) that drove Lorentz, Fitzgerald, Poincaré, and Einstein to the special theory of relativity.

Wigner also observes that *the same mathematical concepts* turn up in entirely unexpected connections. For example , the trigonometric functions which occur in Ptolemy's astronomy turn out to be the functions which are invariant with respect to translation (time invariance). They are also the appropriate functions for linear systems. The enormous usefulness of the same pieces of mathematics in widely different situations has no rational explanation (as yet).

Furthermore, the *simplicity* of mathematics has long been held to be the key to applications in physics. Einstein is the most famous exponent of this belief. But even in mathematics itself the simplicity is remarkable, at least to me; the simplest algebraic equations, linear and quadratic, correspond to the simplest geometric entities, straight lines, circles, and conics. This makes analytic geometry possible in a practical way. How can it be that simple mathematics, being after all a product of the human mind, can be so remarkably useful in so many widely different situations?

Because of these successes of mathematics, there is at present a strong trend toward making each of the sciences mathematical. It is usually regarded as a goal to be achieved, if not today, then tomorrow. For this audience I will stick to physics and astronomy for further examples.

Pythagoras is the first man to be recorded who clearly stated that "Mathematics is the way to understand the universe." He said it both loudly and clearly, "Number is the measure of all things."

Kepler is another famous example of this attitude. He passionately believed that God's

handiwork could be understood only through mathematics. After twenty years of tedious computations, he found his famous three laws of planetary motion—three comparatively simple mathematical expressions that described the apparently complex motions of the planets.

It was Galileo who said, "The laws of Nature are written in the language of mathematics." Newton used the results of both Kepler and Galileo to deduce the famous Newtonian laws of motion, which together with the law of gravitation are perhaps the most famous example of the unreasonable effectiveness of mathematics in science. They not only predicted where the known planets would be but sucessfully predicted the positions of unknown planets, the motions of distant stars, tides, and so forth.

Science is composed of laws which were originally based on a small, carefully selected set of observations, often not very accurately measured originally; but the laws have later been found to apply over much wider ranges of observations and much more accurately than the original data justified. Not always, to be sure, but often enough to require explanation.

During my thirty years of practicing mathematics in industry, I often worried about the predictions I made. From the mathematics that I did in my office I confidently (at least to others) predicted some future events—if you do so and so, you will see such and such—and it usually turned out that I was right. How could the phenomena know what I had predicted (based on human-made mathematics) so that it could support my predictions? It is ridiculous to think that is the way things go. No, it is that mathematics provides, somehow, a reliable model for much of what happens in the universe. And since I am able to do only comparatively simple mathematics, how can it be that simple mathematics suffices to predict so much?

I could go on citing more examples illustrating the unreasonable effectiveness of mathematics, but it would only be boring. Indeed, I suspect that many of you know examples that I do not. Let me, therefore, assume that you grant me a very long list of successes, many of them as spectacular as the prediction of a new planet, of a new physical phenomenon, of a new artifact. With limited time, I want to spend it attempting to do what I think Wigner evaded—to give at least some partial answers to the implied question of the title.

What is Mathematics? Having looked at the effectiveness of mathematics, we need to look at the question, *"What is Mathematics?"* This is the title of a famous book by Courant and Robbins [2]. In it they do not attempt to give a formal definition, rather they are content to show what mathematics is by giving many examples. Similarly, I shall not give a comprehensive definition. But I will come closer than they did to discussing certain salient features of mathematics as I see them.

Perhaps the best way to approach the question of what mathematics is, is to start at the beginning. In the far distant, prehistoric past, where we must look for the beginnings of mathematics, there were already four major faces of mathematics. First, there was the ability to carry on the *long chains of close reasoning* that to this day characterize much of mathematics. Second, there was *geometry*, leading through the concept of continuity to topology and beyond. Third, there was *number*, leading to arithmetic, algebra, and beyond. Finally there was *artistic taste*, which plays so large a role in modern mathematics. There are, of course, many different kinds of beauty in mathematics. In number theory it seems to be mainly the beauty of the almost infinite detail; in abstract algebra the beauty is mainly in the generality. Various areas of mathematics thus have various standards of aesthetics.

The earliest history of mathematics must, of course, be all speculation, since there is not now, nor does there ever seem likely to be, any actual, convincing evidence. It seems, however, that in the very foundations of primitive life there was built in, for survival purposes if for nothing else, an understanding of cause and effect. Once this trait is built up beyond a single observation to a sequence of, "If this, then that, and then it follows still further that . . . ," we are on the path of the first feature of mathematics I mentioned, long chains of close reasoning. But it is hard for me to see how simple Darwinian survival of the fittest would select for the ability to do the long chains that mathematics and science seem to require.

Geometry seems to have arisen from the problems of decorating the human body for various purposes, such as religious rites, social affairs, and attracting the opposite sex, as well as from the problems of decorating the surfaces of walls, pots, utensils, and clothing. This also implies the fourth aspect I mentioned, aesthetic taste, and this is one of the deep foundations of mathematics. Most textbooks repeat the Greeks and say that geometry arose from the needs of the Egyptians to survey the land after each flooding by the Nile River, but I attribute much more to aesthetics than do most historians of mathematics and correspondingly less to immediately utility.

The third aspect of mathematics, numbers, arose from counting. So basic are numbers that a famous mathematician once said, "God made the integers, man did the rest" [3]. The integers seem to us to be so fundamental that we expect to find them wherever we find intelligent life in the universe. I have tried, with little success, to get some of my friends to understand my amazement that the abstraction of integers for counting is both possible and useful. Is it not remarkable that 6 sheep plus 7 sheep make 13 sheep; that 6 stones plus 7 stones make 13 stones? Is it not a miracle that the universe is so constructed that such a simple abstraction as a number is possible? To me this is one of the strongest examples of the unreasonable effectiveness of mathematics. Indeed, I find it both strange and unexplainable.

In the development of numbers, we next come to the fact that these counting numbers, the integers, were used successfully in measuring how many times a standard length can be used to exhaust the desired length that is being measured. But it must have soon happened, comparatively speaking, that a whole number of units did not exactly fit the length being measured, and the measurers were driven to the fractions—the extra piece that was left over was used to measure the standard length. Fractions are not counting numbers, they are measuring numbers. Because of their common use in measuring, the fractions were, by a suitable extension of ideas, soon found to obey the same rules for manipulations as did the integers, with the added benefit that they made division possible in all cases (I have not yet come to the number zero). Some acquaintance with the fractions soon reveals that between any two fractions you can put as many more as you please and that in some sense they are homogeneously dense everywhere. But when we extend the concept of number to include the fractions, we have to give up the idea of the next number.

This brings us again to Pythagoras, who is reputed to be the first man to prove that the diagonal of a square and the side of the square have no common measure—that they are irrationally related. This observation apparently produced a profound upheaval in Greek mathematics. Up to that time the discrete number system and the continuous geometry flourished side by side with little conflict. The crisis of incommensurability tripped off the Euclidean approach to mathematics. It is a curious fact that the early Greeks attempted to make mathematics rigorous by replacing the uncertainties of numbers by what they felt was the more certain geometry (due to Eudoxus). It was a major event to Euclid, and as a result you find in *The Elements* [4] a lot of what we now consider number theory and algebra cast in the form of geometry. Opposed to the early Greeks, who doubted the existence of the real number system, we have decided that there should be a number that measures the length of the diagonal of a unit square (though we need not do so), and that is more or less how we extended the rational number system to include the algebraic numbers. It was the simple desire to measure lengths that did it. How can anyone deny that there is a number to measure the length of any straight line segment?

The algebraic numbers, which are roots of polynomials with integer, fractional, and, as was later proved, even algebraic numbers as coefficients, were soon under control by simply extending the same operations that were used on the simpler system of numbers.

However, the measurement of the circumference of a circle with respect to its diameter soon forced us to consider the ratio called pi. This is not an algebraic number, since no linear

combination of the powers of pi with integer coefficients will exactly vanish. One length, the circumference, being a curved line, and the other length, the diameter, being a straight line, make the existence of the ratio less certain than is the ratio of the diagonal of a square to its side; but since it seems that there ought to be such a number, the transcendental numbers gradually got into the number system. Thus by a further suitable extension of the earlier ideas of numbers, the transcendental numbers were admitted consistently into the number system, though few students are at all comfortable with the technical apparatus we conventionally use to show the consistency.

Further tinkering with the number system brought both the number zero and the negative numbers. This time the extension required that we abandon the division for the single number zero. This seems to round out the real number system for us (as long as we confine ourselves to the processes of taking limits of sequences of numbers and do not admit still further operations) —not that we have to this day a firm, logical, simple, foundation for them; but they say that familiarity breeds contempt, and we are all more or less familiar with the real number system. Very few of us in our saner moments believe that the particular postulates that some logicians have dreamed up create the numbers—no, most of us believe that the real numbers are simply there and that it has been an interesting, amusing, and important game to try to find a nice set of postulates to account for them. But let us not confuse ourselves—Zeno's paradoxes are still, even after 2,000 years, too fresh in our minds to delude ourselves that we understand all that we wish we did about the relationship between the discrete number system and the continuous line we want to model. We know, from nonstandard analysis if from no other place, that logicians can make postulates that put still further entities on the real line, but so far few of us have wanted to go down that path. It is only fair to mention that there are some mathematicians who doubt the existence of the conventional real number system. A few computer theoreticians admit the existence of only "the computable numbers."

The next step in the discussion is the complex number system. As I read history, it was Cardan who was the first to understand them in any real sense. In his *The Great Art or Rules of Algebra* [5] he says, "Putting aside the mental tortures involved multiply $5+\sqrt{-15}$ by $5-\sqrt{-15}$ making $25-(-15)$ . . . ." Thus he clearly recognized that the same formal operations on the symbols for complex numbers would give meaningful results. In this way the real number system was gradually extended to the complex number system, except that this time the extension required giving up the property of ordering the numbers—the complex numbers cannot be ordered in the usual sense.

Cauchy was apparently led to the theory of complex variables by the problem of integrating real functions along the real line. He found that by bending the path of integration into the complex plane he could solve real integration problems.

A few years ago I had the pleasure of teaching a course in complex variables. As always happens when I become involved in the topic, I again came away with the feeling that "God made the universe out of complex numbers." Clearly, they play a central role in quantum mechanics. They are a natural tool in many other areas of application, such as electric circuits, fields, and so on.

To summarize, from simple counting using the God-given integers, we made various extensions of the ideas of numbers to include more things. Sometimes the extensions were made for what amounted to aesthetic reasons, and often we gave up some property of the earlier number system. Thus we came to a number system that is unreasonably effective even in mathematics itself; witness the way we have solved many number theory problems of the original highly discrete counting system by using a complex variable.

From the above we see that one of the main strands of mathematics is the extension, the generalization, the abstraction—they are all more or less the same thing—of well-known concepts to new situations. But note that in the very process the definitions themselves are

subtly altered. Therefore, what is not so widely recognized, old proofs of theorems may become false proofs. The old proofs no longer cover the newly defined things. The miracle is that almost always the theorems are still true; it is merely a matter of fixing up the proofs. The classic example of this fixing up is Euclid's *The Elements* **[4]**. We have found it necessary to add quite a few new postulates (or axioms, if you wish, since we no longer care to distinguish between them) in order to meet current standards of proof. Yet how does it happen that no theorem in all the thirteen books is now false? Not one theorem has been found to be false, though often the proofs given by Euclid seem now to be false. And this phenomenon is not confined to the past. It is claimed that an ex-editor of *Mathematical Reviews* once said that over half of the new theorems published these days are essentially true though the published proofs are false. How can this be if mathematics is the rigorous deduction of theorems from assumed postulates and earlier results? Well, it is obvious to anyone who is not blinded by authority that mathematics is not what the elementary teachers said it was. It is clearly something else.

What is this "else"? Once you start to look you find that if you were confined to the axioms and postulates then you could deduce very little. The first major step is to introduce new concepts derived from the assumptions, concepts such as triangles. The search for proper concepts and definitions is one of the main features of doing great mathematics.

While on the topic of proofs, classical geometry begins with the theorem and tries to find a proof. Apparently it was only in the 1850's or so that it was clearly recognized that the opposite approach is also valid (it must have been occasionally used before then). Often it is the proof that generates the theorem. We see what we can prove and then examine the proof to see what we have proved! These are often called "proof generated theorems" **[6]**. A classic example is the concept of uniform convergence. Cauchy had proved that a convergent series of terms, each of which is continuous, converges to a continuous function. At the same time there were known to be Fourier series of continuous functions that converged to a discontinuous limit. By a careful examination of Cauchy's proof, the error was found and fixed up by changing the hypothesis of the theorem to read, "a uniformly convergent series."

More recently, we have had an intense study of what is called the foundations of mathematics—which in my opinion should be regarded as the top battlements of mathematics and not the foundations. It is an interesting field, but the main results of mathematics are impervious to what is found there—we simply will not abandon much of mathematics no matter how illogical it is made to appear by research in the foundations.

I hope that I have shown that mathematics is not the thing it is often assumed to be, that mathematics is constantly changing and hence even if I did succeed in defining it today the definition would not be appropriate tomorrow. Similarly with the idea of rigor—we have a changing standard. The dominant attitude in science is that we are not the center of the universe, that we are not uniquely placed, etc., and similarly it is difficult for me to believe that we have now reached the ultimate of rigor. Thus we cannot be sure of the current proofs of our theorems. Indeed it seems to me:

The Postulates of Mathematics Were Not on the Stone Tablets that Moses Brought Down from Mt. Sinai.

It is necessary to emphasize this. We begin with a vague concept in our minds, then we create various sets of postulates, and gradually we settle down to one particular set. In the rigorous postulational approach the original concept is now replaced by what the postulates define. This makes further evolution of the concept rather difficult and as a result tends to slow down the evolution of mathematics. It is not that the postulation approach is wrong, only that its arbitrariness should be clearly recognized, and we should be prepared to change postulates when the need becomes apparent.

Mathematics has been made by man and therefore is apt to be altered rather continuously by

him. Perhaps the original sources of mathematics were forced on us, but as in the example I have used we see that in the development of so simple a concept as number we have made choices for the extensions that were only partly controlled by necessity and often, it seems to me, more by aesthetics. We have tried to make mathematics a consistent, beautiful thing, and by so doing we have had an amazing number of successful applications to the real world.

The idea that theorems follow from the postulates does not correspond to simple observation. If the Pythagorean theorem were found to not follow from the postulates, we would again search for a way to alter the postulates until it was true. Euclid's postulates came from the Pythagorean theorem, not the other way. For over thirty years I have been making the remark that if you came into my office and showed me a proof that Cauchy's theorem was false I would be very interested, but I believe that in the final analysis we would alter the assumptions until the theorem was true. Thus there are many results in mathematics that are independent of the assumptions and the proof.

How do we decide in a "crisis" what parts of mathematics to keep and what parts to abandon? Usefulness is one main criterion, but often it is usefulness in creating more mathematics rather than in the applications to the real world! So much for my discussion of mathematics.

Some Partial Explanations. I will arrange my explanations of the unreasonable effectiveness of mathematics under four headings.

1. *We see what we look for.* No one is surprised if after putting on blue tinted glasses the world appears bluish. I propose to show some examples of how much this is true in current science. To do this I am again going to violate a lot of widely, passionately held beliefs. But hear me out.

I picked the example of scientists in the earlier part for a good reason. Pythagoras is to my mind the first great physicist. It was he who found that we live in what the mathematicians call L_2—the sum of the squares of the two sides of a right triangle gives the square of the hypotenuse. As I said before, this is not a result of the postulates of geometry—this is one of the results that shaped the postulates.

Let us next consider Galileo. Not too long ago I was trying to put myself in Galileo's shoes, as it were, so that I might feel how he came to discover the law of falling bodies. I try to do this kind of thing so that I can learn to think like the masters did—I deliberately try to think as they might have done.

Well, Galileo was a well-educated man and a master of scholastic arguments. He well knew how to argue the number of angels on the head of a pin, how to argue both sides of any question. He was trained in these arts far better than any of us these days. I picture him sitting one day with a light and a heavy ball, one in each hand, and tossing them gently. He says, hefting them, "It is obvious to anyone that heavy objects fall faster than light ones—and, anyway, Aristotle says so." "But suppose," he says to himself, having that kind of a mind, "that in falling the body broke into two pieces. Of course the two pieces would immediately slow down to their appropriate speeds. But suppose further that one piece happened to touch the other one. Would they now be one piece and both speed up? Suppose I tied the two pieces together. How tightly must I do it to make them one piece? A light string? A rope? Glue? When are two pieces one?"

The more he thought about it—and the more you think about it—the more unreasonable becomes the question of when two bodies are one. There is simply no reasonable answer to the question of how a body knows how heavy it is—if it is one piece, or two, or many. Since falling bodies do something, the only possible thing is that they all fall at the same speed—unless interfered with by other forces. There is nothing else they can do. He may have later made some experiments, but I strongly suspect that something like what I imagined actually happened. I later found a similar story in a book by Pólya [7]. Galileo found his law not by experimenting but by simple, plain thinking, by scholastic reasoning.

I know that the textbooks often present the falling body law as an experimental observation;

I am claiming that it is a logical law, a consequence of how we tend to think.

Newton, as you read in books, deduced the inverse square law from Kepler's laws, though they often present it the other way; from the inverse square law the textbooks deduce Kepler's laws. But if you believe in anything like the conservation of energy and think that we live in a three-dimensional Euclidean space, then how else could a symmetric central-force field fall off? Measurements of the exponent by doing experiments are to a great extent attempts to find out if we live in a Euclidean space, and not a test of the inverse square law at all.

But if you do not like these two examples, let me turn to the most highly touted law of recent times, the uncertainty principle. It happens that recently I became involved in writing a book on *Digital Filters* [**8**] when I knew very little about the topic. As a result I early asked the question, "Why should I do all the analysis in terms of Fourier integrals? Why are they the natural tools for the problem?" I soon found out, as many of you already know, that the eigenfunctions of translation are the complex exponentials. If you want time invariance, and certainly physicists and engineers do (so that an experiment done today or tomorrow will give the same results), then you are led to these functions. Similarly, if you believe in linearity then they are again the eigenfunctions. In quantum mechanics the quantum states are absolutely additive; they are not just a convenient linear approximation. Thus the trigonometric functions are the eigenfunctions one needs in both digital filter theory and quantum mechanics, to name but two places.

Now when you use these eigenfunctions you are naturally led to representing various functions, first as a countable number and then as a non-countable number of them—namely, the Fourier series and the Fourier integral. Well, it is a theorem in the theory of Fourier integrals that the variability of the function multiplied by the variability of its transform exceeds a fixed constant, in one notation $1/2\pi$. This says to me that in any linear, time invariant system you must find an uncertainty principle. The size of Planck's constant is a matter of the detailed identification of the variables with integrals, but the inequality must occur.

As another example of what has often been thought to be a physical discovery but which turns out to have been put in there by ourselves, I turn to the well-known fact that the distribution of physical constants is not uniform; rather the probability of a random physical constant having a leading digit of 1, 2, or 3 is approximately 60%, and of course the leading digits of 5, 6, 7, 8, and 9 occur in total only about 40% of the time. This distribution applies to many types of numbers, including the distribution of the coefficients of a power series having only one singularity on the circle of convergence. A close examination of this phenomenon shows that it is mainly an artifact of the way we use numbers.

Having given four widely different examples of nontrivial situations where it turns out that the original phenomenon arises from the mathematical tools we use and not from the real world, I am ready to strongly suggest that a lot of what we see comes from the glasses we put on. Of course this goes against much of what you have been taught, but consider the arguments carefully. You can say that it was the experiment that forced the model on us, but I suggest that the more you think about the four examples the more uncomfortable you are apt to become. They are not arbitrary theories that I have selected, but ones which are central to physics.

In recent years it was Einstein who most loudly proclaimed the simplicity of the laws of physics, who used mathematics so extensively as to be popularly known as a mathematician. When examining his special theory of relativity paper [**9**] one has the feeling that one is dealing with a scholastic philosopher's approach. He knew in advance what the theory should look like, and he explored the theories with mathematical tools, not actual experiments. He was so confident of the rightness of the relativity theories that, when experiments were done to check them, he was not much interested in the outcomes, saying that they had to come out that way or else the experiments were wrong. And many people believe that the two relativity theories rest more on philosophical grounds than on actual experiments.

Thus my first answer to the implied question about the unreasonable effectiveness of mathematics is that we approach the situations with an intellectual apparatus so that we can

only find what we do in many cases. It is both that simple, and that awful. What we were taught about the basis of science being experiments in the real world is only partially true. Eddington went further than this; he claimed that a sufficiently wise mind could deduce all of physics. I am only suggesting that a surprising amount can be so deduced. Eddington gave a lovely parable to illustrate this point. He said, "Some men went fishing in the sea with a net, and upon examining what they caught they concluded that there was a minimum size to the fish in the sea."

2. *We select the kind of mathematics to use.* Mathematics does not always work. When we found that scalars did not work for forces, we invented a new mathematics, vectors. And going further we have invented tensors. In a book I have recently written **[10]** conventional integers are used for labels, and real numbers are used for probabilities; but otherwise all the arithmetic and algebra that occurs in the book, and there is a lot of both, has the rule that

$$1+1=0.$$

Thus my second explanation is that we select the mathematics to fit the situation, and it is simply not true that the same mathematics works every place.

3. *Science in fact answers comparatively few problems.* We have the illusion that science has answers to most of our questions, but this is not so. From the earliest of times man must have pondered over what Truth, Beauty, and Justice are. But so far as I can see science has contributed nothing to the answers, nor does it seem to me that science will do much in the near future. So long as we use a mathematics in which the whole is the sum of the parts we are not likely to have mathematics as a major tool in examining these famous three questions.

Indeed, to generalize, almost all of our experiences in this world do not fall under the domain of science or mathematics. Furthermore, we know (at least we think we do) that from Gödel's theorem there are definite limits to what pure logical manipulation of symbols can do, there are limits to the domain of mathematics. It has been an act of faith on the part of scientists that the world can be explained in the simple terms that mathematics handles. When you consider how much science has not answered then you see that our successes are not so impressive as they might otherwise appear.

4. *The evolution of man provided the model.* I have already touched on the matter of the evolution of man. I remarked that in the earliest forms of life there must have been the seeds of our current ability to create and follow long chains of close reasoning. Some people **[11]** have further claimed that Darwinian evolution would naturally select for survival those competing forms of life which had the best models of reality in their minds—"best" meaning best for surviving and propagating. There is no doubt that there is some truth in this. We find, for example, that we can cope with thinking about the world when it is of comparable size to ourselves and our raw unaided senses, but that when we go to the very small or the very large then our thinking has great trouble. We seem not to be able to think appropriately about the extremes beyond normal size.

Just as there are odors that dogs can smell and we cannot, as well as sounds that dogs can hear and we cannot, so too there are wavelengths of light we cannot see and flavors we cannot taste. Why then, given our brains wired the way they are, does the remark, "Perhaps there are thoughts we cannot think," surprise you? Evolution, so far, may possibly have blocked us from being able to think in some directions; there could be unthinkable thoughts.

If you recall that modern science is only about 400 years old, and that there have been from 3 to 5 generations per century, then there have been at most 20 generations since Newton and Galileo. If you pick 4,000 years for the age of science, generally, then you get an upper bound of 200 generations. Considering the effects of evolution we are looking for via selection of small chance variations, it does not seem to me that evolution can explain more than a small part of the unreasonable effectiveness of mathematics.

Conclusion. From all of this I am forced to conclude both that mathematics is unreasonably effective and that all of the explanations I have given when added together simply are not enough to explain what I set out to account for. I think that we—meaning you, mainly—must continue to try to explain why the logical side of science—meaning mathematics, mainly—is the proper tool for exploring the universe as we perceive it at present. I suspect that my explanations are hardly as good as those of the early Greeks, who said for the material side of the question that the nature of the universe is earth, fire, water, and air. The logical side of the nature of the universe requires further exploration.

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GENERALIZING THE NOTION OF A PERIODIC SEQUENCE

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Introduction. Given the first few elements of an infinite integer sequence, we can often inductively infer what the rest of the sequence is. For example, if we see the numbers

$$2, 16, 54, 128, \ldots,$$

we might infer that the kth element should be the number $2k^3$ (see [**1**] or [**2**] for material on inference of integer sequences). Sometimes we feel that a sequence is best described as two or more simpler sequences which have been intertwined, for example,

$$1, 0, 2, 0, 3, 0, \ldots$$

or

$$1, 1, 4, 2, 9, 4, 16, 8, \ldots.$$

We are going to extend the traditional definition of a periodic sequence to include sequences which behave in a pseudo-periodic fashion. Our first three sequences will have *generalized periods* 1, 2, and 2, respectively. The sequence

$$1, 2, 3, 2, 3, 4, 3, 4, 5, \ldots$$

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has generalized period 3. A sequence like

$$1,1,2,1,2,3,2,1,2,3,4,3,2,1,\ldots$$

does not have a generalized period.

A simple programming language has been invented to define our generally periodic sequences. We shall see that by appropriate manipulation of the programming language we are able to prove a number of properties of our sequences with relative ease. This idea of utilizing programming languages to prove theorems may find application in other areas.

The ORVA Language. In order to make precise the notion of generalized periodicity, we have devised a very simple programming language, called ORVA, which generates sequences of numbers. ORVA stands for ORdered VAriable—each variable introduced into a program has a unique positive integer rank. This defines a strict total ordering of all variables, which will limit the way in which variable assignments are made. For most purposes the numerical rank of a variable is unimportant and only its order relative to other variables is considered.

The ORVA language has only three types of statements—assignment, output, and an unconditional "goto." An assignment has the form

$$x_n := c_n x_n + c_{n-1} x_{n-1} + \cdots + c_1 x_1 + c_0$$

where the variables x_1 to x_{n-1} must have a lower rank than that of x_n, and c_1 to c_n are real numbers. The leading coefficient c_n must be non-negative. An output statement has the form "PRINT x_i," outputting only a single value per statement. The "goto" has the form "GO L" where L is a label attached to some preceding statement.

Unless otherwise specified, a subscript for a variable x_i denotes its rank relative to other variables x_j, but says nothing about its relation to a variable w_j, i.e., $i<j$ implies $\operatorname{rank}(x_i)<\operatorname{rank}(x_j)$ but possibly $\operatorname{rank}(x_i)>\operatorname{rank}(w_j)$ for some w_j.

Sample program:

$$
\begin{array}{ll}
 & x_1 := 1 \\
 & x_2 := 0 \\
L: & x_2 := x_2 + x_1 \\
 & x_1 := x_1 + 2 \\
 & \text{PRINT } x_2 \\
 & \text{GO } L.
\end{array}
$$

The resulting output is the sequence $1,4,9,16,25,\ldots$. Typically, the first statements initialize variables and the rest of the code is an infinite loop.

Definition of Generalized Periodicity.

DEFINITION. Suppose an ORVA program generates some sequence S. A *reduced* ORVA program is one which generates S with the minimal number of "PRINT" statements in its loop.

DEFINITION. The *generalized period* (abbreviated g.p.) of a sequence that can be generated by an ORVA program is defined to be the number of "PRINT" statements in the loop of a reduced ORVA program for that sequence. We say that a sequence is *generally periodic* if and only if it can be generated by an ORVA program.

The idea of generalized periodicity and this formulation of its definition are due to Manuel Blum.

We will now describe a construction which allows us to put ORVA programs into a form which is easy to work with.

DEFINITION. The *normal form* for a loop allows only one assignment to each variable, zero or one print statements for each variable, and places the assignment statements first, followed by the output statements, followed by a goto. The assignments are made to variables in order of rank, with the highest ranking variable being assigned first.

Example:

$$
\begin{aligned}
L:\quad & x_n := c_{n,n}x_n + \cdots + c_{n,1}x_1 + c_{n,0}\\
& x_{n-1} := c_{n-1,n}x_{n-1} + \cdots + c_{n-1,1}x_1 + c_{n-1,0}\\
& \vdots\\
& x_1 := c_{1,1}x_1 + c_0\\
& \text{PRINT } x_{i_1}\\
& \vdots\\
& \text{PRINT } x_{i_m}\\
& \text{GO } L.
\end{aligned}
$$

We require $i_j = i_k$ iff $j = k$, and $m \leqslant n$.

THEOREM 1. *Every* ORVA *loop can be put into normal form, without altering the number of* "PRINT" *statements.*

Proof. Given a loop of code, we will produce an equivalent loop in normal form which generates the same output for each cycle of the loop.

First we move all PRINT statements to the end of the loop and eliminate multiple printing of the same variable. For each statement "PRINT x_i" we introduce a previously unused variable w_j with $\text{rank}(w_j) > \text{rank}(x_i)$. Delete the statement "PRINT x_i" and insert the code "$w_j := x_i$; PRINT w_j." If "PRINT x_i" occurs in more than one place in the loop, a different w_j must be used in each instance. Each "PRINT w_j" can be moved toward the end of the loop without altering the value of w_j. Hence, we can now move all PRINT statements to the end of the loop so that they form a block of output statements which immediately precedes the "GO" statement. If we don't shuffle the original order of the PRINT statements, then the result is a program which produces the same output as before.

We now have a loop consisting of a block of assignment statements, followed by a block of output statements, followed by a goto. We must show how to convert an arbitrary block of assignment statements into an equivalent one where the assignments are to variables of successively decreasing rank. The proof is by induction on n, the highest rank of any variable in the block.

Base. For $n = 1$ all assignments are to the same variable, so we have a block

$$
\begin{aligned}
x_1 &:= a_1x_1 + a_0\\
x_1 &:= b_1x_1 + b_0\\
&\vdots\\
x_1 &:= m_1x_1 + m_0.
\end{aligned}
$$

We can combine the first two assignments into the single statement

$$x_1 := b_1(a_1x_1 + a_0) + b_0 \qquad (= b_1a_1x_1 + (b_1a_0 + b_0))$$

so we can delete the first two statements and insert this one in their place. If we continue to combine the first two assignments of each new block we will eventually be left with one

statement (e.g., $x_1 := c_1x_1 + c_0$) which is equivalent to the entire original block of statements. This statement is in the desired form.

Induction Step. By hypothesis assume that any block of statements all of whose variables have rank less than n can be converted into a block where assignments are to variables of successively decreasing rank, $n > 1$. Let $\operatorname{rank}(x_i) = i$.

Consider the statements

$$x_i := a_ix_i + a_{i-1}x_{i-1} + \cdots + a_0 \tag{1}$$

$$x_n := b_nx_n + b_{n-1}x_{n-1} + \cdots + b_0 \tag{2}$$

and the statement

$$\begin{aligned} x_n := {} & b_nx_n + \cdots + b_{i+1}x_{i+1} + b_i(a_ix_i + \cdots + a_0) + b_{i-1}x_{i-1} + \cdots + b_0 \\ & (= b_nx_n + \cdots + b_{i+1}x_{i+1} + b_ia_ix_i + (b_ia_{i-1} + b_{i-1})x_{i-1} + \cdots \\ & \quad + (b_ia_1 + b_1)x_1 + (b_ia_0 + b_0)). \end{aligned} \tag{3}$$

If (1) and (2) occur successively in a block and $i = n$ then we can delete them and insert (3) in their place. If $i < n$, then we insert (3), followed by (1). Using this method, we find the last assignment in a block to x_n and move it upward to the top of the block (by combining or switching with the immediately preceding assignment and making necessary changes in the scalars). When we reach the top, the block will consist of an assignment to x_n, followed by assignments to variables of lower rank. By the induction hypothesis these can be converted into a block of decreasingly ranked assignments, so that the whole block has the required form. □

We will henceforth assume that all ORVA loops initially are in normal form.

Properties of Generally Periodic Sequences. We will now prove some theorems which describe the properties of sequences which are generated by ORVA programs.

Convention. We denote the value of a variable x after t iterations of the loop by $x(t)$. $x(0)$ is the value of x as the loop is about to be entered for the first time. Hence, each variable x in an ORVA loop is associated with a function $x(t)$ over the non-negative integers.

THEOREM 2 (Monotonicity Theorem). *For any variable x in an* ORVA *loop there is an integer t_0 such that for all $t \geq t_0$, $x(t)$ is either constant or strictly monotonic. We call such a function* ultimately monotonic (*abbreviated* u.m.).

Proof. The proof is an induction on n, the highest rank of any variable in an ORVA loop *in normal form.*

Base. $n = 1$. Let x have rank 1. Then the loop has one assignment: $x := ax + b$ ($a \geq 0$).

Case 1. $a = 0$. Then $x(t) = b$ for all t; hence it is constant and therefore u.m.

Case 2. $a = 1$. An easy induction proves that $x(t) = bt + x(0)$, so $x(t)$ is constant if $b = 0$ and strictly monotonic otherwise.

Case 3. $a > 0$, $a \neq 1$.

Claim.

$$x(t) = a^t\left(x(0) + \frac{b}{a-1}\right) - \frac{b}{a-1}.$$

Proof of Claim by induction on t:

If $t = 0$, then $a^0(x(0) + b/(a-1)) - b/(a-1) = x(0)$.

Assuming that the claim holds for $x(t-1)$ we have

$x(t) = ax(t-1) + b$ by definition

$$=a\left(a^{t-1}\left(x(0)+\frac{b}{a-1}\right)-\frac{b}{a-1}\right)+b \quad \text{by inductive hypothesis}$$
$$=a^t\left(x(0)+\frac{b}{a-1}\right)+b-\frac{ab}{a-1}$$
$$=a^t\left(x(0)+\frac{b}{a-1}\right)+\frac{((a-1)-a)b}{a-1}$$
$$=a^t\left(x(0)+\frac{b}{a-1}\right)-\frac{b}{a-1}$$

so the claim is true. Since a function f of the form $f(t)=a^t k_1+k_2$ is strictly monotonic when $a>0$, $x(t)$ is u.m.

Induction Step. $n>1$. Assume by hypothesis that all variables in a loop with rank $<n$ are u.m.

First we will need a

LEMMA. *Fix n. Assume that if a variable has rank $<n$, then it is* u.m. *Let w have rank n and let $x_1, x_2, \ldots, x_{n-1}$ have ranks* 1 *through $n-1$, respectively* (*so they are* u.m. *by assumption*). *Define the function $w(t)$ by*

$$w(t)=c_{n-1}x_{n-1}(t)+c_{n-2}x_{n-2}(t)+\cdots+c_1x_1(t)$$

where the c_i are arbitrary constants. Then we claim that $w(t)$ is u.m.

Proof of Lemma. We will produce a variable z with rank $n-1$ such that $w(t)=z(t)$ for all $t\geqslant 0$. Then, since $z(t)$ is u.m. by assumption, $w(t)$ will also be u.m.

Consider a loop in normal form containing two variables x and y with $n>\text{rank }(y)>\text{rank }(x)$. Let their assignments in the loop be

$$y:=ay+bx+f(x_1,\ldots,x_i)$$
$$x:=cx+g(x_1,\ldots,x_j)$$

where we are using "f" and "g" as a shorthand to denote a sum of other lower-ranking variables. Then we can write

$$y(t+1)=ay(t)+bx(t)+f(t)$$
$$x(t+1)=cx(t)+g(t).$$

We wish to show that $z(t)=py(t)+qx(t)$ is u.m. for any scalars p and q. Without loss of generality assume $z(0)=py(0)+qx(0)$. At the beginning of the loop containing x and y insert the assignment

$$z:=az+(pb+qc-qa)x+pf(x_1,\ldots,x_i)+qg(x_1,\ldots,x_j). \tag{4}$$

Claim. $z(t)=py(t)+q(t)$.

Proof by induction on t: For $t=0$ the claim is true by assumption. Assume the claim holds for $z(t-1)$, so that

$$z(t)=az(t-1)+(pb+qc-qa)x(t-1)+pf(t-1)+qg(t-1)$$
$$=a(py(t-1)+qx(t-1))+(pb+qc-qa)x(t-1)+pf(t-1)+qg(t-1)$$
$$\text{by induction hypothesis}$$
$$=p(ay(t-1)+bx(t-1)+f(t-1))+q(cx(t-1)+g(t-1))$$
$$=py(t)+qx(t)$$

so the claim is true.

Now observe that z does not depend upon the variable y. Hence we can eliminate the variable y and its assignment statement, let rank (z) have the value rank (y), and leave (4) in the

loop. Then, since rank $(z)<n$, the assumption tells us that $z(t)$ is u.m., and hence $py(t)+qx(t)$ is u.m.

Observe that using this method we can successively produce $c_1x_1+c_2x_2$, $(c_1x_1+c_2x_2)+c_3x_3$, $((c_1x_1+c_2x_2)+c_3x_3)+c_4x_4$, etc., with each sum being u.m. Therefore we can produce a variable z such that $z(t)=c_1x_1(t)+c_2x_2(t)+\cdots+c_{n-1}x_{n-1}(t)$ and $\operatorname{rank}(z)=n-1$. This proves the lemma.

Now we wish to show that if

$$w:=a_nw+a_{n-1}x_{n-1}+\cdots+a_1x_0+a_0$$

is the first statement in a loop in normal form and rank $(w)=n$, then $w(t)$ is u.m.

Case 1. $a_n=0$. Then the lemma can be applied to show $w(t)$ is u.m.

Case 2. $a_n>0$. Let $f(t)=a_{n-1}x_{n-1}(t)+\cdots+a_1x_1(t)+a_0$. Because the variables x_1 to x_{n-1} have rank $<n$, the lemma again applies to tell us that $f(t)$ is u.m.

$\underline{A}$. Suppose for all $t>t_0$ we have $f(t)\geqslant f(t-1)$. Fix some $t>t_0$. Then, if $w(t)>w(t-1)$, we get $w(t+1)=a_nw(t)+f(t)>a_nw(t-1)+f(t)\geqslant a_nw(t-1)+f(t-1)=w(t)$; hence

$$w(t)>w(t-1)\Rightarrow w(t+1)>w(t). \tag{i}$$

Similarly $w(t)\geqslant w(t-1)\Rightarrow w(t+1)\geqslant w(t)$.

$\underline{B}$. Suppose for all $t>t_0$ we have $f(t)\leqslant f(t-1)$. Fix some $t>t_0$. Then, if $w(t)<w(t-1)$, we have $w(t+1)=a_nw(t)+f(t)<a_nw(t-1)+f(t)\leqslant a_nw(t-1)+f(t-1)=w(t)$, giving

$$w(t)<w(t-1)\Rightarrow w(t+1)<w(t). \tag{ii}$$

Similarly $w(t)\leqslant w(t-1)\Rightarrow w(t+1)\leqslant w(t)$.

Now it is easy to show that $w(t)$ is u.m. First determine if $\underline{A}$ or $\underline{B}$ holds for $f(t)$. Suppose $\underline{A}$ is true. Then after t_0 steps we inspect $w(t)$. If $w(t+1)>w(t)$ for any $t>t_0$, we know that $w(t)$ is strictly monotone increasing, by (i). Otherwise, if $w(t+1)\leqslant w(t)$ for all $t>t_0$ but, for some $t>t_0$, $w(t+1)\geqslant w(t)$, then $w(t)$ is constant from then on, again using (i). Lastly, it can happen that $w(t+1)<w(t)$ for all $t>t_0$, so that $w(t)$ is strictly monotone decreasing.

Similarly, if $\underline{B}$ holds, then we use (ii) to get an equivalent result.

This proves the induction step. □

COROLLARY. *If a sequence has generalized period one, then it is ultimately monotonic.*

The Monotonicity Theorem is a useful tool in proving properties of generally periodic sequences. It forms the basis for an easy proof of the next theorem.

THEOREM 3. *A sequence of period p has generalized period p.*

Proof. Let $\sigma_0,\sigma_1,\ldots,\sigma_{p-1},\sigma_p,\sigma_{p+1},\ldots$ be a sequence with p as its smallest period $(\sigma_i=\sigma_{i+p})$. We can generate this sequence with an ORVA loop that uses p variables equal to constants $\sigma_0,\ldots,\sigma_{j-1}$, and has p PRINT statements.

Let us assume that an ORVA program exists which generates $\sigma_0,\sigma_1,\sigma_2,\ldots$ using $m<p$ PRINT statements in its loop. Assuming that the loop is in normal form, then there are m different variables $z_1,\ldots,z_m$ which are printed at the end of each cycle in the loop.

Consider any z_i. By the Monotonicity Theorem $z_i(t)$ is u.m. If $z_i(t)$ was ultimately strictly increasing or decreasing, it would take on more than p different values, and hence could not be printing correct values for the sequence $\sigma_0,\sigma_1,\ldots$. We conclude that for all $i=1,\ldots,m$, $z_i(t)$ is constant (ultimately). Hence the sequence $z_1(0),z_2(0),\ldots,z_m(0),z_1(1),z_2(1),\ldots$ has period m because $z_i(t)=z_i(t+1)$ for all $t>t_0$. But $m<p$, and we assumed that we were generating a sequence of period p. Contradiction. □

It is clearly desirable that Theorem 3 be true, if our definition for a generalized period is to be a good one. We can now see why certain constraints were placed upon the ORVA language:

REMARK. If we allow the leading coefficient of the general assignment statement to be negative, then Theorems 2 and 3 don't hold. Example:

$$\begin{aligned} & x:=1 \\ L: \quad & x:=-x+1 \\ & \text{PRINT } x \\ & \text{GO } L. \end{aligned}$$

This program outputs the period two sequence $0,1,0,1,\ldots$ with only one PRINT statement in its loop. $x(t)$ is not u.m.

REMARK. If we eliminate the rankings, we again can find a counterexample to Theorems 2 and 3. Example:

$$\begin{aligned} & x:=1 \\ & y:=2 \\ L: \quad & z:=x \\ & x:=y \\ & y:=z \\ & \text{PRINT } z \\ & \text{GO } L. \end{aligned}$$

This program outputs $1,2,1,2,\ldots$ with only one PRINT statement.

DEFINITION. For any numerical sequence $S=\sigma_0,\sigma_1,\sigma_2,\ldots$ the *sequence of differences* is $\Delta S=\sigma_1-\sigma_0,\sigma_2-\sigma_1,\sigma_3-\sigma_2,\ldots$.

Taking differences is commonly used as an aid for inferring a sequence. The following theorem complements this technique.

THEOREM 4. *If a sequence S has generalized period p, then the corresponding sequence of differences ΔS has* g.p. p. *Conversely, if a sequence T with* g.p. p *is considered a sequence of differences, then any sequence S such that $\Delta S=T$ must have* g.p. p.

Proof. Suppose we are considering a sequence where the (reduced) ORVA loop prints successively the variables $z_1,z_2,\ldots,z_p$ in each iteration. Suppose z_1 is assigned in the loop by the statement

$$z_1:=cz_1+c_nx_n+\cdots+c_0$$

and w.l.o.g. assume that $z_1,\ldots,z_p$ are initialized before entering the loop.

Case 1. $p=1$. We construct an ORVA loop to generate successive differences as follows: Add the statement "$w:=-z_1$" at the top of the loop, where w is a new variable. Insert "$w:=w+z_1$" directly before "PRINT z_1," and change "PRINT z_1" to "PRINT w." The loop now outputs $w(t)=z_1(t)-z_1(t-1)$, as desired.

Case 2. $p>1$. Create p new variables $w_1,\ldots,w_p$ and at the beginning of the loop add

$$\begin{aligned} & w_1:=z_2-z_1 \\ & w_2:=z_3-z_2 \\ & \quad\vdots \\ & w_{p-1}:=z_p-z_{p-1} \\ & w_p:=cz_1+c_nx_n+c_{n-1}x_{n-1}+\cdots+c_0-z_p. \end{aligned}$$

Delete all PRINT statements and insert "PRINT w_1; $\cdots$; PRINT w_p" at the end of the loop. This prints the sequence of differences.

Note that we have shown that g.p. $(\Delta S) \leqslant$ g.p. (S).

Now assume that a sequence of differences ΔS is generated by a reduced ORVA loop printing the variables $w_1, \ldots, w_m$, and the original sequence S started with σ_0. Before the loop insert "$z_i := \sigma_0$; PRINT z_1." Let $z_1, \ldots, z_m$ be new variables. Delete all "PRINT w_i" statements.

Case 1. $m=1$. At the end of the loop insert "$z_1 := z_1 + w_1$; PRINT z_1."

Case 2. $m>1$. At the end of the loop add

$$
\begin{array}{l}
z_2 := z_1 + w_1 \\
\vdots \\
z_m := z_{m-1} + w_{m-1} \\
z_1 := z_1 + w_m + \cdots + w_1 \\
\text{PRINT } z_2 \\
\vdots \\
\text{PRINT } z_m \\
\text{PRINT } z_1
\end{array}
$$

This new loop outputs the original sequence S. We see g.p.$(S) \leqslant m =$ g.p.(ΔS). Hence g.p.$(S)=$ g.p.(ΔS). □

DEFINITION. Let $S = \sigma_0, \sigma_1, \sigma_2, \ldots$. Then $-S = -\sigma_0, -\sigma_1, -\sigma_2, \ldots$. Let $T = \tau_0, \tau_1, \tau_2, \ldots$. Then $S + T = \sigma_0 + \tau_0, \sigma_1 + \tau_1, \sigma_2 + \tau_2, \ldots$.

THEOREM 5. g.p.$(S)=$ g.p.$(-S)$.

Proof. In the program generating S replace each statement "PRINT x_i" by "$w_i := -x_i$; PRINT w_i," where w_i is a new variable. Then g.p.$(-S) \leqslant$ g.p.(S). But then g.p.$(S)=$ g.p.$(-(-S)) \leqslant$ g.p.$(-S) \leqslant$ g.p.(S). □

THEOREM 6. *If* g.p.$(S)=$ g.p.$(T)=p$ *then* g.p.$(S+T) \leqslant p$.

Proof. Assume w.l.o.g. that the programs for S and T do not have any variables in common. Form a new loop consisting of all code from the loop for S and the loop for T. Assuming that variables $s_1, \ldots, s_p$ and $t_1, \ldots, t_p$ are printed, delete all PRINT statements and in place of each PRINT s_i and PRINT t_i we insert "$w_i := s_i + t_i$; PRINT w_i" where w_i is a new variable.

Similarly altering the initial code (before the loop) leads to a program for $S+T$ with p print statements. □

Note that g.p.$(S+(-S))=1$ for all sequences S. We will show later that g.p.$(S+T)$ divides p when g.p.$(S)=$ g.p.$(T)=p$.

COROLLARY 6. *If* g.p.$(S)=$ g.p.$(T)=p$, *then* g.p.$(S-T) \leqslant p$.

THEOREM 7. *If x_i is a variable in an* ORVA *program, then there exist positive real constants* $1, \lambda_1, \ldots, \lambda_m$ *and polynomials* $p_0(t), \ldots, p_m(t)$ *such that* $x_i(t) = p_0(t) + p_1(t)\lambda_1^t + \cdots + p_m(t)\lambda_m^t$. *The numbers* $\lambda_1, \ldots, \lambda_m$ *correspond to non-zero leading coefficients in the assignment statements.*

Proof. Let $\vec{x}(t)$ denote the vector $\langle 1, x_1(t), \ldots, x_n(t) \rangle$ where x_1 to x_n are variables in an ORVA loop in normal form $(\operatorname{rank}(x_i) < \operatorname{rank}(x_{i+1}))$. Each iteration of the loop is equivalent to a matrix transformation $\vec{x} = A \cdot \vec{x}(t) + B$, where A is an $(n+1)$ $(n+1)$ lower triangular matrix. Induction will prove that $A^t \cdot x(0) = x(t)$.

The diagonal elements of A are also its eigenvalues, and since a diagonal element corresponds

to the non-negative leading coefficient of an assignment or equals one, the eigenvalues are non-negative. There exist matrices E and T such that $A=T^{-1}ET$ and E is in Jordan-canonical form, with the eigenvalues of A being the diagonal elements of E. Then $\vec{x}(t)=T^{-1}E^tT\vec{x}(0)$.

Suppose E is a strictly diagonal matrix, say

$$E=\begin{bmatrix} \lambda_1 & & & \bigcirc \\ & \lambda_2 & & \\ & & \ddots & \\ \bigcirc & & & \lambda_n \end{bmatrix} \text{ and } E^t=\begin{bmatrix} \lambda_1^t & & & \bigcirc \\ & \lambda_2^t & & \\ & & \ddots & \\ \bigcirc & & & \lambda_n^t \end{bmatrix}.$$

Then

$$\begin{aligned} x_1(t) &= \langle 0,\ldots,0,\underset{\substack{\uparrow \\ i\text{th place}}}{1},0,\ldots,0\rangle T^{-1}E^tT\vec{x}(0) \\ &= \langle a_1,\ldots,a_n\rangle\cdot E^t\cdot\langle b_1,\ldots,b_n\rangle \quad \text{for some } a_j\text{'s and } b_j\text{'s} \\ &= a_1b_1\lambda_1^t+\cdots+a_nb_n\lambda_n^t \end{aligned}$$

which is in the proper form.

Next take the case that E is a Jordan block, say

$$E=J=\begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \bigcirc \\ & & \ddots & \ddots & \\ & \bigcirc & & \lambda & 1 \\ & & & & \lambda \end{bmatrix}.$$

Using the bionomial identity $\binom{t}{m}=\binom{t-1}{m-1}+\binom{t-1}{m}$ and interpreting $\binom{t}{m}$ as 0 if $m>t$, we can prove by induction that

$$J^t=\begin{bmatrix} \lambda^t & t\lambda^{t-1} & \binom{t}{2}\lambda^{t-2} & \binom{t}{3}\lambda^{t-3} & \binom{t}{n}\lambda^{t-n} \\ & \lambda^t & t\lambda^{t-1} & \binom{t}{2}\lambda^{t-2} & \binom{t}{n-1}\lambda^{t-(n-1)} \\ & & \ddots & & \\ & & & \ddots & \\ \bigcirc & & & & \lambda^t \end{bmatrix}.$$

Then

$$\begin{aligned} x_i(t) &= \langle a_1,\ldots,a_n\rangle\cdot J^t\cdot\langle b_1,\ldots,b_n\rangle \\ &= \langle a_1,\ldots,a_n\rangle\cdot\langle \textstyle\sum_{i=0}^{n-1}b_{i+1}\binom{t}{i}\lambda^{t-i},\sum_{i=0}^{n-2}b_{i+1}\binom{t}{i}\lambda^{t-i},\ldots,b_n\lambda^t\rangle \\ &= p(t)\lambda^t \quad \text{for some polynomial } p(t). \end{aligned}$$

Finally, in the most general case where E contains Jordan sub-blocks, a variable x_i is the sum of elements of the first two types, so $p_1(t)\lambda_1^t+\cdots+p_m(t)\lambda_m^t=x_i(t)$ for some p_j's and λ_j's. □

The outline for this proof comes from another presented by Pravin Varaiya.

THEOREM 8. *Given any function of t of the form* $p_1(t)\lambda_1^t+\cdots+p_m(t)\lambda_m^t$ *where* $\lambda_i>0$ *for*

$i=1,\dots,m$, we can produce an ORVA *program containing a variable w such that $w(t)$ is that function for $t \geqslant 0$.*

Proof. We will prove the theorem for the case $w(t)=t^k\lambda^t$. The more general case follows easily. Proof by induction on k:

Base. $k=0$. Let $w(0)=1$ and place "$w:=\lambda w$" in the loop. Then we have $w(t)=\lambda^t$.

Induction Step. $k>0$. Assume we have variables $x_0,\dots,x_{k-1}$ such that $x_i(t)=t^i\lambda^t$ for $i<k$. Let $w(0)=0$. At the top of the loop containing the x_i, add the statement $w:=\lambda w+\binom{k}{1}\lambda x_{k-1}+\cdots+\binom{k}{k}\lambda x_0$.

Claim. $w(t)=t^k\lambda^t$.

Proof of Claim by induction on t: If $t=0$, then $w(0) = 0 = 0^k\lambda^0$. Assume the claim true for the first t values of w.

$$\begin{aligned} w(t+1) &= \binom{k}{0}\lambda t^k\lambda^t+\binom{k}{1}\lambda t^{k-1}\lambda^t+\cdots+\binom{k}{k}\lambda t^0\lambda^t \quad \text{by hypothesis}\\ &= \lambda^{t+1}\left[\binom{k}{0}t^k+\binom{k}{1}t^{k-1}+\cdots+\binom{k}{k}t^0\right]\\ &= \lambda^{t+1}(t+1)^k. \end{aligned}$$

This proves the claim and hence the theorem. □

For the following definitions let $R=\rho_0,\rho_1,\rho_2,\dots$ and $S=\sigma_0,\sigma_1,\sigma_2,\dots$ be generally periodic sequences.

DEFINITION. R and S are *equivalent* ($R\equiv S$) if there exist constants i_0 and j_0 such that

$$\rho_{i_0+i}=\sigma_{j_0+i} \quad \text{for all } i\geqslant 0.$$

For example, suppose $R=0,1,2,3,\dots$ and $S=2,3,4,\dots$. Then $R\equiv S$.

DEFINITION. R is *contained* in $S (R\subseteq S)$ if there exist constants i_0 and j_0 such that $\rho_{i_0}=\sigma_{j_0}$ and an increasing function f such that $f(0)=0$ and $\rho_{i_0+i}=\sigma_{j_0+f(i)}$.

Example: If $R=1,2,4,8,\dots$ and $S=1,2,3,\dots$, then $R\subseteq S$.

DEFINITION. If the function f above can be represented as $f(t)=mt$ where $m\geqslant 1$, then we say that R is an *m-section* of S. Alternatively, we can say that S is m-times as dense as R ($D(S/R)=m$).

Example: Let $R=1,4,16,64,\dots$ and $S=1,2,4,8,16,\dots$. Then R is a 2-section of S, $D(S/R)=2$.

Notation. Let $(r(t))_{t=0}^{\infty}$ denote the sequence $r(0),r(1),r(2),\dots$ generated by a variable r.

Suppose $R=r(t)_{t=0}^{\infty}$ and $S=(s(t))_{t=0}^{\infty}$. If R is an m-section of S, then we have $r(i_0+t)=s(j_0+mt)$ for all $t\geqslant 0$. We note a corollary to Theorem 7.

COROLLARY 7. *If $S=(s(t))_{t=0}^{\infty}$ has* g.p. 1, *then there exist constants $\lambda_1,\dots,\lambda_n$ and polynomials $p_1(t),\dots,p_n(t)$ such that $s(t)=p_1(t)\lambda_1^t+\cdots+p_n(t)\lambda_n^t$.*

Using Theorem 8 we can derive another result.

COROLLARY 8. *Let S have* g.p. 1. *Then, given an integer $m\geqslant 1$, each m-section of S has* g.p. 1.

Proof. Let $S=(s(t))_{t=0}^{\infty}$ and suppose $R=(r(t))_{t=0}^{\infty}$ is an m-section of S with $r(i_0+t)=s(j_0+mt)$ for all $t\geqslant 0$. Then for $t\geqslant i_0$, $r(t)=s(j_0+m(t-i_0))$. By Corollary 7 there exist $\lambda_1,\dots,\lambda_n$ and $p_1(t),\dots,p_n(t)$ such that

$$r(t)=p_1(j_0+m(t-i_0))\lambda_1^{(j_0+m(t-i_0))}+\cdots+p_n(j_0+m(t-i_0))\lambda_n^{(j_0+m(t-i_0))}$$

$$=p_1'(t)\lambda_1'^t+\cdots p_n'(t)\lambda_n'^t,$$

so R has g.p. 1 by Theorem 8. □

THEOREM 9. *Let R be a sequence with generalized period* 1. *For each integer $m \geqslant 1$ there exists a unique* (*up to equivalence*) *sequence S with* g.p. 1 *such that R is an m-section of S ($D(S/R)=m$).*

Example: If a sequence $1, i_1, 9, i_2, 25, i_3, \ldots$ is known to have g.p. 1, then the sequence $i_1, i_2, i_3, \ldots$ must be equivalent to the sequence $4, 16, 36, \ldots$.

Proof. Let $R=(r(t))_{t=0}^{\infty}$. By Theorem 7 there exist constants $\lambda_1, \ldots, \lambda_n$ and polynomials $p_1(t), \ldots, p_n(t)$ such that $r(t)=p_1(t)\lambda_1^t+\cdots+p_n(t)\lambda_n^t$.

Existence: Fix $m \geqslant 1$. By Theorem 8 there exists a sequence

$$S=\left(p_1\left(\tfrac{t}{m}\right)\lambda_1^{t/m}+\cdots+p_n\left(\tfrac{t}{m}\right)\lambda_n^{t/m}\right)_{t=0}^{\infty}$$

with g.p. 1. Then $D(S/R)=m$.

Uniqueness: Suppose $S=(s(t))_{t=0}^{\infty}$ and $S'=(s'(t))_{t=0}^{\infty}$ both have g.p. 1 and $D(S/R)=D(S'/R)=m$. By definition there exist constants i_0, j_0, and j_0' such that $r(i_0+im)=s(j_0+im)=s'(j_0'+im)$ for all $i \geqslant 0$. Consider the sequence $S-S'$. By the corollary to Theorem 6, $S-S'$ has g.p. 1. Every mth term of this sequence is zero. Since it is ultimately monotonic, it must be equivalent to the sequence $0, 0, 0, \ldots$. Hence $S \equiv S'$. □

Before we prove our final theorem we need a lemma.

LEMMA. *Let R, S, and W be sequences with* g.p. 1, *such that $W \subseteq R$ and $W \subseteq S$. Suppose W is a p-section of R and W is a q-section of S ($D(R/W)=p$ and $D(S/W)=q$). Then p divides q implies $R \subseteq S$.*

Proof. Let $q=mp$ where m is a positive integer. By Theorem 7 W is equivalent to a sequence $(p_1(t)\lambda_1+\cdots+p_n(t)\lambda_n^t)_{t=0}^{\infty}$. By Theorem 8 there exist sequences

$$R'=\left(p_1(t/p)\lambda_1^{t/p}+\cdots+p_n(t/p)\lambda_n^{t/p}\right)_{t=0}^{\infty}$$

and

$$S'=\left(p_1(t/q)\lambda_1^{t/q}+\cdots+p_n(t/q)\lambda_n^{t/q}\right)_{t=0}^{\infty}$$

each having g.p. 1. Since $S'=(p_1(t/mp)\lambda_1^{t/mp}+\cdots+p_n(t/mp)\lambda_n^{t/mp})_{t=0}^{\infty}$, we observe that R' is an m-section of S'; hence $R' \subseteq S'$.

By construction $D(R'/W)=p=D(R/W)$ and $D(S'/W)=q=D(S/W)$; so by Theorem 9 we have $R \equiv R'$ and $S \equiv S'$. But then $R \subseteq S$. □

We may refer to a generalized period of a sequence to mean the number of print statements in the loop of a (possibly) unreduced ORVA program for that sequence. The next theorem shows that all generalized periods of a sequence must be integral multiples of the fundamental (or smallest) generalized period. This is in accord with the similar result for periodic sequences, and hence strengthens our belief that we have a good definition for the generalized period of a sequence.

THEOREM 10. *Let X have a* g.p. *of p and let Y have a* g.p. *of q. Let d be the greatest common divisor of p and q. If $X \equiv Y$, then there exists a sequence Z with a* g.p. *of d such that $Z \equiv X \equiv Y$.*

Proof. We will assume that $X=Y$ and find an appropriate Z such that $Z=X=Y$. The result for equivalence follows.

Let $X=((x_i(t))_{i=0}^{p-1})_{t=0}^{\infty}$ and $Y=((y_i(t))_{i=0}^{q-1})_{t=0}^{\infty}$. Let X_i denote $(x_i(t))_{t=0}^{\infty}$ and $Y_j=(y_j(t))_{t=0}^{\infty}$, so that each X_i is a p-section of X, and Y_j is a q-section of Y. For convenience let $\mu(0),\mu(1),\mu(2),\ldots$ denote the sequence $x_0(0),x_1(0),\ldots,x_{p-1}(0),x_0(1),x_1(1),\ldots=X=Y$. Note that

$$x_i(t)=\mu(pt+i) \qquad \text{for all } i\geqslant 0$$

and

$$y_i(t)=\mu(qt+i) \qquad \text{for all } i\geqslant 0.$$

Next define a sequence $A_0=(a_0(t))_{t=0}^{\infty}$ with $a_0(t)=x_0(qt)=\mu(pqt)=y_0(pt)$ for all $t\geqslant 0$. Intuitively, we chose points where X_0 and Y_0 "intersect." We have $A_0\subseteq X_0$ with $D(X_0/A_0)=q$ and $A_0\subseteq Y_0$ with $D(Y_0/A_0)=p$. Corollary 8 tells us that X_0 and Y_0, and hence A_0, all have g.p. 1. We now use Theorem 9 to find the unique sequence with g.p. 1 $Z_0=(z_0(t))_{t=0}^{\infty}$ which is pq/d times as dense as A_0, $D(Z_0/A_0)=pq/d$. Because p and q both divide pq/d the preceding lemma tells us that $X_0\subseteq Z_0$ and $Y_0\subseteq Z_0$. We would like to show that $X_i\subseteq Z_0$ whenever d divides i, $i=0,\ldots,p-1$.

Pick any such i, setting $i=cd$. There exist constants a' and b' such that $d=a'p+b'q$. Either $a'<0<b'$ or $b'<0<a'$. W.l.o.g. assume the former inequality holds, so we set $a=-ca', b=cb'$ to get $ap+i=bq$ with a and b non-negative.

We now look at points where X_i and Y_0 "intersect." Define $B_0=(b_0(t))_{t=0}^{\infty}$ with $b_0(t)=y_0(b+tp)=\mu(bq+tpq)=\mu(ap+i+tqp)=x_i(a+tq)$. Using the lemma freely we have

$$D(Y_0/A_0)=p \text{ and } D(Z_0/A_0)=\frac{pq}{d} \text{ implies } D(Z_0/Y_0)=\frac{q}{d}$$

$$D(Y_0/B_0)=p \text{ and } D(Z_0/Y_0)=\frac{q}{d} \text{ implies } D(Z_0/B_0)=\frac{pq}{d}$$

$$D(X_i/B_0)=q \text{ and } D(Z_0/B_0)=\frac{pq}{d} \text{ implies } D(Z_0/X_i)=\frac{p}{d}.$$

This requires that $X_i\subseteq Z_0$, as desired.

We now know that $X_i\subseteq Z_0$ for $i=0,d,2d,\ldots,p-d$. For each such X_i, $D(Z_0/X_i)=p/d$, and there are p/d different such X_i's; hence

$$\begin{aligned} Z_0&=X_0(0),X_d(0),X_{2d}(0),\ldots,X_{p-d}(0),X_0(1),\ldots,\\ &=\mu(0),\mu(d),\mu(2d),\ldots. \end{aligned}$$

Therefore the sequence $\mu(0),\mu(d),\mu(2d),\ldots$ has g.p. 1.

In a similar manner we can define a sequence A_1 with $a_1(t)=x_1(qt)=y_1(pt)$, construct Z_1 such that $D(Z_1/A_1)=pq/d$, and eventually conclude that the sequence $\mu(1),\mu(d+1)$, $\mu(2d+1),\ldots$ has g.p. 1. Continuing in this fashion we will eventually have sequences $Z_0,Z_1,\ldots,Z_{d-1}$ each with g.p. 1 such that $Z=((z_i)_{i=0}^{d})_{t=0}^{\infty}=X=Y$. □

COROLLARY 10.1. *If a sequence has a generalized period of q and its fundamental period is p, then p divides q.*

COROLLARY 10.2. *Let S and T be sequences with* g.p.(S)=g.p.$(T)=p$. *Then* g.p.$(S+T)$ *divides p.*

Proof. In Theorem 6 we produced a program for $S+T$ which had p PRINT statements in its loop. By Theorem 10 g.p.$(S+T)$ divides p. □

Conclusion. A significant number of properties of periodic sequences carry over to the generalized case. We feel that they constitute a good justification for choosing our particular definition of a generalized period. The ORVA language has proved to be a useful tool for dealing with generally periodic sequences.

There are sequences, such as the factorial sequence $1,2,6,24,\ldots$, which are not generally periodic but are not particularly complex. Possibly our present definition can be extended to

include a broader class of sequences. Some research in this direction was greeted by a large increase in complexity, so that a further generalization of the definition of periodicity constitutes a non-trivial problem.

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A GEOMETRIC METHOD OF PHASE PLANE ANALYSIS

DUANE W. DeTEMPLE

Introduction. The real first-order linear autonomous homogeneous system of differential equations in the plane is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = Q\begin{pmatrix} x \\ y \end{pmatrix}, \qquad Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{1}$$

where the dot denotes the derivative with respect to the real variable t, and a, b, c, d are real constants. The usual method to study (1) is to introduce a real linear change of variables which transforms the coefficient matrix Q to a canonical (say, real Jordan) form.

Our purpose here is to replace the algebraic approach by a geometric one. The development leads to a construction procedure by which the phase portrait can be quickly and accurately drawn in the original x, y-plane with no calculations whatever required.

In Section 1 the real differential system is recast into an equivalent complex form involving both $z = x + iy$ and the conjugate variable $\bar{z} = x - iy$. Employing conjugate variable methods we derive in Section 2 some geometric results which are important in the sequel. In particular, the invariants of the coefficient matrix Q are identified geometrically in Section 3. In Section 4 the geometric description and construction procedures for phase portraits is illustrated. The concluding Section 5 shows the existence and construction of a homothetic family of conics isogonal to the trajectory system; the principal analytical tool in this section is the Schwarz function [2].

1. Complexification. The real plane is converted into the complex plane $\mathbb{C}$ by assigning the complex number $z = x + iy$ to the point (x,y) in the real plane. To invert this transformation, it is convenient to introduce the complex conjugate $\bar{z} = x - iy$. In matrix form the transformation from real to conjugate coordinates can be written

$$\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = M\begin{pmatrix} x \\ y \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}. \tag{2}$$

The real differential system (1) can be recast into the equivalent complex form

$$\begin{pmatrix} \dot{z} \\ \dot{\bar{z}} \end{pmatrix} = P\begin{pmatrix} z \\ \bar{z} \end{pmatrix}, \qquad P = MQM^{-1} = \begin{pmatrix} \gamma & \rho \\ \bar{\rho} & \bar{\gamma} \end{pmatrix}, \tag{3}$$

The author received his Ph.D. from Stanford in 1970, under the supervision of M. M. Schiffer (to whom he wishes to dedicate this paper with appreciation and admiration). Since 1970 he has taught at Washington State University. His principal fields of interest are complex analysis, elementary geometry, mathematics history, and mathematics education.—*Editors*

where

$$\gamma=\frac{1}{2}[(a+d)+i(c-b)]$$
$$\rho=\frac{1}{2}[(a-d)+i(c+b)]. \tag{4}$$

From (3) we obtain

$$\dot{z}=\gamma z+\rho\bar{z}. \tag{5}$$

The second equation $\dot{\bar{z}}=\bar{\gamma}\bar{z}+\bar{\rho}z$ is redundant, and so (5) is the equivalent complex form of (1).

When $\rho=0$ (i.e., $a=d$, $b=-c$) the solution of (5) is immediate: the trajectory which passes through z_0 at $t=0$ is

$$z(t)=z_0e^{\gamma t}=z_0e^{i(\operatorname{Im}\gamma)t}e^{(\operatorname{Re}\gamma)t}. \tag{6}$$

This is, in general, a family of equiangular spirals about the origin. In the special case $\operatorname{Im}\gamma=0$, the origin is a star.

When $\rho\neq 0$, the differential equation does not integrate easily. The right-hand side of (5) depends upon $\bar{z}$ as well as z, and so we will next consider conjugate variables in more detail.

2. Complex Conjugate Coordinates. The real axis in the complex z-plane is given by the equation $\bar{z}=z$. The z-plane is rotated clockwise through an angle θ by replacing z with $e^{-i\theta}z$ (and hence $\bar{z}$ with $e^{i\theta}\bar{z}$), and so

$$\bar{z}=Az, \qquad A=e^{-2i\theta}, \tag{7}$$

is the equation of the line through the origin which makes an angle θ with the real axis. The quantity A, which gives the orientation (as distinguished from the slope) of the undirected line, is known as the *clinant*.* If a clinant A is located on the unit circle in the w-plane, $w=u+iv$ (see Fig. 1), then the line $\bar{z}=Az$ is the line passing through $w=-1$ and $w=A$ in the $\bar{z}$-plane shown.

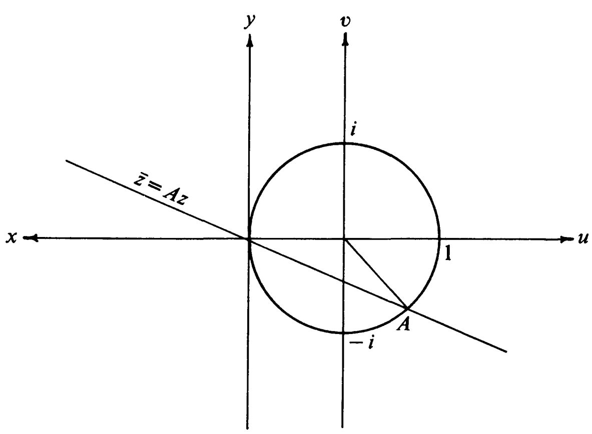


FIG. 1

Translating the z-plane by $z\mapsto z-z_0$, it follows from (7) that

$$\bar{z}=\bar{z}_0+A(z-z_0) \tag{8}$$

is the equation of the line through z_0 with clinant A. The slope m and clinant A of a line are

*This definition agrees with Davis [2, p. 9]; in Carver [1, p. 12] A would be replaced by its negative. Both books provide an attractive introduction to conjugate variable methods of plane geometry.

related by the formulas

$$A=\frac{1-im}{1+im}, \qquad m=-i\frac{1-A}{1+A}. \tag{9}$$

Notice that two lines are perpendicular if and only if their clinants are negatives of one another.

Turning next to circles, the equation $|z|^2=r^2$ becomes $\bar{z}=r^2/z$ in conjugate coordinates. Hence replacing z with $z-\gamma$,

$$\bar{z}=\bar{\gamma}+\frac{r^2}{z-\gamma} \tag{10}$$

is the equation of a circle of radius $r>0$ centered at $\gamma\in\mathbb{C}$. Denote the circle by $C(\gamma;\, r)$.

When combined with the algebra of complex numbers, these representations in complex coordinates provide a convenient method to deduce geometric relationships. This will be illustrated by proving two results, each of which will be useful to us later.

First, suppose the line $\bar{z}=Az$ intersects the circle given by (10) at z_1 and z_2. Eliminating $\bar{z}$ between the two equations we get

$$Az^2-z(\bar{\gamma}+A\gamma)+|\gamma|^2-r^2=0.$$

As z_1 and z_2 are the two roots of this quadratic, it follows that

$$Az_1z_2=|\gamma|^2-r^2.$$

Taking absolute values shows

$$|z_1|\,|z_2|=\Big||\gamma|^2-r^2\Big|, \tag{11}$$

which is independent of A. We have obtained a theorem from classical plane geometry:

Let O be a point not lying on a circle Γ; then the product of the lengths of the segments from O to the points of intersection with Γ is the same for all lines through O which intersect Γ.

For the second result, let $\Gamma=C(\gamma;\, r)$, $r>0$, be a circle which is either tangent to or disjoint from the real axis. This will be the case if and only if $\Delta\leqslant 0$, where

$$\Delta=4\big[r^2-(\operatorname{Im}\gamma)^2\big].$$

For any real ξ it follows from (10) and (11) that

$$\bar{z}=\xi+\frac{|\gamma-\xi|^2-r^2}{z-\xi} \tag{12}$$

is the equation for the circle $C=C(\xi;\, R)$, $R^2=|\gamma-\xi|^2-r^2$, centered at ξ which is orthogonal to Γ. As $[|\gamma-\xi|^2-r^2]-(\operatorname{Re}\gamma-\xi)^2=-\frac{1}{4}\Delta\geqslant 0$, it is clear C will intersect the vertical line L through γ. The clinant of a vertical line is -1, and so the equation of L is

$$\bar{z}=\bar{\gamma}+(-1)(z-\gamma)=2\operatorname{Re}\gamma-z. \tag{13}$$

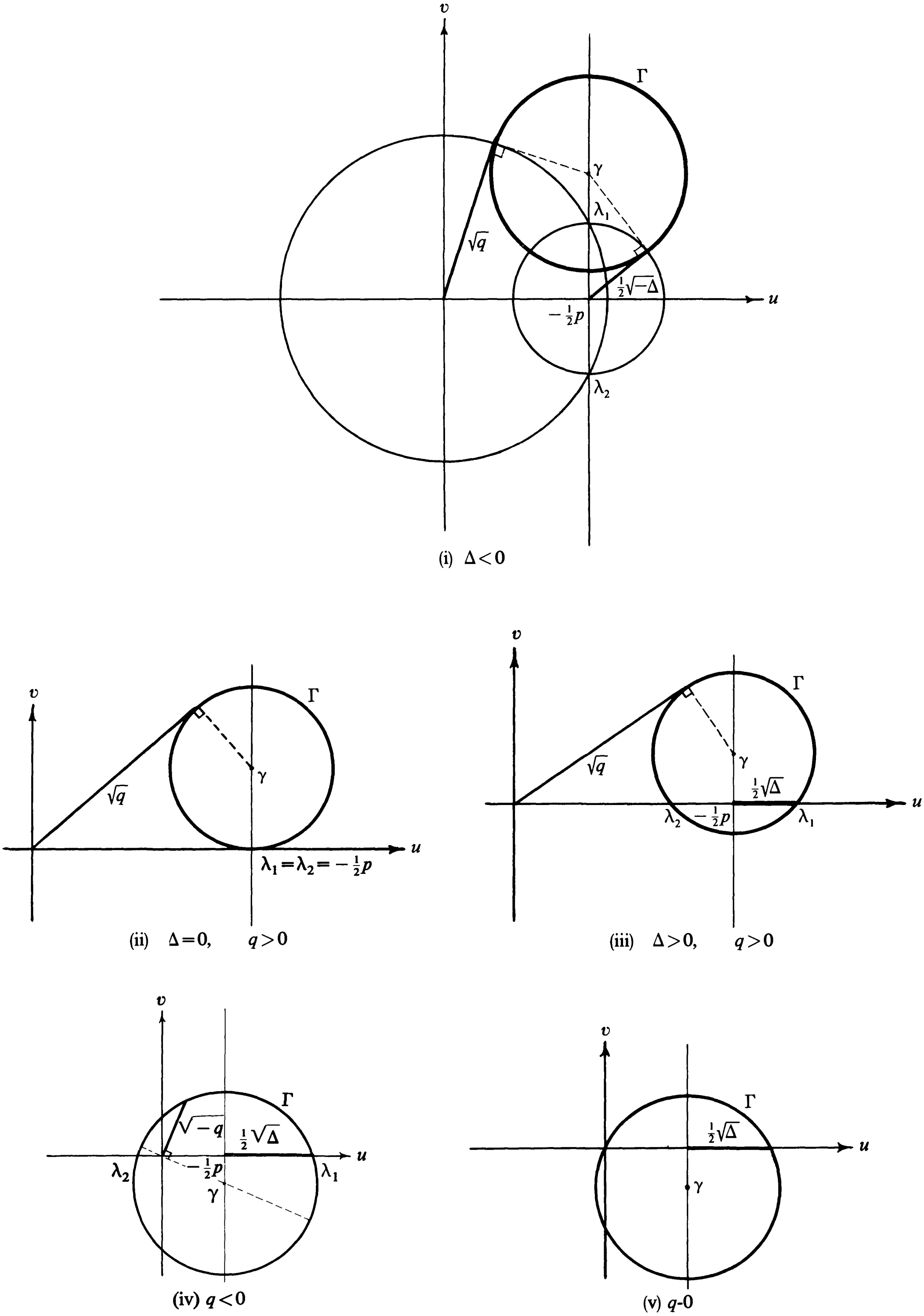
To find the points of intersection of C and L we eliminate $\bar{z}$ between (12) and (13), arriving at the quadratic equation

$$z^2-(2\operatorname{Re}\gamma)z+|\gamma|^2-r^2=0. \tag{14}$$

Since this equation has the same roots

$$\lambda,\bar{\lambda}=\operatorname{Re}\gamma\pm\tfrac{1}{2}i\sqrt{-\Delta}$$

for all ξ, we conclude *there are two points, λ and $\bar{\lambda}$, through which all circles orthogonal to both Γ and the x-axis will pass*. The family of circles through two points (which we will call *common points*), together with its orthogonal family of circles, form the well-known *Steiner circles*. An interesting connection between Steiner circles and linear transformations of the plane will be established in the next section.



(i) $\Delta<0$

(ii) $\Delta=0, \quad q>0$

(iii) $\Delta>0, \quad q>0$

(iv) $q<0$

(v) q-0

FIG. 2

3. Geometric Identification of the Invariants. From (3) we see P and Q are similar matrices and therefore have the same invariants:

discriminant

$$\Delta=(a-d)^2+4bc=4[|\rho|^2-(\operatorname{Im}\gamma)^2],$$

determinant

$$q=ad-bc=|\gamma|^2-|\rho|^2,$$

trace

$$-p=a+d=2\operatorname{Re}\gamma,$$

eigenvalues

$$\lambda_1=\tfrac{1}{2}(-p+\Delta^{\frac{1}{2}}),\lambda_2=\tfrac{1}{2}(-p-\Delta^{\frac{1}{2}}).$$

In the complex w-plane, $w=u+iv$, which we will refer to as the parameter plane, introduce the circle

$$\Gamma=C(\gamma;|\rho|).$$

The two geometric results derived in the preceding section allow us to geometrically represent the invariants in the parameter plane. There are five distinct cases when $\rho\neq 0$, as shown in Figure 2. Notice that the eigenvalues are the common points of the Steiner circles determined by Γ and the u-axis. The determinant q is negative, zero, or positive when the origin is, respectively, interior, on, or exterior to Γ. Similarly, the discriminant Δ is negative, zero, or positive in the respective cases the u-axis does not intersect Γ, is tangent to Γ, or intersects Γ in two distinct points.

4. Phase Portraits. Along the line $\bar{z}=Az$ it follows from the differential equation $\dot{z}=\gamma z+\rho\bar{z}$ that

$$\dot{z}=z(\gamma+\rho A). \tag{15}$$

Thus for $z\neq 0\neq\gamma+\rho A$,

$$\arg\dot{z}=\arg z+\arg(\gamma+\rho A), \tag{16}$$

which shows that lines through the origin are isoclines. If on sufficiently many lines $\bar{z}=Az$ one draws short directed segments in the direction of $\dot{z}$, the phase portrait is easily sketched. To this end suppose a $\bar{z}$-plane is superimposed on the parameter w-plane as shown in Figure 3. The origin of the $\bar{z}$-plane is $w=\gamma-\rho=d-ib$ and the negative x-axis passes through $w=\gamma+\rho=a+ic$. In view of Figure 1, the line $\bar{z}=Az$ is the line through $w=\gamma+\rho A$. Since we have a $\bar{z}$-plane (rather than a z-plane), the angle $\phi=\arg(\gamma+\rho A)$ is taken clockwise to determine the tangent direction $\dot{z}$ along $\bar{z}=Az$. It is perhaps surprising that the direction field is constructible in the Euclidean sense.

A case of special interest occurs when Γ intersects the u-axis. The intersection points are the (necessarily real) eigenvalues λ_j of Q, and the line from $\bar{z}=0$ through $w=\lambda_j$ is an *invariant line*. Indeed, it is the eigenspace corresponding to λ_j. As shown in Figure 3, the tangent directions point outward if $\lambda_j>0$ (since $\phi_j=0$), and inward if $\lambda_j<0$ (since $\phi_j=\pi$). If Γ passes through the origin $w=0$, zero is an eigenvalue and the corresponding invariant line is a line of critical points.

If Γ happens to intersect the vertical v-axis, there are seen to exist *orthogonal lines*; that is, lines which are crossed by the trajectories at right angles.

Let us apply these ideas to the specific case $a=3$, $b=-1$, $c=0$, $d=1$. To give the x,y axes their usual right-handed orientation, we shall carry out the construction in a $\bar{w}$ parameter plane, as shown in Figure 4. The points $(a,\ c)=(3,\ 0)$ and $(d,\ -b)=(1,1)$ are plotted first, and the segment between them is bisected to locate the center of Γ. The circle Γ is then drawn, along with the x,y axes. In this example the x-axis itself is an invariant line, corresponding to the eigenvalue $\lambda_1=3$. A second invariant line corresponds to $\lambda_2=1$. The figure also shows a typical isocline.

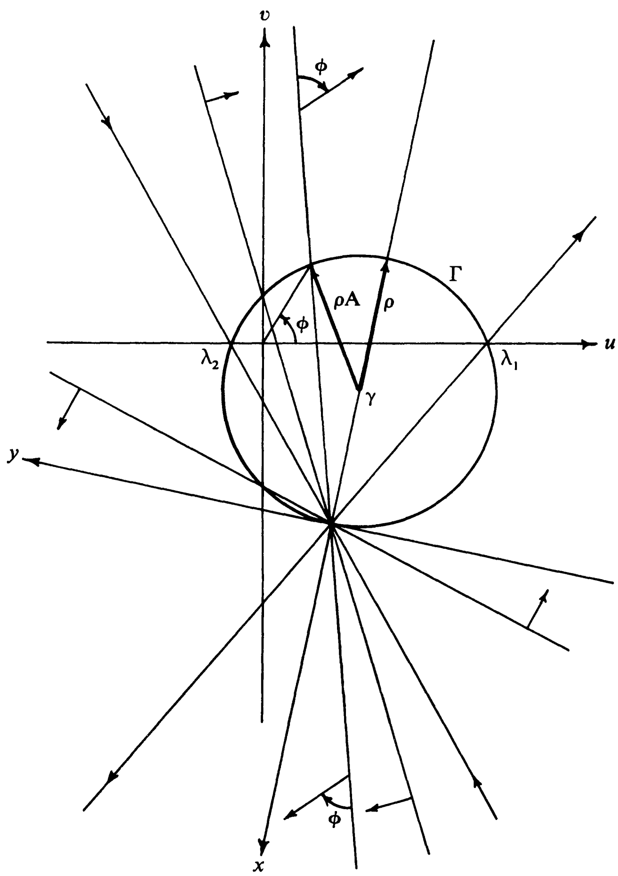


FIG. 3

The phase portrait drawn in Figure 4 is typical of the case $\Delta > 0$, $q > 0$. As shown in Figure 2, this is the case wherein Γ intersects the u-axis in two distinct points of the same sign. There are two invariant lines. With the single exception of the invariant line corresponding to the eigenvalue farther from $w = 0$, the trajectories are tangent to the invariant line corresponding to the eigenvalue nearer to $w = 0$. Thus $z = 0$ is a two-tangent node.

A similar discussion holds for the remaining cases illustrated in Figure 2. If Γ does not intersect the u-axis (i.e., if $\Delta < 0$) and is not centered on the v-axis ($\operatorname{Re}\gamma \neq 0$) the origin $z = 0$ is a focus. The trajectories spiral inwardly toward $z = 0$ when $\operatorname{Re}\gamma < 0$; they spiral outwardly if $\operatorname{Re}\gamma > 0$. For $\operatorname{Re}\gamma = 0$, the trajectories form a family of homothetic ellipses, as can be shown by the following argument. Define

$$V = \operatorname{Im} \bar{z}(\gamma z + \rho \bar{z}). \tag{17}$$

As a surface over the z-plane, this defines an elliptic paraboloid when $\Delta < 0$. (The surface is a parabolic cylinder when $\Delta = 0$, an elliptic hyperboloid when $\Delta > 0$.) Now along a trajectory, V can be considered as a function of t; indeed $V = \operatorname{Im} \bar{z}\dot{z}$, and so

$$\begin{aligned}\dot{V} &= \operatorname{Im} \dot{\bar{z}}\dot{z} = \operatorname{Im} \dot{\bar{z}}(\gamma \dot{z} + \rho \dot{\bar{z}}) \\ &= \operatorname{Im} \dot{z}(\gamma \bar{z} - \bar{\rho} z) = \operatorname{Im} \dot{z}(\gamma \bar{z} - \dot{\bar{z}} + \bar{\gamma} \bar{z}) \\ &= \operatorname{Im} \dot{z}\bar{z}(\gamma + \bar{\gamma}) = (2\operatorname{Re}\gamma) V.\end{aligned}$$

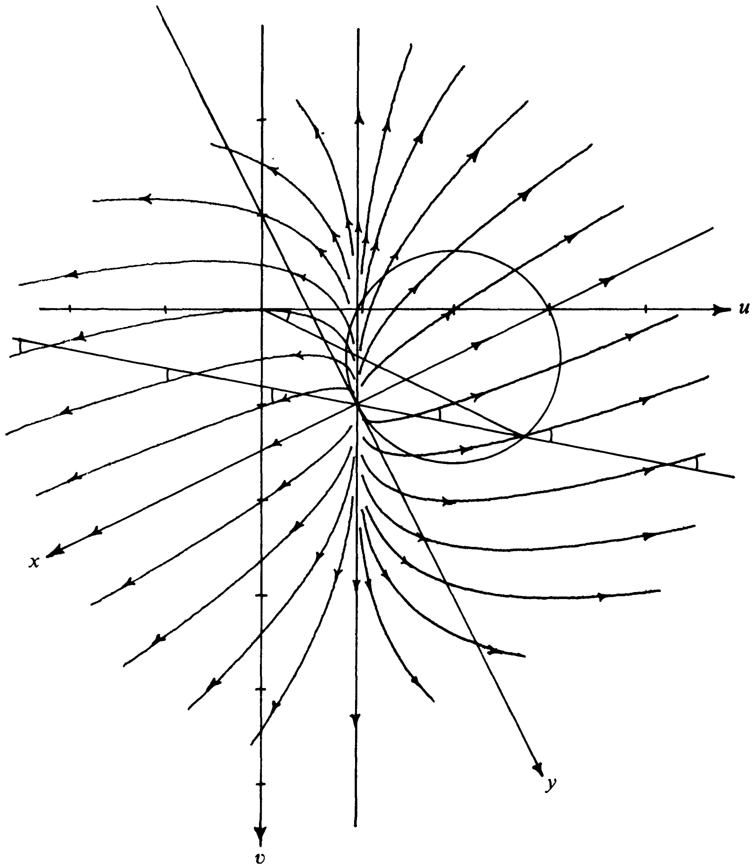


FIG. 4

Thus for arbitrary γ we have $V=V_0e^{(2\operatorname{Re}\gamma)t}$, and for $\operatorname{Re}\gamma=0$ the trajectories are level curves of the V surface. When $\Delta<0$ these are ellipses, and $z=0$ is a center. (For $\Delta=0$ the level curves are parallel lines; for $\Delta>0$ they are conjugate hyperbolas.)

If Γ is a proper circle ($\rho\neq 0$) tangent to the u-axis at a non-zero point ($\Delta=0, q>0$), there is a single invariant line, corresponding to the eigenvalue of multiplicity two, to which all trajectories are tangent. The origin $z=0$ is a one-tangent node.

If Γ contains $w=0$ in its interior ($q<0$), there are two invariant lines corresponding to real eigenvalues of opposite sign. In this case $z=0$ is a saddlepoint.

Finally, if Γ passes through the origin ($q=0$), the phase portrait consists of parallel lines and a line of critical points.

5. A Family of Isogonal Conics. In conjugate coordinates the equations for lines and circles have the same form

$$\bar{z}=S(z). \tag{18}$$

More generally to any analytic arc there exists a unique function S, called the Schwarz function, so that (18) is the equation of the arc in conjugate coordinates. S is analytic (possibly multivalued) in some striplike region about the arc. Schwarz functions are discussed in Davis **[2]**, where many of their remarkable properties are developed. For our purposes, the most important property of a Schwarz function is that $S'(z)$ is the clinant of the tangent line to the arc at z. This

follows from (9), since

$$S'(z)=\frac{d\bar{z}}{dz}=\frac{dx-idy}{dx+idy}=\frac{1-iy'}{1+iy'}=\frac{1-im}{1+im}=A.$$

To obtain the Schwarz function for the ellipse

$$\left(\frac{x}{\alpha}\right)^2+\left(\frac{y}{\beta}\right)^2=1, \qquad \alpha>\beta>0, \tag{19}$$

we use the transformation $x=\frac{1}{2}(z+\bar{z})$, $y=-\frac{1}{2}i\,(z-\bar{z})$ to recast (19) into the conjugate variable form

$$(\alpha^2-\beta^2)(z^2+\bar{z}^2)-2(\alpha^2+\beta^2)z\bar{z}+4\alpha^2\beta^2=0. \tag{20}$$

Solving this quadratic for $\bar{z}$ shows

$$E(z)=\frac{(\alpha^2+\beta^2)z+2\alpha\beta\sqrt{z^2-\alpha^2+\beta^2}}{\alpha^2-\beta^2} \tag{21}$$

is the Schwarz function for the ellipse (19). To find E' it seems simplest to differentiate (20) implicitly with respect to z, and then solve for $d\bar{z}/dz=E'(z)$. We find that

$$E'(z)=-\frac{r\bar{z}-z}{rz-\bar{z}}, \tag{22}$$

where

$$r=\frac{\alpha^2+\beta^2}{\alpha^2-\beta^2}. \tag{23}$$

We observe that $1<r<\infty$.

Replacing z by $\bar{T}z$, where $T=e^{i\tau}$, we obtain $E_T(z)=\bar{T}E(\bar{T}z)$ as the Schwarz function for the ellipse whose major axis is inclined to the real axis at an angle τ. Moreover

$$E_T'(z)=-\frac{r\bar{z}-\bar{T}^2z}{rz-T^2\bar{z}}. \tag{24}$$

The Schwarz function H for the conjugate hyperbolas

$$\left(\frac{x}{\alpha}\right)^2-\left(\frac{y}{\beta}\right)^2=\pm 1, \qquad \alpha\geqslant\beta>0, \tag{25}$$

can be derived analogously. For the rotated hyperbolas with Schwarz function $H_T(z)=\bar{T}H(\bar{T}z)$ we find

$$H_T'(z)=-\frac{s\bar{z}-\bar{T}^2z}{sz-T^2\bar{z}}, \tag{26}$$

where

$$s=\frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}. \tag{27}$$

Here we note that $0\leqslant s<1$.

The clinant of the tangent line to a trajectory of $\dot{z}=\gamma z+\rho\bar{z}$ is

$$S'(z)=\frac{d\bar{z}}{dz}=\frac{\dot{\bar{z}}}{\dot{z}}=\frac{\bar{\gamma}\bar{z}+\bar{\rho}z}{\gamma z+\rho\bar{z}}. \tag{28}$$

When $\rho=0$, this shows the trajectories intersect all radial lines at a constant angle; that is, concentric circles form an isogonal family of curves, as we knew already in Section 1. When $\rho\neq 0$, there still exists a family of isogonal conics, as we are now in a position to demonstrate.

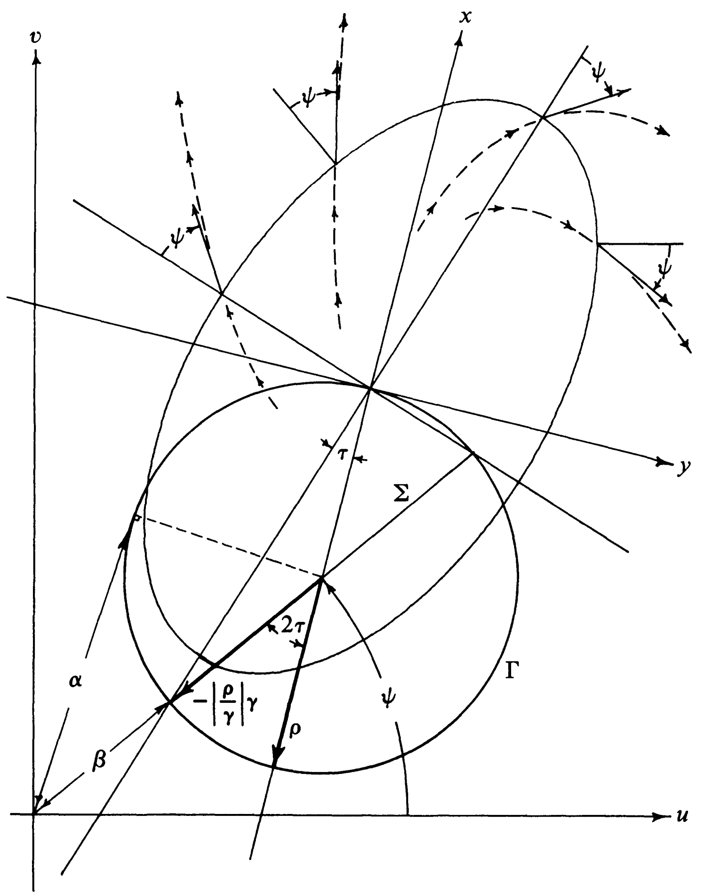


FIG. 5

Case I: $q>0$. In this case the origin $w=0$ lies outside the Γ circle in the parameter plane. Let us consider any ellipse homothetic to one whose semi-axes are

$$\alpha=\sqrt{|\gamma|^2-|\rho|^2}\,, \qquad \beta=|\gamma|-|\rho|. \tag{29}$$

With this choice

$$r=\frac{\alpha^2+\beta^2}{\alpha^2-\beta^2}=\frac{|\gamma|}{|\rho|}. \tag{30}$$

Next choose T to satisfy

$$T^2=-r\frac{\rho}{\gamma}. \tag{31}$$

(this determines $\tau=\arg T$ up to a multiple of π). Then

$$rz-T^2\bar{z}=r\left(z-\frac{T^2}{r}\bar{z}\right)=\frac{r}{\gamma}(\gamma z+\rho\bar{z})$$

and so from the equation (24) the Schwarz function E_T of the rotated ellipse satisfies

$$E_T'(z)=-\frac{\gamma}{\bar{\gamma}}\left(\frac{\bar{\gamma}\bar{z}+\bar{\rho}z}{\gamma z+\rho\bar{z}}\right).$$

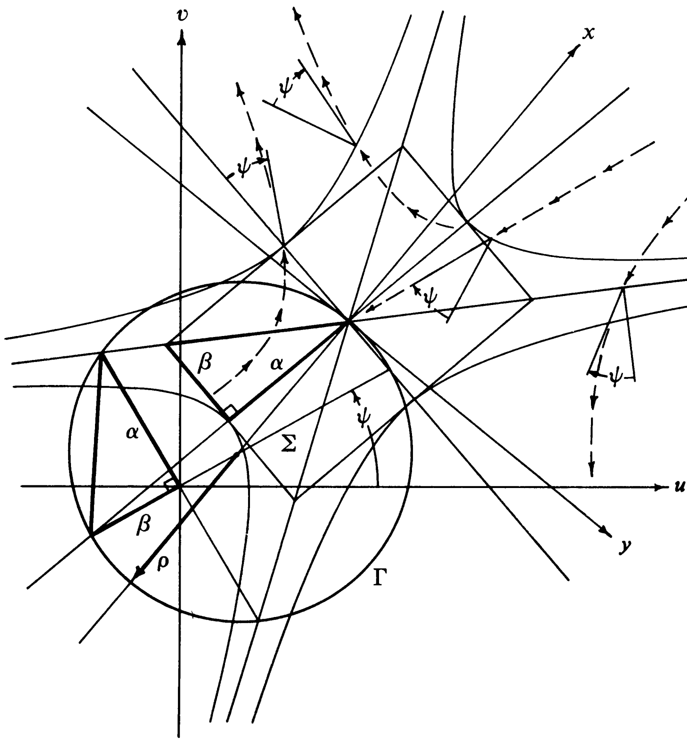


FIG. 6

From (28) it follows that

$$-E_T'(z)=\left(\frac{\gamma}{|\gamma|}\right)^2 S'(z) \tag{32}$$

where S is the Schwarz function for a trajectory of the differential equation. Thus *there is a family of central ellipses with common axes which intersect all trajectories of $\dot{z}=\gamma z+\rho\bar{z}$, $|\gamma|>|\rho|$, at a constant angle*. In particular, there exists a unique family of closed isogonal trajectories.

Equations (29)–(32) allow one to find a typical isogonal ellipse in the $\bar{z}$-plane. Let Σ denote the radial segment in the w-plane which extends from the origin $w=0$, passes through the center of the circle Γ, and terminates on Γ (see Fig. 5). From (29) we observe we may choose the semi-major axis α as the length of a segment tangent to Γ, in which case β is the distance along Σ to its first intersection point with Γ. From (30) and (31) we see

$$\rho=T^2\left(-\frac{\gamma}{|\gamma|}|\rho|\right),$$

from which it is a simple matter to verify that the ellipse's axes are determined by the intersection points of Σ with Γ: the point nearer $w=0$ determines the major axis, and the endpoint of Σ opposite γ determines the minor axis. Finally, we note that the left-hand side of (32) is the clinant of the line normal to the ellipse, and so any trajectory of the differential equation crosses a normal at the same angle $\psi=\arg\gamma$ (which is the angle Σ makes to the u-axis). Portions of typical trajectories are shown as dashed arcs in Figure 5.

Case II: $q<0$. The origin $w=0$ of the parameter plane is now interior to Γ, and $|\gamma|<|\rho|$. We now show the existence of a family of homothetic hyperbolas which are isogonal to the trajectories of the differential equation. The calculations are nearly identical to the case above. Choose

$$\alpha=\sqrt{|\rho|^2-|\gamma|^2}\,,\qquad \beta=|\rho|-|\gamma| \tag{33}$$

and

$$T^2=\begin{cases} -s\dfrac{\rho}{\gamma}, & \gamma\neq 0,\\ -i\rho/|\rho|, & \gamma=0. \end{cases} \tag{34}$$

Then

$$s=\frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}=\frac{|\gamma|}{|\rho|} \tag{35}$$

and, from (26) and (28),

$$H_T'(z)=\begin{cases} -\left(\dfrac{\gamma}{|\gamma|}\right)^2 S'(z), & \gamma\neq 0,\\ S'(z), & \gamma=0. \end{cases} \tag{36}$$

If $\operatorname{Re}\gamma=0$, we see from (32) and (36) that the trajectories of the differential equations are, respectively, homothetic ellipses and hyperbolas. They are of course the level curves of the function V defined by (17). Figure 6 shows the geometric determination of an isogonal hyperbola in the case $\operatorname{Re}\gamma\neq 0$. The congruent right triangles with legs α and β show that the common asymptotes of the hyperbolas are determined by the points on Γ intercepted by the chord through $w=0$ which is perpendicular to the diameter Σ. It should also be noticed that the asymptotes are isoclines which are intersected at the same angle as the hyperbolas themselves. Again portions of the trajectories are indicated in Figure 6 by dashed arcs.

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INTERNATIONAL OLYMPIAD XXI

SAMUEL L. GREITZER

The Twenty-first International Olympiad was held in England. The delegates met at Churchill College of the University of Bristol. On Friday, June 29, the delegates had just about completed the selection of the problems to be used in the Olympiad. Of course, there was still the question of precise wording, and, for some of the problems, that took a long time. The completed Olympiad will be found at the end of this article. A look at No. 6 will show how this problem was added to, padded, and extended until everyone accepted it. We had similar difficulty with No. 3, of which more later.

This year, there were actually 22 nations represented. Don Rollins attended as our "observer" (observers observe the activities, and report back, so that their countries might take further action); Jim Williams was an observer from Australia. Professors Gillis and Marcus represented Israel, who participated for the first time; and Professor Barone Netto represented Brazil, also a newcomer. A list of nations with their scores appears later on. On Saturday, June 30, the

problems were finally complete. A surprise to me was the appearance of Professor Ioannes Merminghis, of Greece. Final preparations for the Olympiad were completed on Sunday, and we were ready.

On Monday, the jury traveled to Westfield College, and after a simple opening ceremony the students went at the problems for the first day. Meanwhile, the delegates traveled back to Bristol. On Tuesday, we again traveled to Westfield College, where the teams were given the second day's problems to work on. Murray Klamkin and I started grading the first day's results for the USA. They were not good, as I, at least, had expected.

Each country grades its own papers, but teams of "coordinators" look over the papers, sometimes lowering and sometimes raising grades a point or so. The coordinating began on Wednesday, July 4. We had trouble with Problem 3 from the start. This problem has so many solutions, and they are so varied, that the coordinators had decided that the point to be located should "have some relation to the figure." We didn't understand what this means (we still don't), and we could get nowhere. We tried—as late as 10:30 PM, Thursday, July 5—but to no avail. When we left, there were two other delegations also concerned about Problem 3. Our final score was 199—the lowest score we ever had—and we were in fifth place. A list of the nations with their scores is presented.

Country	Problem No.						Total	No. in team (8 unless otherwise specified)
	1	2	3	4	5	6		
Austria	12	12	28	38	30	32	152	
Belgium	0	8	15	20	10	13	66	
Bulgaria	14	28	22	35	17	34	150	
Brazil	5	2	0	2	5	5	19	5
Cuba	5	6	6	10	5	3	35	4
Czechoslovakia	14	27	40	38	37	22	178	
BDG(W. Germany)	36	46	50	37	17	49	235	
Finland	0	14	25	29	11	10	89	
France	14	18	34	39	26	35	155	
GDR(E. Germany)	19	27	34	39	26	35	180	
Gt. Britain	10	33	48	40	35	52	218	
Greece	2	11	12	18	8	6	57	
Hungary	6	24	36	45	29	36	176	
Israel	3	16	26	33	17	24	119	
Netherlands	5	10	38	37	16	24	130	
Poland	1	28	28	48	19	36	160	
Rumania	31	34	44	42	40	49	240	
Sweden	6	26	23	41	23	24	143	
USA	9	40	32	28	38	52	199	
USSR	36	45	49	48	41	48	267	
Vietnam	18	22	27	23	24	20	134	4
Yugoslavia	26	35	15	35	30	31	172	

The reader is invited to examine this list, then look over the problems at the end of this article, and draw conclusions as to which are the more difficult and which we should have in our classes for gifted students of mathematics. Can it be that all the nations are out of step except the USA?

As for the performance of our team, here are the results:

		Problem No.						
		1	2	3	4	5	6	Total
Richard Agin	Chicago, IL	0	7	0	0	7	3	17
Randy Ekl	N. Huntingdon, PA	0	1	0	0	1	7	9
Michael Finn	Annandale, VA	0	4	3	6	7	7	27
Ronald Kaminsky	Albany, NY	1	0	7	0	2	7	17
Michael Larsen	Lexington, MA	6	7	7	5	7	7	39
Laurence Penn	Kings, Pt., NY	2	7	1	6	1	7	24
Mark Pleszkoch	Manassas, VA	0	7	7	5	6	7	32
Bruce Smith	San Rafael, CA	0	7	7	6	7	7	34

Again, should not a gifted student know something of number theory?

On Friday, July 6, we had our meeting, at which scores were made final and prizes awarded. At the reception that night, Michael Larsen received a First Prize; Bruce Smith and Mark Pleszkoch, Second Prizes; and Michael Finn and Laurence Penn, Third Prizes.

Two items of interest remain to be reported. First, as of this time, the Olympiad for 1980 is scheduled for Ulan Bator, in Mongolia. There is absolutely no contact between the United States and Mongolia. Second, when I returned home, it was to discover that we had been given a grant to hold the International Olympiad in the United States in 1981.

The Olympiad problems are appended. For those who would like to have solutions, a pamphlet with these solutions is obtainable from:

Professor Walter E. Mientka
Department of Mathematics
917 Oldfather Hall
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Lincoln, Nebraska 68588

Pamphlets are 50¢ each. Receipts go to help the Olympiad Fund.

International Mathematical Olympiad XXI

1. Let p and q be natural numbers such that

$$\frac{p}{q}=1-\frac{1}{2}+\frac{1}{3}-\cdots-\frac{1}{1318}+\frac{1}{1319}.$$

Prove that p is divisible by 1979. (W. Germany)

2. A prism with pentagons $A_1A_2A_3A_4A_5$ and $B_1B_2B_3B_4B_5$ as top and bottom faces is given. Each side of the two pentagons and each of the line-segments A_iB_j, for all $i,j=1,\ldots,5$, is colored either red or green. Every triangle whose vertices are vertices of the prism and whose sides have all been colored has two sides of a different color. Show that all 10 sides of the top and bottom faces are the same color. (Bulgaria)

3. Two circles in a plane intersect. Let A be one of the points of intersection. Starting simultaneously from A two points move with constant speeds, each point traveling along its own circle in the same sense. The two points return simultaneously after one revolution. Prove that there is a fixed point P in the plane such that, at any time, the distances from P to the moving points are equal. (USSR)

4 Given a plane π, a point in this plane and a point Q not in π, find all points R in π such that the ratio $(QP+PR)/QR$ is a maximum. (USA)

5. Find all real numbers a for which there exist non-negative real numbers x_1, x_2, x_3, x_4, x_5 satisfying the relations

$$\sum_{k=1}^{5} kx_k = a, \qquad \sum_{k=1}^{5} k^3 x_k = a^2, \qquad \sum_{k=1}^{5} k^5 x_k = a^3. \qquad \text{(Israel)}$$

6. Let A and E be opposite vertices of a regular octagon. A frog starts jumping at vertex A. From any vertex of the octagon except E, it may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there. Let a_n be the number of distinct paths of exactly n jumps ending at E.

Prove that

$$a_{2n-1} = 0, \qquad a_{2n} = \frac{1}{\sqrt{2}}(x^{n-1} - y^{n-1}), \qquad n = 1,2,3,\ldots,$$

where $x = 2+\sqrt{2}$ and $y = 2-\sqrt{2}$. (W. Germany)

*Note: A path of n jumps is a sequence of vertices $(P_0, \ldots, P_n)$ such that
(i) $P_0 = A$, $P_n = E$;
(ii) for every i, $0 \leqslant i \leqslant n-1$, P_i is distinct from E;
(iii) for every i, $0 \leqslant i \leqslant n-1$, P_i and P_{i+1} are adjacent.

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PROGRESS REPORTS

Edited by P. R. Halmos

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It is easy to be too busy to pay attention to what anyone else is doing, but not good. All of us should know, and want to know, what has been discovered since our formal education ended, but new words, and relations between them, are growing too fast to keep up. It is possible for a person to learn of the title of a recent work and of the key words used in it and still not have the faintest idea of what the subject is.

Progress Reports is to be an almost periodic column intended to increase everyone's mathematical information about what others have been up to. Each column will report one step forward in the mathematics of our time. The purpose is to inform, more than to instruct: what is the name of the subject, what are some of the words it uses, what is a typical question, what is the answer, who found it. The emphasis will be on concrete questions and answers (theorems), and not on general contexts and techniques (theories). References will be kept minimal: usually they will include only one of the earliest papers in which the answer appears and a more recent exposition of the discovery, whenever one is easily available.

Everyone is invited to nominate subjects to be reported on and authors to prepare the reports. The ground rules are that the principal theorem should be old enough to have been published in the usual sense of that word (and not just circulated by word of mouth or in preprints); it should be of interest to more than just a few specialists; and it should be new enough to have an effect on the mathematical life of the present and near future. In practice most reports will probably be on progress achieved somewhere between 5 and 15 years ago.

THE p-ADIC APPROACH TO SOLUTIONS OF EQUATIONS OVER FINITE FIELDS

Neal Koblitz

Suppose we have a set of r polynomials $\{f_i\}$ in n variables with integer coefficients and we would like to know how many solutions $(x_1, \ldots, x_n)$ with integer coordinates x_j exist (if any) to

the simultaneous system $f_i(x_1,\ldots,x_n)=0$, $i=1,\ldots,r$. This is usually a very difficult problem, even when $r=1$, $n=2$, and $f_1(x_1,x_2)$ is only, say, a cubic polynomial. However, a much more approachable problem is to study the number of solutions to the simultaneous *congruences* $f_i(x_1,\ldots,x_n)\equiv 0 \pmod p$, where p is any prime number which we fix once and for all. Put another way, we consider the coefficients of the f_i as belonging to the finite field $\mathbb{F}_p$ of integers modulo p, and we count the solutions $(x_1,\ldots,x_n)$ with x_j in $\mathbb{F}_p$.

More generally, let $\mathbb{F}_q$, where q is some prime power p^f, be any finite field, and let $\{f_i(x_1,\ldots,x_n)\}_{i=1,\ldots,r}$ be polynomials with coefficients in $\mathbb{F}_q$. For each $s=1,2,\ldots$, we are interested in the number N_s of solutions to

$$f_i(x_1,\ldots,x_n)=0, \qquad i=1,\ldots,r, \tag{1}$$

where the coordinates are allowed to take values in the finite field having q^s elements. We then study the sequence of numbers $\{N_s\}$—the numbers of solutions in the sequence of extension fields of $\mathbb{F}_q$.

The simplest example occurs when $r=0$, i.e., there is no equation which must be satisfied, in which case all $(q^s)^n$ n-tuples of $x_j\in\mathbb{F}_{q^s}$ are "solutions": $N_s=q^{ns}$. Here the solution set is all of n-dimensional affine space $(\mathbb{F}_{q^s})^n$. It is often useful to work in projective space (which is the set of lines through the origin in affine $(n+1)$-dimensional space, i.e., the set of non-zero homogeneous $(n+1)$-tuples $(x_0,x_1,\ldots,x_n)$, where two $(n+1)$-tuples are identified if they differ by a scalar multiple). In this case if we take $r=0$ we find that the number of such homogeneous $(n+1)$-tuples with $x_j\in\mathbb{F}_{q^s}$ is given by:

$$N_s=\frac{q^{s(n+1)}-1}{q^s-1}=q^{sn}+q^{s(n-1)}+\cdots+q^s+1. \tag{2}$$

The last example generalizes in a remarkable way to the general case of r equations in n unknowns (in either affine or projective space). Namely, there always exist numbers (in fact, algebraic integers) $\alpha_1,\alpha_2,\ldots,\alpha_h$ and $\beta_1,\beta_2,\ldots,\beta_k$ depending only on our original $\{f_i\}$ such that

$$N_s=\alpha_1^s+\alpha_2^s+\cdots+\alpha_h^s-\beta_1^s-\beta_2^s-\cdots-\beta_k^s. \tag{3}$$

This striking fact forms part of the "Weil conjectures" (1949) concerning the "zeta-function" of $\{f_i\}$. The zeta-function is defined by

$$Z(T)=\exp\left(\sum_{s=1}^{\infty}N_sT^s/s\right),$$

i.e., it is a formal power series in a variable T (having integer coefficients) which depends on the sequence $\{N_s\}$. For example, in the case of projective n-space, where N_s is given by (2), we readily compute

$$Z(T)=\frac{1}{(1-T)(1-qT)(1-q^2T)\cdots(1-q^nT)}.$$

It is a straightforward exercise to show that the existence of α's and β's for which (3) holds is equivalent to $Z(T)$ being a rational function. In fact, the α's will be the reciprocals of the roots of the polynomial in the denominator, and the β's will be the reciprocal roots of the numerator.

The rest of the Weil conjectures discuss the number and location of the α's and β's in the complex plane, which turn out to depend on geometrical properties of the surface ("algebraic variety") in n-space determined by the equations (1). For example, if we have a curve of genus g (i.e., over the complex numbers this would mean that the corresponding Riemann surface has g "handles"), then $h=2$, $\alpha_1=1$, $\alpha_2=q$; $k=2g$; $\beta_1,\ldots,\beta_{2g}$ occur in pairs whose product is q; and all β's have complex absolute value equal to $\sqrt{q}$. The last of these assertions is called the "Riemann hypothesis" (by analogy with the famous conjecture on the location of zeros of the Riemann zeta-function), and its proof by P. Deligne in 1973 was the last and most difficult step in the proof of the Weil conjectures.

Returning to the rationality of $Z(T)$, this part of the conjectures was first proved in 1960 by B. Dwork using "p-adic analysis," which is the study of functions of "p-adic" numbers (rather than real or complex numbers). For our fixed prime p, the set $\mathbb{Q}_p$ of p-adic numbers consists of all formal expressions of the form

$$a_{-m}p^{-m}+a_{-m+1}p^{-m+1}+\cdots+a_0+a_1p+a_2p^2+a_3p^3+\cdots \quad (\text{with } a_i\in\{0,1,\ldots,p-1\}),$$

which are added and multiplied from left to right much like decimals (but note that infinitely many positive powers of p are allowed and only finitely many negative powers—just the opposite of decimals). Another way to think of p-adic numbers is the following. Consider two rational numbers $a,\ b\in\mathbb{Q}$ to be close if $a-b$ is divisible (i.e., its numerator is divisible) by a large power of p. This radically new notion of distance gives us a topology on $\mathbb{Q}$. Completing $\mathbb{Q}$ ("filling in the holes") in this topology gives $\mathbb{Q}_p$—just as completing $\mathbb{Q}$ in the usual topology, where closeness means $|a-b|$ is small, gives the real numbers $\mathbb{R}$.

We can define functions on $\mathbb{Q}_p$ much as on $\mathbb{R}$, for example by power series. But their behavior may be very different. For example, $e^x=\Sigma x^n/n!$ converges everywhere as a function of $x\in\mathbb{R}$ but only on a fairly small disc when considered as a function of $x\in\mathbb{Q}_p$. A modification of e^x which is useful in the p-adic case because of its better convergence is the "Artin-Hasse exponential," obtained by formally expanding

$$\exp(x+x^p/p+x^{p^2}/p^2+x^{p^3}/p^3+\cdots)=\sum\frac{1}{n!}\left(x+\frac{x^p}{p}+\frac{x^{p^2}}{p^2}+\frac{x^{p^3}}{p^3}+\cdots\right)^n.$$

The Artin-Hasse exponential is used to construct a p-adic power series $\Theta(x)$ with the following property: if $x_{a_0}=a_0+a_1p+\cdots$ happens to be a $(p-1)$th root of 1 in $\mathbb{Q}_p$ (it turns out that for each $a_0=1,2,\ldots,p-1$ there is one and only one $(p-1)$th root of 1 of the form $x_{a_0}=a_0+a_1p+a_2p^2+\cdots\in\mathbb{Q}_p$), then the map

$$a_0\mapsto\Theta(x_{a_0})\quad\text{for}\quad a_0=1,\ldots,p-1,\quad\text{and}\quad 0\mapsto\Theta(0)=1$$

is a homomorphism (called a "character") from the additive group $\mathbb{F}_p=\{0,1,\ldots,p-1\}$ to the multiplicative group of pth roots of 1. Θ is called a "p-adic lifting" of the character of $\mathbb{F}_p$. Such characters and their p-adic liftings are used to count the number N_s of solutions to $\{f_i\}$.

The advantage of working in a p-adic field rather than a finite field is that continuous techniques from other branches of mathematics, such as analysis and topology, can often be used. Dwork constructed certain infinite-dimensional p-adic vector spaces and continuous linear maps on them and showed that N_s can be expressed in terms of the traces of these maps. The idea for such an expression comes from the Lefschetz fixed-point theorem in topology, which says, roughly speaking, that the number of fixed points of a map F from a space X to itself is equal to the alternating sum of the traces of the map induced by F on certain vector spaces—called "cohomology" spaces—associated to X.

By expressing the zeta-function $Z(T)$ in terms of maps of p-adic vector spaces, Dwork showed explicitly that as a p-adic function $Z(T)$ is meromorphic, i.e., a ratio of two everywhere convergent p-adic power series. It remained to use a lemma which says that, in view of simple estimates on the size of the coefficients of $Z(T)$, this can only happen if $Z(T)$ is a rational function.

The rationality of $Z(T)$ was later proved (by A. Grothendieck) using techniques of algebraic geometry without p-adic analysis, and Deligne's proof of the rest of the Weil conjectures also uses algebra-geometric rather than p-adic techniques. However, Dwork's explicit p-adic methods have deepened our understanding of the zeta-function in many directions and illustrate how p-adic analysis can shed light on number theoretic questions which at first glance do not seem to be related to p-adic numbers. A recent example of such an unexpected dividend is E. Bombieri's use of work of Dwork on p-adic solutions to differential equations to obtain irrationality results such as the following: if $m>10^{317}$, then the number

$$\operatorname{Dilog}\left(\frac{1}{m}\right) \underset{\text{def}}{=} \sum_{n=1}^{\infty} n^{-2} m^{-n}$$

is irrational.

Reference

1. B. Dwork, On the rationality of the zeta-function of an algebraic variety, Amer. J. Math., 82 (1960) 631–648.

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MATHEMATICAL NOTES

EDITED BY DEBORAH TEPPER HAIMO AND FRANKLIN TEPPER HAIMO

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Advice to prospective authors: The editors have recently been receiving about **ten times** as many Mathematical Notes as can be used. It will simplify our work if authors will submit only especially interesting manuscripts. Mathematical Notes should be short papers of one to four printed pages which give new insights, new and improved proofs of old theorems, brief bits of mathematical folklore that have not found a home in the literature, or (occasionally!) new results that are not too technical. The topics should be of wide current interest. Papers that have already been rejected by a research journal are only very rarely suitable as Mathematical Notes.
R.P.B.

UNIFORM CONTINUITY OF DERIVATIVES IN CONVEX SETS

J. W. NIENHUYS

Introduction. For f' to be uniformly continuous in an open subset A of a Banach space, a necessary condition is that for any $\epsilon > 0$ there exists a $\delta > 0$ such that the relations $\|h\| \leqslant \delta$, $s \in A$, $s + \xi h \in A$ for $0 \leqslant \xi \leqslant 1$ imply $\|f(s+h) - f(s) - f'(s)\cdot h\| \leqslant \epsilon\|h\|$ (cf. [**1**, §8.6, Problem 3]). If A is uniformly non-oblate convex, then this condition is sufficient. An example is given that shows (in $\mathbb{R}^3$) that mere convexity does not suffice. The intuitive meaning of non-oblate is "does not have arbitrarily sharp edges." Any bounded open convex set is uniformly non-oblate.

1. Notations and conventions. E and F are Banach spaces. A is an open subset of E and f: $A \to F$ is a continuously differentiable mapping. We shall say that f has property (U) if for any $\epsilon > 0$ there exists a $\delta > 0$ such that the relations $\|h\| \leqslant \delta$, $s \in A$, $s + \xi h \in A$ for $\xi \in [0, 1]$ imply

$$\|f(s+h) - f(s) - f'(s)\cdot h\| \leqslant \epsilon\|h\|. \tag{1}$$

The expression on the left-hand side of this inequality will be called a two-point difference from s to $s+h$.

$B_{x,r}$ denotes the open ball with center x and radius r.

2. Preliminaries. If f' is uniformly continuous in A, then f has property (U). This can be seen from

$$\|f(s+h) - f(s) - f'(s)\cdot h\| \leqslant M\|h\|,$$

where $M = \sup\{\|f'(s+\xi h) - f'(s)\| \mid \xi \in [0, 1]\}$.

However, the inverse implication is not true generally. Property (U) implies that f' is uniformly continuous if the shape of A satisfies certain conditions. Not only has A to be convex,

but also A cannot have arbitrarily sharp edges, in a sense that we will explain now.

3. Main result.

DEFINITION. An open convex set A in a Banach space E is said to be uniformly non-oblate, with positive constants k and δ, if for all $x \in A$ there exists an $x' \in A$ such that $B_{x',k\delta} \subset A \cap B_{x,\delta}$.

Now, if a subset S of a convex set A is shrunk toward a point of A, then the resulting set is still contained in A. By shrinking toward a point we mean applying a map $z \to x + t(z-x)$, for some t, $0<t<1$. Hence it is easy to see that the following holds:

LEMMA. *Each bounded open convex set is uniformly non-oblate. If A is open, convex and uniformly non-oblate with constants k and δ, then for all d such that $0<d \leqslant \delta$, A is uniformly non-oblate with constants k and d.*

The terminology "non-oblate" is taken from **[2]**. However, there it is applied to cones only, and open cones are trivially non-oblate, in the same manner as bounded convex open sets are trivially non-oblate. In our discussion the emphasis is on *uniformly*.

THEOREM. *Let A be an open convex uniformly non-oblate subset of a Banach space E and f: $A \to F$ a differentiable mapping with property* (U). *Then f' is uniformly continuous.*

Proof. Let A be uniformly non-oblate with constants k and δ'. Let $\epsilon>0$ be given. Choose δ, $0<\delta \leqslant \delta'$, such that for all $s \in A$ and $h \in A$ with $s+h \in A$

$$\|f(s+h)-f(s)-f'(s)\cdot h\| \leqslant \frac{\epsilon k}{6}\|h\|, \quad \text{if } h \leqslant 2\delta.$$

Suppose s and t are in A and $\|s-t\|=d<\delta$. Choose $s' \in A$ such that $B_{s',kd} \subset B_{s,d} \cap A$. This is possible because of the Lemma.

Suppose $y \in E$ and $\|y\|=kd$. Then both $s_1=s'+\frac{1}{2}y$ and $s_2=s'-\frac{1}{2}y$ are in $B_{s,d} \cap A$. We will now express $(f'(s)-f'(t))\cdot y$ in terms of the four two-point differences from s and t to s_1 and s_2.

$$\begin{aligned}(f'(t)-f'(s))\cdot y = &(f(s_1)-f(s)-f'(s)\cdot(s_1-s))\\ &-(f(s_2)-f(s)-f'(s)\cdot(s_2-s))\\ &-(f(s_1)-f(t)-f'(t)\cdot(s_1-t))\\ &+(f(s_2)-f(t)-f'(t)\cdot(s_2-t)).\end{aligned}$$

Because $\|s-s_i\|<d<2\delta$ and $\|t-s_i\| \leqslant \|s-t\|+\|s-s_i\| \leqslant 2d<2\delta$ for $i=1,2$, we obtain

$$\|(f'(t)-f'(s))\cdot y\| \leqslant \frac{\epsilon k}{6}(\|s-s_1\|+\|s-s_2\|+\|t-s_1\|+\|t-s_2\|) \leqslant \epsilon k d = \epsilon\|y\|.$$

Hence $\|f'(t)-f'(s)\|<\epsilon$; this proves that f' is uniformly continuous.

4. A Counterexample. We have seen already that every bounded convex subset of a Banach space is uniformly non-oblate. Every open convex set in $\mathbb{R}^2$ (and $\mathbb{R}^1$) is uniformly non-oblate too. Observe that the tangent to the set cannot make more than four sharp ($>\pi/2$) turns.

However, the set

$$P=\left\{(x,y,z) \in \mathbb{R}^3 | x>0 \text{ and } y> \max(4,x^2) \text{ and } 0<z<1-\frac{x^2}{y}\right\}$$

is not uniformly non-oblate.

This can be seen as follows. The bounding surfaces meet at points of the parabolic curve

$$Q=\{(x,x^2,0) \in \mathbb{R}^3 | x \geqslant 2\}$$

at an angle of $\arctan((4+x^{-2})^{\frac{1}{2}}/x)$, which is less than $3/x$, for $x \geqslant 2$.

Note that $\bar{P}$ is the closed convex hull of

$$\{(0,4,0)\} \cup Q \cup \{(x,4,1-\tfrac{1}{4}x^2) \in \mathbb{R}^3 | 0 \leqslant x \leqslant 2\}.$$

We shall now define a function f on P with a continuous but not uniformly continuous derivative. Moreover, f will have property (U). Let g be a two times continuously differentiable function on $[0,\infty[$, positive on $[0,1[$, zero on $[1,\infty[$ and such that $g(0)=1$. An example of such a function is $g(t)=\max(0,(1-t)^3)$ for $t\geqslant 0$. We define f on P by $f(x,y,z)=zg((y^{\frac{1}{2}}-x)y^{\frac{1}{4}})$. Clearly f is twice continuously differentiable on P. We denote the derivative of f with respect to z by f_z. Now $\lim f_z(x,y,z)=1$ if (x,y,z) approaches Q, but $f_z(a^{\frac{1}{2}}-a^{-\frac{1}{4}},a,a^{-\frac{3}{4}})=0$ if $a\geqslant 4$. So f_z is not uniformly continuous. Indeed, f_z increases very steeply when the point moves to Q in the x-direction. This, combined with the fact that the z-components of the difference vectors h in the relevant two-point differences from s to $s+h$ are very small, makes the example work, as we shall see now. We estimate a two-point difference by applying Taylor's formula directly (cf. [**1**, 8.14.3]). If s and $s+h$ are both in P then

$$f(s+h)-f(s)-f'(s)\cdot h=\left(\int_0^1(1-\zeta)f''(s+\zeta h)d\zeta\right)\cdot(h,h). \tag{2}$$

Now observe that for s in the support of f (the closure in P of $\{s\in P\,|\,f(s)>0\}$), $y=x^2+O(x^0)$, we have $z=O(x^{-3/2})$ if we write $s=(x,y,z)$. The notation $O(x^\alpha)$ used here is shorthand for a number m for which there exists an estimate $|m|<Cx^\alpha$ for all $x>4$ and a constant C independent of x. Computation of the second partial derivatives of f shows that the absolute value of the right-hand side of (2) is smaller than

$$O(x^{-\frac{1}{2}})h_1^2+O(x^{-2})h_2^2+O(x^{-3/2})|h_1h_2|+O(x^{-\frac{1}{2}})|h_2h_3|+O(x^{\frac{1}{2}})|h_1h_3|,$$

where $h=(h_1,h_2,h_3)$.

If we suppose $\|h\|\leqslant 1$ (euclidean norm), then the right-hand side of (2) equals zero or $h_3=O(x^{-1})$. We shall prove this now.

If the right-hand side of (2) does not equal zero, then at least one of s, $s+h$ must be in the support of f; hence both must be within a distance 1 of the support of f.

Now suppose that the distance of $a=(a_1,a_2,a_3)$ to an element $b=(b_1,b_2,b_3)$ of the support of f is smaller than 1 and that $a\in P$.

Let now $c=(c_1,c_1^2,0)$ be the point in Q that is nearest to $a'=(a_1,a_2,0)$. Then the distance from a' to c is less than 2. Hence $0<a_3<2(3/c_1)$. It follows that the z-coordinates of s and $s+h$ are both positive and both smaller than $6/\max(2,x-2)$. Hence $h_3=O(x^{-1})$. Consequently, the right-hand side of (2) is smaller than $O(x^{-\frac{1}{2}})\|h\|$. So for all $\epsilon>0$ there exists an M such that (1) holds for $x>M$ and $\|h\|<1, s=(x,y,z)$ and both s and $s+h$ in P.

By uniform continuity of the derivative in $P\cap\{(x,y,z)\in\mathbb{R}^3|x<M+1\}$, there exists a δ, $0<\delta<1$ such that (1) holds in P if $\|h\|<\delta$ and both s and $s+h$ in P.

5. Remarks. If we combine the above with the Hahn-Banach theorem, we can obtain a counterexample in any real Banach space of dimension larger than three. The counterexample above shows also that 8.6, problem 3, in [**1**] contains an error.

Suppose that A is an arbitrary open convex subset of a Banach space and suppose A is not uniformly non-oblate. There remains the problem of constructing on A an f with property (U) but with f' not uniformly continuous.

This problem seems harder, because in a general Banach space there is a lack of suitable differentiable functions with bounded support.

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AN EXAMPLE OF A LOCALLY FINITE FAMILY

V. L. N. SARMA, NAND LAL, AND V. K. SINGH

1. A family $\mathfrak{B}$ of subsets of a topological space $(X, \mathfrak{T})$ is *locally finite at a point* $x \in X$ if x has a neighborhood meeting only finitely many members of $\mathfrak{B}$. It is known that a sufficient condition for the union and closure operations on $\mathfrak{B}$ to commute—in the sense that the equality

$$\cup\{\bar{B}: B \in \mathfrak{B}\} = \overline{\cup B} \tag{1}$$

holds—is that $\mathfrak{B}$ be locally finite at each $x \in X$ (see, for instance, John L. Kelley, *General Topology*, p. 126). The purpose of this article is to provide examples of (i) a locally finite non-σ-discrete family, and (ii) a non-locally finite family for every subfamily of which the equality (1) holds. The general problem of characterizing the families for which (1) holds is covered by the following theorem, which we do not find explicitly stated in textbooks on topology.

THEOREM 1. *A necessary and sufficient condition that equality* (1) *hold for a family*, $\mathfrak{B}$, *of subsets of a topological space*, $(X, \mathfrak{T})$, *is that* $\mathfrak{B}$ *be locally finite at each point of* $X \setminus \cup\{\bar{B}: B \in \mathfrak{B}\}$.

The easy proof is omitted.

2. Let $\{u_1, u_2, \ldots, u_n\}$ be a finite orthonormal set in an inner product space, H, and $v_n = (1/n)\sum_{j=1}^{n} u_j$. For any $x \in H$ we have

$$\|x - v_n\|^2 = \frac{1}{n}\sum_{j=1}^{n}\|x - u_j\|^2 - \frac{n-1}{n}. \tag{2}$$

Let H be infinite dimensional, U an infinite orthonormal set in H, r a non-negative real number, and $\mathfrak{B}(r)$ the family of all closed spheres $B(u,r)$ of radius r and centers $u \in U$. It is obvious that the family $\mathfrak{B}(r)$ is discrete for $0 \leqslant r < 1/\sqrt{2}$.

THEOREM 2. *For each r with $0 \leqslant r < 1$, there is a positive integer k such that every k (or fewer) members of $\mathfrak{B}(r)$ have a non-empty intersection and every $k+1$ (or more) distinct members of $\mathfrak{B}(r)$ have an empty intersection.*

Proof. Pick the unique positive integer k such that $(k-1)/k \leqslant r^2 < k/(k+1)$. Given any n elements, $u_1, u_2, \ldots, u_n$ of U, with v_n as above, we have, for $1 \leqslant j \leqslant n \leqslant k$,

$$\|u_j - v_n\|^2 = \frac{n-1}{n} \leqslant r^2,$$

so that $v_n \in \cap_{j=1}^{n} B(u_j, r)$.

On the other hand, if $n > k$ and x is in the intersection of n distinct members of $\mathfrak{B}(r)$, say $x \in \cap_{j=1}^{n} B(u_j, r)$, then the identity (2) implies that

$$0 \leqslant \|x - v_n\|^2 \leqslant r^2 - \frac{n-1}{n} \leqslant r^2 - \frac{k}{k+1} < 0,$$

which is impossible.

3. THEOREM 3. *The family $\mathfrak{B}(r)$ is locally finite for $0 \leqslant r < 1$.*

Proof. Fix k as in the proof of Theorem 2, let $0 < \delta < \sqrt{k/(k+1)} - r$, and let $x \in H$. Let $B(u_j, r)$ $(j = 1, 2, \ldots, k+1)$ be any $k+1$ distinct members of $\mathfrak{B}(r)$. Using the identity (2) again, we have

$$\frac{k}{k+1}\sum_{j=1}^{k+1}\|x - u_j\|^2 = \frac{k}{k+1} + \|x - v_{k+1}\|^2 \geqslant \frac{k}{k+1} > (r+\delta)^2.$$

Hence $\|x - u_j\| > r + \delta$ for at least one value of j, and the open sphere $S(x, \delta)$ can meet at most k members of $\mathfrak{B}(r)$.

COROLLARY. *If H admits an uncountable orthonormal system, and if $\frac{1}{2} \leqslant r^2 < 1$, then $\mathfrak{B}(r)$ is locally finite but not σ-discrete.*

The proof follows from the observation that any non-empty discrete subfamily of $\mathfrak{B}(r)$ is a singleton.

THEOREM 4. (i) *For $r \geqslant 1$, $\mathfrak{B}(r)$ is not locally finite*; (ii) $\cup \mathfrak{B}(1)$ *is closed.*

Proof. The origin of H is in each member of $\mathfrak{B}(r)$ when $r \geqslant 1$; and (i) is a trivial consequence. Let $x \in H \setminus \cup \mathfrak{B}(1)$ be arbitrary. Then $x \neq 0$. Take $\delta = \frac{1}{2}[(1+\|x\|^2)^{\frac{1}{2}} - 1]$. If we write

$$U_x = \{u \in U: \|x-u\| < 1+\delta\},$$

it is easily seen that

$$\begin{aligned} U_x &= \{u \in U: (x,u)+(u,x) > 2\delta + 3\delta^2\} \\ &\subset \{u \in U: |(x,u)|^2 > \delta^2\} = V_x, \text{ say.} \end{aligned}$$

Bessel's inequality shows that the number of elements in V_x cannot exceed $\|x\|^2/\delta^2$. It follows that $\mathfrak{B}(1)$ is locally finite at each $x \in H \setminus \cup \mathfrak{B}(1)$. An appeal to Theorem 1 establishes (ii).

It is left to the reader to show that (1) does not hold for $\mathfrak{B}(r)$ with $r > 1$.

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THE PROBABILITY THAT NEIGHBORS REMAIN NEIGHBORS AFTER RANDOM REARRANGEMENTS

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Suppose n people are arranged in a row and then rearranged at random in another row. What is the probability that no two people who were neighbors in the original row will be neighbors in the new row? In this note we will show that, for large n, the probability is approximately e^{-2}. In fact we show that, for large n, the probability that k of the original $n-1$ neighboring pairs are again neighbors is approximately $(2^k/k!)e^{-2}$.

To be more precise let $A(n,k)$ be the number of permutations π of $\{1,2,\dots,n\}$ such that for exactly k values of i, $i=1,\dots,n-1$, i and $i+1$ occur consecutively, in either order, in the list $\pi(1),\pi(2),\dots,\pi(n)$. We derive an expression for $A(n,k)$ and then use it to prove that

$$\lim_{n\to\infty} \frac{A(n,k)}{n!} = \frac{2^k}{k!}e^{-2}. \tag{1}$$

1. Derivation of the Formula for $A(n,k)$. For $i=1,2,\dots,n-1$ let S_i denote the set of permutations π such that i and $i+1$ occur consecutively in either order in the list $\pi(1),\pi(2),\dots,\pi(n)$. Then $A(n,k)$ is the set of permutations that belong to exactly k of the S_i.

For $i=1,2,\dots,n-1$ let $g_i = \sum \#(\cap_{j\in B} S_j)$ where the sum is taken over all i-element subsets B of $\{1,2,\dots,n-1\}$. Then, according to the principle of inclusion and exclusion, given, for example, in Liu [1],

$$A(n,k) = \sum_{i=k}^{n-1} (-1)^{i-k}\binom{i}{k} g_i. \tag{2}$$

[Note that in the calculation of $A(n,0)$ g_0 should be taken to be $n!$.] It will suffice to evaluate the g_i and, for this purpose, we now evaluate the numbers $\#(\cap_{j\in B} S_j)$. Assume that B is an i-element subset of $\{1,2,\dots,n-1\}$. By a *component* of B we mean a maximal set of consecutive integers contained in B. We show that $\#(\cap_{j\in B} S_j)$ depends only on i and the number of components of B.

Suppose B has c components and that $\{k,k+1,\ldots,k+r\}$ is one of them. If $\pi\in\bigcap_{j\in B}S_j$, then the numbers $k,k+1,\ldots,k+r,k+r+1$ must occur consecutively, in either increasing or decreasing order, in the list $\pi(1),\pi(2),\ldots,\pi(n)$. Thus, to form a permutation $\pi\in\bigcap_{j\in B}S_j$, we begin by forming the c *blocks* $\{k,k+1,\ldots,k+r+1\}$ corresponding to the components of B, and then decide whether each block is to appear in increasing or decreasing order, a total of 2^c choices. Since each block contains one more element than its corresponding component in B, the number of elements in all the blocks is $c+i$. This leaves $n-c-i$ elements of $\{1,2,\ldots,n\}$ that are not in any block. Finally, to assemble the permutation π, we choose an ordering of the $n-i$ objects consisting of the $n-c-i$ elements not in any block and the c blocks. This gives $(n-i)!$ choices. Thus we have found that, if B is an i-element subset of $\{1,2,\ldots,n-1\}$ with c components, then

$$\#\left(\bigcap_{j\in B}S_j\right)=2^c(n-i)!.$$

For any integers m,i,c, $m\geqslant i\geqslant c\geqslant 1$, let $F(m,i,c)$ denote the number of i-element subsets of $\{1,2,\ldots,m\}$ that have c components. Then we have found that

$$g_i=\sum_{c=1}^{i}F(n-1,i,c)2^c(n-i)!,\qquad i\geqslant 1. \tag{3}$$

[Recall that $g_0=n!$.]

Next we calculate $F(m,i,c)$. A c-component, i-element subset of $\{1,2,\ldots,m\}$ is determined by $2c+1$ numbers, $j_0,i_1,j_1,i_2,j_2,\ldots,i_c,j_c$, where the i's are the component sizes and the j's give the sizes of the spaces around the components. Since the components are not empty, we know $i_1,i_2,\ldots,i_c>0$; and since there are gaps between components, we must have $j_1,\ldots,j_{c-1}>0$. We also have $j_0\geqslant 0$ and $j_c\geqslant 0$. In addition we know that

$$i_1+i_2+\cdots+i_c=i, \tag{4}$$

$$j_0+j_1+\cdots+j_c=m-i. \tag{5}$$

The number of solutions to (4) in positive integers is well known to be

$$\binom{i-1}{c-1}=\binom{i-1}{i-c}.$$

The number of solutions to (5) with its restrictions is easily seen to be

$$\binom{m-i+1}{c}.$$

We conclude that

$$F(m,i,c)=\binom{i-1}{i-c}\binom{m-i+1}{c}. \tag{6}$$

Combining equations (2), (3), and (6), we find

$$A(n,k)=\sum_{i=k}^{n-1}\left[(-1)^{i-k}\binom{i}{k}\sum_{c=1}^{i}\binom{i-1}{i-c}\binom{n-i}{c}2^c(n-i)!\right]. \tag{7}$$

This is the desired formula. [In view of the comments above about g_0, we need to take the second sum to be $n!$ if $i=0$.] From (7) we can construct a short table of $A(n,k)$ as follows:

$A(n,k)$

n^k	0	1	2	3	4	5
1	1					
2	0	2				
3	0	4	2			
4	2	10	10	2		
5	14	40	48	16	2	
6	90	230	256	120	22	2

2. Limiting Behavior. To calculate $\lim_{n\to\infty}(A(n,k)/n!)$, let

$$a_{ik}(n)=(-1)^{i-k}\binom{i}{k}\sum_{c=1}^{i}\binom{i-1}{i-c}\binom{n-i}{c}2^c\frac{(n-i)!}{n!},\qquad k\leqslant i<n,\tag{8}$$

$$a_{ik}(n)=0,\qquad i\geqslant n.\tag{9}$$

From (7), $A(n,k)/n!=\sum_{i=k}^{\infty}a_{ik}(n)$. Granting for a moment the moving of the limit inside the summation sign, we have

$$\lim_{n\to\infty}\frac{A(n,k)}{n!}=\sum_{i=k}^{\infty}\lim_{n\to\infty}a_{ik}(n).\tag{10}$$

To evaluate $\lim_{n\to\infty}a_{ik}(n)$ we may assume $n>i$ and use expression (8) for $a_{ik}(n)$. Each term in the sum in (8) is a rational function of n. Those with $c<i$ have negative degree and approach zero. The term with $c=i$ approaches $2^i/i!$. Thus

$$\lim_{n\to\infty}a_{ik}(n)=(-1)^{i-k}\binom{i}{k}\frac{2^i}{i!}=\frac{2^k}{k!}(-1)^{i-k}\frac{2^{i-k}}{(i-k)!}.\tag{11}$$

We can obtain (1) now by substituting (11) in (10).

Finally note that

$$|a_{ik}(n)|\leqslant\binom{i}{k}2^i\frac{(n-i)!}{n!}\sum_{c=1}^{i}\binom{i-1}{i-c}\binom{n-i}{c}=\binom{i}{k}2^i\frac{(n-i)!}{n!}\binom{n-1}{i}$$
$$=\frac{2^i}{k!(i-k)!}\frac{n-i}{n}\leqslant\frac{2^k}{k!}\frac{2^{i-k}}{(i-k)!},$$

so that $\sum_{i=k}^{\infty}a_{ik}(n)$ is dominated by the convergent series $(2^k/k!)\sum_{i=k}^{\infty}(2^{i-k})/(i-k)!)$. This justifies moving the limit to obtain (10).

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CLASSROOM NOTES

EDITED BY DEBORAH TEPPER HAIMO AND FRANKLIN TEPPER HAIMO

Material for this department should be sent to Professor Deborah Tepper Haimo, Department of Mathematical Sciences, University of Missouri, St. Louis MO 63121.

Advice to prospective authors: The editors have recently been receiving about **ten times** as many Classroom Notes as can be used. It will simplify our work if authors will submit only especially interesting manuscripts.

R.P.B.

ESTIMATING THE ERROR IN THE TRAPEZOIDAL RULE

EDWARD ROZEMA

Beginning calculus students usually have no difficulty in either understanding the basic

geometric idea behind the Trapezoidal Rule or applying it to a specific integration problem. However, they usually are not given an error estimate or are given one in a form which is difficult to use. It is the purpose of this note to express an estimate for the error in terms of the values of the first derivative at the endpoints (see equation (3) below). The proof combines the standard error estimate with a simple observation about Riemann sums. This result is surely not new, but it does not seem to be very well known or to be exploited to its full potential.

THEOREM. *If f is in $C^2[a,b]$, if $a=x_0<x_1<\cdots<x_n=b$ is a regular partition of $[a,b]$, and if*

$$T_n=\frac{b-a}{2n}[f(x_0)+2f(x_1)+\cdots+2f(x_{n-1})+f(x_n)],$$

then

$$\int_a^b f(x)\,dx=T_n+E_n$$

where

$$E_n=-\frac{(b-a)^3}{12n^2}C_n. \tag{1}$$

Moreover,

$$\operatorname*{limit}_{n\to\infty} C_n=\frac{f'(b)-f'(a)}{b-a}. \tag{2}$$

Hence, from (1) *and* (2),

$$E_n\simeq-\frac{(b-a)^2}{12n^2}[f'(b)-f'(a)]. \tag{3}$$

Proof. In the standard derivation of the Trapezoidal Rule [**1**, p. 284 ff.], it is shown that for each $i=1,2,\ldots,n$, there exists a z_i in $[x_{i-1},x_i]$ such that

$$\int_a^b f(x)\,dx=T_n-\frac{(b-a)^3}{12n^2}\sum_{i=1}^n\frac{f''(z_i)}{n}. \tag{4}$$

Thus the error can be written as in equation (1) with

$$C_n=\sum_{i=1}^n\frac{f''(z_i)}{n}. \tag{5}$$

Notice that C_n is a Riemann sum for $f''(x)\,(b-a)$. Since $\{x_0,x_1,\ldots,x_n\}$ is a regular partition of $[a,b]$, it follows that

$$\begin{aligned}\operatorname*{limit}_{n\to\infty} C_n&=\frac{1}{b-a}\operatorname*{limit}_{n\to\infty}\sum_{i=1}^n f''(z_i)\frac{b-a}{n}\\&=\frac{1}{b-a}\int_a^b f''(x)\,dx\\&=\frac{f'(b)-f'(a)}{b-a},\end{aligned}$$

which establishes (2) and ends the proof.

From (1) and (2) it is easy to derive the following extremely useful relationship between E_n and E_{2n}:

COROLLARY. *With the assumptions of the previous theorem,*

$$\operatorname*{limit}_{n\to\infty}\frac{E_n}{E_{2n}}=4. \tag{6}$$

Hence $E_{2n} \simeq (1/4)E_n$.

The standard form for the error E_n is usually derived from (5) by pointing out that C_n falls between the minimum and maximum value of $f''(x)$ for all x in $[a,b]$; hence, by the Intermediate Value Theorem, $C_n = f''(\zeta_n)$ for some ζ_n in $[a,b]$. Thus

$$E_n = -\frac{(b-a)^3}{12n^2} f''(\zeta_n). \tag{7}$$

The usual bound on E_n follows immediately:

$$|E_n| \leqslant \frac{(b-a)^3}{12n^2} M_2 \tag{8}$$

where $M_2 = \max\{|f''(x)| \,|\, x \text{ is in } [a,b]\}$.

TABLE 1

Integral	Desired E_n	From (3)		From (8)	
		n	Actual $\|E_n\|$	n	Actual $\|E_n\|$
$\int_0^1 \exp(-x^2)\,dx$	$5(10^{-7})$	351	$(4.9)10^{-7}$	579	$(1.6)10^{-7}$
$\int_1^2 (1/x)\,dx$	10^{-3}	8	$(.98)10^{-3}$	13	$(.37)10^{-3}$

Let us look at two examples illustrating the use of (3) and the standard estimate (8). (The results are summarized in Table 1.) Suppose we wish to compute $I = \int_0^1 \exp(-x^2)\,dx$ using the Trapezoidal Rule. (Note: $I = .74682413$ to eight correct digits.) In order to use (3) we compute $f'(x) = -2x\exp(-x^2)$, $f'(1) = -2e^{-1} = -.73575888$, $f'(0) = 0$; therefore, from (3),

$$E_n \simeq \frac{1}{n^2}(.06131324).$$

If we wish $|E_n| \leqslant 5(10^{-7})$, then we must choose n to satisfy

$$\frac{1}{n^2}(.06131324) \leqslant 5(10^{-7})$$

or

$$n \geqslant 351.$$

Using $n = 351$ and double precision on an IBM 360/65 computer, we obtain $T_{351} = .74682363$. Therefore $E_{351} = I - T_{351} = 4.9(10^{-7})$. (The simple details have been included here precisely because they are simple enough to be done by freshmen.) To carry through the standard estimate (8), one needs the third derivative plus some analysis, by no means a trivial task. This same problem was worked using (8) in [**1**, p. 293] where the value $n > 578$ was obtained, a considerable overestimate for n. The use of the maximum of $|f''(x)|$ over $[a,b]$ as in (8) instead of the mean value of $f''(x)$ as in (3) usually results in a similar overestimate of n with the resulting waste in computational effort and increase in roundoff pollution.

Certainly (3) works well for large values of n (or small desired values of E_n); it usually works as well for the small values of n often seen in calculus texts. For example, if we wish to approximate $\ln 2 = \int_1^2 (1/x)\,dx$ with $|E_n|$ no larger than 10^{-3}, then (3) yields $n \geqslant 8$. If the computations are performed with $n = 8$, the actual error will be seen to be less than $(.98)10^{-3}$. For this example, equation (8) is also easy to use and gives $n \geqslant 13$. The actual error will be less than $(.4)10^{-3}$. (These calculations were done on a TI-30 hand-held calculator.)

Some other comments are in order here. Since we have an estimate for the error E_n, we could

add it to T_n to obtain

$$\int_a^b f(x)\,dx \simeq T_n - \frac{(b-a)^2}{12n^2}(f'(b)-f'(a)). \tag{9}$$

This formula is known as the corrected trapezoidal rule; it is known to have an error $E(CT_n)$ given by

$$E(CT_n) = \frac{f^{(4)}(\zeta)(b-a)^5}{720n^4}$$

for some ζ in $[a,b]$. (See Conte and de Boor [**1**, p. 288 and 293].) This indicates that the convergence in (2) is very rapid. Since the derivation of the error term $E(CT_n)$ is similar to that of the error term for the Trapezoidal Rule given above, we can also write

$$E(CT_n) \simeq \frac{(b-a)^4}{720n^4}(f^{(3)}(b)-f^{(3)}(a)). \tag{10}$$

This correction could be added to the formula in (9) to give a Twice Corrected Trapezoidal Rule. This formula can be used to achieve accurate results for small values of n. The penalty we pay, of course, is the need to evaluate f' and $f^{(3)}$ at both a and b. For example, it is easy to show that if we use the Twice Corrected Trapezoidal Rule to estimate $\int_0^1 \exp(-x^2)\,dx$ with $n=2$, we will get .7468 263. The error here is less than $(1/4)\ 10^{-5}$. Although this is interesting, the main idea is that approximation (3) gives a simple way of estimating the error in the Trapezoidal Rule and, only secondarily, a way of correcting it.

There are times when (3) will fail to return a useful estimate for E_n: if $f'(b)=f'(a)$, then (3) implies that $E_n \simeq 0$; in such a case we may have to use equation (8). This will happen whenever f is periodic on $[a,b]$. However, the error given by equations (1) and (2) does indicate the known fact that the Trapezoidal Rule returns better than expected results for periodic functions (see Table 2 for an example); so the error bound in (8) will be even more pessimistic than usual.

TABLE 2

Comparison of the Trapezoidal Rule and Simpson's Rule applied to the integration of the periodic function $1/(\cos(6x)+2)$ from 0 to π: $\int_0^\pi 1/(\cos(6x)+2)\,dx = 1.8138$

n	Trapezoidal Rule	Simpson's Rule
4	1.8326	1.7453
8	1.8139	1.8077
16	1.8138	1.8138

One may also apply these ideas to Simpson's Rule:

$$S_n = \frac{b-a}{3n}[f(x_0)+4f(x_1)+2f(x_2)+\cdots+4f(x_{n-1})+f(x_n)].$$

If this is done, then we get, for a function f in $C^4[a,b]$ and for n even,

$$\int_a^b f(x)\,dx = S_n + E(S_n),$$

$$E(S_n) = -\frac{(b-a)^5}{180n^4}C(S_n) \tag{11}$$

$$\lim_{n\to\infty} C(S_n) = \frac{f^{(3)}(b)-f^{(3)}(a)}{b-a}, \tag{12}$$

and

$$E(S_n) \simeq -\frac{(b-a)^4}{180n^4}(f^{(3)}(b)-f^{(3)}(a)). \tag{13}$$

(This approximation can be found in Jensen and Rowland [2, p. 222].) In the Trapezoidal Rule, $C_n = f''(\zeta_n)$ for some ζ_n in $[a,b]$; for Simpson's Rule, $C(S_n)=f^{(4)}(s_n)$ for some s_n in $[a,b]$. Equation (13) may be used to obtain a corrected Simpson's Rule:

$$\int_a^b f(x)\,dx \simeq S_n - \frac{(b-a)^4}{180n^4}\left[f^{(3)}(b)-f^{(3)}(a)\right]. \tag{14}$$

See Table 3 for one comparison of the various methods.

TABLE 3

Estimates of $\int_0^1 \exp(-x^2)\,dx = .7468\ 2413\ldots.$

Method	n	Estimate	Estimated Error		True Error
			Eq'n	Value	
Trapezoidal Rule	2	.7313 7025	(3)	.0153 2831	.0154 5393
Corrected Trapezoidal Rule	2	.7466 9856	(10)	.0001 2774	.0001 2557
Twice-Corrected Trapezoidal Rule	2	.7468 2630	--	------	.0000 0217
Simpson's Rule	2	.7471 8043	(13)	-.0005 1094	-.0003 5630
Corrected Simpson's Rule	2	.7466 6949	--	------	.0001 5464
Simpson's Rule	4	.7468 5538	(13)	-.0000 3193	-.0000 3125
Corrected Simpson's Rule	4	.7468 2345	--	------	.0000 0068

Finally, we return to the relationship between $E(S_{2n})$ and $E(S_n)$. From (1) and (2) we saw that $E_{2n} \simeq (1/4)E_n$. Similarly, from (11) and (12) it is easy to see that $E(S_{2n}) \simeq (1/16)E(S_n)$.

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SUMMATION OF SERIES

GABRIEL KLAMBAUER

The purpose of this note is to sum some interesting infinite series by use of elementary methods.

I. Let f and g be functions in the variable x such that

$$f(x) = af(bx) + cg(x),$$

where a, b, c are given constants. Then

$$a^{k-1}f(b^{k-1}x) = a^k f(b^k x) + a^{k-1}cg(b^{k-1}x),$$

A TRICK WITH REDUNDANT INFORMATION

M. STOJAKOVIĆ

Consider the system of linear equations

$$AX = B \tag{1}$$

where A is an $n\times n$ matrix, $X=[x_1,\dots,x_n]^t$, and $B=[b_1,\dots,b_n]^t$. The question "Why not just take the determinant of both sides in (1) to obtain the solution X (if it exists)?" is not as silly as it may seem.

For simplicity, let us write (1) in the case $n=3$:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{2}$$

The amount of information given by (2) is the same as that given by

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 0 & 0 \\ x_3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} b_1 & 0 & 0 \\ b_2 & 0 & 0 \\ b_3 & 0 & 0 \end{bmatrix}. \tag{3}$$

Using the identity

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & b & c \\ 0 & e & f \\ 0 & h & i \end{bmatrix},$$

we obtain by matrix addition

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & b & c \\ b_2 & e & f \\ b_3 & h & i \end{bmatrix}. \tag{4}$$

We may now take the determinant of both sides of (4) to obtain

$$(\det A)x_1 = \det A_1$$

(with the obvious meaning of A_1), which is Cramer's rule for finding x_1!

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOVI SAD, PB 239, NOVI SAD 21000, YUGOSLAVIA.

MATHEMATICAL EDUCATION

EDITED BY W. E. MASTROCOLA

Material for this department should be sent to W. E. Mastrocola, Department of Mathematics, Colgate University, Hamilton, NY 13346.

ON "AN EXPERIMENTAL EVALUATION OF RETESTING"

KEVIN GALLAGHER

Deatsman [1] discussed a program in which college freshman algebra students were permitted (within certain limits) to retake exams whose results were unsatisfactory. Retesting for the

comprehensive final exam was not permitted. To help determine whether retesting during the term affects a student's performance on the final exam, Deatsman conducted a two-tailed statistical hypothesis test comparing the mean score of the Retesters (students permitted to retest) with the mean score of the Non-retesters (students not permitted to retest) on the final exam. The null hypothesis was: "There is no difference between the mean final exam scores of freshman algebra students who are allowed to take retests and the scores of those who are not."

In the slower class of the two groups analyzed, the mean on the final of the Retesters was 71.7 and the mean on the final of the Non-retesters was 76.2. The calculation of the test statistic yielded $t = 1.019$. Deatsman concluded:

> The probability of obtaining a value of t as large as or larger than the one obtained for the slower class was between 0.3 and 0.4. Thus the null hypothesis could be rejected at the .60 level of confidence, and we may conclude that there is at least a 60% probability that the mean final exam scores of the slower group were actually decreased by allowing retests. [1, p. 53]

The problem here is twofold. Let $\underline{\alpha}$ be the smallest α-level at which one could reject the null hypothesis with the calculated test statistic. Let H_0 be the condition that the null hypothesis is true. Deatsman fails to interpret $\underline{\alpha}$ as a conditional probability and he confuses $\underline{\alpha}$ in a hypothesis test with the choice of α in a $(1-\alpha)$ confidence interval. Since the test in question is two-tailed, in truth, $\frac{1}{2}\underline{\alpha} = p(t > 1.019 | H_0)$. Deatsman interprets $\underline{\alpha}$ as $p(t > 1.019)$. He also interprets $1 - \underline{\alpha}$ as $p(\sim H_0)$, a measure of how confident he is that the null hypothesis is false. In reality, $1 - \underline{\alpha} = p(|t| < 1.019 | H_0)$. With $t = 1.019$, $\underline{\alpha} \cong .31$. So there was a somewhat greater than .30 probability of obtaining $|t| > 1.019$ under a true null hypothesis.

Few people are willing to risk such a large probability of rejecting the null hypothesis when it is true. The only sensible interpretation of Deatsman's statistical test calculations is that they fail to contribute any information to help determine the usefulness of retesting.

Reference

1. G. A. Deatsman, An experimental evaluation of retesting, this MONTHLY, 86 (1979) 51–53.

DEPARTMENT OF MATHEMATICS, NORTHERN MICHIGAN UNIVERSITY, MARQUETTE, MI 49855.

PROBLEMS AND SOLUTIONS

EDITED BY VLADIMIR DROBOT

The editors receive more problems than they can use, but they are always on the lookout for interesting and novel problems. The task of selection will be much easier if proposers will keep the following guidelines in mind:

Send all proposed problems, in duplicate if possible, to Vladimir Drobot, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053. Please include solutions and any information that will help the editors, including reasons why the problem is interesting. Problems in well-known textbooks and results in generally accessible sources are not acceptable.

Solutions should be sent to the addresses given at the head of each problem set.

An asterisk (*) indicates that the proposer did not supply a solution. If you submit a problem without a solution, you should tell the editors whether you know (or somebody else knows) how to solve the problem. If you

are uncertain, the problem might be more suitable for the Research Problems section; see the instructions given there.

Proposers are asked to aim for the same audience as for the rest of the MONTHLY; *a rule of thumb is to think of people who have had at least a year of graduate work in Mathematics, very likely some years ago. Proposals that can be both understood and solved by high school students, or by those in the first two years of college, are usually more appropriate for the Mathematics Magazine or the Two-Year College Mathematics Journal. No precise line can be drawn between elementary and advanced problems, but in general an elementary problem will involve notions no more difficult than those encountered in the first year of graduate work and should be solvable by methods common at the same level. An advanced problem may deal with more advanced concepts or require more sophisticated techniques for its solution.*

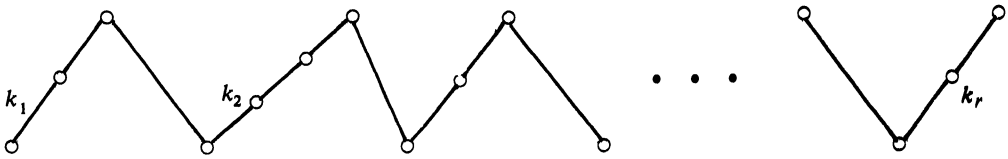
A problem that takes half a page or more to state, or that involves several ad hoc definitions, is usually unsuitable. The statement of a problem should avoid specialized jargon and symbols whenever the English language is more readily understood: for example, "f is a continuous function" is preferable to "$f \in C$."

Before submitting a problem, be sure that it says exactly what you intend it to say. Solvers are not telepathic and will usually solve what was proposed, not what you had in mind.

PROBLEMS DEDICATED TO EMORY P. STARKE

Solutions of these problems dedicated to E. P. Starke should be mailed to Prof. A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, N.M. 87131, by June 30, 1980. To facilitate consideration, type with double spacing in the format used below. If acknowledgment is desired, include a self-addressed card, label, or envelope.

S 26. *Proposed by L. Carlitz, Duke University.*



Here $\pi=(a_1,a_2,\ldots,a_n)$ is a permutation of $1,2,\ldots,n$ composed of r increasing subsequences (rises) placed one after the other with the last term of each (except the final) rise larger than the following term. The number of terms in the ith rise is denoted by k_i. Let $g_2(n,r)$ be the number of π having each $k_i \geqslant 2$ and let $h_1(n,r)$ be the number of π satisfying the conditions $k_1=1$, $k_r>1$, and $k_i \geqslant 2$ for $2 \leqslant i \leqslant r-1$. Show that

$$h_1(n,r)=g_2(n,r-1).$$

A combinatorial proof would be preferred. (The proposer's solution is not purely combinatorial.)

SOLUTIONS OF PROBLEMS DEDICATED TO EMORY P. STARKE

Converse and Analogues of a Binomial Coefficient Property

S 1 [1979, 54]. *Proposed by George Pólya, Stanford University.*

Consider the integer $n, n>2$, the three sequences

$$\binom{n}{1},\binom{n}{2},\binom{n}{3},\ldots,\binom{n}{n-1}, \tag{1}$$

$$s_2^n, s_3^n, \ldots, s_{n-1}^n, \tag{2}$$

$$S_2^n, S_3^n, \ldots, S_{n-1}^n \tag{3}$$

(binomial coefficients, Stirling numbers of the first and second kind, respectively), and the two statements:

(I) If n is prime, all terms of the sequence are divisible by n.

(II) If n is composite, there is in the sequence a term (there may be several terms) non-divisible by n.

Statement (I) has been proved; prove statement (II) for all three sequences.

(For the notation used, see G. Pólya and G. Szegö, *Problems and Theorems in Analysis*, vol. 2, problem VIII 247.1, p. 153, also its solution on p. 358 and the passages there quoted.)

(1) *Solution by Lajos Takács, Case Western Reserve University*. Let n be composite and let p be a prime divisor of n. Then

$$\binom{n}{p}\frac{1}{n}=\frac{(n-1)(n-2)\cdots(n-p+1)}{p!}$$

is not an integer because none of the $p-1$ factors in the numerator is divisible by p. This proves (II) for (1).

(2) and (3). *Solution by Duane M. Broline, Auburn University*. (2) *Stirling numbers of the first kind.* Now $\Sigma_{k=1}^{n} s_k^n x^k = x^{(n)} = (x)(x-1)\cdots(x-n+1)$. Since $s_1^n = (-1)^{n-1}(n-1)!$ *and* $s_n^n = 1$, putting $x=1$ in the above equation yields

$$0=(-1)^{n-1}(n-1)!+\sum_{k=2}^{n-1} s_k^n+1.$$

If $s_k^n \equiv 0 \pmod{n}$, for $k=2,\ldots,n-1$, then $0\equiv(-1)^{n-1}(n-1)!+1 \pmod{n}$. Therefore n and $(n-1)!$ are relatively prime and n is prime.

(3) *Stirling numbers of the second kind.* Now $x^n = \Sigma_{k=1}^{n} S_k^n x^{(k)}$. Since $S_1^n = S_n^n = 1$, dividing the above equation by x yields:

$$x^{n-1}=\frac{x^{(n)}}{x}+\sum_{k=2}^{n-1} S_k^n \frac{x^{(k)}}{x}+1.$$

For every k, $x^{(k)}/x$ is a polynomial in x which when evaluated at 0 gives $(-1)^{k-1}(k-1)!$. Hence

$$0=(-1)^{n-1}(n-1)!+\sum_{k=2}^{n-1} S_k^n(-1)^{k-1}(k-1)!+1.$$

As in the case of the Stirling numbers of the first kind, the assumption $S_k^n \equiv 0 \pmod{n}$, $k=2,\ldots,n-1$ implies that n is prime.

Also solved by Anon, Ken Brown, L. Carlitz, Eli L. Isaacson, Barry Powell, J. M. Stark, and the proposer.

Wordless Solution

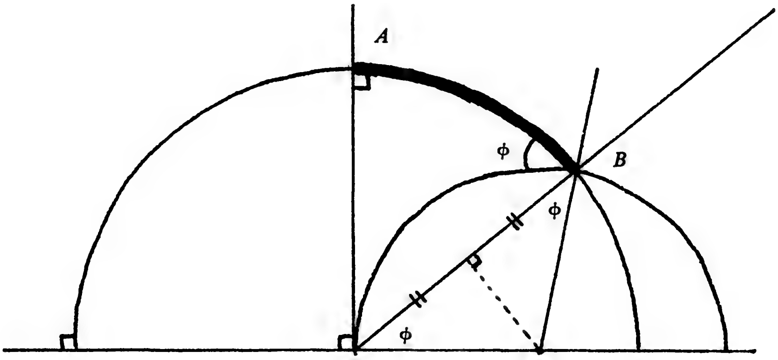
S 2 [1979, 55]. *Proposed by H. S. M. Coxeter, University of Toronto.*

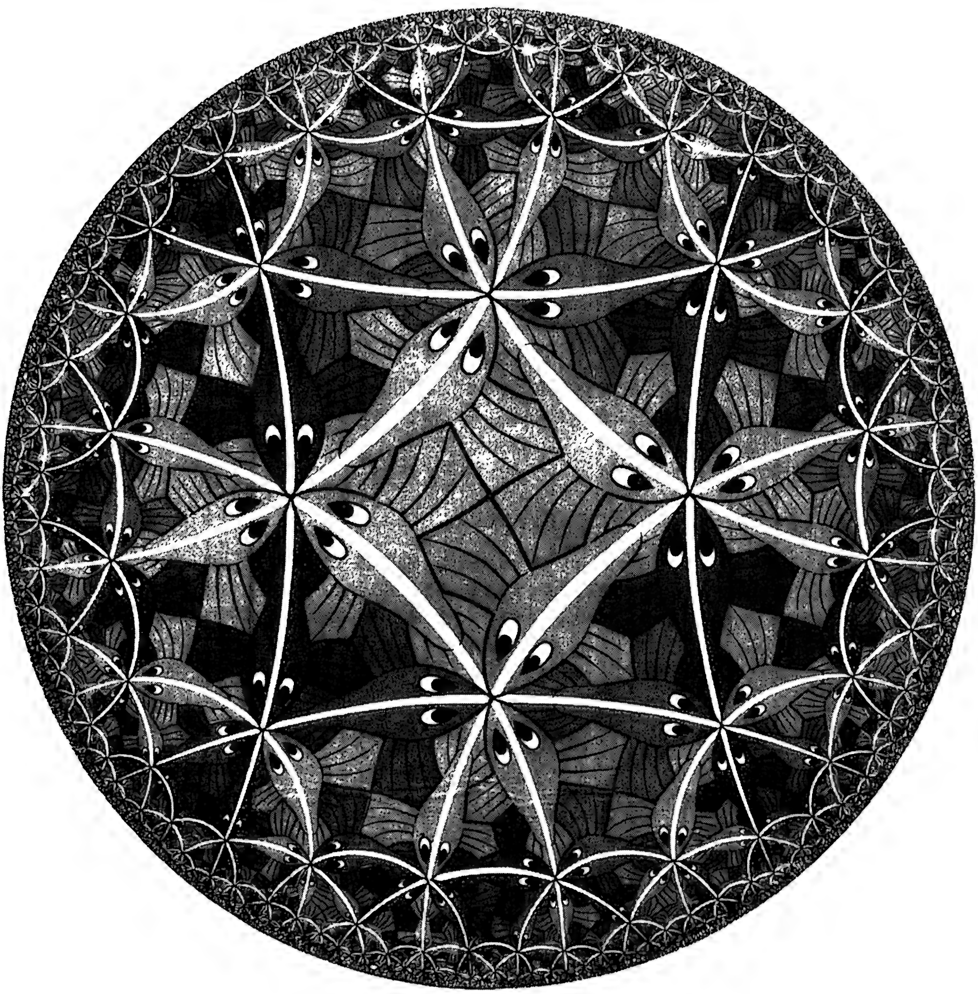
In the hyperbolic plane, the locus of a point at constant distance δ from a fixed line (on one side of the line) is one branch of an "equidistant curve" (or "hypercycle"). In Poincaré's half-plane model, this curve can be represented by a ray making a certain angle with the bounding line of the half-plane. Show that this angle is equal to $\Pi(\delta)$, the angle of parallelism for the distance δ.

Solution by Jan van de Craats, University of Leiden, the Netherlands. $\Pi(AB)=\phi$. (See p. 135.)

Note by the proposer. It follows, after a considerable amount of computation, that each of the white arcs in M. C. Escher's "Circle Limit III" cuts the peripheral circle at an angle of

$$\operatorname{arcsec}(2^{7/4}+2^{5/4}),$$





M. C. Escher's Circle Limit III, reproduced with the permission of the Escher Foundation, Gemeentemuseum, The Hague.

which is almost 80°. (See p. 135.) For details, see H. S. M. Coxeter's "The Non-Euclidean Symmetry of Escher's Picture 'Circle Limit III,'" *Leonardo*, 12 (no. 1) (1979) 19–25.

Also solved by Anon, C. L. Belna, R. W. Chamberlain, David C. Kay, R. K. Oliver, George D. Parker, Richard S. Millman, J. M. Stark, and the proposer.

Asymptotic to $\pi(n)$

S 3 [1979, 55]. *Proposed by Albert A. Mullin, Huntsville, Alabama.*

Prove that *any* strictly positive real-valued arithmetical function f satisfying the functional equation

$$(f(n+1)/(n+1))+n=(n+1)f(n)/f(n+1)$$

for every integer n exceeding some prescribed positive integer m is *necessarily* asymptotic to $\pi(n)$, the number of prime numbers not exceeding n; i.e, $f(n)\sim\pi(n)$.

Solution by Douglas A. Hensley, Texas A & M University. Let $y(n)=(1/n)f(n)$. Then with $\Delta y=y(n+1)-y(n)$ we have

$$n\Delta y+y^2(n+1)=0 \tag{1}$$

By analogy with the Bernoulli equation $xy'+y^2=0$ we should let $v=1/y$ and expect $v\sim\log n$. With this substitution, (1) becomes

$$\Delta v=v(n)/nv(n+1). \tag{2}$$

Now $\Delta v>0$ so v is increasing, and $\Delta v<1/n$. For larger n, $nv>1$ so $\Delta v=v/(nv+n\Delta v)>v/(nv+1)>1/2n$. Thus $v\to\infty$ and for large n, $v>1$. Thus for large n, $v/(nv+1)>1/(n+1)$ so

$$\frac{1}{n+1}<\Delta v<\frac{1}{n}. \tag{3}$$

Summing, we have $v\sim\log n$ and $f(n)=n/v\sim n/\log n\sim\pi(n)$. □

Also solved by Anon, Emil Grosswald, Ellen Hertz, Eli L. Isaacson, O. P. Lossers (Netherlands), L. E. Mattics, R. W. K. Odoni (England), J. M. Stark, Lajos Takács, and the proposer.

ELEMENTARY PROBLEMS

Solutions of these Elementary Problems should be mailed in duplicate to Dr. J. L. Brenner, 10 Phillips Road, Palo Alto, CA 94303 (U.S.A.), by June 30, 1980. Please type with double spacing and place the solver's name and mailing address on each sheet. If acknowledgment is desired, include a self-addressed card or label.

E 2815. *Proposed by Leon Gerber, St. John's University, Jamaica, N.Y.*

Establish the inequality

$$\cos x+\cos y\leqslant 1+\cos xy$$

for $0\leqslant x^2+y^2\leqslant\pi$.

E 2816. *Proposed by R. Bojanic, Ohio State University.*

Consider a circular segment AOB with $\measuredangle AOB<\pi$. Let C be the orthogonal projection of the point B on the line $\overline{OA}$. Suppose that the arc $\widehat{AB}$ and the segment $\overline{CA}$ are each divided into n equal parts. If M is the point of the partition of the arc AB closest to B, and N the point of the partition of the segment $\overline{CA}$ closest to C, show that the projection of the midpoint of the arc $\widehat{MB}$ onto the line $\overline{OA}$ is always contained in the interval (C,N).

E 2817. *Proposed by Jeffrey Shallit, undergraduate, Princeton University.*

For k in $\{1,2,\ldots,n\}$, let $R(k,n)$ be the remainder in the division of n by k. Thus $n=qk+R(k,n)$ with q an integer and $0 \leqslant R(k,n) < k$. Set $T(n)=\sum_{k=1}^{n} R(k,n)$.

(i) Express $T(n)$ in terms of the function σ, where $\sigma(n)$ is the sum of the positive integral divisors of n.
(ii) Show that $T(2^n)=T(2^n-1)$.

E 2818. *Proposed by J. Linkovskiĭ-Condé, Moscow, U.S.S.R.*

Let $T_n=2^n+1$ for all positive integers. Let ϕ be the Euler totient function, let k be any positive integer, and $m=n+k\phi(T_n)$. Show that T_m is divisible by T_n.

E 2819. *Proposed by C. Notari, Mégrine-Coteaux, Tunisia.*

Let $T(z)$ be a polynomial with integral coefficients, having a root in common with $Q(z)=z^n+1$. Supposing that for each root v_i of $Q(z)$ we have $|T(v_i)| \leqslant 1$, prove that $Q(z)$ divides $T(z)$.

E 2820. *Proposed by C. Notari, Mégrine-Coteaux, Tunisia.*

Let $T(z)$ be a non-constant polynomial with integral coefficients and $T(0)\neq 0$. If for each root v_i of $Q(z)=z^n+1$, we have $T(v_i)=1$, prove the existence of a unique integer k $(0 \leqslant k < n)$, such that $T(z)+z^k$ or $T(z)-z^k$ is divisible by z^n+1.

SOLUTIONS OF ELEMENTARY PROBLEMS

$\mathrm{Det}[C_j^{im+j-1}]$

E 2729 [1978, 594]. *Proposed by John Goth, Austin, Texas.*

Evaluate $\det(A)$ where $A=(a_{ij})$ is the $n\times n$ matrix given by

$$a_{ij}=\binom{im+j-1}{j} \qquad (i,j=1,\ldots,n),$$

m being a fixed positive integer.

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, Netherlands, and Albert Nijenhuis, University of Pennsylvania (independently). Consider the more general $n\times n$ matrix B given by $b_{ij}=p_j(x_i)$, where p_j is a polynomial of degree j and with leading coefficient c_j and zero constant term, $1\leqslant j\leqslant n$, and $x_1,\ldots,x_n$ are indeterminates. We claim $\det(B)=(c_1\cdots c_n)(x_1\cdots x_n)\Pi_{i>j}(x_i-x_j)$.

Proof. Det(B) is a polynomial in the x_i of degree $1+2+\cdots+n=n(n+1)/2$. Det$(B)=0$ if any $x_i=0$, or if $x_i=x_j$ for some $i\neq j$. Hence $\det(B)=a(x_1\cdots x_n)\Pi_{i>j}(x_i-x_j)$ for some constant a. We now compare the coefficients of the term $x_1x_2^2\cdots x_n^n$ and find $a=c_1\cdots c_n$. □

In the problem we take $p_j(x)=x(x+1)\cdots(x+j-1)/j!$ and $x_i=im$. Then $c_j=1/j!$. A computation yields $\det(A)=m^{n(n+1)/2}$.

Also solved by the Bennett College team, Robert Breusch, Robert C. Carson, W. Glenn Clark, Francis Clarke, Lorraine L. Foster, Michael Goldberg, Eli L. Isaacson, Thomas Jager, A. A. Jagers (Netherlands), L. Kuipers (Switzerland), Peter W. Lindstrom, S. C. Locke (Canada), J. G. Mauldon, Gerhard Metzen (Canada), Jose Luis de Miguel (Spain), N. Miku (Netherlands), Roger B. Nelsen, Robert Patenaude, Otto G. Ruehr, St. Olaf College Problems Group, Thomas E. Salazar, K. R. P. Singh (India), Paul Smith (Canada), University of South Alabama Problem Group, Allen Stenger, Michael Vowe (Switzerland), Victor K. W. Wei, Gregory Wulczyn, Paul J. Zwier, and the proposer.

Editor's comments. Goldberg and Kuipers located a similar problem which was solved in par. 731, pp. 679–680, Thomas Muir, *A Treatise on the Theory of Determinants*, Dover reprint, 1960. Smith used a related identity (z.9) p. 82, H. W. Gould, *Combinatorial Identities*, Morgantown, W.V., 1972.

Euler's Constant as a Limit

E 2748 [1978, 824]. *Proposed By Lance Littlejohn, Pennsylvania State University.*

If $f(x)=x^n\log x$ find

$$\lim_{n\to\infty}\frac{f^{(n)}(1/n)}{n!}.$$

Solution by Zachary Franco (sophomore), Stuyvesant High School, Brooklyn, N.Y. Since

$$f^{(k)}(x)=x^{n-k}n(n-1)\cdots(n-k+1)\left[n^{-1}+(n-1)^{-1}+\cdots+(n-k+1)^{-1}+\log x\right]$$

which may easily be verified by induction, it follows that

$$f^{(n)}(1/n)=n!\left[\Sigma_{i=1}^{n-1}i!-\log n\right],$$

so the desired limit is equal to Euler's constant $\gamma=.577\ldots$.

Also solved by 67 other readers.

Equal Sums of Powers of Primes

E 2749 [1979, 55]. *Proposed by Leo J. Alex, SUNY at Oneonta.*

(i) Show that neither of the equations

$$3^a+1=5^b+7^c,\quad 5^a+1=3^b+7^c$$

has a solution in integers a, b, c other than $a=b=c=0$.

(ii) Show that the only solutions to the equation

$$7^a+1=3^b+5^c$$

in integers a, b, c are $(a,b,c)=(0,0,0)$ or $(1,1,1)$.

Solution by Lorraine L. Foster, California State University, Northridge. Clearly $a,b,c\geqslant 0$.

(i) If the exponent of 3 is not zero, neither equation is solvable modulo 3.

(ii) Suppose (a,b,c) is a third solution. Clearly $c\neq 0$, $7^a\equiv 3^b+4 \pmod{20}$, so that $a\equiv b\equiv 1 \pmod 4$. Hence $5^c\equiv 5 \pmod{16}$ and $c\equiv 1 \pmod 4$. Examining possibilities modulo 13 we easily conclude $b\equiv 1 \pmod 3$, $a\equiv 1 \pmod{12}$. Therefore $b\equiv 1 \pmod 6$, $5^c\equiv 5 \pmod 7$, and hence $c\equiv 1 \pmod 6$. Thus our equation is absurd modulo 9 if $b>1$, and absurd modulo 25 if $b=1$, $c>1$. Hence no third solution exists.

Also solved by Theodore S. Bolis, Milton Eisner [part (i)], Eli L. Isaacson [part (i)], L. Kuipers (Switzerland), Man Kam Kwong, David Leep, L. E. Mattics, Paul Monsky, Dalton E. Orr, Victor Pambuccian (Rumania) [part (i)], and the proposer.

Sum of Powers of Primes

E 2750 [1979, 55]. *Proposed by the editors (based on E 2749).*

Find all solutions in integers a,b,c of the equation

$$9+5^a=3^b+7^c.$$

Solution by Lorraine L. Foster, California State University, Northridge. The only solutions in integers are $(a,b,c)=(0,1,1)$, $(0,2,0)$ or $(2,3,1)$.

Proof. (Obviously $a,b,c\geqslant 0$.) Let (a,b,c) be a fourth solution. Clearly $a\neq 0$ so that $14\equiv 3^b+7^c(\text{mod}\,20)$, $b\equiv 3$, $c\equiv 1(\text{mod}\,4)$. Hence $5^a\equiv 9(\text{mod}\,16)$, $a\equiv 2(\text{mod}\,4)$. Therefore $5^a\equiv 8$, $7^c\equiv 7$, 11 or 8, $3^b\equiv 3$, 9 or $1(\text{mod}\,13)$. We easily conclude that $3|b, c\equiv 1(\text{mod}\,12)$. Thus $b\equiv 3(\text{mod}\,6)$, $5^a\equiv 4(\text{mod}\,7)$, $a\equiv 2(\text{mod}\,6)$. Suppose that $c>1$ and $9+5^a\equiv 3^b(\text{mod}\,49)$. Then $(a,b)\equiv(2,9)$, $(8,27)$, $(14,3)$, $(20,21)$, $(26,39)$, $(32,15)$, or $(38,33)(\text{mod}\,42)$. However, in each case we find

$9+5^a-3^b$ is not a power of 7(mod 43). (Note: 3,5 are primitive roots modulo 49 and 43.) Therefore $c=1$, $b>3$, $5^a\equiv -2(\text{mod}\,81)$, $a\equiv 20(\text{mod}\,54)$, $3^b\equiv 5^{20}+2\equiv 37(\text{mod}\,109)$, again a contradiction. (Note 3,5 have order 27 modulo 109).

Also solved by Leo J. Alex, Theodore S. Bolis, L. Kuipers (Switzerland), David Leep, L. E. Mattics, Paul Monsky, and Dalton E. Orr.

Multiplicative Group Z_p

E 2753 [1979, 56]. *Proposed by Haim Rose, Kiriat Shmonah, Israel.*

Let p be a prime and $g=\{r_1,r_2,\dots,r_k\}$ be any group under multiplication modulo p, where the r_i are integers with $0<r_i<p$. Let P be the product of all the r_i and Q be the product of those r_i satisfying $0<r_i<p/2$. Prove:

(i) $P\equiv -(-1)^k\ (\text{mod}\,p)$. [An extension of Wilson's Theorem.]

(ii) If $k=2h$, with h an odd integer, then $Q\equiv \pm 1\ (\text{mod}\,p)$.

(iii) If $1\leqslant r_i\leqslant (p-1)/2$ for $1\leqslant i\leqslant k$, then $P\equiv 1\ (\text{mod}\,p)$. Can this situation actually occur?

(iv) If $k=2h, h\geqslant 2$, then p^2 is an integral divisor of the numerator of the sum

$$\frac{1}{r_1}+\frac{1}{r_2}+\cdots+\frac{1}{r_k}.$$

[An extension of Wolstenholme's Theorem.]

Solution by Lorraine L. Foster, California State University, Northridge. Since g is of order k, $x^k-1\equiv\prod_{i=1}^k(x-r_i)(\text{mod}\,p)$. Hence $(-1)^k\prod_{i=1}^k r_i\equiv -1\ (\text{mod}\,p)$ and we have (i). Let $S=\{x\in g|0<x<p/2\}$. If $2|k$ then $p-1\in g$, $g=S\cup\{p-x|x\in S\}$, $(-1)^{k+1}\equiv P\equiv(-1)^{k/2}Q^2$ and (ii) follows easily. Also, since $\prod_{i=1}^k(p-r_i)=\prod_{i=1}^k r_i$, if $k/2>1$ the proof of Wolstenholme's Theorem in [1] is easily modified and we have (iv). Suppose now that $g=S$. Then clearly $p-1\notin g$, $2\nmid k$, $P\equiv 1\ (\text{mod}\,p)$. To complete (iii) note that if $g=\langle 8\rangle\ (\text{mod}\,151)$ then $g=S$.

Reference

1. Hardy and Wright, The Theory of Numbers, Oxford, 1960, pp. 88f.

Also solved by Anders Bager (Denmark), Ken Brown, Milton Eisner, David Hammer, Eli L. Isaacson, Michael Josephy (Costa Rica), L. Kuipers (Switzerland), David Leep, Mark Merriman, and Lawrence Somer.

Editorial Note. The example $\langle 8\rangle$ mod 151, with 5 members, was given by Emma Lehmer. No example can have three members. The following examples were found by David Hammer: $\langle 8\rangle$ mod 151, $\langle 48\rangle$ mod 541, $\langle 89\rangle$ mod 691, $\langle 75\rangle$ mod 1381, $\langle 123\rangle$ mod 1831, $\langle 191\rangle$ mod 1871, $\langle 279\rangle$ mod 2081, $\langle 385\rangle$ mod 3271, all of order 5; and $\langle 204\rangle$ mod 3389, $\langle 12\rangle$ mod 4943, of order 7. It seems likely that any odd order >3 can occur.

Pattern of Intersection of Lines

E 2754 [1979, 56]. *Proposed by Jim Fickett, University of Colorado.*

Given n arbitrary lines $k_1,\dots,k_n$ in the plane, need there exist another n lines $h_1,\dots,h_n$ having the same intersection pattern but with all intersection points rational? The first condition means that for every subset S of $\{1,\dots,n\}$ we have

$$\bigcap_{i\in S} k_i\neq\varnothing \Leftrightarrow \bigcap_{i\in S} h_i\neq\varnothing.$$

Editorial note. Peter Renz and Branko Grünbaum point out that this problem is solved (negatively) in Grünbaum's book *Convex Polytopes*, Interscience (Wiley), 1967, pp. 93–94, as well as in *Arrangements and Spreads*, CBMS Regional Conference Series in Mathematics no. 10, American Math. Soc., 1972, pp. 34–35.

A solution was received from L. E. Mattics.

ADVANCED PROBLEMS

Solutions of these Advanced Problems should be sent to Professor Roger C. Lyndon, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, by June 30, 1980. To facilitate their consideration, please type (in duplicate, with double spacing) and place the solver's mailing address on the solution sheets. If acknowledgment is desired, include a self-addressed card.

6288. *Proposed by Jim Fickett and Arlan Ramsay, University of Colorado, Boulder.*

Let S be a separable set in real separable Hilbert space which is "fat" in the sense that, for each u of norm 1,

$$\sup\{(u,s)|s\in S\}-\inf\{(u,s)|s\in S\}\geqslant 1.$$

Must the interior of S be nonempty?

6289. *Proposed by James W. Fickett, University of Colorado, Boulder.*

Let A be a subset of the metric space M. Show that the two following conditions are equivalent.

(1) A is a G_δ in M.

(2) Any continuous function f mapping A into another metric space N has an extension F: $M\to N$ which is continuous at each point of A (considered as a point of M).

6290. *Proposed by Rudolph Ahlswede, Bielefeld University, Germany, and David E. Daykin, Reading University, England.*

Find a good constant k with the following property: If P is the power set of all 2^n subsets of $\{1,2,\ldots,n\}$ and for $i=1,2,3$ each of the functions f_i: $P\to\{-1,1\}$ has $f(a)\leqslant f(b)$ whenever a is a subset of b and

$$\sum_{a\in P} f_i(a)\geqslant 2^n k$$

then

$$\sum_{a\in P} f_1(a)f_2(a)f_3(a)\geqslant 0.$$

SOLUTIONS OF ADVANCED PROBLEMS

A Characterization of Integers That Differ by Two

6200 [1978, 203]. *Proposed by Brian Conrey, David Leep, and Gerry Myerson, University of Michigan.*

Define $\left(\frac{a}{b}\right)_r$ to be the least positive integer x such that $bx\equiv a \pmod r$. Let k,m,n be positive integers with $(m,n)=1, k<n, m<n$. Show

$$(1)\ m<\left(\frac{k}{m}\right)_n+\left(\frac{k}{n}\right)_m\leqslant n,\qquad (2)\ \left(\frac{1}{m}\right)_n+\left(\frac{1}{n}\right)_m=\frac{m+n}{2}\quad \text{if and only if } n-m=2.$$

Thus, for m,n prime, (2) characterizes twin primes.

Solution by H. William Oliver, Williams College. Putting $x=(k/m)_n$, we know that (x,y) is the unique solution of the linear Diophantine equation $mx-ny=k$, which satisfies $1\leqslant x\leqslant n-1$, $0\leqslant y\leqslant m-1$. We also have $n(m-y)-m(n-x)=k$ with $1\leqslant m-y\leqslant m$, $1\leqslant n-x\leqslant n-1$; so $m-y=(k/n)_m$. We wish to show that $m<x+(m-y)\leqslant n$, or that $0<x-y\leqslant n-m$.

Set $n=m+d$ with $d\geqslant 1$, and $k=m+q$, with $q<d$. The equation $mx-ny=k$ then becomes $m(x-y)=m+q+dy$, showing that $x-y>0$. From $y\leqslant m-1$, we deduce: $m(x-y)=m+q+dy$ $\leqslant m(d+1)+q-d<m(d+1)$, whence $x-y\leqslant d=n-m$.

Suppose now that $k=1$ and $x+(m-y)=(n+m)/2$, or $x=y+(n-m)/2$. Obviously, $n-m$ is even. Substituting this last value of x into $mx-ny=1$ gives $m(y+(n-m)/2)-ny=1$, or $y=m/2-1/(n-m)$. Since $n-m$ is even and y is an integer, $n-m=2$.

Also solved by Robert Breusch, L. Carlitz, L. E. Clarke (England), Columbia University Problem Group, L. Kuipers (Switzerland), L. E. Mattics, Roger B. Nelsen, Blair Spearman, and the proposer.

A Limit of Matrices

6209 [1978, 283]. *Proposed by Marcel F. Neuts, Purdue University, Lafayette.*

Let A be a primitive nonnegative matrix of order m and let B be a finite real matrix of order m. Denote the spectral radius of A by ρ. Show that

$$\lim_{n\to\infty}\frac{1}{n}\rho^{-(n-1)}\sum_{\nu=0}^{n-1}A^{\nu}BA^{n-1-\nu}$$

exists and identify the limit.

Solution by H. Kestelman, University College, London. Set

$$C=\frac{1}{\rho}A\quad\text{and}\quad M_n=\frac{1}{n}\sum_{\nu=0}^{n-1}C^{\nu}BC^{n-1-\nu}.$$

Then C is primitive and has spectral radius 1; hence $\{C^n\}$ converges to C^∞ where $C^\infty=vw^T$, $Cv=v, w^TC=w^T$ and $w^Tv=1$. If we show that $\lim_{n\to\infty}M_n=C^\infty BC^\infty$, this will solve the problem. Now

$$M_n=\frac{1}{n}\sum_{\nu=0}^{n-1}C^{\nu}BC^{\infty}+\frac{1}{n}\sum_{\nu=0}^{n-1}C^{\nu}B(C^{n-1-\nu}-C^{\infty})=p_n+q_n,$$

say. Since $\lim_{\nu\to\infty}C^{\nu}BC^{\infty}=C^{\infty}BC^{\infty}$, $\{p_n\}$ converges to $C^\infty BC^\infty$; it remains to show that $q_n\to 0$ as $n\to\infty$. If we choose any norm on $m\times m$ matrices, then

$$n\|q_n\|\leqslant\sum_{\nu=0}^{n-1}\|C^{\nu}B\|\,\|C^{n-1-\nu}-C^{\infty}\|.$$

Since $\{C^{\nu}B\}$ converges, $\{\|C^{\nu}B\|\}$ is bounded, and finally it is enough to show that $\sum_{\nu=0}^{n-1}\|C^{n-1-\nu}-C^{\infty}\|$ is a bounded function of n. For this we can appeal to a standard result on matrices, namely, that if $\{P^j\}$ converges to P^∞ then $\sum_j\|P^j-P^\infty\|$ converges.

Also solved by L. E. Clarke (England), Joel Levy, and the proposer.

Nonnormal Numbers

6219 [1978, 500]. *Proposed by M. J. Pelling, Balliol College, Oxford, England.*

Construct an uncountable class of real numbers not normal in the scales of 3 and 5.

Solution by Andrew Odlyzko, Bell Laboratories. Let (β_n) be any infinite sequence of 0's and 1's. We construct a real number $x\in(0,1)$ as follows. Let the 10th through 30th digits of x in base 3 be β_1. This defines only about $30\cdot\log_5 3$ digits of x in base 5. Then let the 100th through 300th digits of x in base 5 be β_2. This defines only about $300\cdot\log_3 5$ digits of x in base 3. Proceed in

this way, so that digits

$$10^{2k} \text{ through } 3\cdot 10^{2k} \text{ in base 5 are } \beta_{2k},$$

$$10^{2k+1} \text{ through } 3\cdot 10^{2k+1} \text{ in base 3 are } \beta_{2k+1}.$$

This procedure constructs an x for each sequence (β_n), and each such x is clearly not normal in either of the bases, hence it solves the problem. This procedure can be easily generalized to construct uncountably many real numbers that are not normal to any base.

A much more interesting question is the following. Are there uncountably many real numbers which are normal in base 3 but not in base 5? This result (and a lot more) follows from a theorem of W. Schmidt (Acta Arith., 7 (1962) 299–309), but I do not know of any simple proof.

Also solved by the proposer.

Expected Perimeter Length

6230 [1978, 686]. *Proposed by Gérard Letac, Université Paul Sabatier, Toulouse, France.*

$X(t)$ is the perimeter length of the convex hull of $b(s)_{0\leqslant s\leqslant t}$, where b is the standard brownian motion in the euclidean plane. Compute $\mathbf{E}(X(t))$.

Solution by Lajos Takács, Case Western Reserve University. By definition $b(u)=(\xi(u),\eta(u))$ where $\{\xi(u),\ 0\leqslant u<\infty\}$ and $\{\eta(u),\ 0\leqslant u<\infty\}$ are independent, standard Brownian motion processes (having continuous sample functions). Write $\zeta_\alpha(u)=\xi(u)\cos\alpha+\eta(u)\sin\alpha$. It follows from a theorem of A. Cauchy (1832) that

$$X(t)=\int_{-\pi}^{\pi}\Big[\max_{0\leqslant u\leqslant t}\zeta_\alpha(u)\Big]\,d\alpha. \tag{*}$$

The process $\{\zeta_\alpha(u),\ 0\leqslant u<\infty\}$ is a standard Brownian motion process for each α. If we form the expectation of (*), then we get

$$\mathbf{E}\{X(t)\}=2\pi\mathbf{E}\Big\{\max_{0\leqslant u\leqslant t}\zeta_\alpha(u)\Big\}.$$

By the reflection principle we obtain that

$$P\Big\{\max_{0\leqslant u\leqslant t}\zeta_\alpha(u)\leqslant x\sqrt{t}\Big\}=2\Phi(x)-1 \text{ for } x\geqslant 0$$

where $\Phi(x)$ is the normal distribution function. Thus $\mathbf{E}\{\max_{0\leqslant u\leqslant t}\zeta_\alpha(u)\}=\sqrt{2t/\pi}$ and, consequently,

$$\mathbf{E}\{X(t)\}=\sqrt{8\pi t}\ .$$

References

1. A. Cauchy, Note sur divers théorèmes relatifs à la rectification des courbes, et à la quadrature des surfaces, Comptes Rendus Acad. Sci. Paris, 13 (1841) 1060–1065.

2. ______, Mémoire sur la rectification des courbes et la quadrature des surfaces courbes, Mémoires de l'Académie des Sciences, Paris, 22 (1850) 3–15. [Oeuvres Complètes d'Augustin Cauchy, Ser. I, Vol. 2, Gauthier-Villars, Paris, 1908, pp. 167–177.]

The proposer supplies the following more recent references:

3. L. Nachbin, The Haar Integral, Van Nostrand, 1976.

4. L. Santaló, "Integral Geometry," in Studies in Global Geometry and Analysis, S. S. Chern, ed., MAA Studies in Mathematics, Vol. 4, pp. 147–193.

REVIEWS

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with the assistance of the mathematics departments of St. Olaf, Carleton, and Macalester Colleges
COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

Overcoming Math Anxiety. Sheila Tobias. Norton, New York, 1978. 278pp. $10.95. (Telegraphic Review, May 1979.)

Mind Over Math. Stanley Kogelman, Joseph Warren. Dial Press, New York, 1978. xii + 239 pp. $8.95. (Telegraphic Review, June-July 1979.)

Today the phrases "math anxiety," "math avoidance," and "mathophobia" are commonplaces of the educational scene, although they were virtually unknown a few years ago. Many of the most enterprising of our educators are devoting themselves, almost exclusively, to attempts to remedy a situation in which many of our citizens seek to avoid any contact with mathematical activity and thus handicap themselves very seriously both in their day-to-day lives and in their job opportunities. The problem of the best treatment to remedy the behavior of math avoidance is a complicated one, since the origins of the behavior are themselves deep-seated and complex. The major thrust of the remedial work being undertaken has been the mathematical rehabilitation of women, but nobody suggests that the problem is confined to females. It is true, however, that there are especially adverse features of our social and cultural environment which militate against a woman's appetite for and success in mathematics.

The two books under review are written with the intention of trying to understand the problem and alleviate the situation it produces. In each case, the authors have chosen as the title of their book the name of the organization that they founded in order to contribute to the solution of the difficult problem of math avoidance. Both books have attracted substantial attention and have been extensively reviewed in the lay and educational press. We believe, however, that they also merit the attention of mathematicians and teachers of mathematics at the post-secondary level. The problem they discuss has become increasingly obvious in our pre-calculus and remedial courses, and it is one we will meet more frequently in the future as we come to be more concerned professionally with continuing education in mathematics.

The purpose of the book by Sheila Tobias is described in her own words:

> I have examined the myths surrounding mathematics, tried out some intervention techniques in an experimental clinic at my university, conferred with others more familiar than I am with how and why people learn, and looked at the implications of math avoidance on American education and values.
>
> This book contains the results of my inquiry. . . . The book is mainly a discussion of how intimidation, myth, misunderstanding, and missed opportunities have affected a large proportion of the population. My principal purpose in writing this book is to convince women and men that their fear of mathematics is the result and not the cause of their negative experiences with mathematics, and to encourage them to give themselves one more chance.
>
> Much of what I have to say stems from introspection for I am a victim as well as a reporter of math anxiety.

Thus Tobias's book is to be seen as a personal statement. It records her growing awareness of the nature of her own problem and that of many other adults and her attempts to deal with that problem for herself and for others. The first part of the book is devoted to her thoughts about

the nature and origins of the problem in terms of the way in which mathematics is taught and the way in which society is structured. Certain portions are documented and several useful references are cited. The second part of the book deals with her thoughts about some of the most common mathematical problems encountered at the pre-college level. She is particularly concerned here with the solving of word problems and with fractions, negative numbers, and the calculus.

Many mathematicians have complained that the book is full of mathematical errors. With this view we must concur. However, we do not deduce from this, as some critics and reviewers have, that the significance of the book is entirely vitiated by these errors. Indeed, we believe that the courageous way in which Tobias has exposed her own fumblings and gropings toward an understanding of difficult mathematical concepts is to be admired and saluted. Moreover her book, viewed as a personal statement—and this is clearly what the book is—should be of substantial value to adults and, particularly, to prospective and in-service elementary and secondary teachers of mathematics. Of course, anybody who reads the book under the impression that the author is offering a definitive treatment of the mathematical topics discussed is in some danger of being rendered even more anxious! This danger is perhaps increased by the fact that, so far, no review we have seen has drawn much attention to the inadequacies of the mathematics. Thus we feel that we should do so in our review, it being clearly understood that our purpose in doing so is to supplement the value of the text (and perhaps lead to certain improvements in subsequent editions!).

Thus we will mainly concentrate on the mathematical parts of Tobias's text. In any case this is what we should do, since we are less expert in the matters discussed in the earlier sections of the book. We can, however, record our conviction that the earlier sections are well thought out and well argued and will indeed encourage the adult reader who has experienced these symptoms of aversion to mathematics. In particular, we believe that the book reveals the important truth that avoidance of mathematics is a natural response to a type of instruction that most people have had in the name of mathematics education. There is nothing pathological in avoiding that which is unpleasant, dreary, and pointless.

Without attempting to be comprehensive, let us now list some of the more serious mathematical shortcomings. The first occurs on page 91 in the discussion of the ball-player's average. The problem as stated does not admit a solution, since inadequate information is given; one cannot infer the ball-player's average by knowing only that it was .233 going into the game and that he got three hits in four times at bat. Moreover the average of .252 is puzzling and we can find no reasonable way of arriving at it. It would be reasonable to suppose that the ball player had actually got 28 hits out of 120 prior to the game, so that his experience on the day in question would bring his average to .250. A thoroughly confusing example!

The discussion of the cannon-ball problem, viewed as a description of personal experience, was very instructive; but there are remarks left in the text which would mislead. In particular, on page 144 it is stated that "we can dismiss the set of four not weighed"; of course we should not dismiss them, since we know that they are all true. In the following paragraph we are told that the first weighing tells us that the odd ball "either lies in the set we have not weighed or in one of the two sets we have weighed." We do not need a weighing to tell us this! The weighing tells us *whether* the odd ball lies "in the set. . . ."

The discussion of this problem in the text reveals that the author does not fully understand the importance of using appropriate mathematical notation. This is also revealed elsewhere in the book. Notation poses a very subtle question and it is not surprising that Tobias has difficulty with it—this is not at all to her discredit. However, it is plain that the question of what makes a notation (like x^{-2}) useful and natural lies at the very heart of the development of mathematics itself. In our choice of notation and terminology we convey our understanding (or misunderstanding) of mathematics.

Closely related to these questions is the issue of the *meaning* of mathematical terms. Tobias discusses meaning a great deal but fails to distinguish between the *mathematical meaning* of a mathematical term and the set of real-life interpretations of that term. This confusion shows up very clearly (page 161) in the discussion of the meaning of fractions. For example, in the treatment on page 163 of the division of $3/4$ by $2/3$ there is no mention of how the problem arose, of its real-life context, so that it is pointless to discuss which is the most self-explanatory notation. Further, we find, on page 168, that $16/x$ is a fraction, although on page 179 we are told that x is not a number! Both these statements are far too categorical and, of course, their juxtaposition has led to contradiction. What we would have liked to see the author discuss is the important pedagogical question of what makes the study of fractions hard in the standard curriculum. It is our view that the answer to this question is that they are so often used inappropriately.

The same problem of meaning bedevils Tobias's discussion of the minus sign. On page 173 we are told that one can approach the idea of -4 via subtraction but that this does not tell us what it means. If we approach it as $0-4$ then this certainly tells us the *mathematical* meaning; but, of course, -4 has many meanings, in the sense that negative numbers can model a wide variety of different real-life situations. On page 176 we find that "the minus sign has gained another meaning." It is important here to understand, as we suspect Tobias does not understand, that we are dealing with a new real-life situation which can be modeled with the use of the minus sign. This is not the sense in which we usually understand, in day-to-day life, that a word has gained another meaning. If we say "James has contracted," then we may mean that James has entered into an agreement, or we may mean that James has become smaller. If we only knew the first meaning originally and were then informed of the second meaning, we would say that the word "contract" had gained another meaning for us and, by doing so, had rendered the phrase ambiguous to us. When in mathematics we enlarge the significance of a symbol we do *not* introduce ambiguity. This crucial point never becomes clear in Tobias's text.

There is much else we would like to say about the treatment of negative numbers and of their arithmetic. The problem of the interpretation of the product of two negative numbers can certainly be handled very well using simple ideas of graphs of linear functions. In the discussion of these issues Tobias has not fulfilled her intention of demystifying the subject for the reader.

This must also be said about the treatment of the calculus. Tobias appears to have fallen into the hands of a mathematician, in 1976, who was possessed of a somewhat whimsical sense of humor, since that mathematician apparently convinced her that there was a strong connection between the basic ideas of the differential calculus and the problem of medieval theology of how many angels fit on the head of a pin. It would be idle to detail all the consequences of this somewhat adventitious connection. However, it is only fair to state that Tobias admits that she has not mastered the calculus. We repeat that her discussion provides an extremely instructive personal reminiscence and should form the basis of a salutary exercise in the real understanding of the calculus. Properly used, these frank disclosures of the nature of the thinking of one very intelligent lay person in the presence of the subtleties of the calculus are valuable. Perhaps, however, one should protest that the statement on page 129 that "mathematicians conceptualize this a little differently" is a trifle disingenuous.

We would not wish to end this review of Tobias's book on a carping note, but we must draw attention to the fact that the text bears strong evidence that it was put together in a great hurry. There are glaring discrepancies between the text and examples on page 165 and page 181. The important mathematical discussion in the footnote to page 139 is wrecked by two serious errors. There is a strange discrepancy between Figures 8 and 9 on pages 226 and 227. Certainly the second graph cannot be displaying the speed of the particle whose path is shown in the first graph. Perhaps it is intended to display the vertical velocity, in which case it should cross the time axis at $7\frac{1}{2}$ and not $2\frac{1}{2}$.

However, as we have said, it is not our intention to carp. The book by Sheila Tobias, although flawed, is valuable. Its value must, of course, depend upon the use to which it is put; and we believe that, properly used, it can help a great deal in the struggle of individuals to overcome their distaste for and fear of mathematics. It can also, as we have indicated, be of great value to actual and prospective teachers. If supplemented by an acceptable treatment of the mathematical topics discussed, its value would be furthur increased. But it is not at all our view that Tobias's own attempts to understand those topics should be suppressed.

We come now to the second book under review. This book is really very different in nature from *Overcoming Math Anxiety*. *Mind Over Math* is also, to be sure, not a mathematics book, although it does contain a mathematical supplement. However, there the resemblance largely ends. *Mind Over Math* is an account of a workshop program designed by Stanley Kogelman and Joseph Warren to help various people overcome their feelings of anxiety toward mathematics. The book describes the activities of the workshops. These workshops begin with discussions and the sharing, at length, of the participants' negative feelings toward mathematics. In subsequent meetings the students participate in exercises and receive advice to help them cope with and eliminate, if possible, the stress they feel when they are faced with mathematical activities. In the later parts of the book some specific hints are given about how to read mathematics and how to attack mathematical problems. The last chapter is a testimonial to the effectiveness of the workshops. A supplemental part contains some mathematics. We will have more to say about that later.

When we started reviewing *Mind Over Math* we had an almost irresistible urge to check to see if our Blue Cross and Blue Shield cards were in our wallets. Perhaps it was the use of words like "clinical" in the advertising material and the designation of "Dr." before each of the authors' names on the front and back covers of the book. Perhaps, too, it was the case histories in Chapter 1 that contained statements like: "I can't stand to add my check book. I avoid it. It's crazy. I get nauseous."

There are personal statements from students about their feelings of anxiety and (p. 12) such comments from the authors as: "We might begin to wonder what it is about math that can evoke such emotions"; and "It is not surprising that someone would want to avoid the painful feelings that math evokes." Now, even though these remarks are followed by an explanation that to do mathematics in the presence of anxiety is all but impossible, they might be taken to imply that the subject of mathematics is inherently unpleasant and that the purpose of the book is to help the reader to *endure* if he or she must face mathematics. There is no mention (and, possibly it may appear to the anxious reader, no hope) of ever really *enjoying* mathematics.

In Chapter 1, following some personal descriptions of how anxious people feel about mathematics, the authors say, "Clearly these people are not functioning normally when it comes to math." What they do not tell us is what it would mean "to function normally when it comes to math." In particular, there is no indication that there might exist some normal person who thoroughly enjoys doing mathematics. And, in fact, they state that "all mathematicians we have known recall having had feelings similar to ours that interfered with their ability to do math." It seems possible then that the anxious reader may feel, at the outset, that he or she fits into a remarkably large group of abnormal human beings and that what comes next is a guided tour toward some undefined behavioral normality, through exposure to anecdotal accounts of the authors' friendly nonthreatening workshops.

There was a ray of light at the end of Chapter 1 in the following statement: "Although math does require a more focused, convergent approach than other subjects, imagination, emotions, and intuition play a critical role in all mathematics." We were hoping that remark would be followed by, "There is an emotional satisfaction in doing problems in original, imaginative ways." But that statement doesn't occur until Chapter 10! What actually followed the first remark was, "Many of the feelings people have about the cold, logical, rigid qualities of math are related to firmly entrenched myths. The next chapter is devoted to examining these myths so

that math can be approached with more comfort and realistic expectations." We feel we should offer our students more that just freedom from pain. It may be that the authors really intend this, too, and just felt it was too risky to mention "The Joy of Mathematics"—but we wish they had said so. We also feel that the reader is being encouraged to replace admittedly dangerous myths by their opposites; we would prefer a more rational, if less exciting, approach to the development of new attitudes.

Despite our qualms we believe the book contains a great deal of useful information for teachers of mathematics at all levels. The personal accounts, read by a sensitive person, would surely make him or her a better teacher. One could deduce, by reading these case histories, a lot of implied advice about how to handle many routine situations that can easily be psychologically damaging to students if not handled tactfully. Teachers may be able to sense why they should avoid statements like: "That's easy," "Let me show you a 'better' way to do the problem," and "You should have learned that in ninth grade." Kogelman and Warren have particularly good suggestions for handling some standard situations we all encounter in our teaching. For example:

> To further encourage the use of intuition, we recommend that math teachers not ask students to explain how they got an answer when the answer is right. In a math class, we may ask if anyone knows the answer to a problem. When someone offers the correct answer we then ask if anyone else can explain it. The person who gave the answer may offer an explanation if he or she wishes, but it is not necessary to do so. When you know the answer but don't know how to justify it, being asked for an explanation robs you of the pleasure of getting the answer right. These answers exemplify mathematical intuition at work.

This approach would certainly maximize student participation while minimizing the stress.

Pólya fans may note that scattered throughout the book are suggestions for understanding the problem; restating the problem; doing the problem another way to gain more insight; using your intuition; constructing a similar, but simpler, problem; and finding a related problem. These Pólya-like pieces of advice made us check to see if any of Pólya's books might be on their reference list. They are not, for a very simple reason: there is no reference list!

Mathematical readers will be particularly interested in Part 6 of the book, entitled "Supplemental Math." The purpose of this part is to give the reader a "feeling for fractions, decimals, percentages, metric measurements, algebra, and the calculus." However, in this part the authors reveal their own approach to the teaching of mathematics, and that approach seems to us to be well designed to induce the very symptoms the authors intend to cure. Where Sheila Tobias is tentative and speculative, Kogelman and Warren are authoritarian. For example, we read on page 193, "When working with fractions, the translation of the word 'of' is 'times.' " And on page 198 decimals are defined as "fractions written in another way." There are also mysterious conventions, introduced as if they were natural. Thus on page 194 the question, "How many quarters are in one half?" is, according to the authors, "written symbolically as

$$\frac{\frac{1}{2}}{\frac{1}{4}} = 2."$$

Moreover, in the monstrous expression on the left, $\frac{1}{2}$ is referred to as the numerator and $\frac{1}{4}$ as the denominator, so that we have a fraction with a fraction in the numerator and a fraction in the denominator. Numerators have numerators, sometimes! Yet the reader is intended to be enlightened.

It is plain that Kogelman and Warren belong to the school of thought, all too popular in this country, which identifies elementary mathematics with computational skill, and computational skill with mechanical efficiency. Problems arrive on the suffering student's desk entirely divorced from all context. "Suppose you wanted to multiply 2.31 times 1.7," suggest the authors.

Nobody should *want* to multiply these two numbers, but some might *need* to if the calculation modeled an interesting real-life situation. It could only do so if a meaning had already been attached to the multiplication of decimals and that meaning would be determined by utility. Of this crucial aspect of arithmetic there is little trace. Even where, as in computing percentages, this connection is established, it is lost by the authors themselves in their solutions to the problems they state. Thus on page 207, to "compute $\frac{1}{4}$ of \$80.00," a horribly mechanical and artificial procedure is presented and the answer turns out to be the *number* 20. This last example also illustrates the subservience of the authors to the tyranny of algorithms. Apparently one cannot compute $\frac{1}{4}$ of \$80.00 by dividing \$80.00 by 4! Instead we must replace "of" by "$\times$", then replace "80" (which has mysteriously lost the "\$" sign at that point) by "$\frac{80}{1}$," and, finally, multiply the two fractions $\frac{1}{4}$ and $\frac{80}{1}$, getting $\frac{80}{4}$. How do we simplify $\frac{80}{4}$? Remember that we are apparently incapable of dividing 80 by 4!

Readers who believe it when they read, "When dealing with fractions, however, multiplication results in quantities getting smaller," will experience more author-induced anxiety when they encounter, a few pages later, several examples that involve the multiplication of two fractions each of which is larger than one.

The authors have, as a matter of policy, avoided almost entirely anything recognizable as mathematical pedagogy in the conduct of their workshops. We conclude that they are well advised.

Still, the book can serve several useful purposes. It will, no doubt, be of some comfort and possibly quite helpful to many people who have math anxiety. It will give teachers more understanding of the real psychological problems many of their students have toward mathematics. Teachers who offer courses for liberal arts students and remedial courses will find some effective classroom techniques scattered throughout the book.

Both of these books demonstrate, perhaps implicitly and in very different ways, the existence of a real problem in the teaching of pre-college mathematics and the necessity for the involvement of professional mathematicians in the effort to deal with it.

Added in Proof. We were recently sent a copy of the third impression of the book by Sheila Tobias, and it is pertinent to report that many errors have been corrected in this printing. Of those to which we draw attention in our review, the matter of the ball-player's average (page 91) has been completely cleared up, and the difficulties connected with the status of $16/x$ on page 168 and with Figures 8 and 9 on pages 226 and 227 have been partially eliminated (by removing the statement on page 179 that x is not a number, and by removing the numerical labels from the graphs). Other errors, of a proofreading nature, on pages 139 and 181 have also been corrected.

The removal of these mistakes, of course, improves the text. But many serious conceptual defects remain uncorrected, and it is with these that we were principally concerned. We had hoped, for example, to see some change on page 216, where there is a grave confusion between the "number of angels," meaning the size of the angel-set, and "the number of an angel," meaning her position in the angel-sequence; and where the figure carries the phrase "total area all angels" and the footnote explains that our angels are one-dimensional. The continued presence of such conceptual errors serves to underline the importance of the last sentence of our review.

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TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, S*, L***, *Mathematical Circus: More Games, Puzzles, Paradoxes, and Other Mathematical Entertainments from Scientific American*. Martin Gardner. Alfred A. Knopf, 1979, xiii + 272 pp, $9.95. [ISBN: 0-394-50207-8] The ninth collection of Gardner's *Scientific American* columns, mostly from the years 1968-1971, supplemented by commentary from readers and afterthoughts by Gardner. 20 chapters, including triangle theorems, random walks, induction games, sphere packing and a dazzling assortment of illusions, magic and conundrums. LAS

GENERAL, P, *Matematicheskaya Entsiklopediya*. Sovetskaya Entsiklopediya (Moscow), V. 1. 1977, 1152 columns. 7.10 rubles. The first volume (A-G) of a projected five-volume encyclopedic dictionary based on a survey of terms used in books roughly at the level of graduate work in mathematics. The entries are mostly phrases (like algebraic K-theory, Gaussian distribution, Weierstrass-Stone theorem, context-free language, Green's theorem). All in Russian, but should be useful to non-Russians who are reading Russian mathematics in the original language. RPB

GENERAL, S*, L*, *The Dancing Wu Li Masters: An Overview of the New Physics*. Gary Zukav. William Morrow, 1979, 352 pp, $5.95 (P). [ISBN: 0-688-08402-8] A fascinating, personal interpretation of quantum physics ("Wu Li" in Chinese), devoid not only of equations and mathematics ("stripped of mathematics, physics is pure enchantment"), but also of the traditional tone of authoritarian scientific objectivity. Layman Zukav shares with the reader his own sense of wonder and excitement as he discovers both the mysteries and insights of modern physics. LAS

GENERAL, S*, *The Tokyo Puzzles*. Kobon Fujimura. Trans: Fumie Adachi. Scribner's, 1978, 184 pp, $3.95 (P). [ISBN: 0-684-15537-0] Paperback edition of the 1978 hard cover English translation of a unique Japanese collection of problems. (TR, August/September 1978.) LAS

GENERAL, S, P, L*, *Papers in Mathematics*. Ed: Paul R. Meyer. NY Acad of Sci, 1979, 101 pp, $20. [ISBN: 0-89766-026-9] A diverse collection of excellent expository papers--addressed to mathematicians who are not specialists in the field of the papers--originally presented at the Mathematics Section meeting of the New York Academy of Science. Topics include vibrations in fluid dynamics, algorithms with fewest operations, undecidable topological statements and much more. LAS

EDUCATION, L, *Manipulative Activities and Games in the Mathematics Classroom*. Ed: Lee E. Vochko. NEA, 1979, 92 pp, $6 (P). [ISBN: 0-8106-1706-4] Part of N.E.A.'s Curriculum Series, this volume includes a collection of activities and games arranged by grade level (primary through secondary) together with statements of rationale for their use. Materials required are easy and inexpensive to make. JNC

EDUCATION, P, *The Radio Mathematics Project: Nicaragua 1976-1977*. Ed: Patrick Suppes, Barbara Searle, Jamesine Friend. IMSSS (Inst. for Math. Studies in the Social Sci., Stanford U.), 1978, xi + 356 pp. Detailed description and evaluation of the use of instructional radio to teach elementary school mathematics in rural Nicaragua. LAS

HISTORY, S(15-16), P, *Archimedes: Ingenieur, Naturwissenschaftler und Mathematiker*. Ivo Schneider. Wissenschaftliche Buchgesellschaft, 1979, viii + 209 pp, (P). [ISBN: 3-534-06844-0] An account, based on all the available sources, of Archimedes' life and his work in science, engineering and mathematics. JD-B

HISTORY, T(13-16: 1, 2), S*, P, L**, *The Historical Development of the Calculus*. C.H. Edwards, Jr. Springer-Verlag, 1979, xii + 351 pp, $28. [ISBN: 0-387-90436-0] A careful, comprehensive history of calculus, from Babylonian geometry to Robinson's infinitesimals. Each section includes sufficient detail (in modern notation) to enable the reader to penetrate the thought processes of the giants of the past; exercises designed to replicate key calculations (with the tools of the time) enable the book to be used as a text. Should be on the bookshelf of every calculus teacher. LAS

FOUNDATIONS, P, *Chance, Cause, Reason: An Inquiry into the Nature of Scientific Evidence*. Arthur W. Burks. U of Chicago Pr, 1977, xvi + 694 pp, $12.50 (P). [ISBN: 0-226-08088-9] Paperback version of original hardcover edition (TR, April 1978). LAS

COMBINATORICS, T(18), S, P, *Orthogonal Designs: Quadratic Forms and Hadamard Matrices*. Anthony V. Geramita, Jennifer Seberry. Lect. Notes in Pure and Appl. Math., V. 45. Dekker, 1979, x + 460 pp, $35 (P). [ISBN: 0-8247-6774-8] Orthogonal designs, which are generalizations of Hadamard matrices, are square matrices with indeterminate entries whose rows are orthogonal under Euclidean inner product. Their relationship to rational quadratic forms is the novel feature here. There are numerous consequences implied for the theory as well as for applications to combinatorics, statistics, coding theory and other areas. SS

COMBINATORICS, S(17-18), P, *Graph Theory and Related Topics*. Ed: J.A. Bondy, U.S.R. Murty. Acad Pr, 1979, xxxii + 371 pp. $40. [ISBN: 0-12-114350-3] Collection of thirty papers from the proceedings

of a conference held in honor of W.T. Tutte, on his 60th birthday. The papers and collection of unsolved problems that appear relate to the varied contributions of Tutte to graph theory. SS

COMBINATORICS, S(18), P. *Difference Sets in Elementary Abelian Groups.* Paul Camion. Séminaire de Mathé. Supérieures. Pr U Montreal, 1979, 96 pp, $10 (P). [ISBN: 2-7606-0450-0] Using group characters and Fourier transforms, difference sets in an abelian group are constructed and studied. Beyond the intrinsic properties of general difference sets, there are two chapters on their interesting relationship to linear codes. SS

COMBINATORICS, T(14: 1), S, P, L*. *Introduction to Graph Theory, Second Edition.* Robin J. Wilson. Longman, 1979, viii + 163 pp, $8 (P). [ISBN: 0-582-44397-0] This book is a solid introduction to the basics of graph theory. The *Second Edition* has expanded the sections on applications, reorganized the material on connectivity and edge-colorings and the exercises have been substantially rewritten. Along with the many exercises, it would be nice if some hints and solutions were included. (*First Edition,* TR, June/July 1973; ER, June/July 1974.) CEC

NUMBER THEORY, S(18), P. *Lectures on Forms of Higher Degree.* J.-I. Igusa. Tata Inst, 1978, 175 pp, $9.90 (P). The author's main concern is polynomial forms of degree $m > 2$ and the extension or appropriate generalization to these forms of some of the classical results on quadratic forms, particularly the work of C.L. Siegel. Topics include Mellin transformations, asymptotic expansions, arithmetic theory, and Poisson formulas. Bibliography. JS

ALGEBRA, P. *Lecture Notes in Mathematics-734: Ring Theory, Waterloo 1978.* Ed: David Handelman, John Lawrence. Springer-Verlag, 1979, xi + 352 pp, $17.80 (P). [ISBN: 0-387-09529-2] 16 papers from a summer research institute at Waterloo, emphasizing lower K-theory, artinian and noetherian rings, actions and representations of groups on rings. LAS

ALGEBRA, T(17-18: 1), S, P. *Gruppentheoretische Methoden und ihre Anwendung.* Eduard Stiefel, Albert Fässler. Teubner Stuttgart, 1979, 256 pp, (P). [ISBN: 3-519-02348-2] Theory of group representation, with applications, largely to physics. Intended for mathematicians, physicists and engineers. Exercises with solutions. JD-B

ALGEBRA, T(16-18: 1, 2), S, P, L. *An Introduction to Homological Algebra.* Joseph J. Rotman. Pure and Appl. Math., V. 85. Acad Pr, 1979, xi + 376 pp, $26.50. [ISBN: 0-12-599250-5] A reworking and updating of notes from a 1968 course, this book gives a very readable introduction to homological algebra. The beginning section on line integrals, explicit references to the sources of the subject, and material on localization and on Quillen's solution of Serre's problem present a broad view of which reflects both the history and present interests of the subject. This very readable account includes a goodly number of exercises and a good index. JAS

REAL ANALYSIS, T(16-18: 1, 2), S. *Integración: Teoría y Técnicas.* M. de Guzman, B. Rubio. Editorial Alhambra, 1979, x + 171 pp, (P). [ISBN: 84-205-0631-1] Classical introduction (in Spanish) to Lebesgue measure and integration theory, including L^p spaces, convolutions, and differentiation theory. LAS

COMPLEX ANALYSIS, P. *Lecture Notes in Mathematics-743: Romanian-Finnish Seminar on Complex Analysis.* Ed: Cabiria Andreian Cazacu, *et al.* Springer-Verlag, 1979, xvi + 713 pp, $33 (P). [ISBN: 0-387-09550-0] Proceedings of the meeting held in Bucharest, Romania, June 27-July 2, 1976. JAS

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Control and Information Sciences-16: Stochastic Control Theory and Stochastic Differential Systems.* Ed: M. Kohlmann, W. Vogel. Springer-Verlag, 1979, xii + 615 pp, $33 (P). [ISBN: 0-387-09480-6] Twelve survey lectures and a large number of research reports which comprise the proceedings of the workshop which took place in January 1979 at Bad Honnef. The aim is to provide an introductory compendium for non-experts with a glimpse at the research frontiers. JAS

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Mathematics-684: Distributions and Nonlinear Partial Differential Equations.* Huzihiro Araki, Elemer E. Rosinger. Springer-Verlag, 1978, xi + 146 pp, $9.80 (P). [ISBN: 0-387-08951-9] Solutions are obtained in algebras which contain distributions. RBK

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Mathematics-730: Functional Differential Equations and Approximations of Fixed Points.* Ed: Heinz-Otto Peitgen, Hans-Otto Walther. Springer-Verlag, 1979, xv + 503 pp, $23 (P). [ISBN: 0-387-09518-7] About three-fourths of the lectures from the conference held at the University of Bonn, July 17-22, 1978. JAS

DIFFERENTIAL EQUATIONS, P. *Seminar on Singularities of Solutions of Linear Partial Differential Equations.* Ed: Lars Hörmander. Annals of Math. Stud., No. 91. Princeton U Pr, 1979, ix + 283 pp, $18.50; $7.50 (P). Modified and augmented notes from the seminar held at the Institute for Advanced Study in 1977-78. JAS

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Mathematics-703: Equadiff IV.* Ed: Jiří Fábera. Springer-Verlag, 1979, xix + 441 pp, $21.30 (P). [ISBN: 0-387-09116-5] The proceedings consist of the texts of about two-thirds of the plenary addresses and the invited lectures from the conference held in Prague, August 22-26, 1977. JAS

NUMERICAL ANALYSIS, P. *Numerische Integration.* G. Hämmerlin. Int. Ser. Num. Math., V. 45. Birkhäuser, 1979, 320 pp, $29.80 (P). [ISBN: 3-7643-1014-6] Twenty-four papers developed from lectures given at the numerical integration conference held at Oberwolfach, October 1-7, 1978. JAS

NUMERICAL ANALYSIS, P. *Sparse Matrix Proceedings 1978.* Ed: Iain S. Duff, G.W. Stewart. SIAM, 1979, xvi + 334 pp, $21.50. [ISBN: 0-89871-160-6] Proceedings of the symposium held in Knoxville, Tennessee on November 2-3, 1978. The emphasis was on applications-oriented research. JAS

NUMERICAL ANALYSIS, P, *Proceedings of the 1979 Army Numerical Analysis and Computers Conference.* US Army Research Office (P.O. Box 12211, Research Triangle Park, NC), 1979, xiii + 482 pp, (P). The proceedings of the 1979 conference held at White Sands Missile Range, New Mexico, February 14-16, 1979. JAS

FUNCTIONAL ANALYSIS, T(17-18), P, *Natural Function Algebras.* Charles E. Rickart. Springer-Verlag, 1979, xiii + 240 pp, $14.80 (P). [ISBN: 0-387-90449-2] A "natural" function algebra is a general Banach algebra of functions (defined on the spectrum of the algebra) that exhibits analytic properties (e.g., the local maximum modulus principle) without necessarily possessing any relevant analytic structure. The main purpose of this monograph is to "provide a systematic account of some of [these] algebraically induced analytic phenomena." LAS

NUMERICAL ANALYSIS, P, *Numerische Methoden bei graphentheoretischen und kombinatorischen Problemen, Band 2.* L. Collatz, G. Meinardus, W. Wetterling. Int. Ser. Num. Math., V. 46. Birkhäuser, 1979, 255 pp, $32.50 (P). [ISBN: 3-7643-1078-2] Sixteen papers from the meeting held at Oberwolfach, May 7-12, 1978. Volume one (not submitted for review) apparently contains only organizational material. JAS

FUNCTIONAL ANALYSIS, P, *Lecture Notes in Mathematics-720: The Structure of Nuclear Fréchet Spaces.* Ed Dubinsky. Springer-Verlag, 1979, 187 pp, $9.80 (P). [ISBN: 0-387-09504-7] A study of nuclear Fréchet spaces which centers on problems relating to approximation properties (e.g., bases, finite dimensional approximations of the identity) and questions of when a nuclear Fréchet spaces is a subspace, quotient space or complemented subspace of a given nuclear Fréchet space. TRS

FUNCTIONAL ANALYSIS, T(18: 1), S, P, *Einführung in die Theorie der eindimensionalen singulären Integraloperatoren.* I. Gohberg, N. Krupnik. Birkhäuser, 1979, 379 pp, $59.50. [ISBN: 3-7643-1020-0] A monograph on the theory of linear, one-dimensional singular integral equations. JD-B

FUNCTIONAL ANALYSIS, P, *Lecture Notes in Mathematics-735: Propriêtês Spectrales des Algèbres de Banach.* Bernard Aupetit. Springer-Verlag, 1979, xii + 192 pp, $13.80 (P). [ISBN: 0-387-09531-4] A comprehensive account of the spectrum in Banach algebras. Topics: general properties, continuity and uniform continuity of spectrum, characterizations of commutative, finite dimensional, and symmetric Banach algebras. Emphasizes the role of subharmonic functions. TRS

FUNCTIONAL ANALYSIS, P, *Multiparameter Spectral Theory in Hilbert Space.* B.D. Sleeman. Research Notes in Math., No. 22. Fearon-Pitman, 1978, 118 pp, $12.25 (P). [ISBN: 0-273-08414-3] An up-to-date account of developments in multiparameter spectral theory. Intended for a wide audience, the author recounts the subject's origins in partial differential equations and presents introductory material on tensor products of Hilbert spaces. TRS

FUNCTIONAL ANALYSIS, P, *Operator Colligations in Hilbert Spaces.* Mikhail S. Livshits, Artem A. Yantsevich. Trans: AMS. V.H. Winston, 1979, xiii + 212 pp, $19.95. [ISBN: 0-470-26541-8] An operator colligation extends the notion of a dissipative operator. The first part of this book develops the notion of an operator colligation and the latter part treats nonstationary stochastic processes using results on dissipative operators. JAS

FUNCTIONAL ANALYSIS, T(18: 1), S, P, *Classical Banach Spaces II, Function Spaces.* Joram Lindenstrauss, Lior Tzafriri. Ergebnisse der Math., B. 97. Springer-Verlag, 1979, x + 243 pp, $39. [ISBN: 0-387-08888-1] Prerequisites for this second volume on Banach spaces "include beside standard material from function analysis and measure theory only a superficial knowledge" of Volume I. The major topics are Banach Lattices (Part 1) and Rearrangement Invariant Function Spaces (Part 2). Applications are developed. Bibliography, index. JS

OPTIMIZATION, T(17-18: 1, 2), P, L, *Mathematics of Finite-Dimensional Control Systems: Theory and Design.* David L. Russell. Lect. Notes in Pure and Appl. Math., V. 43. Dekker, 1979, viii + 553 pp, $45 (P). [ISBN: 0-8247-6869-8] Intended for a general audience, these lecture notes present the fundamentals of modern control theory, emphasizing those aspects of linear-quadratic methodology which have actually been used, along with explanations of why they meet a need and how they are justified mathematically. Presumes differential equations and linear algebra. Exercises. TRS

OPTIMIZATION, P, *Lecture Notes in Control and Information Sciences-14: International Symposium on Systems Optimization and Analysis.* Ed: A. Bensoussan, J.L. Lions. Springer-Verlag, 1979, viii + 332 pp, $16 (P). [ISBN: 0-387-09447-4] Proceedings of the symposium held at Rocquencourt, France, December 11-13, 1978 with the sponsorship of IRIA LABORIA. Sessions were devoted to economic models; identification, estimation, filtering; adaptive control; numerical methods in optimization; and distributed systems. JAS

OPTIMIZATION, S(17-18), P, *Suchprobleme.* Rudolf Ahlswede, Ingo Wegener. Teubner Stuttgart, 1979, 328 pp, (P). [ISBN: 3-519-02058-0] A study, taking account of much recent work, of a number of different types of search problems. JD-B

OPTIMIZATION, T(16-17: 1), S, *Spieltheorie.* Burkhard Rauhut, Norbert Schmitz, Ernst-Wilhelm Zachow. Teubner Stuttgart, 1979, 400 pp, (P). [ISBN: 3-519-02351-2] An introductory but fairly sophisticated text on game theory. Exercises. JD-B

OPTIMIZATION, T*, P, L*, *Applied Linear Programming for the Socioeconomic and Environmental Sciences.* Michael R. Greenberg. Acad Pr, 1978, xv + 327 pp, $19. [ISBN: 0-12-299650-X] The authors' three goals are to stimulate new applications of linear programming, to prevent the mathematics of linear programming from acting as a barrier to a user, and to get students to a computer quickly. After discussing the essential theory (including post optimality), the text presents a wide variety of applications, probably wider than in any other text. TAV

OPTIMIZATION, T?, P, *Optimal Control of Discrete Systems.* V.G. Boltyanskii. Trans: Ron Hardin. Halsted Pr, 1978, x + 392 pp, $57.50. [ISBN: 0-7065-1578-1] The material in the first two-thirds of this text should be covered in a good advanced calculus sequence. The final third, primarily on convexity conditions to optimization, hardly justifies the cost (note price). TAV

ANALYSIS, P, *Seminar on Micro-Local Analysis.* Victor W. Guillemin, Masaki Kashiwara, Takahiro Kawai. Annals of Math. Stud., No. 93. Princeton U Pr, 1979, vii + 137 pp, $17; $6.50 (P). Micro-local analysis, the study of generalized functions as local objects on the cotangent bundle, makes possible the study in "meticulous detail" of singularities of solutions of partial differential equations. This brief volume contains six lectures, beginning with a general introduction, and concluding with applications to symmetric spaces, holonomic systems and Feynman integrals. LAS

ANALYSIS, T(17: 1), L, *Fourier Series, A Modern Introduction, Volume 1, Second Edition.* R.E. Edwards. Grad. Texts in Math., V. 64. Springer-Verlag, 1979, xii + 224 pp, $16.80. [ISBN: 0-387-90412-3] The author presumes knowledge of metric and normed linear spaces, some point set topology and a "fair degree of familiarity with Lebesgue integration." He exploits the topological group structure of R to motivate expansion in trigonometric series. Topics include convolutions, Cesàro summability, series in L^2, and pointwise convergence. Appendices on metric spaces, linear spaces, weak completeness and Runge's theorem. Bibliography. TAV

ANALYSIS, P, *Lecture Notes in Mathematics-729: Ergodic Theory.* Ed: M. Denker, K. Jacobs. Springer-Verlag, 1979, xii + 209 pp, $12.50 (P). [ISBN: 0-387-09517-9] The majority of the talks (those not published elsewhere) from the conference held at Oberwolfach, June 11-17, 1978. JAS

ANALYSIS, P, *Lecture Notes in Mathematics-725: Algèbres d'Opérateurs.* Ed: P. de la Harpe. Springer-Verlag, 1979, 309 pp, $17.60 (P). [ISBN: 0-387-09512-8] Lectures from the seminar held at Les Plans-sur-Bex, Switzerland, March 13-18, 1978. Includes bibliographic material on the lectures not included in this volume. JAS

ANALYSIS, P, *Lecture Notes in Mathematics-737: Volterra Equations.* Ed: Stig-Olaf Londen, Olaf J. Staffans. Springer-Verlag, 1979, viii + 314 pp, $16 (P). [ISBN: 0-387-09534-9] Proceedings of the Helsinki Symposium on Integral Equations held at Helsinki University of Technology, Otaniemi, Finland on August 11-14, 1978. The emphasis is on the qualitative theory, with results in both the finite and infinite dimensional cases. JAS

ANALYSIS, S(17-18), P, *Ein Konstruktiver weg zur Masstheorie und Funktionalanalysis.* Peter Zahn. Wissenschaftliche Buchgesellschaft, 1978, 350 pp, DM 42 (P). [ISBN: 3-534-07767-9] A careful and complete constructivist approach to standard topics in analysis. The presentation is formal and without exercises, but probably readable by a graduate student (rather than only by an active researcher). JAS

ANALYSIS, P, *Lecture Notes in Mathematics-713: Séminaire de Théorie du Potentiel Paris, No. 4.* M. Brelot, G. Choquet, J. Deny. Springer-Verlag, 1979, vi + 281 pp, $15.70 (P). [ISBN: 0-387-09252-8] The proceedings of the seminar during the academic year 1977-1978. JAS

ALGEBRAIC GEOMETRY, P, *Lectures on Curves on Rational and Unirational Surfaces.* M. Miyanishi. Tata Inst, 1978, 307 pp, $9.90 (P). Notes prepared from lectures given at the Tata Institute from January to March 1978. JAS

ALGEBRAIC GEOMETRY, P, *Lecture Notes in Mathematics-732: Algebraic Geometry.* Ed: K. Lønsted. Springer-Verlag, 1979, vi + 658 pp, $31.90 (P). [ISBN: 0-387-09527-6] The papers from the Copenhagen summer meeting in algebraic geometry held August 7-12, 1978 at the H.C. Øroted Institute of the University of Copenhagen. Includes a few additional related papers. JAS

DIFFERENTIAL GEOMETRY, S(18), P, *Differential Geometry of Instantons.* J.H. Rawnsley. Dublin, 1978, v + 60 pp, (P). An instanton is a finite-action solution of the Yang-Mills field equations, or more concretely, an event which is isolated in space and time. This monograph is a high level but mostly self-contained exposition of the calculation by Atiyah, Hitchin, and Singer of the dimension of the space of self-dual instantons by means of the Index Theorem applied to a suitable complex. This presentation is an expanded version of the one given in the spring of 1977 at the Dublin Institute for Advanced Studies. JAS

GEOMETRY, T(16-17: 1, 2), S, L*, *Geometry and Convexity, A Study in Mathematical Methods.* Paul J. Kelly, Max L. Weiss. Wiley, 1979, x + 261 pp, $21.95. [ISBN: 0-471-04637-X] A thorough and tasteful integration of the methods of metric topology, linear algebra, classical analysis and geometry for purposes of studying convexity, convex bodies and surfaces, and applications. Pedagogical concerns are apparent in choice of content, in organization, care of exposition and in the numerous exercises. An unusual and noteworthy textbook. SS

ALGEBRAIC TOPOLOGY, P, *Lecture Notes in Mathematics-741: Algebraic Topology, Waterloo 1978.* Ed: Peter Hoffman, Victor Snaith. Springer-Verlag, 1979, xi + 654 pp, $30.80 (P). [ISBN: 0-387-09545-4] Proceedings of a conference held at the University of Waterloo in June 1978. The papers are divided into four classifications: L-theory and algebraic K-theory, group actions, homotopy and cohomology, and loop and H-spaces. JAS

TOPOLOGY, T(13-14), *Beginner's Topology.* F. Burton Jones. J. of Undergraduate Mathematics (Dept. of Math., Guilford Coll., Greensboro, NC 27410), 40 pp, $5 (P). This is one chapter of a set of notes used to teach topology with the class participating in development of the material. AWR

TOPOLOGY, P, *Lecture Notes in Mathematics-722: Topology of Low-Dimensional Manifolds.* Ed: Roger Fenn. Springer-Verlag, 1979, 154 pp, $9.80 (P). [ISBN: 0-387-09506-3] The proceedings of the second topology seminar of the University of Sussex held July 8-11, 1977. JAS

TOPOLOGY, P. *H-spaces with Torsion.* John R. Harper. Memoirs No. 223. AMS, 1979, viii + 72 pp, $6 (P). [ISBN: 0-8218-2223-3] The author constructs for each odd prime p a simply-connected finite H-space with p-torsion in its homology. JAS

TOPOLOGY, P. *The Classifying Spaces for Surgery and Cobordism of Manifolds.* Ib Madsen, R. James Milgram. Princeton U Pr, 1979, xii + 279 pp, $20; $8.50 (P). An expository survey of the significant recent (last twenty years) successful attack on the homology theory of classifying spaces for various sphere bundles. "Much of the material covered in this book appears in print with detailed proofs for essentially the first time." JAS

TOPOLOGY, P. *Lecture Notes in Mathematics-719: Categorical Topology.* Ed: H. Herrlich, G. Preuß. Springer-Verlag, 1979, xii + 420 pp, $19.50 (P). [ISBN: 0-387-09503-9] Proceedings of the conference held at Berlin, August 27 to September 2, 1978. JAS

TOPOLOGY, P. *Seifert Fibered Spaces in 3-Manifolds.* William H. Jaco, Peter B. Shalen. Memoirs No. 220. AMS, 1979, viii + 192 pp, $8 (P). [ISBN: 0-8218-2220-9] The main theorem classifies all maps of a Seifert fibered space into a sufficiently large 3-manifold with certain restrictions. Applications of this result yield a refined version of Waldhausen's torus-annulus theorem and a new canonical decomposition for the above class of 3-manifolds. TLS

PROBABILITY, T(16-17: 1), L. *The Algebra of Random Variables.* M.D. Springer. Wiley, 1979, xix + 470 pp, $26.95. [ISBN: 0-471-01406-0] An important exposition of the use of Fourier, Laplace, and particularly Mellin transform methods to obtain probability density functions of rational functions of random variables with given distributions. Examples are t, F and chi-square distributions. Many important distributions are examples of H-functions, a class of distributions closed under rational operations. Emphasizes explicit calculation (numerical and analytical) of inverse transform. Is suitable for advanced undergraduates or college faculty seminar. 420 references. RBK

PROBABILITY, P*, L*, *Probabilities and Potential.* Claude Dellacherie, Paul-André Meyer. Math. Stud., V. 29. North-Holland, 1978, 189 pp, $29 (P). [ISBN: 0-7204-0701-X] Following the lead of N. Bourbaki whose *Structures fondamentales de l'analyse* contains no analysis, the authors admit "this volume contains little enough probability and no potential theory whatsoever." Instead this very readable book contains those preliminaries on measure and stochastic processes that will make a discussion of the title topics possible in a later volume. Contains a very useful index of notation and an extensive bibliography. TAV

PROBABILITY, P. *Analytic Function Methods in Probability Theory.* Ed: B. Gyires. North-Holland, 1980, 379 pp, $63.50. [ISBN: 0-444-85333-2] Most of the papers from the colloquium held at the Kossuth L. University of Debrecen, Hungary from August 29 to Septmeber 2, 1977 together with a few later papers. JAS

PROBABILITY, P. *Probabilistic Analysis and Related Topics, Volume 2.* Ed: A.T. Bharucha-Reid. Acad Pr, 1979, ix + 207 pp, $26.50. [ISBN: 0-12-095602-0] The second of a number of volumes which will appear at "irregular intervals." This volume contains: "Optimal control of stochastic systems" by N.U. Ahmed, "Gleason measures" by Ryszard Jajte, "An introduction to nonstandard analysis" and "Hyperfinite probability theory" by Peter A. Loeb, and "Limit theorems: Stochastic matrices, ergodic markov chains, and measures on semigroups" by Arunava Mukherjea. JAS

PROBABILITY, P. *Infinitely Divisible Point Processes.* Klaus Matthes, Johannes Kerstan, Joseph Mecke. Wiley, 1978, xii + 532 pp, $42.50. [ISBN: 0-471-99460-X] A translation and reordering of 1974 German edition. The reworking of the topics was intended to make the text more readable. It remains, however, a difficult, sometimes obtuse, treatment of a difficult, sometimes obtuse, subject. TAV

PROBABILITY, P. *Geometric Problems in the Theory of Infinite-Dimensional Probability Distributions.* V.N. Sudakov. Proc. of Steklov Inst. of Math., No. 141. AMS, 1979, v + 178 pp, $40 (P). [ISBN: 0-8218-3041-4] An exposition of previously published results dealing with extension of a generalized random process to a measure in a Banach space and with the existence of an independent complement to a pair of given measurable decompositions of a measure space. JAS

PROBABILITY, P. *Lecture Notes in Mathematics-706: Probability Measures on Groups.* Ed: H. Heyer. Springer-Verlag, 1979, xiii + 348 pp, $17.80. [ISBN: 0-387-09124-6] Proceedings of the fifth conference on this subject held at Oberwolfach on January 29 to February 4, 1978. JAS

PROBABILITY, P. *Lecture Notes in Mathematics-709: Probability in Banach Spaces II.* Ed: A. Beck. Springer-Verlag, 1979, 205 pp, $12.50 (P). [ISBN: 0-387-09242-0] Proceedings of the second conference held June 18-24, 1978 in Oberwolfach. JAS

COMPUTER PROGRAMMING, P, L. *Compact Numerical Methods for Computers: Linear Algebra and Function Minimisation.* J.C. Nash. Halsted Pr, 1979, ix + 227 pp, $27.50. [ISBN: 0-470-26559-0] Algorithms selected according to shortness of program, general utility and proven reliability for a "small" computer, which is any computing device allowing 6000 characters of memory for program and data. An excellent guide for engineers, scientists and programmers of all disciplines. Many examples. TRS

COMPUTER PROGRAMMING, S(13-18). *PASCAL with Style, Programming Proverbs.* Henry F. Ledgard, John F. Hueras, Paul A. Nagin. Hayden, 1979, 210 pp, $6.95 (P). [ISBN: 0-8104-5124-7] This is a guide to good programming style in any language, not a primer in Pascal. All examples and exhortations are focused around Pascal code, but programmers of all languages can benefit. Main thrust of the text is the set of rules of style, the proverbs. Each one is stated, explained, and applied. Special attention given to the top-down approach to problem solving and to a set of standards for program writing. Many examples and chapter exercises. Appendices. Bibliography. Index. RJA

COMPUTER PROGRAMMING, S(13-18), L. *The BYTE Book of Computer Music*. Ed: Christopher P. Morgan. BYTE Pub, 1979, v + 144 pp, $10 (P). [ISBN: 0-931718-11-2] Covers the gamut from getting your computer to play a tune, through interfacing your home organ or player piano, to machine language programs for the fast Fourier transform on two popular microcomputers. JAS

COMPUTER PROGRAMMING, S(13-18). *Introduction to T-BUG: The TRS-80 Machine Language Monitor*. Don Inman, Kurt Inman. Dilithium Pr, 1979, 120 pp, $6.95 (P). [ISBN: 0-918398-33-9] An introductory guide via examples to the use of machine language instructions for the Z80 microprocessor chip at the heart of the Radio Shack TRS-80 computer. T-BUG is Radio Shack's machine language monitor which simplifies the use of machine language with their computer. Prerequisites: enough background that "hexagesimal" makes sense, interest in learning machine language for a Z80; and ownership of a TRS-80 computer with T-BUG programming (this is not a theory book). JAS

COMPUTER PROGRAMMING, S(13). *Sixty Challenging Problems with BASIC Solutions, Second Edition*. Donald D. Spencer. Hayden, 1979, 128 pp, $6.95 (P). [ISBN: 0-8104-5180-0] The 60 cartoon-embellished problems will be challenging for beginners, dealing with GCD's, Pythagorean triples, rolling dice, Fibonacci numbers, various kinds of primes, factorials, and magic squares. Background material included as needed. TRS

COMPUTER PROGRAMMING, *BASEX: A Simple Language and Compiler for 8080 Systems*. Paul Warme. BYTE Pub, 1979, iii + 97 pp, $8 (P). [ISBN: 0-931718-05-8]

COMPUTER PROGRAMMING, S(13-14), L. *Basic with Style: Programming Proverbs*. Paul A. Nagin, Henry F. Ledgard. Hayden, 1978, 134 pp, $5.95 (P). [ISBN: 0-8104-5115-8] This book offers short rules and guidelines for writing more accurate programs. Strong emphasis on documentation and top-down programming in Basic. Not an introduction to the language, but a guide to better programming. Lots of examples of good and bad programs. Includes exercises. Very readable. CEC

COMPUTER PROGRAMMING, T(14-15: 1), S. *Assembly Language Fundamentals 360/370 OS/VS DOS/VS*. Rina Yarmish, Joshua Yarmish. A-W, 1979, xvi + 768 pp, $17.95. [ISBN: 0-201-08798-7] A slowly-paced introduction. Computer arithmetic, instruction set, editing, branching and looping and some operating concepts. Intended especially for business and data processing students. RWN

COMPUTER PROGRAMMING, T(13: 1), S. *The ABC's of Fortran Programming*. Michael J. Merchant. Wadsworth, 1979, ix + 357 pp, $10.95 (P). [ISBN: 0-534-00634-5] An introduction to Fortran which presents the basics of the language and elements of good programming. Presentation is plain and simple. List directed I/O used in early chapters. Neither previous computer experience nor mathematics beyond high school algebra required. Exercises. TRS

COMPUTER SCIENCE, S(13-16), L**. *The Computer Age: A Twenty-Year View*. Ed: Michael L. Dertouzos, Joel Moses. MIT Pr, 1979, xvi+ 491 pp, $25. [ISBN: 0-262-04055-7] Are we nearer the beginning or the end of the computer revolution? 21 renowned "veterans" of the revolution attempt to predict, for the rest of this century, its continued impact on science, industry, education and society. John McCarthy and Joseph Weizenbaum provide critical commentary. LAS

COMPUTER SCIENCE, S(15-18), P, L. *Foundations of Computer Science III*. Ed: J.W. de Bakker, J. van Leeuwen. Math Centrum, 1979. *Part 1: Automata, Data Structures, Complexity*. Math. Centre Tracts, No. 108. iii + 112 pp, Dfl. 14 (P) [ISBN: 90-6196-176-9]; *Part 2: Languages, Logic, Semantics*. Math. Centre Tracts, No. 109. i + 164 pp, Dfl. 20 (P). [ISBN: 90-6196-177-7] Contains most lectures given at the Third Advanced Course on the Foundations of Computer Science held from August 21 to September 1, 1978, in Amsterdam. RJA

COMPUTER SCIENCE, T(16-17: 1), S, L*. *Assemblers, Compilers, and Program Translation*. Peter Calingaert. Computer Sci Pr, 1979, xiv + 270 pp, $17.95. [ISBN: 0-914894-23-4] An introduction to translation-related topics. Presumes some familiarity with high- and low-level languages and usage of common data structures. Includes one- and two-pass assembly, storage management, macro-processing, interpretation and generation, syntatic and lexical analysis, compiler implementation, loaders and linkers. Exercises. RWN

COMPUTER SCIENCE, P. *Computer Aided Design of Digital Systems, A Bibliography, Volume IV, 1977-79*. W.M. vanCleemput. Computer Sci Pr, 1980, viii + 196 pp, $30 (P). [ISBN: 0-914894-61-7] Lists technical publications in hardware and software design grouped in 10 subject categories. Includes author and subject and keywords indexes. (Previous volumes cover 1960-1974 (TR, November 1977), 1975-1976 (TR, November 1977), 1976-1977; updates are issued annually.) LAS

COMPUTER SCIENCE, T*(14-18: 1, 2), S, P, L. *The Design of Well-Structured and Correct Programs*. Suad Alagić, Michael A. Arbib. Springer-Verlag, 1978, x + 292 pp, $14.80. [ISBN: 0-387-90299-6] Weaves together techniques of top-down design and verification of program correctness. Provides many examples of correctly designed algorithms. Carefully develops analytic tools for design, and presents proof rules for various program structures. An exciting work, in an attractive format, rich in bibliographic remarks. Teaches the reader Pascal in the environment of algorithm design. Chapter exercises. References. Glossary. Appendices. Index of algorithms. Author and subject indexes. RJA

SYSTEMS THEORY, P. *Lecture Notes in Control and Information Sciences-19: Global and Large Scale System Models*. Ed: B. Lazarević. Springer-Verlag, 1979, viii + 232 pp, $12.50 (P). [ISBN: 0-387-09637-X] Twelve lectures on corporate, national and world models, from an August 1978 seminar in Dubrovnik, Yugoslavia. LAS

SYSTEMS THEORY, P. *Lecture Notes in Control and Information Sciences-13: Polynomial Response Maps*. Eduardo D. Sontag. Springer-Verlag, 1979, viii + 168 pp, $9.80 (P). [ISBN: 0-387-09393-1] Elaboration of the author's doctoral thesis: just as realization in the field of finite-dimensional linear dynamical systems depends on linear algebra, so realization theory for non linear systems may depend on algebraic geometry and commutative algebra. AWR

SYSTEMS THEORY, P. *Lecture Notes in Control and Information Sciences-11: Modeling, Estimation, and Their Applications for Distributed Parameter Systems.* Y. Sawaragi, T. Soeda, S. Omatu. Springer-Verlag, 1978, vi + 269 pp, $14.30 (P). [ISBN: 0-387-09142-4] Estimation and control problems for stochastic partial differential equations. Background material in probability and stochastic processes in Hilbert space is provided. RBK

APPLICATIONS, P. *Information Linkage Between Applied Mathematics and Industry.* Ed: Peter C.C. Wang, *et al.* Acad Pr, 1979, xiv + 661 pp, $35. [ISBN: 0-12-734250-8] Proceedings of a February 1978 symposium held at the Naval Postgraduate School, bringing together engineers and applied mathematicians to share problems of mutual interest. Contributions cover differential equations, finite mathematics, optimization and stochastic modeling. LAS

APPLICATIONS, T(16-17: 1, 2), S, L. *Integral Transforms in Science and Engineering.* Kurt Bernardo Wolf. Math. Concepts and Methods in Sci. and Eng., V. 11. Plenum Pr, 1979, xiii + 489 pp, $32.50. [ISBN: 0-306-39251-8] An encyclopedic volume which gives detailed examinations of both theoretical and some applied aspects of the Fourier, LaPlace, Bessel, Hankel, Bargmann and other transform techniques. Includes lots of high quality illustrations and lots of exercises. CEC

APPLICATIONS, T(16-18), L. *Mathematical Statistical Mechanics.* Colin J. Thompson. Princeton U Pr, 1972, x + 278 pp, $5.95 (P). [ISBN: 0-691-08220-0] Paperback edition of 1972 hardcover original (TR, January 1973); an introductory text on the mathematical aspects of phase transitions and the Ising model, with applications to biology. LAS

APPLICATIONS (ART), L. *Visual Art, Mathematics and Computers: Selections from the Journal Leonardo.* Ed: Frank J. Malina. Pergamon Pr, 1979, xiv + 325 pp, $50. [ISBN: 0-08-021854-7] 53 papers from *Leonardo* on applications of mathematics and computers to pictorial and three-dimensional art. An excellent anthology and serious contemporary computer art. LAS

APPLICATIONS (ARTIFICIAL INTELLIGENCE), S, L. *The Computer and the Brain, Tenth Printing.* John von Neumann. Yale U Pr, 1979, xiv + 82 pp, $3.45 (P); $9. First published in 1958, this "unfinished and fragmentary" manuscript of von Neumann's undelivered Silliman Lectures continues to pique the scientific imagination. LAS

APPLICATIONS (ARTIFICIAL INTELLIGENCE), P. *Artificial Intelligence and Pattern Recognition in Computer Aided Design.* Ed: Jean-Claude Latombe. North-Holland, 1978, x + 510 pp, $60. [ISBN: 0-444-85229-8] Proceedings of the third Working Conference of an IFIP Working Group that was organized with the International Association for Pattern Recognition and held in Grenoble, March 17-19, 1978. RJA

APPLICATIONS (BIOLOGY), T*(14-15: 1), S*, P*, L*. *Mathematics for Biomedical Applications.* Stanton A. Glantz. U of Calif Pr, 1979, xii + 423 pp, $29.50. [ISBN: 0-520-03599-2] Loaded with marvelous real examples in physiology, pharmacology, and biomedical instrumentation. Peppered with authoritative, insightful comments. If the introduction, brief review of calculus and various other topics, which are better done elsewhere, were omitted, the text would be a gem and, perhaps, reasonably priced. Still, as is, the book is highly recommended for purchase by anyone interested in biomedical applications of differential equations. JK

APPLICATIONS (BIOLOGY), S(18), P. *Nonlinear Analysis and Mechanics: Heriot-Watt Symposium, Volume III.* Ed: R.J. Knops. Fearon-Pitman, 1979, 173 pp, $16.50 (P). [ISBN: 0-273-08432-1] A collection of three papers in applied mathematics, M.E. Gurtin and R.C. McCamy on "Population dynamics with age dependence," A. Pazy on "Semi-groups on non-linear contractions and their asymptotic behavior," and M. Slemrod on "Damped conservation laws in continuum mechanics." Bibliography. JS

APPLICATIONS (DATA PROCESSING), P. *Prelucrarea si Transmiterea Numerică A Datelor si Conducerea Processelor cu Ajutorul Calculatoarelor.* Gh. Cartianu. Editura Academiei (Romania), 269 pp, Lei 17 (P). Twenty-four papers mostly on the technology of data transmision and digital control. JAS

APPLICATIONS (DECISION THEORY), T(14-17: 1, 2), S, P, L*. *Group Choice.* Boris G. Mirkin. Trans: Yelena Oliker. V.H. Winston (Distr: Wiley), 1979, xxx + 252 pp, $22.50. [ISBN: 0-470-26702-X] This volume presents group choice as the general problem of aggregating individual data, whether from experts, voters, consumers, or decision makers. Written by a member of the Soviet Academy of Sciences, it offers a unique Soviet view of subjectively oriented aspects of decision theory: preference analysis, theory of voting (including Arrow's paradox), game theory and economic equilibria. Uses only elementary finite mathematics. LAS

APPLICATIONS (ECONOMICS), S(14-16), P, L. *Cost-Benefit Analysis, A Handbook.* Peter G. Sassone, William A. Schaffer. Acad Pr, 1978, xiv + 182 pp, $14.50. [ISBN: 0-12-619350-9] Addressed to practicing cost-benefit analysts, government officials, concerned citizens, and students who wish to understand cost-benefit analysis. Minimal mathematical background (some calculus) is required. AWR

APPLICATIONS (ECONOMICS), P. *Control Theory in Mathematical Economics.* Ed: Pan-Tai Liu, Jon G. Sutinen. Lect. Notes in Pure and Appl. Math., V. 47. Dekker, 1979, viii + 241 pp, $26.50 (P). [ISBN: 0-8247-6852-3] This is a collection of papers presented at the Third Kingston Conference on Differential Games and Control Theory. Mathematical control of dynamic systems is the unifying theme of papers on managing fishing resources, selling assets with a deadline, modelling taxi service at an airport, and more. AWR

APPLICATIONS (LANGUAGE TRANSLATION), P. *Translating and the Computer.* Ed: Barbara M. Snell. North-Holland, 1979, xi + 189 pp, $29.25. [ISBN: 0-444-85302-2] Proceedings of a one day seminar between language translators and computer scientists held in London, November 14, 1978. Purpose was to familiarize translators with advances in machine translation and to inform the designers of computer systems of the translator's practical needs. Lists of speakers and participants. RJA

APPLICATIONS (MICROPROCESSORS), T*(13-18; 1, 2), S. *Microprocessors, Theory and Applications.* Gene A. Streitmatter, Vito Fiore. Reston Pub, 1979, xx + 456 pp, $15.95. [ISBN: 0-8359-4371-2] Basic text on microprocessor design, organization, and programming. Includes binary arithmetic, microprocessor electronics, instruction sets, bus control, I/O, memory, software, and descriptions of complete 8080 and 6800 systems. Beautiful format with numerous illustrations, tables, diagrams, and photos. Appendices. Glossary. Bibliography. Index. RJA

APPLICATIONS (MODELLING), P. *Lecture Notes in Control and Information Sciences-18: Modelling and Optimization of Complex System.* Ed: G.I. Marchuk. Springer-Verlag, 1979, vi + 293 pp, $14.30 (P). [ISBN: 0-387-09612-4] These proceedings contain most of the papers presented at an IFIP Working Conference held at Novosibirsk, USSR, July 3-9, 1978. Modelling problems in immunology and evolution, as well as more abstract modelling problems are considered. JMS

APPLICATIONS (PATTERN RECOGNITION), T(16-18: 1, 2), S, P, L. *Picture Languages, Formal Models for Picture Recognition.* Azriel Rosenfeld. Comp. Sci. and Appl. Math. Acad Pr, 1979, xiii + 225 pp, $21. [ISBN: 0-12-597340-3] Text treats questions regarding the transformation of digital pictures, i.e., arrays of numbers representing gray levels, from the standpoint of automata theory. Includes digital geometry, string, cellular, array, and cellular array automata, special automaton models, and array grammars. The approach is mathematical but not to the point of obscuring understandability. Chapter references. Index. RJA

APPLICATIONS (PHYSICS), P. *Lecture Notes in Physics-106: Feynman Path Integrals.* Ed: S. Albeverio, *et al.* Springer-Verlag, 1979, xi + 451 pp, $21.30 (P). [ISBN: 0-387-09532-2] Proceedings of the international colloquium held in Marseille in May 1978. The contents are divided into seven sections indicating the scope of this conference and the areas of research importance; these are mostly in quantum mechanics, geometrophysics, and gauge fields. JAS

APPLICATIONS (PHYSICS), P. *Selected Papers (1937-1976) of Julian Schwinger.* Ed: M. Flato, C. Fronsdal, K.A. Milton. Math. Physics & Appl. Math., V. 4. Reidel (Distr: Kluwer Boston), 1979, xxvii + 413 pp, Dlf. 56. [ISBN: 90-277-0974-2] A sixtieth birthday retrospective selection (by Schwinger) of some fifty papers. "Great weight was assigned to papers that seem to be less widely known or appreciated than they deserve;" so many important papers have been omitted. This volume includes a complete bibliography and brief comments on the importance of the papers here reprinted. JAS

APPLICATIONS (PHYSICS), P. *Fundamental Problems in Statistical Mechanics IV.* Ed: E.G.D. Cohen, W. Fiszdon. PWN, 1978, 541 pp. All the lectures and some seminars from the summer school held from September 14-24, 1977 at Jadwisin, Poland. JAS

APPLICATIONS (PHYSICS), S(16-17). *Mathematical Principles of Mechanics and Electromagnetics.* C.-C. Wang. Plenum Pr, 1979. *Part A: Analytical and Continuum Mechanics.* Math. Concepts and Methods in Sci. and Eng., V. 16. xix + 198 pp, $29.50. [ISBN: 0-306-40211-4]; *Part B: Electromagnetism and Gravitation.* Math. Concepts and Methods in Sci. and Eng., V. 17. xix + 187 pp, $29.50. [ISBN: 0-306-40212-2] An advanced undergraduate treatment of the topics listed in the titles. Physical intuition and vector and tensor algebra are assumed; the presentation is moderately classical; and the result is a bridge to modern geometric approaches to the material. The mathematics is quite rigorous (modulo the inherent assumptions of physics) but the emphasis is on physical intuition rather than on mathematical elegance. No exercises, but a reasonably good index. The bibliography emphasizes reliable classics rather than pointing to the future. JAS

APPLICATIONS (PHYSICS), T(17-18), S, P. *A Course in Elasticity.* B.M. Fraeijs de Veubeke. Trans: F.A. Ficken. Appl. Math. Sci., V. 29. Springer-Verlag, 1979, xi + 330 pp, $16.80 (P). [ISBN: 0-387-90428-X] A pleasant translation based on lecture notes of the late Professor de Veubeke. Highly technical, some illustrations, but no exercises. MU

APPLICATIONS (PHSYICS), P. *Methods of Modern Mathematical Physics III: Scattering Theory.* Michael Reed, Barry Simon. Acad Pr, 1979, xv + 463 pp, $42. [ISBN: 0-12-585003-4] Scattering phenomena (tomography, sonar, particle physics, blue sky) are modelled by spectral analysis of wave operators. Completes a four-part work (*V. I,* TR, October 1972; ER, December 1973; *V. II,* December 1975; *V. IV,* TR, February 1979) that records applied functional analysis from its origins in the thirties to contemporary research. ("Some of the starred problems summarize the contents of research papers!") LAS

APPLICATIONS (PHYSICS), P. *Fracture Mechanics.* Ed: Robert Burridge. SIAM-AMS Proc., V. 12. AMS, 1979, vi + 169 pp, $13.60. [ISBN: 0-8218-1332-3] Ten of twelve invited papers from an AMS/SIAM symposium held in March 1978 in New York on the stability of cracks and the mathematical modelling of earthquakes. LAS

APPLICATIONS (PHYSICS), P. *Lecture Notes in Physics-98: Nonlinear Problems in Theoretical Physics.* Ed: A.F. Rañada. Springer-Verlag, 1979, x + 216 pp, $12.50 (P). [ISBN: 0-387-09246-3] Proceedings of the ninth G.I.F.T. International Seminar on Theoretical Physics, held at Jaca, Spain in June 1978. JAS

APPLICATIONS (SIMULATION), P, L. *Current Issues in Computer Simulation.* Ed: Nabil R. Adam, Ali Dogramaci. Acad Pr, 1979, xviii + 292 pp, $24. [ISBN: 0-12-044120-9] 18 independently authored chapters on computer simulation languages (GPSS, GASP, etc.), simulation applications, and theoretical models, both analytical and statistical. A comprehensive subject index helps make this a particularly valuable "state of the art" survey. LAS

Reviewers Whose Initials Appear Above

Richard J. Allen, St. Olaf; Ralph P. Boas, Northwestern; Judith N. Cederberg, St. Olaf; Clifton E. Corzatt, St. Olaf; John Dyer-Bennet, Carleton; Roger B. Kirchner, Carleton; Joseph Konhauser, Macalester; R.W. Nau, Carleton; A. Wayne Roberts, Macalester; Thomas R. Savage, St. Olaf; John Schue, Macalester; Seymour Schuster, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; Timothy L. Strotman, St. Olaf; Milton Ulmer, Carleton; T.A. Vessey, St. Olaf.

NEWS AND NOTICES

EDITED BY PAUL A. HAEDER, University of Nebraska at Omaha

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036

SUGGESTION BOX

Members of the MAA are encouraged to send in suggestions, questions, etc., about the operations of the Association. Communications will be referred to the appropriate officer of the Association for answering; from time to time, those of general interest may also be answered in one or both of the official journals. Communications should be addressed to: Suggestion Box, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036

NEW JOURNAL OF UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS

The Mathematical Association of America is cooperating with the Undergraduate Mathematics and its Applications Project (UMAP) in the publication of the *UMAP Journal*. Subject to NSF approval, the Journal will be published quarterly by Birkhäuser, Boston, Inc., with the first issue appearing in early 1980. Ross L. Finney, University of Illinois, will be the editor. The Journal will publish modules that provide instruction in applications of undergraduate mathematics, or which make a contribution to instruction in undergraduate mathematics in some other way. These materials, which have been both peer- and student-reviewed, are intended to facilitate the use of applications and modeling in the college classroom. They are multidisciplinary and written for a readership which includes students as well as faculty and other professionals.

The Journal will also contain a variety of additional articles, including: surveys of the applications of mathematics in specific fields; historical perspectives on the development of subject areas; descriptions of innovative educational programs and their implementation. It will contain a review section with annotated bibliographies, as well as a letters-to-the-editor department which hopefully will become a forum for all those concerned with mathematics education at the undergraduate level. The editorial board represents both UMAP and the MAA.

In subsequent years, MAA members will be able to subscribe to the *UMAP Journal* with their regular MAA dues billing. The press of time makes this impossible for the first volume. Further information may be obtained from: EDC/UMAP, 55 Chapel Street, Newton, Massachusetts 02160.

PROFESSOR ALDER ELECTED

Professor Henry Alder, former President of the Mathematical Association of America, has been elected vice-chairman of the Council of Scientific Society Presidents for 1979. After serving as vice-chairman for 1979 he will become chairman in 1980. CSSP is an organization founded in 1973 consisting of the presidents of most major professional organizations in the natural sciences. It maintains close ties on science policy questions with the executive and legislative branches of the federal government. Alder is a CSSP member as past president of the MAA.

SCIENCE AND TECHNOLOGY IN THE U.S.S.R.

The general characteristics and organization of science and technology in the Soviet Union are presented in a report prepared by Ursula M. Kruse-Vaucienne, former program officer of the National Science Foundation's Division of International Programs, and by political scientist John M. Logsdon of George Washington University.

The 90-page study, *Science and technology in the Soviet Union: a profile*, illustrated with charts and tables, describes in nontechnical language the organization of science within the formal institutions of the Soviet government, including the prestigious U.S.S.R. Academy of Sciences, the industrial ministries, and the educational system. The report also describes the number and disciplinary composition of the scientific personnel working in research and development, as well as the general patterns of financing research and development, and the status of the flow of science information. Comparisons are made between the Soviet Union and the United States.

Among many observations, the report points out:

"Most of the basic research in the Soviet Union—about 80 to 85 percent—is conducted in the elaborate network of Academy laboratories, while Soviet higher educational institutes, unlike their U.S. counterparts, conduct very little research but focus on teaching.

Soviet scientists continue to be accorded the historically high prestige that they have always had within the Russian culture. However, engineering continues to be a relatively low-status profession."

The report is available in limited numbers from the Graduate Program in Science, Technology and Public Policy, The George Washington University, Washington, D.C. 20052.

-NSF Announcement

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

ASSOCIATE EXECUTIVE DIRECTOR

The Mathematical Association of America has an opening for an Associate Executive Director at its headquarters in Washington, D.C., beginning July 1, 1980 or as soon as possible thereafter. Applicants should have interest and ability in administration and should be willing to exercise independent initiative. A Ph.D. in mathematics or a mathematical science is desirable. Salary will be commensurate with the experience and training of the candidate.

Applicants should send a curriculum vitae, and should also arrange to have three letters of recommendation sent directly to: Professor Dorothy L. Bernstein, President
Mathematical Association of America
Division of Applied Mathematics
Brown University
Providence, Rhode Island 02912

The Mathematical Association of America is an equal opportunity employer.

OCTOBER MEETING OF THE OHIO SECTION

The Ohio Section of MAA held its Fall meeting at The College of Wooster, Wooster, Ohio, October 26-27, 1979. One hundred and thirty people registered for the meeting. Section Chairman D.O. Koehler presided: H.W. Vayo was the Program Chairman.

Invited addresses included: *Catastrophe Theory*, by Y-C Lu, Ohio State University; and *Mathematical Modeling and Existence Theorems*, by Dorothy L. Bernstein, Brown University.

The following contributed papers were also presented:

Applications of Complex Analysis to Operator Theory, J.J. Buoni, Youngstown State University and B.L. Wadhwa, Cleveland State University
Mathematics - 'In State Nascendi', K. Cummins, Kent State University
An Analysis of Variance for Exponential Random Variables, C. Davis, University of Toledo
Operators Defined by Multiplication by Analytic Functions, J.A. Deddens, University of Toledo
On the Monotonicity of a Class of Exponential Sequences, T.P. Dence, Bowling Green State University, Firelands Campus
Calculus and the Hinterlands, D.O. Koehler, Miami University
An Example of Complex Variables in Operator Theory, R. Lange, Youngstown State University
Computer Controlled Milling of Models of Surfaces, C.A. Long, Bowling Green State University
Nahuatl Mathematics, S.E. Payne, Miami University.
Life Insurance - A Computer Project for Students, L.J. Schneider, John Carroll University
Nth Order Tensor Operators, G.L. Szoke, University of Akron
Analytic Functions and Range Inclusion of the Operator $X \to (AX-XB)$, R.E. Weber, Pennsylvania State University, Sharon campus.

The meeting agenda also included meetings of The Executive Committee and of *ad hoc* committees - Committee on Cooperation Among Colleges and Universities, Committee on Curriculum, and Committee on Teacher Training and Certification. A special Business Meeting was scheduled to review proposed changes in the By-Laws of The Section.

Meeting highlights included discussion sessions and special presentations. *A Panel Discussion: Modeling Courses in the Curriculum* was led by D.L. Bernstein, Brown University; D. Hull, Ohio State University (moderator); and Z. Karian, Denison University. *A 'Swap' Session: Freshman Placement* was moderated by D.J. Horwath, John Carroll University. *Special Sessions: Operator Theory* were moderated by J.J. Buoni, Youngstown State University and B.L. Wadhwa, Cleveland State University. Also, a special 'Heroes of Mathematics' lecture was presented: *The Mystery of Ancient Britain's Mathematics*, by L. Peck, Miami University.

The officers and committee chairs for academic year 1979-80 include: *Executive Committee* - D.O. Koehler (Miami University), Section Chairman; D.L. Deever (Otterbein College), Section Chairman-Elect; M.D. Wetzel (Denison University), Section Past-Chairman; G. Mavrigian (Youngstown State University), Secretary-Treasurer; S.W. Hahn (Wittenberg University), Sectional Governor; and H.W. Vayo (University of Toledo), Program Committee Chairman. *Program Committee* - H.W. Vayo, Chairman; D.J. Horwath (John Carroll University); and A.G. Poorman (Ashland College). *Ad Hoc Committee Chairmen* - Committee on Co-operation Among Colleges and Universities: K.E. Eldridge (Ohio University) and J.D. Faires (Youngstown State University), co-chairmen. Committee on Curriculum: F.W. Carroll (Ohio State University). Committee on Teacher Training and Certification: H.W. Brockman (Capital University). Committee on Computing: Z. Karian (Denison University). Public Informations Officer and Newsletter Editor: R.A. Little (Baldwin-Wallace College). Representative to the Two-Year College Mathematics Journal: C.P. Yang (Miami University-Middletown). High School Math Competition Supervisor: L.J. Schneider (John Carroll University).

Professors C. Hampton and P. Brown were in charge of meeting arrangements for the host institution.

GUS MAVRIGIAN, *Secretary-Treasurer*

OCTOBER MEETING OF THE INDIANA SECTION

The fall meeting of the Indiana Section of the MAA was held at Wabash College in Crawfordsville on Saturday, October 27, 1979, with 65 people present. The chairman of the Section, Underwood Dudley of DePauw University, presided.

The following papers were presented:

Cancellation in Arithmetic and Topology, P. Hilton, Case Western Reserve University
Rearranging the Alternating Harmonic Series, C. Cowen, Purdue University
How to Paint a High-Dimensional Triangle, P. Halmos, Indiana University
Can Remedial Students Be Saved?, a panel discussion was presented by M. Gemignani, J. Kuczkowdki, and I. Boodt, Indiana University-Purdue University Indianapolis.

P.T. Mielke read a statement of gratitude toward Professor J. Crawford Pauley on the anniversary of his fiftieth year at Wabash College. It was warmly supported by the entire section.

R. R. PATTERSON, *Secretary-Treasurer*

EASTERN PENNSYLVANIA AND DELAWARE SECTION

The annual section meeting was held at Drexel University on November 17, 1979. These elected with terms:

Chair - Howard Anton (1980)
Vice Chair - Bing Wong (1980)
Secretary Treasurer - Willard Baxter (1982)
At Large Executive Committee - Peter Jessup (1982)
Pat Overdeer (1982)
Bruce Scranton (1981)

Invited talks with title were:

Problems and Results in Unimodal Sequences, Curtis Greene
Rings with Involution - an Overview, Willard Baxter
Mathematical precocity - identifying and developing the potential, W. C. George
and a panel discussion: *The machine in the garden - the relationship of computer sciences and the undergraduate mathematics major*, with panelists: John Kellett, John Koch, and Walter Brown.

WILLARD E. BAXTER, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Sixtieth Summer Meeting, University of Michigan, Ann Arbor, August 18–20, 1980.

Sixty-fourth Annual Meeting, San Francisco, California, January 9–11, 1981.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, West Virginia Wesleyan College, Buckhannon, April 25–26, 1980.

EASTERN PENNSYLVANIA AND DELAWARE, Cedar Crest College, Allentown, Pennsylvania, April 26, 1980.

FLORIDA, Jacksonville University, Jacksonville, March 7–8, 1980.

ILLINOIS, John A. Logan College, Carterville, April 25–26, 1980.

INDIANA, Valparaiso University, Valparaiso, April 26, 1980.

INTERMOUNTAIN, Utah State University, Logan, Utah, late April or early May 1980.

IOWA, Simpson College, Indianola, April 18–19, 1980.

KANSAS, Kansas State University, Manhattan, April 12, 1980.

KENTUCKY, Western Kentucky University, Bowling Green, April 11–12, 1980.

LOUISIANA–MISSISSIPPI, Louisiana Tech University, Ruston, February 15–16, 1980.

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, University of Richmond, Richmond, Virginia, April 12, 1980.

METROPOLITAN NEW YORK, Mercy College, Dobbs Ferry, May 4, 1980.

MICHIGAN, Hope College, Holland, May 2–3, 1980.

MISSOURI, Westminster College, Fulton, April 25–26, 1980.

NEBRASKA, Doane College, Crete, April 18–19, 1980.

NEW JERSEY, Hyatt House, Cherry Hill, March 15, 1980.

NORTH CENTRAL, Gustavus Adolphus College, St. Peter, Minnesota, April 25–26, 1980.

NORTHEASTERN, Saturday after Thanksgiving, and third week in June in odd-numbered years.

NORTHERN CALIFORNIA, Naval Postgraduate School, Monterey, February 23, 1980.

OHIO, Wittenberg University, Springfield, April 25–26, 1980.

OKLAHOMA–ARKANSAS, Westark Community College, Fort Smith, Arkansas, March 28–29, 1980.

PACIFIC NORTHWEST, Central Washington University, Ellensburg, Washington, June 20–21, 1980.

ROCKY MOUNTAIN, University of Colorado, Boulder, March 28–29, 1980.

SEAWAY, Herkimer Community College, Herkimer, New York, May 2–3, 1980.

SOUTHEASTERN, Appalachian State University, Boone, North Carolina, April 11–12, 1980.

SOUTHERN CALIFORNIA, California State University, Northridge, March 8, 1980.

SOUTHWESTERN, Northern Arizona University, Flagstaff, spring 1980.

TEXAS, East Texas State University, Commerce, April 11–12, 1980.

WISCONSIN, University of Wisconsin, Milwaukee, March 28–29, 1980.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Toronto, January 3–8, 1981.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES

AMERICAN MATHEMATICAL SOCIETY, University of Michigan, Ann Arbor, August 19–22, 1980.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Amherst, Massachusetts, June 23–26, 1980.

ASSOCIATION FOR COMPUTING MACHINERY, Kansas City, Missouri, February 12–14, 1980.

ASSOCIATION FOR SYMBOLIC LOGIC

ASSOCIATION FOR WOMEN IN MATHEMATICS

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHÉMATIQUES

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS, University of Michigan, Ann Arbor, August 18–21, 1980.

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Seattle, Washington, April 16–19, 1980.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Shoreham Hotel, Washington, D. C., May 5–7, 1980.

PI MU EPSILON

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS

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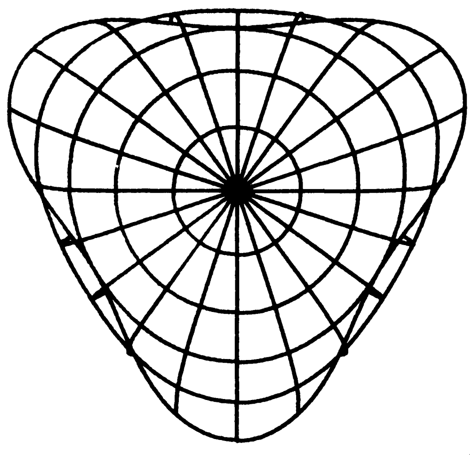

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OFFBEAT INTEGRAL GEOMETRY

LAWRENCE ZALCMAN

1. Introduction. The integral geometry of the title is what is sometimes called stereology or stereometry. This is the subject that deals with determining properties of a geometric object from information concerning its sections by lower-dimensional manifolds and integrals over those manifolds. Current interest and activity in this area have been considerable, primarily in the direction of applications. These range, truly, from heaven above to the earth beneath and the water under the earth: from radioastronomy, through geophysics **[19]**, to the search for oil and (other) buried treasure. There is even a connection with the design of cores for nuclear reactors.

What is perhaps the most spectacular of all applications concerns brain surgery or, more precisely, radiology (x-ray tomography). Here the object is to determine the size, extent, and location of tumors in the brain by means of a finite number of x-rays taken from a finite number of different angles. (It turns out—theoretically speaking—that such data are *never* sufficient to provide the required information; on the other hand, the methods work rather well in practice.) Both the practical and theoretical aspects of this enterprise are described in the interesting survey articles of Smith, Solmon, and Wagner **[27]** and Shepp and Kruskal **[24]**.

My own interest in the subject stems from a very different direction and lies entirely in the theoretical sphere. Accordingly, I shall have nothing more to say about applications in this article. And so, let us turn our attention to less practical concerns.

2. Stereology. Suppose G is an open region in the plane which is bounded by a Jordan curve. Is G determined uniquely by the lengths (linear Lebesgue measure) of its intersections with all lines? Questions of this sort seem first to have been considered (in a somewhat more general form) by J. Radon **[18]** in 1917. That *some* geometric information can be squeezed from this data is immediately evident: for instance, the convex hull of G is precisely the complement of the union of all lines whose intersection with G have *zero* length. It is less obvious how one can piece together knowledge of the lengths (but not the positions!) of the intersections to recapture the domain G. While the following theorem does not provide a recipe for such a reconstruction, it does answer our original question, and in an extraordinarily strong form.

THEOREM (H. Cramér–H. Wold [7]). *Let $f \in L^1(\mathbb{R}^2)$. Suppose that*

$$\int_L f\,ds = 0 \tag{1}$$

for almost every line in each direction. Then $f = 0$ almost everywhere.

The assumption that f is Lebesgue (area) integrable assures via Fubini's theorem that, for each fixed direction, f is integrable on almost every line having that direction. Thus the hypothesis (1) makes sense. On the other hand, the mere existence and vanishing of the (one-dimensional) integrals in (1) is very far from forcing f to vanish identically. This is shown by a famous example of Sierpiński **[25]**, who constructed a *nonmeasurable* set S in the plane whose intersection with each line consists of at most two points. Taking $f = \chi_S$, the characteristic function of S, we see at once that (1) is satisfied; but, of course, χ_S does not vanish almost everywhere.

To apply the Cramér-Wold Theorem to our original situation, let S_1 and S_2 be two measurable sets having finite area in the plane and suppose that, for every line L, the lengths of $S_1 \cap L$ and $S_2 \cap L$ coincide. Putting $f = \chi_{S_1} - \chi_{S_2}$, we see that (1) is satisfied so that $\chi_{S_1} = \chi_{S_2}$

Lawrence Zalcman received his Ph.D. under the direction of Kenneth Hoffman at MIT, has taught at Stanford and (since 1972) at the University of Maryland; he has held numerous visiting positions. He was awarded the Chauvenet Prize in 1976. Further details of his career and interests are given in this MONTHLY, 83 (1976) 84–85.—*Editors*

almost everywhere; i.e., S_1 and S_2 differ at most by a set of measure zero. In case S_1 and S_2 are actually Jordan domains, this implies that $S_1 = S_2$.

There are three excellent reasons for presenting the proof of this theorem: it is short, extremely elegant, and highly suggestive. There is also one further reason. Every mathematical paper should contain at least one complete proof, and this is the only one I intend to give.

Accordingly, let

$$\hat{f}(\xi,\eta) = \int\int f(x,y) e^{i(\xi x + \eta y)} dx\, dy \tag{2}$$

be the Fourier transform of f. We shall need only two facts about $\hat{f}$: it is continuous, and the correspondence between f and $\hat{f}$ is one-to-one. In particular, if $\hat{f}=0$ then $f=0$ (almost everywhere). I claim that if f satisfies (1) then $\hat{f}(0,\eta)\equiv 0$. In fact,

$$\begin{aligned}\hat{f}(0,\eta) &= \int\int f(x,y) e^{i\eta y} dx\, dy \\ &= \int\left(\int f(x,y)\, dx\right) e^{i\eta y} dy \\ &= 0,\end{aligned}$$

since the inner integral vanishes for almost every fixed value of y. A moment's thought now reveals that $\hat{f}(\xi,\eta)=0$ for *all* (ξ,η). Indeed, what the calculation above actually shows is that $\hat{f}$ vanishes on a line through the origin (a direction) whenever the integral of f over each perpendicular line (i.e., each line having the orthogonal direction) vanishes. Thus $\hat{f}$ vanishes on *every* line through the origin and so vanishes identically.

With a little more effort one can conclude much more. For instance, since $\hat{f}$ is continuous, it is sufficient to know that (1) holds for almost every line in a (possibly countable) *dense* set of directions. This in itself is not terribly surprising, but it points the way to grander things. Suppose, for instance, that f has compact support. Then we can expand the exponential in (2) and integrate term by term to obtain for $\hat{f}(\xi,\eta)$ a power series expansion convergent for all (real and complex) values of (ξ,η). Thus $\hat{f}$ is an entire function of the variables (ξ,η). We have already seen that if (1) holds for the family of lines having a fixed direction, then $\hat{f}$ vanishes on the line through the origin orthogonal to this family. Thus, if (1) holds for almost all lines in each of a countable number of directions, $\hat{f}$ must vanish on a countable number of lines through the origin. It is easy to see that this forces all terms in the Taylor expansion of $\hat{f}$ to vanish. Thus $\hat{f}\equiv 0$, and so $f=0$. This seems genuinely surprising, since the set of directions for which (1) is assumed to hold can be quite thin. In fact, it is hard to imagine how one would go about proving such a result by purely geometric means.

The study of functions by means of their integrals over lines, planes, or higher dimensional affine subspaces has given rise to the extensive theory of the Radon transform. We shall not pursue that theory here, since our vaunted purpose is the examination of the "offbeat." Instead, we direct the interested reader to **[17]** and the references given there.

3. Kreis und Kugel. Lines and planes are not the only geometrically natural sets over which one integrates functions; circles and spheres are every bit as good. Unfortunately, the analogue of the Cramér-Wold theorem for circles is *too easy*. Indeed, suppose that $f\in C(\mathbb{R}^2)$ and

$$\int_C f\, ds = 0 \tag{3}$$

for all circles C. Then

$$f(z) = \lim_{r\to 0}\frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})\, d\theta = 0,$$

so f vanishes identically. This conclusion remains true—with "identically" replaced by "almost

everywhere"—if we assume only that f is measurable and locally integrable, a condition we write as $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$.

Thus, in order to obtain an interesting theory, we are forced to do the honorable thing and abandon the assumption that (3) holds for *small* circles. What can we conclude if (3) is known to hold only for circles whose radii are bounded away from 0, say $r > r_0$?

Confronted now by a nontrivial question, we may begin to appreciate some additional wrinkles in the problem for circles. For one thing, circles do not fit together as neatly as lines. For another, condition (3) does not assume any limitation on the growth of f at infinity; and, in the absence of such a limitation, it is not at all clear how one may bring in the mechanism of the Fourier transform or allied devices. In short, the problem for circles now looks *harder* than the corresponding problem for lines.

It is, but the solution has been known for over forty years. You can read about it in Fritz John's lovely little monograph **[15]**. Briefly, if (3) holds for all circles whose radii lie in some interval $r_1 < r < r_2$, then f must vanish. Most reasonable people would be satisfied with such an answer. Since our interest here is in finding *minimal* hypotheses, let us ask (boldly?, naively?) whether we can obtain the same conclusion under the assumption that (3) holds for all circles having *a single fixed radius r*. To answer that question, it will be helpful to know something about Bessel functions.

4. A Very Short Course on Bessel Functions. Virtually everything we need to know about Bessel functions can be read off from Schlömilch's generating formula

$$\exp\left(\frac{w}{2}\left[t-\frac{1}{t}\right]\right)=\sum_{n=-\infty}^{\infty} J_n(w)t^n,$$

where the series on the right converges uniformly on each compact set in $C_w \times (C_t \backslash \{0\})$. A little algebra then yields

$$J_\alpha(w)=\sum_{k=0}^{\infty}\frac{(-1)^k w^{\alpha+2k}}{2^{\alpha+2k}k!\Gamma(\alpha+k+1)};$$

while this formula follows from the generating function only for α integral, it can be used to define Bessel functions for arbitrary (real) index α. Aside from the (possibly) multi-valued factor w^α, these are all entire functions having infinitely many zeros. When $\alpha > -1$, the case which will be of interest to us, these zeros are all real and are distributed in an asymptotically regular fashion along the real axis. (This also holds true when α is a negative integer, since by Schlömilch's formula $J_{-n}(w) = J_n(-w)$.) For α equal to half an odd integer, $J_\alpha(z)$ can be expressed in terms of trigonometric functions. In particular,

$$J_{-1/2}(z)=\left(\frac{2}{\pi z}\right)^{1/2}\cos z,\quad J_{1/2}(z)=\left(\frac{2}{\pi z}\right)^{1/2}\sin z.$$

For more on Bessel functions, see **[30]**.

5. An Example. Let $g(\zeta)=g(\xi+i\eta)=e^{i\eta}$ and let $z=x+iy$. Then

$$\begin{aligned}\int_{|\zeta-z|=r} g(\zeta)\,ds &= \int_0^{2\pi} g(z+re^{i\theta})\,r\,d\theta=\int_0^{2\pi} e^{i(y+r\sin\theta)}\,r\,d\theta\\ &= re^{iy}\int_0^{2\pi} e^{ir\sin\theta}\,d\theta\\ &= re^{iy}\int_0^{2\pi}\exp\left[\frac{r}{2}\left(e^{i\theta}-\frac{1}{e^{i\theta}}\right)\right]d\theta\\ &= re^{iy}\int_0^{2\pi}\sum_{-\infty}^{\infty} J_n(r)e^{in\theta}\,d\theta\\ &= 2\pi\cdot r\cdot e^{iy}\cdot J_0(r).\end{aligned}$$

Thus the integral of g over a circle of radius r (and arbitrary center) vanishes precisely when $J_0(r)=0$. It follows that (3) holds for all circles having radii belonging to a certain infinite set (the positive zeros of J_0). Thus, the answer to the question posed at the end of Section 3 is a resounding no. Nor can the example be dismissed as mere pathology: the function $g(\zeta)$ is bounded and real-analytic.

6. A Positive Result. Of course, that is not the whole story; otherwise I shouldn't be writing this article. If, as Morris Kline has written **[16]**, "the virtue of a logical proof is not that it compels belief, but that it suggest doubts; and the proof tells us where to concentrate our doubts," it is no less true that the virtue of a counterexample is not that it compels doubt, but that it suggests belief; and the counterexample tells us where to concentrate our belief. In the counterexample of the previous section, the ratio r_1/r_2 of any two radii for which (3) holds is a quotient of zeros of $J_0(z)$. And so, in consonance with the principle enunciated above, we have the following result **[10]**, **[12]**, **[26]**, **[32]**.

THEOREM. *Let $f\in C(\mathbb{R}^n)$, $r_1,r_2>0$. Suppose that the integral of f (against $(n-1)$-dimensional surface area) over every sphere of radius r_1 and r_2 is 0. Then $f\equiv 0$ so long as r_1/r_2 is not a quotient of zeros of $J_{(n-2)/2}(z)$.*

This is not the place to enter into the details of the proof; later on, I shall give some general indications of the ideas and techniques involved. For the moment, let me instead mention some amplifications and illustrations.

The regularity assumption on f can be weakened considerably; it is enough, in fact, for f to belong to $L^1_{\text{loc}}(\mathbb{R}^n)$. It then follows (though this is not quite obvious) that f is integrable on almost all spheres of a fixed radius, where "almost all" is taken naturally in the sense of the measure of the centers. Similarly, one need require only that the integral of f vanish over *almost* all spheres having radius r_1 and r_2. While these are real variable refinements of only technical interest, they do make the connection with integral geometry immediate. Let S be a measurable set in the plane. Then S is uniquely determined, up to a set of measure 0, by the lengths of its intersections with (almost) all circles of radius r_1 and r_2, so long as r_1/r_2 is not a quotient of zeros of $J_0(z)$.

Much more striking is the rigidity or, better, instability of our result. While the quotients of zeros of the Bessel functions form a countable (and hence "small") set, they are, by virtue of the asymptotically regular distribution of those zeros, dense in the positive reals. Thus, the slightest perturbation of one of the radii can destroy the conclusion. This would seem to rule out any possibility of a purely geometric proof.

Finally, it should be stressed that the "exceptional set" really does exist. The example of Section 5 shows this to be the case for $n=2$, and similar examples can be constructed for arbitrary n.

There are two cases in which the set of quotients of zeros of $J_{(n-2)/2}(z)$ can be identified more explicitly. When $n=3$, we have $J_{1/2}(z)=(2/\pi z)^{1/2}\sin z$, so the exceptional set consists simply of all quotients $\pi m/\pi n$, i.e., of all rational numbers. When $n=1$, we have the set of quotients of zeros of $J_{-1/2}(z)=(2/\pi z)^{1/2}\cos z$, so the exceptional set consists of all rationals m/n, m and n both odd. Thus, a function on the line which satisfies $f(x+r_j)+f(x-r_j)\equiv 0$ $(j=1,2)$ must vanish identically so long as $r_1/r_2\neq m/n$, where m and n are odd integers. It is an amusing exercise to try to prove this last result, which makes contact with Problem A3 of the 1977 Putnam Competition, directly.

7. An Embarrassing Question. People ask, "Where do the Bessel functions come from?" I used to think I knew the answer to that question; now I'm not so sure. There are by now a number of different approaches to the theorem, and in each approach the Bessel functions arise in a slightly different way: as Fourier transforms, as eigenfunctions of the Laplacian, as spherical functions, or as solutions to certain ordinary differential equations. On one level, of

course, these are all the same. What is mildly troubling is that the theorem extends in a number of different directions (some of which we explore below) and that now one, now another (but not all) of the techniques available provides the key to the appropriate extension. No doubt there is a final synthesis; since we have not yet attained it, it seems best to answer humbly that, like Kronecker's integers, the Bessel functions have been granted us by Providence: they are simply a fact of nature.

8. An Inverse Problem and a Problem of Inversion. Let $u \in C^2(\mathbb{R}^n)$ and denote by $U(x,r)$ the average of u over the sphere of radius r ($\geqslant 0$) about the point $x \in \mathbb{R}^n$. Then U satisfies the Euler-Poisson-Darboux equation

$$\frac{\partial^2 U}{\partial r^2} + \frac{n-1}{r}\frac{\partial U}{\partial r} = \Delta_x U$$

$$U(x,0) = u(x) \qquad \frac{\partial U}{\partial r}(x,0) = 0.$$

The theorem of Section 6 can be reformulated as a kind of inverse theorem for the EPD equation: suppose $U(x,r)$ vanishes on the "planes" $r = r_1$ and $r = r_2$; then $u \equiv 0$ so long as r_1/r_2 is not a quotient of zeros $J_{(n-2)/2}(z)$. This bears a striking resemblance to certain uniqueness theorems for boundary value problems for hyperbolic equations; cf. **[28]** and the references cited there.

Dropping the (technical) condition $u \in C^2(\mathbb{R}^n)$, and assuming only that $u \in C(\mathbb{R}^n)$, we may summarize the results of the previous sections in the assertion that the map $u \to (u_1, u_2)$ is one-to-one, where $u_j(x) = U(x, r_j)$ and r_1 and r_2 have been chosen so that the theorem holds. This formulation suggests at once a natural problem: find an explicit inversion formula. In other words, given (u_1, u_2), how do we recover u? In certain very special cases it is easy enough to do this, but the real point of the problem is to produce a formula which works for *all u* and all admissible (r_1, r_2). In this generality, the problem is open even for functions on the line. Of course, such a formula would yield a new (and, almost certainly, the "best") proof of our theorem. It is natural to assume that the exceptional set of bad ratios will correspond to the vanishing of certain Bessel functions in denominator terms of the hypothesized inversion formula.

9. Areal Analogues and Volume Variations. While the hypothesis of the Cramér-Wold Theorem was stated as a condition on integrals over lines, it could equally well have been (and, in fact, originally was) written in terms of area integrals:

$$\int_H f\,dA = 0 \tag{4}$$

for every half-plane H. The passage from (1) to (4) and back is purely routine, just standard integration and differentiation.

Unfortunately, as we have already noted, circles (or spheres) of a fixed radius do not fit together so nicely, and there is no obvious way to integrate the two-circle (sphere) theorem to obtain a two-disc (ball) theorem. Nonetheless, an areal (volume) analogue of that result does hold **[6]**, **[26]**, **[32]**.

THEOREM. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $r_1, r_2 > 0$. Suppose the integral of f (against volume) over (almost) every ball of radius r_1 and r_2 is zero. Then $f = 0$ almost everywhere so long as r_1/r_2 is not a quotient of zeros of $J_{n/2}(z)$.*

As before, this is an extremely unstable result. Of particular interest is the fact that the exceptional set for this theorem differs from that of the two-sphere theorem, final evidence of the impossibility of passing directly from one to the other. On the other hand, it will not have escaped the attention of the careful reader that the exceptional set for volume in $\mathbb{R}^n$ coincides

with the exceptional set for surface area in $\mathbb{R}^{n+2}$. This is no accident: it actually *is* possible to deduce the volume theorem in $\mathbb{R}^n$ from the surface area theorem in $\mathbb{R}^{n+2}$ (though not vice versa). We shall not pursue this point further, since it is not particularly profitable; but the reader may wish to discover for himself the (purely geometric) connection between the two results.

10. Pompeiu's Problem. So far, we have concentrated on what may be discovered about functions from the knowledge of their integrals over certain sets. One may adopt the dual point of view and ask instead for which sets does knowledge of the appropriate integrals determine the function. This leads directly to the Pompeiu problem.

Let $K \subset \mathbb{R}^n$ be a compact set of positive n-dimensional measure. We say that K has the Pompeiu property if whenever $f \in C(\mathbb{R}^n)$ and

$$\int_{\alpha(K)} f\,dV = 0 \tag{5}$$

for all rigid motions α of K, then f vanishes identically.

Most familiar figures in $\mathbb{R}^2$ (ellipses, polygons, etc.) possess the Pompeiu property [6]. Discs don't; in fact, the theorem of the previous section provides a kind of substitute result for this failure. Intuitively, the explanation for this difference lies in the fact that, for sets which lack the rotational symmetry of a disc, the use of rotations α in (5) contributes important additional information beyond what is available when only translations are taken into account. Unfortunately, it is not easy to make this into a mathematically compelling argument; and the extremely interesting question of whether the disc is the *only* closed Jordan domain in the plane for which the Pompeiu property fails remains open. For smoothly bounded domains, this turns out to be equivalent to the following "free boundary" problem [2].

Let $\mathfrak{D}$ be a Jordan domain and suppose that the boundary value problem

$$\begin{aligned} &\Delta u + \lambda u = 0 && \text{on } \mathfrak{D} \\ &u = c, \quad \frac{\partial u}{\partial n} = 0 && \text{on } \partial\mathfrak{D} \end{aligned} \tag{6}$$

has a positive eigenvalue λ. Must $\mathfrak{D}$ be a disc? (When there are *infinitely* many eigenvalues, the answer is yes [2].) In case $\mathfrak{D}$ is the disc $\{(x,y)\colon\ x^2+y^2 \leqslant R\}$, the function $u(x,y) = J_0\big(\sqrt{\lambda(x^2+y^2)}\,\big)$, where $\lambda = (\mu/R)^2$ for some zero μ of $J_1(z)$, satisfies (6). In general, of course, neither $\mathfrak{D}$, nor u, nor λ is assumed to be known.

This question seems strikingly similar to the following characterization of discs in terms of Poisson's equation, with boundary conditions complementary to (6). Suppose there exists on the smoothly bounded Jordan domain $\mathfrak{D}$ a function u such that

$$\begin{aligned} &\Delta u = -1 && \text{on } \mathfrak{D} \\ &u = 0, \quad \frac{\partial u}{\partial n} = c && \text{on } \partial\mathfrak{D}. \end{aligned} \tag{6'}$$

Then $\mathfrak{D}$ is a disc [23], [31]. When $\mathfrak{D} = \{(x,y)\colon\ x^2+y^2 \leqslant R^2\}$, the function $u(x,y) = \frac{1}{4}(R^2 - x^2 - y^2)$ satisfies (6′) with $c = -R/2$. Since Bessel functions are more complicated objects than quadratics, one assumes that problem (6) poses difficulties beyond those encountered in dealing with (6′).

All the matters mentioned above, the open question, its equivalence with the free boundary problem (6), and the result on Poisson's equation (6′) have obvious formulations for $\mathbb{R}^n$: one simply replaces the assumption that $\mathfrak{D}$ is a Jordan domain by the hypothesis that it is a homeomorphic (or smooth) image of the unit ball $\mathbb{B}^n$.

11. Extensions. In previous sections, we have concerned ourselves with conditions which imply that a function vanishes identically. Actually, it is no more difficult to obtain theorems whose conclusions are more arresting. For instance, we have the following two-radius variation on the classical theorem of Morera ([26], [32]).

THEOREM. *Let $f \in C(\mathbb{R}^2)$. Suppose that*

$$\int_\Gamma f(z)\,dz = 0 \tag{7}$$

for all circles Γ of radius r_1 and r_2. Then f is entire (holomorphic) so long as r_1/r_2 is not a quotient of zeros of $J_1(z)$.

Actually, this is a consequence of the two-disc theorem of Section 9, since for smooth functions

$$\int_\Gamma f(z)\,dz = 2i \int\int_\Delta \frac{\partial f}{\partial \bar{z}}\,dx\,dy,$$

where Δ is the disc bounded by Γ and $\partial/\partial\bar{z} = \frac{1}{2}(\partial/\partial x + i(\partial/\partial y))$; one then passes to the general case in routine fashion. This connection, however, is in a certain sense accidental and certainly not representative. At any rate, the result illustrates quite clearly the importance of avoiding assumptions of global integrability: while *every* entire function satisfies (7), the only entire function which satisfies $f \in L^1(\mathbb{R}^2)$ is $f(z) \equiv 0$.

A related theorem characterizes harmonic functions in terms of a two-radius mean-value condition. Let Ω_n denote the uniform distribution of mass on the unit sphere $S^{n-1} \subset \mathbb{R}^n$, normalized to have total measure one. Recall that a function u continuous on a domain $\mathfrak{D} \subset \mathbb{R}^n$ is harmonic on $\mathfrak{D}$ if (Koebe) and only if (Gauss)

$$u(x) = \int_{S^{n-1}} u(x+rt)\,d\Omega_n(t) \tag{8}$$

for all $x \in \mathfrak{D}$ and all $0 < r < \operatorname{dist}(x, \partial\mathfrak{D})$. In particular, when u is harmonic on all of $\mathbb{R}^n$, (8) holds for all $x \in \mathbb{R}^n$ and all $r > 0$.

THEOREM. *Let $u \in C(\mathbb{R}^n)$, $r_1, r_2 > 0$. Suppose (8) holds for all $x \in \mathbb{R}^n$ and $r = r_1, r_2$. Then $\Delta u = 0$ so long as r_1/r_2 is not a quotient of zeros of*

$$\frac{2^{(n-2)/2}\Gamma(n/2)J_{(n-2)/2}(z)}{z^{(n-2)/2}} - 1. \tag{9}$$

This result [8], [9], [10], [12], [32] is the earliest of all two-radius theorems; it was discovered by Jean Delsarte as far back as 1957. It differs from the results of previous paragraphs in that the exceptional set is very sparse or even nonexistent. Indeed, if the positive number r_1/r_2 equals a quotient of zeros of (9), these zeros must lie on a ray through the origin. The asymptotics of the Bessel functions can be used to show that there are, for each $n > 1$, at most a finite number of excluded ratios. (The case $n = 1$ is special: (9) reduces to $\cos z - 1$, and the exceptional set consists of the rationals.) When $n = 3$, *no* ratios need be excluded [9]. Whether the exceptional set is nonempty for *any* $n > 1$ remains an open question. We shall see another example of an unconditional theorem, one in which there is no exceptional set, in Section 12.

For over a decade, Delsarte's theorem remained a curiosity, a kind of "freak theorem," to use the terminology of an even earlier pioneer in the field (cf. Section 14). Part of the reason, perhaps, was that it was widely assumed to be a result specific to harmonic functions; so there was really nothing more to say. We who have seen the theorems of the preceding paragraphs know better. But how are we to fit Delsarte's theorem and our version of Morera's theorem and the sphere and ball theorems of offbeat integral geometry into a single general framework?

The answer is both surprising and surprisingly simple. Let $P(\xi) = P(\xi_1, \xi_2, \ldots, \xi_n)$ be a homogeneous polynomial, i.e., a (finite) sum of monomials, all of which have the same total degree. Denote by D the symbolic vector $(\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n)$. We can define a differential operator $P(D)$ by replacing each ξ_k by $\partial/\partial x_k$ in the expression for P. A function $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ is said to be

a weak, or distributional, solution to $P(D)u=0$ if

$$\int_{\mathbb{R}^n}(P(D)\phi)u\,dV=0$$

for all C^∞ functions ϕ of compact support. (This coincides with the usual notion of solution when u has the smoothness necessary for $P(D)u$ to make classical sense.)

It turns out that *for any homogeneous polynomial P there exist* (*infinitely many distinct*) *two-radius theorems which characterize the* (*weak*) *solutions of* $P(D)u=0$. Associated to each such theorem is a (possibly empty) exceptional set of real ratios of zeros of a certain entire function. Different theorems will, in general, have different exceptional sets—even when they characterize solutions of the same equation.

What seems most remarkable about this result is that no hypothesis of ellipticity is involved; in fact, the *type* of the equation plays no role at all! There is a two radius theorem—in fact, infinitely many—for *any* differential equation $P(D)u=0$, no matter how complicated, so long as P comes from a homogeneous polynomial. We shall forego the recipe for producing these theorems and content ourselves with a single example.

THEOREM. *Let* $u\in C(\mathbb{R}^2)$, $r_1,r_2>0$. *Suppose that*

$$\int u(x+r\cos\theta,y+r\sin\theta)\cos 2\theta\,d\theta=0 \qquad r=r_1,r_2 \tag{10}$$

for all $(x,y)\in\mathbb{R}^2$. *Then u satisfies d'Alembert's equation* $\Box u\equiv u_{xx}-u_{yy}=0$ *so long as* r_1/r_2 *is not a quotient of zeros of* $J_2(z)$. *Conversely, each global solution of* $\Box u=0$ *satisfies* (10) *for all* $r>0$.

So far as the results of previous paragraphs are concerned, they may be set in the present framework as follows:

P	$P(D)u=0$
1	$u=0$
$\frac{1}{2}(x+iy)$	$\dfrac{\partial u}{\partial \bar{z}}=0$
$x_1^2+x_2^2+\cdots+x_n^2$	$\Delta u=0$

Note in particular that the integral geometry theorems of Sections 6 and 9 come from the (trivial!) choice $P(\xi_1,\xi_2,\ldots,\xi_n)=1$, corresponding to $P(D)=I$, the identity operator. Who says trivial things can't be interesting?

12. Less Is More, More or Less. While the formulation of the previous section attains a satisfying generality, it by no means exhausts the possibilities of what may be proved along similar lines. To illustrate the point, let us consider a *one-radius* theorem, which makes explicit contact with classical Fourier analysis.

Suppose f is a continuous function on the plane. For each $z\in\mathbb{C}$, the restriction of f to the circle of radius 1 centered at z has a Fourier series expansion

$$f(z+e^{i\theta})\sim\sum_{n=-\infty}^{\infty}a_n(z)e^{in\theta},$$

where

$$a_n(z)=\frac{1}{2\pi}\int_0^{2\pi}f(z+e^{i\theta})e^{-in\theta}\,d\theta$$

$$=\frac{1}{2\pi i}\int_{|\zeta-z|=1}f(\zeta)(\zeta-z)^{-n-1}\,d\zeta. \tag{11}$$

Now if f is entire, then $a_n(z)\equiv0$, $n=-1,-2,\ldots$. In fact, if f is analytic on the closed disc

$\{|\zeta - z| \leq 1\}$ it follows from (11) (and Cauchy's theorem) that $a_n(z) = 0$ for $n = -1, -2, -3, \ldots$. Conversely, suppose that for some fixed z, $a_n(z) = 0$ for all $n < 0$. Then it is easy to see that f can be extended continuously from the circle $\{|\zeta - z| = 1\}$ to the disc $\{|\zeta - z| \leq 1\}$ so as to be analytic in $\{|\zeta - z| < 1\}$.

Question. Suppose $a_n(z) \equiv 0$, $n = -1, -2, -3, \ldots$. Must f be entire?

The answer is yes [**1**]. This does not seem obvious (at least not to me): while the assumption $a_n(z) \equiv 0$ $(n < 0)$ ensures that f has an analytic extension from each circle $\{|\zeta - z| = 1\}$ to $\{|\zeta - z| < 1\}$, we do *not* know that these extensions agree on overlapping discs. At any rate, the conclusion can be obtained under a (literally!) infinitely weaker hypothesis.

THEOREM [**33**]. *Let $f \in C(\mathbb{R}^2)$ and let $r > 0$, $n > 1$ be fixed. Suppose that*

$$\int_0^{2\pi} f(z + re^{i\theta}) \left\{ \begin{matrix} e^{i\theta} \\ e^{in\theta} \end{matrix} \right\} d\theta = 0 \tag{12}$$

for all $z \in \mathbb{C}$. Then f is an entire function.

Thus, under the assumption that just two negative Fourier coefficients, $a_{-1}(z)$ and $a_{-n}(z)$ (for some n), vanish identically, we can conclude that f is analytic everywhere. As usual, one can weaken the regularity assumed of f to $L^1_{\text{loc}}(\mathbb{R}^n)$ and require only that (12) hold for *almost* all z. What is most striking about this result, however, is the absence of any exceptional set. On a superficial level, this may be explained by the fact, easy to see, that *if* the theorem holds for a single value of r, then it holds for *all* $r > 0$. On a much deeper level, the absence of a set of excluded values is accounted for by the fact that the Bessel functions $J_1(z)$ and $J_n(z)$ have no common zeros (other than $z = 0$). And this last fact depends (at least when $n > 4$) on a deep result of Carl Ludwig Siegel in the theory of transcendental numbers: if $J_\alpha(z) = 0$, $z \neq 0$, and α is rational, then z is transcendental; cf. [**30**, pp. 484–485].

13. Et in Saecula . . . There is one more direction in which the results we have discussed can be extended: to a wider class of *spaces*. This can be done in considerable generality. In fact, the theorems we have discussed have analogues for functions defined on symmetric spaces of rank one [**4**]. It is not my purpose here to enter into a description of the most general result possible, but I should be remiss (and you would be shortchanged) if I did not at least indicate the flavor of the results obtained.

A happy compromise is achieved by focusing attention on spaces of constant curvature. These spaces constitute a natural generalization of $\mathbb{R}^n$ having immediate geometric appeal. Moreover, since they include (in addition to the euclidean spaces of previous sections) spheres and balls in euclidean space, they provide a direct link with the classical tradition of considering functions defined not only on the plane but on the circle and the disc as well.

Accordingly, let $S(n,k)$ denote a complete, simply connected n-dimensional Riemannian manifold of constant curvature k. For fixed n and k, $S(n,k)$ is unique up to isometric equivalence, a fact which allows us to choose a realization *ad lib.* When $k = 0$, we may identify $S(n,0)$ with $\mathbb{R}^n$. For $k = \alpha^2 > 0$, $S(n,\alpha^2)$ can be identified with the n-sphere $S^n(1/\alpha) \subset \mathbb{R}^{n+1}$ of radius $1/\alpha$. For $k = -\alpha^2 < 0$, $S(n, -\alpha^2)$ is n-dimensional hyperbolic space; a realization is given by the ball $B^n(1/\alpha) \subset \mathbb{R}^n$ with the noneuclidean metric

$$ds^2 = \lambda(r)^2 |dx|^2 = \frac{4R^4}{(R^2 - |x|^2)^2} |dx|^2,$$

where $R = 1/\alpha$, $r^2 = |x|^2 = x_1^2 + \cdots + x_n^2$, and $|dx|^2 = dx_1^2 + \cdots + dx_n^2$. In particular, the hyperbolic plane $S(2, -1)$ is the unit disc with the usual noneuclidean (Poincaré) geometry.

Attached to $S(n,k)$ is a natural second-order differential operator invariant under isometries, the Laplace-Beltrami operator Δ_2. Of course, for $\mathbb{R}^n$, $\Delta_2 = \nabla^2$, the ordinary euclidean Laplacian.

When $S(n,k)$ is a sphere (i.e., when $k>0$), Δ_2 is the restriction of the euclidean Laplacian in $\mathbb{R}^{n+1}$ to the sphere $S^n(1/\alpha)$; it is obtained by writing the euclidean Laplacian in spherical coordinates $(r,\theta_1,\dots,\theta_n)$ and then ignoring those terms involving differentiations with respect to r. For the hyperbolic spaces $S(n,-\alpha^2)=B^n(1/\alpha)$, we have

$$\Delta_2=\lambda(r)^{-2}\left[\nabla^2+\frac{n-2}{r}\frac{\lambda'(r)}{\lambda(r)}\sum_{i=1}^{n}x_i\frac{\partial}{\partial x_i}\right].$$

In particular, for $S(2,-1)$

$$\Delta_2=\tfrac{1}{4}(1-|z|^2)^2\nabla^2,$$

so that the "harmonic" functions on $S(2,-1)$ are just the usual harmonic functions.

The theorems of Sections 6, 9, and 11 have very precise analogues for the hyperbolic spaces $S(n,-\alpha^2)$. Let us illustrate this analogy in the case of the Delsarte two-radius theorem of Section 11.

THEOREM [3]. *Let u be a continuous function on $S(n,-\alpha^2)$. Denote by $U(x,r)$ the average of u over the geodesic sphere of radius r about the point $x\in S(n,-\alpha^2)$. Suppose there exist numbers $r_1,r_2>0$ such that*

$$U(x,r_j)\equiv u(x) \qquad j=1,2. \tag{13}$$

Then $\Delta_2 u=0$, so long as the equations

$$\frac{2^{(n-2)/2}\Gamma(n/2)P_z^{-(n-2)/2}(\cosh\alpha r)}{(\sinh\alpha r)^{(n-2)/2}}=1 \qquad r=r_1,r_2 \tag{14}$$

have no common solution $z\in\mathbb{C}$ other than $z=(n-2)/2,\ -n/2$.

Here, the function $P_z^{-(n-2)/2}$ denotes the Legendre function of the first kind, *which we regard as a function of its (lower) index.* For fixed values of the parameter z, the left-hand side of (14) is thought of most simply as the (unique) solution of the eigenvalue problem.

$$\Delta_2F=\alpha^2\left[z-\left(\frac{n-2}{2}\right)\right]\left[z+\frac{n}{2}F\right] \qquad F(0)=1 \tag{15}$$

that depends on the radial variable r alone. In fancier language, the functions of (14) are the *spherical functions* for the symmetric space $S(n,-\alpha^2)$.

To make the analogy with Delsarte's theorem more precise, let us observe that condition (9) may be reformulated as the requirement that the equations

$$\frac{2^{(n-2)/2}\Gamma(n/2)J_{(n-2)/2}(rz)}{(rz)^{(n-2)/2}}=1 \qquad r=r_1,r_2 \tag{16}$$

have no common roots other than $z=0$. Actually, more than an analogy is involved: (16) is a limiting form of (14) obtained by rescaling the variable z and making α tend to 0. Indeed, we have (cf. [**11**, 7.8(1)])

$$\lim_{\alpha\to 0}\frac{P_{-iz/\alpha}^{-(n-2)/2}(\cosh\alpha r)}{(\sinh\alpha r)^{(n-2)/2}}=\frac{J_{(n-2)/2}(rz)}{(rz)^{(n-2)/2}}.$$

Let us illustrate this theorem in the case of greatest concrete interest, the (open) unit disc $\mathfrak{D}=S(2,-1)$. Let $u\in C(\mathfrak{D})$. Written in euclidean coordinates, the condition that $U(x,r_j)\equiv u(x)$ becomes

$$\frac{1}{2\pi\sinh r}\int_{C_r(w)}u(\zeta)\frac{2|d\zeta|}{1-|\zeta|^2}=u(w)$$

for all $w\in\mathfrak{D}$ and $r=r_1,r_2$, where

$$C_r(w)=\left\{\zeta:\left|\frac{\zeta-w}{1-\bar{w}\zeta}\right|=\tanh\frac{r}{2}\right\}$$

is the noneuclidean circle having (noneuclidean) center w and (noneuclidean) radius r. We conclude that u is harmonic so long as the equations

$$P_z(\cosh r_j)=1 \qquad j=1,2 \tag{17}$$

have no common solution other than $z=0,-1$.

To see that this result is sharp, suppose that z *is* a common solution of (17). Fix an origin of coordinates and let $u(w)=P_z(\cosh r)$, where (r,θ) are the geodesic polar coordinates of $w\in\mathfrak{D}$. A little geometric reasoning shows that if ζ has coordinates (r',θ') then

$$u(\zeta)=P_z(\cosh r')=P_z(\cosh r\cosh\rho+\sinh r\sinh\rho\cos(\phi-\theta)),$$

where (ρ,ϕ) are the coordinates of ζ with respect to a new coordinate system centered at w. The addition theorem for Legendre functions [**14**, p. 369] gives

$$P_z(\cosh r\cosh\rho+\sinh r\sinh\rho\cos\psi)=P_z(\cosh r)P_z(\cosh\rho)+2\sum_{m=1}^{\infty}P_z^m(\cosh r)P_z^{-m}(\cosh\rho)\cos m\psi.$$

Performing the required integration, we obtain

$$\begin{aligned}U(w,\rho)&=P_z(\cosh r)P_z(\cosh\rho)\\&=P_z(\cosh r)\qquad \rho=r_1,r_2\\&=u(w)\end{aligned}$$

so (13) holds. However, by (15),

$$\Delta_2P_z(\cosh r)=z(z+1)P_z(\cosh r);$$

hence $u(w)$ is *not* harmonic unless $z=0,-1$.

The analogues of the theorems of Sections 6 and 9 are similar. For instance, a function in $L^1(S(n,-\alpha^2))$ which integrates to zero over every (geodesic) ball of radius r_1 and r_2 must vanish almost everywhere so long as

$$P_z^{-n/2}(\cosh\alpha r_j)=0 \qquad j=1,2$$

has no common solutions $z\in\mathbb{C}$. Examples like the one above show that these results, too, are sharp.

14. . . . Saeculorum. The case of functions defined on the spaces of positive curvature $S(n,\alpha^2)$ (i.e., on spheres) exhibits new features arising from the fact that these spaces are compact. In this case, the notions of local and global integrability coincide, the only harmonic functions are constants, and the mechanism of expansion in series of spherical harmonics is available.

The principal ideas are already evident in the case of functions on the circle S^1. Suppose that $u\in L^1(S^1)$ and that the integral of u over any interval of length $2r(0<r<\pi)$ vanishes, i.e.,

$$\int_{\theta-r}^{\theta+r}u(e^{it})\,dt=0 \qquad 0\leqslant\theta<2\pi. \tag{18}$$

Writing $\chi(e^{it})$ for the characteristic function of the interval (e^{-ir},e^{ir}), we have

$$u*\chi(e^{i\theta})=\int_0^{2\pi}u(e^{i(\theta-t)})\chi(e^{it})\,dt=0. \tag{19}$$

Now u and χ have Fourier series expansions

$$u(e^{i\theta})=\sum_{n=-\infty}^{\infty}a_ne^{in\theta} \qquad \chi(e^{i\theta})=\sum_{n=-\infty}^{\infty}\frac{2}{n}(\sin nr)e^{in\theta},$$

where the constant term in the last expansion is understood to be $2r$. Taking Fourier series on

both sides of (19) and making use of Parseval's formula, we obtain

$$\sum_{n=-\infty}^{\infty}\left(\frac{2}{n}\sin nr\right)a_n e^{in\theta}=0.$$

Therefore, $(2/n)(\sin nr)a_n=0$ for all n, so $a_n=0$ whenever $\sin nr\neq 0$ $(n\neq 0)$. In particular, if r is not a rational multiple of π, $a_n=0$ for all n, so $u\equiv 0$. On the other hand, if $r=k\pi/l$, then any integral power of $e^{il\theta}$ satisfies (18).

Thus, a single condition is sufficient to ensure an affirmative solution to the analogue of the Pompeiu problem: we are dealing with a *one*-radius theorem. The set of "bad" radii is again a (countable) dense set, one more indication of the instability of such results. Both of these features persist in higher dimensions.

THEOREM. *Let $u\in L^1(S(n,\alpha^2))$ and suppose that*

$$\int_B u\,d\Omega=0 \tag{20}$$

for every geodesic ball (i.e., "spherical cap") of (fixed) radius r. Then $u=0$ so long as r is not a zero of any of the functions

$$P_{m+n/2}^{-n/2}(\cos\alpha r)\qquad m=1,2,3,\ldots.$$

In case r is a zero of one of these functions, appropriately chosen (ultra)spherical harmonics will satisfy (20).

This result appears in **[21]** (cf. **[20]**); it was rediscovered in **[3]**. Actually, the case $n=2$ (which contains all essential ideas) was known to Peter Ungar **[29]** as far back as 1952. Ungar found his result surprising (who wouldn't?) and dubbed it a "freak theorem." Freak or forerunner, it is, so far as we know, the oldest result in the area of offbeat integral geometry.

15. Words of (Re)Proof. The patient reader who has successfully battled his way through the preceding sections may now find it difficult to contain his irritation, if not his curiosity. How, after all, does one go about *proving* these results? Fortunately, the main ideas can be sketched rather briefly, and anyone who finds his interest piqued can refer to **[32]** for the necessary details.

Suppose, to fix ideas, that $f\in C(\mathbb{R}^n)$ and

$$\int_S f\,d\Omega=0 \tag{21}$$

for each $(n-1)$-sphere S having radius r_1 or r_2. Equation (21) can be rewritten as a pair of convolution equations

$$f*\mu_1=0\qquad f*\mu_2=0, \tag{22}$$

where μ_j is normalized surface area on the sphere $S_j=\{x:|x|=r_j\}$. It is thus natural to try to apply the mechanism of Fourier analysis. Unfortunately, since f can grow in an arbitrary fashion, its transform need not exist in the ordinary sense. It is more profitable, therefore, to attend to the harmonic analysis of the measures μ_j.

Now it is characteristic of modern analysis that a great deal of unnecessary effort can often be avoided if only one chooses the right space of functions in which to work. This is precisely the case in the present situation. It turns out that the "right" space for our problem is $\mathcal{E}'(\mathbb{R}^n)$, the familiar space of (Schwartz) distributions having compact support. The Fourier transform has a natural extension to $\mathcal{E}'(\mathbb{R}^n)$, which agrees with the n-dimensional analogue of (2) on functions of compact support; and the corresponding space $E'(\mathbb{R}^n)$ of transforms consists of entire functions of n complex variables that obey certain additional growth conditions.

Calculating the Fourier transforms in question presents no problems; one obtains

$$\hat{\mu}(z_1,\ldots,z_n)=\frac{2^{(n-2)/2}\Gamma(n/2)J_{(n-2)/2}\left(r_j\sqrt{z_1^2+\cdots+z_n^2}\right)}{\left(r_j\sqrt{z_1^2+\cdots+z_n^2}\right)^{(n-2)/2}}.$$

(This is an entire function since $J_\alpha(z)/z^\alpha$ is even.)

The condition that r_1/r_2 not be a quotient of zeros of $J_{(n-2)/2}(z)$ now reveals itself as the assumption that $\hat{\mu}_1$ and $\hat{\mu}_2$ have no common zeros. Thus, there exist entire functions H_1, H_2 such that

$$\hat{\mu}_1 H_1+\hat{\mu}_2 H_2=1.$$

(In general, the existence of such functions follows from an important result in several complex variables known as Cartan's Theorem B. In the present case, a much simpler one-variable argument is available; see below.) If we could choose H_1 and H_2 to satisfy appropriate growth conditions, we should have $H_j=\hat{T}_j\in E'(\mathbb{R}^n)$ so that

$$\hat{\mu}_1\hat{T}_1+\hat{\mu}_2\hat{T}_2=1. \tag{23}$$

This translates back to

$$\mu_1 * T_1+\mu_2 * T_2=\delta_0 \tag{24}$$

which, together with (22), would give

$$\begin{aligned} f=f*\delta_0&=f*(\mu_1*T_1+\mu_2*T_2)\\ &=(f*\mu_1)*T_1+(f*\mu_2)*T_2\\ &=0.\end{aligned}$$

(Note that this last equation no longer takes place in $\mathcal{E}'(\mathbb{R}^n)$, but rather in the space $\mathcal{D}'(\mathbb{R}^n)$ of *all* distributions.)

Unfortunately, equation (23) need not hold, even when $n=1$! However, at least for $n=1$, one has a consolation prize: there exist sequences $\{T_{1,k}\}$, $\{T_{2,k}\}$ in $\mathcal{E}'(\mathbb{R})$ such that

$$\hat{\mu}_1\hat{T}_{1,k}+\hat{\mu}_2\hat{T}_{2,k}\to 1$$

in the topology of $E'(\mathbb{R})$. From this it follows that

$$\mu_1 * T_{1,k}+\mu_2 * T_{2,k}\to\delta_0$$

in the topology of $\mathcal{E}'(\mathbb{R})$ and hence, as before, that $f=f*\delta_0=0$. The existence of such sequences is a consequence of a deep function-theoretic result of Laurent Schwartz **[22]**, **[9]**, often called the Fundamental Theorem of Mean Periodic Functions of One Variable.

Schwartz's theorem is quite definitely a one-variable result. Whether it extends to functions of several variables was, until recently, an open question; now we know that it does not **[13]**. And so our road seems blocked by a mountain.

In this particular case, it is actually possible to tunnel through the mountain **[26]**. However, it is easier (and, in the final analysis, more profitable) to observe instead that $\hat{\mu}_1$ and $\hat{\mu}_2$ *can be viewed as functions of the single complex variable* $z=\sqrt{z_1^2+\cdots+z_n^2}$. This suggests that perhaps Schwartz's one-variable theorem can be applied to obtain the desired conclusion after all. And so it can. Fitting the problem into the one-variable setting and keeping track of the necessary details involve just enough effort to keep things respectable mathematics.

Similar reasoning applies in the general case. One again views the two-radius condition as a pair of convolution equations. The only new wrinkle is the possibility that the Fourier transforms may have a common zero at the origin. It is precisely this possibility that allows conclusions more general than $f=0$. In fact, one can easily read the properties f must have directly from the lead terms in the Taylor expansions of the corresponding transforms. Indeed, one of the chief advantages of the approach via Fourier transforms is that it provides, in addition to a method of proof, a *means of discovery*. For the spaces $S(n,k)$, an analogous

reasoning based on a noneuclidean Fourier transform (when $k<0$) or its discrete analogue (for $k>0$) applies.

In all cases, the interplay between complex function theory and harmonic analysis occupies a central role. These two areas, which have already given mathematics so much, continue to be unceasing sources of profound and penetrating insights. (My friend Ian Richards once opined that the three most effective problem-solving devices in mathematics are calculus, complex variables, and the Fourier transform. I concur.) "The close relation between analytic functions and harmonic analysis on Euclidean groups" of which Professor Beurling wrote in [5] is getting closer all the time; it is even (as we have hinted) beginning to embrace spaces which are neither euclidean nor groups. Perhaps some reader who has been charmed or fascinated by one or another exotic flower in the garden of offbeat integral geometry will be inspired to try his hand at cultivating a few blossoms of his own.

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INFINITESIMAL DEFORMATIONS OF PORTIONS OF THE STANDARD SPHERE IN E^3

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1. Introduction. Consider a smooth surface S in Euclidean 3-space E^3. Suppose that we allow S to change smoothly with time t, $0 \leqslant t \leqslant 1$. Each point $p \in S$ moves along a curve $\gamma_p(t)$, and the surface at time t will be $S_t = \{\gamma_p(t) : p \in S\}$. If C is a curve in S, we let C_t be the corresponding curve in S_t. If, for all curves C in S, the length of C_t is independent of t, then $t \to S_t$ is called an *isometric deformation* (ID) of S. If S_t is congruent to S, then the ID is called trivial. A simple example of a nontrivial ID is witnessed when a piece of paper unrolls. If there is no nontrivial ID of S, then S is called rigid. Hilbert, Liebmann, Minkowski, and Weyl proved in the early 1900's that the sphere is rigid. The rigidity of regular convex surfaces was established by Cohn-Vossen around 1930. The interested reader will find the bibliography of Efimov **[2]** a useful guide to the history of rigidity problems.

The sphere minus a polar cap of positive radius is not rigid. This fact is by no means obvious, but it is a consequence of Alexandrov's theorem on the gluing of convex surfaces (see **[2**, p. 317]). The same argument proves that any convex surface minus an open nonvoid set is nonrigid. For a very intuitively plausible argument in the case of the sphere, read the interesting discussion in **[3**, pp. 228–230].

Although it is difficult to find an explicit formula that describes how a sphere minus a polar cap deforms under a prescribed deformation of its boundary, we can gain some insight by considering a linearized version of this problem. Consider a smooth vector field V on a surface S in E^3. The vector field V assigns to each point $p \in S$ a vector v_p based at p. Note that v_p is not assumed to be tangent to S at p, nor do we assume v_p has unit length. Now move each point of S with velocity v_p along the line through p in the direction of v_p. We obtain some deformation S_t of S which is not necessarily isometric. If, for any curve C in S, we have d/dt (length $(C_t)) = 0$ *when* $t = 0$, then V is called an infinitesimal isometric deformation, abbreviated IID. Thus V is an IID,

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if the above deformation of S preserves lengths of curves to first order in t about $t=0$. The study of IID's also has a long history (see [2] again). For some very readable material on IID's consult [4] and [6].

In this article we shall derive some explicit formulas for certain IID's on the ordinary unit sphere minus a point, $S^2-\{(0,0,-1)\}$. These IID's form a basis for the infinite-dimensional vector space of sufficiently smooth IID's (see Theorem II). We also will find an explicit integral formula for the components of an IID on S^2-polar cap. This formula is analogous to Poisson's integral formula for harmonic functions on the disk. As a consequence, an IID on S^2-polar cap which is continuous on the border of S^2-polar cap will then be real-analytic on the rest of S^2-polar cap.

2. Conventions, Definitions, and Statement of Main Results. We let $f(r,\theta)=(\sin r\cos\theta, \sin r\sin\theta, \cos r)$. That is, f is just the parameterization of $S^2\subset E^3$ in terms of geodesic polar coordinates (r,θ) at $(0,0,1)\in S^2$. We let $e_r=\partial f/\partial r$, $e_\theta=(\partial f/\partial\theta)/\|(\partial f/\partial\theta)\|$, $e_n=f$ be the usual unit frame field on $S^2-\{(0,0,1)\}$ associated with the above coordinates. Any vector field on $S^2-\{(0,0,1)\}$ may be expressed as $V=v_1e_r+v_2e_\theta+v_3e_n$. A deformation of S^2 along V may be written in terms of the original coordinates (r,θ) as $G(r,\theta,t)=f+tV$. Assuming that V is C^1, the line element at time t_0 is $\|G_r\|^2dr^2+2G_r\cdot G_\theta\,dr\,d\theta+\|G_\theta\|^2d\theta^2$ at time t_0. The condition that V be an infinitesimal deformation is clearly that $\partial/\partial t\|G_r\|^2=(\partial/\partial t)2G_r\cdot G_\theta=(\partial/\partial t)\|G_\theta\|^2=0$ at $t=0$. In terms of V and f, these become

$$\frac{\partial f}{\partial r}\cdot\frac{\partial V}{\partial r}=0,\qquad \frac{\partial f}{\partial r}\cdot\frac{\partial V}{\partial\theta}+\frac{\partial f}{\partial\theta}\cdot\frac{\partial V}{\partial r}=0,\qquad \frac{\partial f}{\partial\theta}\cdot\frac{\partial V}{\partial\theta}=0.$$

Furthermore, in terms of the components of $V=v_1e_r+v_2e_\theta+v_3e_n$, these equations finally, after a small computation, become

$$\frac{\partial v_1}{\partial r}+v_3=0;\tag{1}$$

$$\frac{\partial v_1}{\partial\theta}-v_2\cos r+\frac{\partial v_2}{\partial r}\sin r=0;\tag{2}$$

$$v_1\cos r+\frac{\partial v_2}{\partial\theta}+v_3\sin r=0.\tag{3}$$

Now upon eliminating v_3 by using (1) we get

$$\frac{\partial v_1}{\partial\theta}=v_2\cos r-\frac{\partial v_2}{\partial r}\sin r;\tag{2$'$}$$

$$\frac{\partial v_2}{\partial\theta}=-v_1\cos r+\frac{\partial v_1}{\partial r}\sin r.\tag{3$'$}$$

If we assume V is C^2, we may reduce the above system to a single equation in v_1 as follows: Differentiate (2)$'$ with respect to θ and in the resulting equation replace the $\partial v_2/\partial\theta$ and $\partial v_2/\partial\theta\,\partial r=\partial v_2/\partial r\partial\theta$ by equivalent expressions given by (3)$'$ and the derivative of (3)$'$ with respect to r. If we write u for v_1 and denote partial differentiation by subscripts we then obtain

$$\sin^2 r u_{rr}+u_{\theta\theta}-\sin r\cos r u_r+u=0.\tag{*}$$

Before stating the principal results of our forthcoming derivations, it should be noted that any constant vector field in E^3 restricted to S^2 is an IID. Thus we might as well assume that V vanishes at $(0,0,1)$ (i.e., $r=0$), since any IID can be obtained from such a V by adding a constant vector field. Assuming $V=0$ at $r=0$, not only do we have $u(0,\theta)=0$, but also $u_r(0,\theta)=(\partial v_1/\partial r)(0,\theta)=-v_3(0,\theta)=0$ by equation (1). With this understanding we state:

THEOREM I. *The fundamental product solutions of* (*) *satisfying* $u(0,\theta)=u_r(0,\theta)=0$, *and* $u(r,\theta+2\pi)=u(r,\theta)$ *are*:

$$u_n^c(r,\theta) \equiv \tan^n\left(\frac{r}{2}\right)\sin r \cos n\theta \quad n \geqslant 1$$

$$u_n^s(r,\theta) \equiv \tan^n\left(\frac{r}{2}\right)\sin r \sin n\theta \quad n \geqslant 0.$$

These yield the following IID's *of* $S^2 - \{(0,0,-1)\}$.

$$V_n^c = \tan^n\left(\frac{r}{2}\right)[\sin r \cos n\theta e_r + \sin r \sin n\theta e_\theta - (n + \cos r)\cos n\theta e_n]$$

$$V_n^s = \tan^n\left(\frac{r}{2}\right)[\sin r \sin n\theta e_r - \sin r \cos n\theta e_\theta - (n + \cos r)\sin n\theta e_n].$$

THEOREM II. *Let D_a be a closed geodesic disk centered on* $(0,0,1)$ *and of radius $a < \pi$ on the unit sphere. Let V be any C^2* IID *on D_a which vanishes at $r=0$ and for which $\partial v_2/\partial r(0,\theta) = 0$ (see remark at end of next section). Then $V(r,\theta) = \sum_{n=1}^{\infty}(a_n V_n^c + b_n V_n^s)$ where a_n and b_n are determined by*

$$\begin{Bmatrix} a_n \\ b_n \end{Bmatrix}\tan^n\left(\frac{a}{2}\right)\sin a = \frac{1}{\pi}\int_0^{2\pi} u(a,\theta)\begin{Bmatrix} \sin n\theta \\ \cos n\theta \end{Bmatrix} d\theta$$

where $u(r,\theta) = v_1(r,\theta)$ = radial (i.e., $\partial/\partial r$) component of V.

THEOREM III. *Let $g(\theta)$ be any continuous function such that $\int_0^{2\pi} g(\theta)d\theta = 0$. Then there is a unique function $u(r,\theta)$ which is continuous for $r \leqslant a$, C^2 for $r < a$, satisfies (*) for $r < a$, and is such that $u(0,\theta) = u_r(0,\theta) = 0$ and $u(a,\theta) = g(\theta)$. Moreover for $r < a$, we have the following integral formula for u:*

$$u(r,\theta) = \frac{\sin r}{\sin a}\frac{\rho(r,a)}{\pi}\int_0^{2\pi} g(\psi)\frac{\cos(\theta-\psi) - \rho(r,a)\,d\psi}{1 + \rho(r,a)^2 - 2\rho(r,a)\cos(\theta - \psi)} \tag{**}$$

where $\rho(r,a) = \tan\frac{r}{2}/\tan\frac{a}{2}$.

COROLLARY. *Given a function $g(\theta)$, as in Theorem* III, *there is a unique* IID, *say, $V = v_1 e_r + v_2 e_\theta + v_3 e_n$, defined and C^2 for $r < a$ such that $V(0,\theta) = 0$, $(\partial v_2/\partial r)(0,\theta) = 0$, and v_1 has a continuous extension to D_a such that $v_1(a,\theta) = g(\theta)$. Moreover $v_1(r,\theta) = u(r,\theta)$, $v_2(r,\theta) = -\sin r \int_0^r \sin^{-2} r u_\theta(r,\theta)dr$, and $v_3(r,\theta) = -u_r$ where u is given by (**).*

In closing this section, the author thought it would be of interest to include some illustrations of what surfaces result if one moves a hemisphere along (i.e., to the end of) an infinitesimal deformation V. These figures were plotted by a Hewlett-Packard 9100B curve plotter. Figures 1 and 2 show an undeformed hemisphere from horizontal and vertical directions. The separation of the longitudinal lines and latitudes is 18°. Figures 3 and 4 are two horizontal views (along the x and y axes, respectively) of the hemisphere deformed along $\frac{1}{10}V_2^c$ (see Theorem I). Figure 5 is a vertical view of the same deformation. Figure 6 is a vertical view of the deformation along $(\sqrt{2}/20)V_3^c$. These illustrations bear a striking resemblance to what one gets by applying inward pressure (with the fingers) at two or three equally spaced points on the equator of half a tennis ball, as can be verified and as might be expected.

3. Derivation of Theorem I. If $u(r,\theta) = R(r)\Theta(\theta)$, then separation of variables in (*) leads to the equations

$$\sin^2 r R'' - \sin r \cos r R' + (1 - m^2)R = 0 \qquad R(0) = R'(0) = 0 \qquad \Theta'' + m^2\Theta = 0$$

where m is an integer since $\Theta(\theta + 2\pi) = \Theta(\theta)$. Now, to prove Theorem I, we could merely verify that a solution to the top equation is $R_m(r) = \tan^m(r/2)\sin r$; and since the roots of the indicial equation are $1 \pm m$, we see that this is the only (up to multiplicative constant) solution satisfying $R(0) = R'(0) = 0$. However the aim here is to show how the solutions $R_m(r) = \tan^m(r/2)\sin r$ are derived.

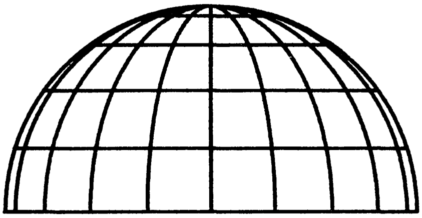

FIG. 1

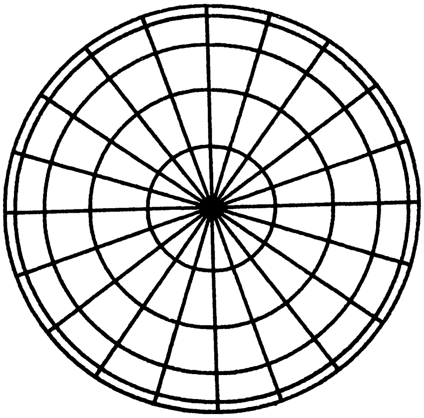

FIG. 2

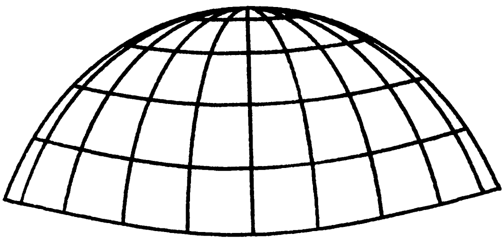

FIG. 3

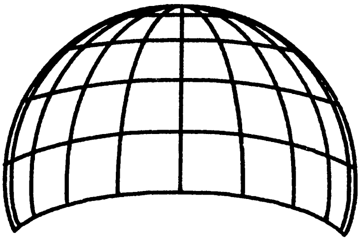

FIG. 4

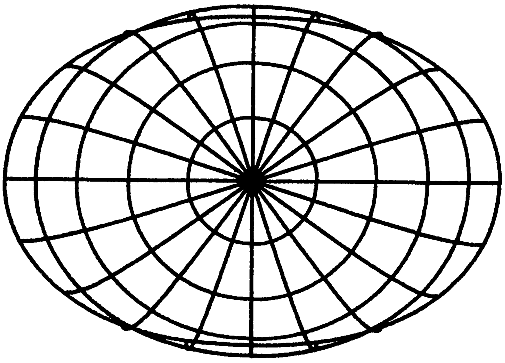

FIG. 5

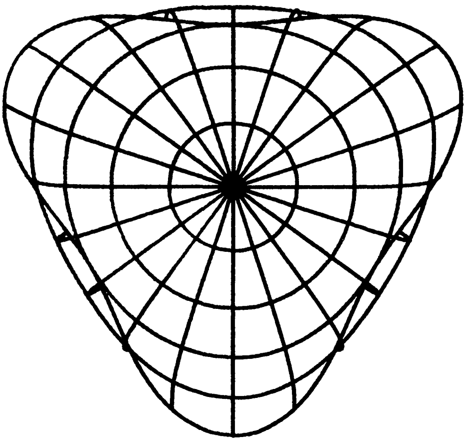

FIG. 6

To make the differential equation for R more amenable to solution by the method of Frobenius, we let $x=\sin r$ and define $y(x)=y(\sin r)=R(r)$. Then under this change of variable the equation becomes

$$x^2(1-x^2)y''-xy'+(1-m^2)y=0 \qquad y(0)=y'(0)=0.$$

We assume a series solution of the form $y(x)=\sum_{n=0}^{\infty}a_n x^{n+q}$ since $x=0$ is a regular singular point. We find from the indicial equation that $a_0\neq 0 \Rightarrow q=1\pm m$, and from $y(0)=y'(0)=0$ we must have $q=1+m$, $m\geqslant 1$. Continuing in the standard fashion we arrive at the solution

$$y(x)=x^{m+1}\sum_{k=0}^{\infty}\frac{\left(\frac{m+1}{2}\right)_k\left(\frac{m}{2}\right)_k}{(m+1)_k k!}(x^2)^k$$

where $(\lambda)_0\equiv 1$ and $(\lambda)_k\equiv\lambda(\lambda+1)\cdots(\lambda+k-1)$ for $k\geqslant 1$. Now the hypergeometric functions

$$F(\alpha,\beta;\gamma;z)\equiv\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!}z^k;$$

where $\gamma\neq 0,-1,-2,\ldots$; satisfy the relations [5]

$$F\left(\alpha,\beta;\alpha+\beta+\tfrac{1}{2};z\right)=\left(\frac{1+\sqrt{1-z}}{2}\right)^{\frac{1}{2}-\alpha-\beta}F\left(\alpha-\beta+\tfrac{1}{2},\beta-\alpha+\tfrac{1}{2};\alpha+\beta+\tfrac{1}{2};\frac{1-\sqrt{1-z}}{2}\right).$$

Hence

$$\begin{aligned}y(x)&=x^{m+1}F\left(\frac{m+1}{2},\frac{m}{2};m+1;x^2\right)\\&=x^{m+1}\left(\frac{1+\sqrt{1-x^2}}{2}\right)^{-m}F\left(1,0;m+1;\frac{1+\sqrt{1-x^2}}{2}\right)\\&=x^{m+1}\left(\frac{1+\sqrt{1-x^2}}{2}\right)^{-m}\quad\text{since } F\left(1,0;m+1;\frac{1+\sqrt{1-x^2}}{2}\right)=1.\end{aligned}$$

So

$$\begin{aligned}R(r)&=\sin^{m+1}r\left(\frac{1+\cos r}{2}\right)^{-m}=2^m\sin r\left(\frac{\sin r}{1+\cos r}\right)^m\\&=2^m\sin r\tan^m\left(\frac{r}{2}\right)\end{aligned}$$

and the derivation is complete.

To obtain v_2 and v_3 in the formulas for V_n^c and V_n^s we use equations (1), (2), and (3) of Section 2 as follows. We immediately have $v_3=-\partial v_1/\partial r$. To get v_2 we solve equation (2) for v_2 and get $v_2(r,\theta)=-\sin r[\int_0^r\sin^{-2}r(\partial v_1/\partial\theta)dr+g(\theta)]$ where $g(\theta)$ is some C^1 function. Substituting this value for $v_2(r,\theta)$ into equation (3) and using the fact that v_1 satisfies (*), we arrive at the condition that $g'(\theta)=0$ or $g(\theta)=c$. Although c is completely arbitrary, in the formulas for V_n^c and V_n^s we have chosen $c=0$. This is equivalent to requiring that $(\partial v_2/\partial r)(0,\theta)=0$, as is seen from the formula for $v_2(r,\theta)$. Actually we might as well restrict ourselves to IID's where $(\partial v_2/\partial r)(0,\theta)=0$, since any IID may be brought to this form by subtracting an IID of the form $c\sin r e_\theta$ which corresponds to an IID generated by rotating S^2 about the z-axis in E^3.

4. Proof of Theorem II. Since $u(r,\theta)$ is C^2 in θ for each value of r,

$$u(r,\theta)=\frac{A_0(r)}{2}+\sum_{n=1}^{\infty}[A_n(r)\cos n\theta+B_n(r)\sin n\theta]$$

where

$$A_n(r)=\frac{1}{\pi}\int_0^{2\pi}u(r,\theta)\cos n\theta\,d\theta,\qquad B_n(r)=\frac{1}{\pi}\int_0^{2\pi}u(r,\theta)\sin n\theta\,d\theta.$$

Now

$$\sin^2 r A_n''(r) - \sin r \cos r A_n'(r) + (1-n^2)A_n(r)$$
$$= \frac{1}{\pi}\int_0^{2\pi}(\sin^2 r u_{rr} - \sin r \cos r u_r + u - n^2 u)\cos n\theta\, d\theta$$
$$= \frac{-1}{\pi}\int_0^{2\pi}(u_{\theta\theta} + n^2 u)\cos n\theta\, d\theta = \frac{-1}{\pi}\int_0^{2\pi}\frac{\partial}{\partial\theta}(n\sin n\theta u + u_\theta \cos n\theta)\, d\theta$$
$$=0.$$

Differentiating under the integral is permitted since $u(r,\theta)$ is C^2. By the same argument $B_n(r)$ satisfies the same equation. Moreover, since $u(0,\theta)=u_r(0,\theta)=0$ we have $A(0)=B(0)=A'(0)=B'(0)=0$. Hence by results in Section 3, we have that $A_n(r)$ and $B_n(r)$ are constant multiples of $\tan^n(r/2)\sin r$ for $n \geqslant 1$ and $A_0(r)=0$. Thus

$$u(r,\theta) = \sum_{n=1}^{\infty}\tan^n\left(\frac{r}{2}\right)\sin r(a_n\cos n\theta + b_n \sin n\theta)$$

where the a_n and b_n are determined by the formula in Theorem II.

To complete the proof we need to show that in the formulas $v_2 = -\sin r \int_0^r \sin^{-2} r(\partial u/\partial\theta)\,dr$ (see Section 3) and $v_3 = -\partial u/\partial r$ the differentiation and integration may be carried out termwise. Since $v_2(r,\theta)$ and $v_3(r,\theta)$ are C^2 in θ, the Fourier series in $\sin n\theta$ and $\cos n\theta$ with coefficients being functions of r converge to v_2 and v_3. For v_3, the coefficient of, say, $\cos n\theta$, is

$$\frac{1}{\pi}\int_0^{2\pi}\frac{-\partial u}{\partial r}\cos n\theta\, d\theta = \frac{-\partial}{\partial r}\frac{1}{\pi}\int_0^{2\pi}u\cos n\theta\, d\theta$$

since $\partial u/\partial r$ is C^0 (in fact C^1). This establishes the validity of termwise differentiation in $v_3 = -\partial u/\partial r$. For v_2, the coefficient of, say, $\cos n\theta$, is

$$\frac{-1}{\pi}\int_0^{2\pi}\left[\sin r\int_0^r \sin^{-2} r\frac{\partial u}{\partial\theta}dr\right]\cos n\theta\, d\theta = \frac{-\sin r}{\pi}\int_0^{2\pi}\left[\int_0^r \sin^{-2} r\frac{\partial u}{\partial\theta}\cos n\theta\, dr\right]d\theta$$
$$= \frac{-\sin r}{\pi}\int_0^r\int_0^{2\pi}\sin^{-2} r\frac{\partial u}{\partial\theta}\cos n\theta\, d\theta\, dr \quad \text{by Fubini}$$
$$= -\sin r\int_0^r \sin^{-2} r\left[\frac{1}{\pi}\int_0^{2\pi}\frac{\partial u}{\partial\theta}\cos n\theta\, d\theta\right]dr$$
$$= -\sin r\int_0^r \sin^{-2} r\left[\frac{1}{\pi}\int_0^{2\pi} - un\sin n\theta\, d\theta\right]dr$$
$$= n\sin r\int_0^r \sin^{-2} r\left[\frac{1}{\pi}\int_0^{2\pi}u\sin n\theta\, d\theta\right]dr$$

which shows that the termwise operation is valid in the formula for v_2.

5. Derivation and Proof of Theorem III and Corollary. In the case where $g(\theta)$ is C^2, we prove first that

$$u(r,\theta) = \sum_{n=1}^{\infty}\tan^n\left(\frac{r}{2}\right)\sin r[a_n\cos n\theta + b_n\sin n\theta]$$

satsifies (*) the conditions imposed on $u(r,\theta)$ in Theorem III where

$$\begin{Bmatrix} a_n \\ b_n \end{Bmatrix}\tan^n\left(\frac{a}{2}\right)\sin a = \frac{1}{\pi}\int_0^{2\pi}g(\theta)\begin{Bmatrix}\sin n\theta \\ \cos n\theta\end{Bmatrix}d\theta.$$

First, since $g(\theta)$ is C^2, we know that the nth Fourier coefficients of g are bounded in absolute value by M/n^2 for some constant M independent of n [1]. Thus the nth term in the above

infinite sum for $u(r,\theta)$ is bounded by $2M/n^2$, and the sum therefore converges uniformly to a continuous function on D_a. Also $(1/\pi)\int_0^{2\pi}g(\theta)\,d\theta=0$ implies that the sum for $r=a$ is $g(\theta)$. It is not difficult to verify that termwise differentiation of the sum yields sums absolutely uniformly convergent on any subdisk $0\leqslant r\leqslant r_0<a$. This is basically since $\sum_{n=0}^{\infty}P(n)k^n<\infty$ provided $k<1$ and $P(n)$ is a polynomial. Thus, u not only satisfies (*) (since each term does) but u is C^∞ and $u(0,\theta)=u_r(0,\theta)=0$. Thus we have the existence of the desired u in the case where $g(\theta)$ is C^2, and the uniqueness follows from the first part of the proof of Theorem II in Section 4.

Now we show how the integral formula can be derived from justifiable manipulations of the infinite sum:

$$u(r,\theta)=\sum_{n=1}^{\infty}\frac{\tan^n\frac{r}{2}\sin r}{\tan^n\frac{a}{2}\sin a}\left(\frac{1}{\pi}\int_0^{2\pi}g(\psi)\cos n\psi\,d\psi\cos n\theta+\frac{1}{\pi}\int_0^{2\pi}g(\psi)\sin n\psi\,d\psi\sin n\theta\right)$$

$$=\frac{1}{\pi}\frac{\sin r}{\sin a}\sum_{n=1}^{\infty}\rho(r,a)^n\int_0^{2\pi}g(\psi)\cos[n(\psi-\theta)]\,d\psi$$

$$=\frac{1}{\pi}\frac{\sin r}{\sin a}\int_0^{2\pi}g(\psi)\sum_{n=1}^{\infty}\rho(r,a)^n\cos[n(\psi-\theta)]\,d\psi.$$

Now note that

$$\sum_{n=1}^{\infty}\rho^n\cos n\omega=\mathrm{RE}\left(\sum_{n=1}^{\infty}[\rho e^{i\omega}]^n\right)$$

$$=\mathrm{RE}\big([1-\rho e^{i\omega}]^{-1}-1\big)=\rho(\cos\omega-\rho)/1+\rho^2-2\rho\cos\omega.$$

Thus the above becomes

$$u(r,\theta)=\frac{\sin r}{\sin a}\frac{\rho(r,a)}{\pi}\int_0^{2\pi}g(\psi)\frac{\cos(\psi-\theta)-\rho(r,a)\,d\psi}{1+\rho(r,a)^2-2\rho(r,a)\cos(\psi-\theta)}.\qquad (**)$$

Now we have assumed thus far that $g(\theta)$ is C^2, but the formula above for u makes sense for $g(\theta)$ merely continuous as long as $r<a$. Moreover (**) defines a solution to (*) for $r<a$, provided we can show that

$$K(r,\theta)=\frac{\sin r}{\sin a}\frac{\rho(r,a)[\cos(\psi-\theta)-\rho(r,a)]}{1+\rho(r,a)^2-2\rho(r,a)\cos(\psi-\theta)}$$

satisfies (*). This may be verified by direct computation or more easily as follows. Suppose $A(\psi,r,\theta)\equiv\sin^2 rK_{rr}+K_{\theta\theta}-\sin r\cos rK_r+K$ and assume $A(\psi_0,r_0,\theta_0)\neq0$ for some $r_0<a$. Let $g(\psi)$ be some C^∞ function positive in (and vanishing outside) a neighborhood of ψ_0 where $h(\psi)=A(\psi,r_0,\theta_0)$ is of constant sign. Then note that (**) defines a function which does not satisfy (*) at (r_0,θ_0), contradicting the fact that we know this function satisfies (*) for a C^2 function $g(\theta)$.

It remains to show that the function $u(r,\theta)$ defined by (*) for $r<a$ extends to a function which is continuous for $r\leqslant a$, such that $u(a,\theta)=g(\theta)$. This amounts to showing that for $P=(r,\theta)$, $r<a$, and $P_0=(a,\theta_0)$ we have:

$$\lim_{P\to P_0}\frac{\sin r}{\sin a}\frac{\rho(r,a)}{\pi}\int_0^{2\pi}\frac{g(\psi)[\cos(\psi-\theta)-\rho(r,a)]\,d\psi}{1+\rho(r,a)^2-2\rho(r,a)\cos(\psi-\theta)}$$

$$=\lim_{P\to P_0}\frac{1}{\pi}\int_0^{2\pi}\frac{g(\psi)[\cos(\psi-\theta)-\rho(r,a)]\,d\psi}{1+\rho(r,a)^2-2\rho(r,a)\cos(\psi-\theta)}=g(\theta_0)$$

$$=\lim_{P\to P_0}\frac{1}{2\pi}\int_0^{2\pi}\frac{g(\psi)\left[1-\rho(r,\theta)^2\right]d\psi}{1+\rho(r,a)^2-2\rho(r,a)\cos(\psi-\theta)}$$

where the last equality is due to Poisson's formula for harmonic functions on the unit disk. If we show that the limit of the difference of the integrals I_1 and I_2 inside the limits of the second and last expressions is 0, then we are clearly done. Since $\int_0^{2\pi} g(\theta)\,d\theta = 0$ we may add $g(\psi)/2$ to the integrand of I_1 without changing I_1. Then, after some computation, we get

$$I_1 - I_2 = \frac{1}{\pi}\int_0^{2\pi} \frac{g(\psi)[1-\rho(r,a)][\cos(\psi-\theta)-\rho(r,a)]\,d\psi}{1+\rho(r,a)^2 - 2\rho(r,a)\cos(\psi-\theta)}.$$

Now for each (r,θ) (with $r<a$) the integrand is bounded in absolute value by the function $|g(\psi)|$, since the function $f(\omega) = ((1-\rho)(\cos\omega - \rho))/(1+\rho^2 - 2\rho\cos\omega)$ for $\rho<1$ has extrema at $\omega = 0$ and π and $-1 \leqslant (\rho-1)/(\rho+1) = f(\pi) \leqslant f(\psi) \leqslant f(0) = 1$. Hence by the Lebesgue dominated convergence theorem $\lim_{P\to P_0}(I_1 - I_2) = 0$ since the limit of the integrand is 0 except at $\psi = \theta_0$ (a set of measure 0). It is clear that for $u(r,\theta)$ defined by equation (**), $u(0,\theta) = u_r(0,\theta) = 0$ and thus $u(r,\theta)$ satisfies all the desired properties in Theorem III. Uniqueness is clear since the difference between two candidates would be a candidate for the case where $g(\theta) \equiv 0$, and the only candidate in this case is identically 0 by the uniqueness in the case $g(\theta)$ is C^2.

The Corollary is immediate from Theorem III once we check back to the last part of Section 3 and note that $(\partial v_2/\partial r)(0,\theta) = 0$ implies $v_2(r,\theta) = -\sin r \int_0^r \sin^{-2} r\, u_\theta(r,\theta)\,dr$, $r<a$.

6. Further Questions. Many open questions in the area of rigidity of surfaces are in [2]. Of particular interest in relation to this article is the following. Is there a parameterized family of surfaces $G(t,r,\theta)$, $r \leqslant a$, such that $\|G_r\|^2 = 1$, $G_r \cdot G_\theta = 0$, $\|G_\theta\|^2 = \sin^2 r$, $G(0,r,\theta) = f(r,\theta)$ (defined in 2) and $G_t(0,r,\theta) = V(r,\theta)$ for a given IID V? In other words, can V be realized as the initial velocity of an actual isometric deformation? Other questions arise if we ask whether the general behavior of IID's on geodesic disks of surfaces of positive variable Gaussian curvature is the same as that for spherical segments. Then again we might consider regions other than geodesic disks and try to find suitable integral formulas for IID's in terms of boundary data. It seems that perhaps many of these other questions might be handled (if they haven't been already) by the general theory of elliptic equations or potential theory.

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MISCELLANEA

31. One is always glad to obtain a second proof of a new theorem, especially in the Theory of Sets of Points, where the reasoning is so subtle that almost all who have written on it have at one time or another stumbled.

—W. H. Young, *Proc. London Math. Soc.*, (2) 8 (1910), p. 117

POSITIVE TEMPERATURES WITH PRESCRIBED INITIAL HEAT DISTRIBUTIONS

CALVIN H. WILCOX

Introduction. The temperature at time t and position x on an infinite uniform insulated rod, measured in degrees above absolute zero, is a non-negative solution of the heat equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}. \tag{1}$$

Moreover

$$Q(a,b;t)=\int_a^b u(t,x)\,dx, \qquad a<b, \tag{2}$$

is the quantity of heat at time t in the portion (a,b) of the rod. Thus it is natural to pose the problem of constructing a non-negative solution of (1) whose initial heat distribution $Q(a,b;0+)$ exists and coincides with a prescribed Borel measure μ on R. The purpose of this note is to show that D. V. Widder's theory of positive temperature functions [5], [6] provides a complete solution of this problem.

Formulation of the Problem. The class of non-negative solutions of (1) in the domain $\Omega_T=(0,T)\times R, 0<T\leqslant +\infty$, will be denoted by

$$H_T^+=C^\infty(\Omega_T)\cap\{u\,|\,u(t,x)\geqslant 0 \quad \text{and} \quad (1) \text{ holds in } \Omega_T\}. \tag{3}$$

A solution $u\in H_T^+$ will be said to have a Borel measure μ on R as its initial heat distribution if

$$Q(a,b;0+)=\mu(a,b) \quad \text{for all} \quad a<b \quad \text{such that} \quad \mu\{a\}=\mu\{b\}=0, \tag{4}$$

where $\mu\{x\}$ is the μ-measure of the point x. Necessary and sufficient conditions on T and μ for the existence of such a solution are derived below from the following two theorems of Widder.

WIDDER'S REPRESENTATION THEOREM [6, p. 136]. *Let*

$$k(t,x)=(4\pi t)^{-1/2}\exp\{-x^2/4t\} \tag{5}$$

denote the fundamental solution of the heat equation. Then $u\in H_T^+$ if and only if there exists a non-decreasing function $\beta(x)$ on $-\infty<x<\infty$, normalized by $\beta(0)=0$ and $\beta(x+)=\beta(x)$, such that

$$\int_{-\infty}^{\infty}\exp\{-x^2/4t\}\,d\beta(x)<\infty \quad \textit{for all } t\in(0,T) \tag{6}$$

and

$$u(t,x)=\int_{-\infty}^{\infty}k(t,x-\xi)\,d\beta(\xi) \quad \textit{for all } (t,x)\in\Omega_T. \tag{7}$$

WIDDER'S INVERSION THEOREM [6, p. 69]. *If $u\in H_T^+$ has the representation* (7), *then for all real a,b with $a<b$*

$$\lim_{t\to 0+}\int_a^b u(t,x)\,dx=\frac{1}{2}[\beta(b)+\beta(b-)-\beta(a)-\beta(a-)]. \tag{8}$$

It is well known that there is a one-to-one correspondence between non-decreasing functions

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$\beta(x)$ on R, normalized as above, and Borel measures ν on R. If β is given, then ν is the unique Borel measure such that

$$\nu(a,b]=\beta(b)-\beta(a) \tag{9}$$

for all half-open intervals $(a,b]$. If ν is given, then β is the function

$$\beta(x)=\begin{cases} \nu(0,x], & x>0 \\ 0, & x=0 \\ -\nu(x,0], & x<0 \end{cases} \tag{10}$$

and the correspondences defined by (9) and (10) are inverse to each other. By using the relation (9) the improper Riemann-Stieltjes integrals in (6), (7) may be interpreted as Lebesgue-Stieltjes integrals with respect to ν. This leads to the following reformulation of Widder's theorems.

COROLLARY 1. *$u\in H_T^+$ if and only if there exists a Borel measure ν on R such that*

$$\int_{-\infty}^{\infty}\exp\{-x^2/4t\}\nu(dx)<\infty \quad \textit{for } 0<t<T \tag{11}$$

and

$$u(t,x)=\int_{-\infty}^{\infty}k(t,x-\xi)\nu(d\xi) \quad \textit{for all } (t,x)\in\Omega_T. \tag{12}$$

COROLLARY 2. *If $u\in H_T^+$ has the representation* (12), *then for all real a,b with $a<b$*

$$Q(a,b;0+)=\nu(a,b)+\frac{1}{2}\nu\{a\}+\frac{1}{2}\nu\{b\}. \tag{13}$$

Equation (13) follows from (8) and the observation that (9) implies $\nu(a,b)=\beta(b-)-\beta(a),\nu\{a\}=\beta(a)-\beta(a-)$ and $\nu\{b\}=\beta(b)-\beta(b-)$. A complete solution of the initial heat distribution problem is given by the following

EXISTENCE AND UNIQUENESS THEOREM. *There exists a solution $u\in H_T^+$ whose initial heat distribution is the Borel measure μ if and only if*

$$\int_{-\infty}^{\infty}\exp\{-x^2/4t\}\,\mu(dx)<\infty \quad \textit{for all } t\in(0,T). \tag{14}$$

If μ satisfies (14), *then the solution is given by*

$$u(t,x)=\int_{-\infty}^{\infty}k(t,x-\xi)\mu(d\xi) \quad \textit{for all } (t,x)\in\Omega_T \tag{15}$$

and is unique.

Proof. Widder's representation theorem implies that any solution must have a representation (12) where ν satisfies (11). Widder's inversion theorem and (4) then imply that $\mu(a,b)=\nu(a,b)$ for all a and b $(>a)$ such that $\mu\{a\}=\mu\{b\}=\nu\{a\}=\nu\{b\}=0$. But this implies that μ and ν are identical Borel measures on R. Hence (11) implies that $\mu=\nu$ must satisfy (14). Conversely, (14) implies that (15) defines a $u\in H_T^+$ that satisfies (4), by Corollaries 1 and 2. Finally, (15) is the unique solution because any other solution would have the form (12), by Corollary 1, and then $\nu=\mu$ by Corollary 2.

Uniqueness of the Fundamental Solution. The uniqueness statement of the theorem implies that $u(t,x)=k(t,x-x_0)\in H_\infty^+$ is characterized by the initial heat distribution

$$\mu_0(a,b]=\begin{cases} 1 & \text{if } x_0\in(a,b] \\ 0 & \text{if } x_0\notin(a,b] \end{cases} \tag{16}$$

corresponding to a unit atom of heat at x_0.

Behavior of $Q(a,b;0+)$ at the Atoms of μ. The solution $u\in H_T^+$ with initial heat distribution

μ is characterized by the requirement that $Q(a,b;0+)=\mu(a,b)$ for all a and b ($>a$) that are not atoms of μ. Corollary 2 then implies that $Q(a,b;0+)$ exists for all real a and $b>a$ and satisfies

$$Q(a,b;0+)=\mu(a,b)+\frac{1}{2}\mu\{a\}+\frac{1}{2}\mu\{b\}. \tag{17}$$

Positive Temperatures with Finite Heat. If the initial heat distribution is finite,

$$\mu(-\infty,\infty)<\infty, \tag{18}$$

then $u\in H_\infty^+, u$ has finite total heat and the law of conservation of heat holds:

$$Q(-\infty,\infty;t)=\int_{-\infty}^{\infty} u(t,x)dx=\mu(-\infty,\infty) \quad \text{for all } t>0. \tag{19}$$

To show this let $\mu(a,b]=\alpha(b)-\alpha(a)$ where $\alpha(x)$ is a normalized nondecreasing function. Then integration by parts gives (see [**6**, p. 70])

$$\int_a^b u(t,x)dx=\int_{-\infty}^{\infty} k(t,x)\alpha(b-x)dx-\int_{-\infty}^{\infty} k(t,x)\alpha(a-x)dx \tag{20}$$

for all finite a and b. Moreover, $\alpha(b-x)\leqslant\alpha(+\infty)<\infty$ for all x and b. Hence, $k(t,x)\alpha(b-x)\leqslant k(t,x)\alpha(+\infty)$ for all $t>0,b$ and x and $\lim_{b\to\infty}k(t,x)\alpha(b-x)=k(t,x)\alpha(+\infty)$ for all $t>0$ and x. Thus

$$\lim_{b\to\infty}\int_{-\infty}^{\infty} k(t,x)\alpha(b-x)dx=\alpha(+\infty) \quad \text{for all } t>0 \tag{21}$$

by Lebesgue's dominated convergence theorem. Similarly,

$$\lim_{a\to-\infty}\int_{-\infty}^{\infty} k(t,x)\alpha(a-x)dx=\alpha(-\infty) \quad \text{for all } t>0. \tag{22}$$

Combining (20), (21), and (22) gives (19), since $\mu(-\infty,\infty)=\alpha(+\infty)-\alpha(-\infty)$.

Absolutely Continuous Initial Heat Distributions. If μ is absolutely continuous with respect to Lebesgue measure, then

$$\mu(a,b)=\int_a^b u_0(x)dx \quad \text{for all } a<b \tag{23}$$

where u_0 is locally Lebesgue integrable and $u_0(x)\geqslant 0$ almost everywhere. In this case the existence condition (14) becomes

$$\int_{-\infty}^{\infty}\exp\{-x^2/4t\}u_0(x)dx<\infty \quad \text{for all } t\in(0,T) \tag{24}$$

and the corresponding solution

$$u(t,x)\int_{-\infty}^{\infty} k(t,x-\xi)u_0(\xi)d\xi \tag{25}$$

satisfies

$$Q(a,b;0+)=\int_a^b u_0(x)dx \quad \text{for all } a<b. \tag{26}$$

Moreover,

$$\lim_{t\to0+} u(t,x)=u_0(x) \tag{27}$$

whenever x is a Lebesgue point of u_0 [**6**, p. 67]. In particular, (27) holds almost everywhere. These results provide a natural generalization of the classical Cauchy problem for the heat equation to a large class of discontinuous temperatures.

Generalizations. It is natural to conjecture, on both physical and mathematical grounds, that the theory presented above should extend to heat diffusion in homogeneous and inhomogeneous

media filling domains in n-dimensional Euclidean space and other Riemannian manifolds. Indeed, Widder's representation theorem has been extended to normal parabolic equations in $\mathbb{R}^n$ by M. Krzyżański [4] and to the class of linear parabolic equations of divergence structure in $\mathbb{R}^n$ by D. G. Aronson [1]. Moreover, for the same class of equations, Aronson has proved a weak form of Widder's inversion theorem which states that [2]

$$\lim_{t\to 0+}\int_{\mathbb{R}^n} u(t,x)\psi(x)dx=\int_{\mathbb{R}^n}\psi(x)\nu(dx) \tag{28}$$

for all continuous ψ that vanish sufficiently rapidly at ∞. More recently, Aronson [3] has shown that if $S\subset\mathbb{R}^n$ is a Borel set such that $\nu(\partial S)=0$, then (28) holds when $\psi=\chi_S$, the characteristic function of S. This is a partial generalization of Widder's inversion theorem. No generalizations to Borel sets such that $\nu(\partial S)>0$ appear to be known. The search for such generalizations should lead to challenging problems in the theories of measure and diffusion.

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CIRCULAR PRIORITIES IN SECURED TRANSACTIONS

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Suppose that an inadequate fund of size X is to be distributed among three "creditors" K_i, $i=1,2,3$. Suppose further that the rules which govern the distribution appear to be inconsistent in that K_1 is to be paid his full claim on the fund before K_2 receives any payment, K_2 before K_3, and K_3 before K_1. How should the fund be distributed?

Such a situation may arise under the laws relating to lending; policies which in themselves appear reasonable or even necessary collectively produce the above results. As we shall see, a first approach to the problem is to define a quantity E_i which we shall call the ith party's "entitlement"; and then to pay the ith party P_i, where P_i is chosen so as to best approximate E_i in the mean-square sense, subject to the constraint that the P_i must sum to X. Unfortunately, such a solution runs foul of other legal policies and we are driven to the use of optimal control theory to find an acceptable solution.

It is this interplay between the legal policies and the resulting mathematical structures which is the theme of this paper. We begin with the legal story, observing that the "circular priorities"

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problem is, in an historical sense, an inevitable result of conflicting interests and policies.

A security interest is an interest in or a right over another's property which may be exercised to guarantee the performance of some obligation, usually the repayment of a loan. One of the oldest forms of a security interest, familiar to most young houseowners, is the mortgage over land and buildings. A consideration of the main features of such a mortgage will illustrate the main principles involved.

When the prospective homeowner wishes to borrow money in order to purchase his new property, he is required to promise to repay. This promise is legally enforceable, but it can, as a practical matter, only be enforced if the debtor has the money to pay. In the vast majority of defaults on loans, the debtor would like to pay but cannot. In such circumstances the legal enforceability of the promise to repay is of little consequence. Thus, the lender will require that he be given a security interest in the property to be purchased. There is an actual splitting of the rights of ownership. The borrower has the right to possession and to use the property so long as he does not default on the loan. The lender has the right, upon default, to seize and sell the property at a fair price and to repay himself from the proceeds.[1]

The borrower has another very important right, the right of *redemption*, i.e., the right to repay the lender according to their agreement and to thereby terminate the lender's interest in the property. Historically, this was a right which could itself serve as security for a further loan; the security interest so created was known as a second mortgage; third and subsequent mortgages could be created.

The way the system worked in practice was that the property would be sold at auction. The first mortgagee would be paid from the proceeds; the remainder, if any, would be applied to the debt owed the second mortgagee, and so on. The first mortgagee is said to have *priority* over the second.[2]

Once the advantages of the mortgage over real property were realized, it was natural to attempt to apply the same principle to goods; if I can borrow money by means of a mortgage over my land, why not borrow a smaller sum by offering my horse and cart as security? This reasonable (to us) suggestion was the subject of heated judicial opposition. There is, said that opinion, a fundamental difference between land and goods: goods are highly mobile and may be transferred from one person to another with a minimum of formality. A prospective lender who was interested in taking a security interest over land could investigate the deeds of ownership of the land. There was no comparable documentation for determining the ownership of a horse and cart. The early judges felt that the only fair way of coping with this state of affairs was to prohibit the transfer of *any* interest in goods *unless* it was accompanied by a transfer of possession of the goods.[3] The legal framework in which this prohibition found its expression was that of fraud: transferring an interest in goods while retaining possession would enable a debtor to falsely represent that he was well-off, thereby obtaining even more credit. The principle is very old and was a settled part of the common law from the time of *Twyne's Case* in 1601.[4]

Notice that the prohibition was *not* against using goods as security for a loan; it has always been possible to split the rights in goods by the ancient institution of the *pledge*, but in a pledge the rights which were *transferred* included the right to possession.

The pledge, which still enjoys a healthy existence in pawn shops all over the world, involves handing over possession of the goods to the creditor, the debtor retaining the right of redemption of the goods upon paying off the loan according to the contractual terms. This type of security over goods was suitable for small loans where the goods pledged by the debtor were items which he could live without. That it must have been reasonably satisfactory for the time is evidenced by the fact that the rule in *Twyne's Case* survived for well over two hundred years; there are few legal rules which have enjoyed such a long and stable existence.

But historical forces were at work which would make the rule unsuitable. As the industrial revolution gained momentum, the real wealth shifted from land to manufacturing. The new industries had an insatiable appetite for credit, for which the mortgage over land was inadequate. The real wealth of industry was in its machinery and its raw material, that is, its goods.

And the only means of security over goods, the pledge, was wholly unsuitable for financing. The whole point of the loan, from the point of view of both the lender and the borrower, was that the borrower could more efficiently operate his business. Parting from the possession of his machinery and raw materials was madness. It was in this context that the legal institutions of the various states and countries finally were compelled to reconsider the rule in *Twyne's Case*.

The resulting Chattels Mortgages Acts differed in detail, but a general scheme was evolved which allowed security interests in goods to be transferred while allowing the debtor to retain possession. The answer was seen to be a system of public registration of such interests; if the transferred interest was so registered, it would be impossible for the debtor to deceive new would-be lenders into advancing "false credit."

Policing of the registration requirement was seen to be a major obstacle to its success. The transactions still were presumed to be fraudulent, the only difference now being that it would be possible to lead evidence showing good faith by the parties.[5] But this presumption of fraud meant that lenders would rather not register. Debtors did not like the registration system for fear that their credit would be damaged; given the views of the business community at the time, this fear was undoubtedly justified. Both lender and borrower disliked the expenses associated with the system of registration.

The policing problem was solved by making the statutes, in a sense, self-enforcing. Unregistered security interests in goods which remained in the possession of the debtor should be ineffective; but not totally ineffective, for that would put the statute in a position of encouraging frauds of a different type. The debtor should not, for example, be able to escape his obligations to the lender merely because the lender had failed to register his interests.

The rule generally adopted is that an unregistered interest is of no effect against the trustee in bankruptcy. Since the major reason for demanding a security interest when granting a loan is to guard against the effect of the debtor's bankruptcy, this provides a powerful incentive for lenders to register their security interests.

A second rule is that an unregistered security interest should be subordinated to later registered interests in the same goods. In the language of priority, a registered interest should have priority over an unregistered interest, even though created later in time.

This last rule was seen to present certain dangers. Suppose that A has an unregistered interest in goods which is *known* to B. It should not be possible for B to gain priority over A merely by rushing to the registration office. Consequently, the rule was modified: Later registered interests could gain priority over earlier unregistered interests if and only if the later interest was registered without knowledge (without notice, as the lawyers say) of the unregistered interest.

It is from these conflicting policies that the phenomenon of circular priorities arises. Suppose that A has an unregistered interest. Suppose that B knows of this, but takes an interest in the same goods and registers. A has priority over B. Further suppose that C later takes and registers an interest in the goods in ignorance of A's unregistered interest. B has priority over C by registration. C has priority over A, being an interest registered without notice of A's unregistered interest. The reader should note carefully that, although the words "prior" and "priority" are used, the relation is *not* transitive.

The problem then is: How should the fund X, realized from the sale of the goods, be distributed among the creditors when the claims of each are C_1, C_2, and C_3, respectively? The only interesting case is when X is insufficient to cover all of the claims.

The simplest judicial solutions to the problem have been merely to break the chain according to some rule.[6] The "first in time" rule is the one which ordinarily applies when all else is equal; in this context, the application of the rule is merely treating the statute as nonexistent. Another approach which has enjoyed some popularity is based on "fault" principles; since the whole mess is caused by A's failing to register, he should go to the back of the queue. The fallacy is apparent: It makes as much, or as little, sense to say that it was B's fault for registering while having notice of A's interest. In short, any "chain breaking" solution is bound to fail to give effect to some of the policies built into the Acts.

Other judicial solutions are based on the legal doctrine of subrogation, a concept which would carry us too far afield at this point. The end result of this approach is subject to the same criticisms as the simple chain-breaking models, namely, that some of the policies of the Acts are sacrificed for one reason or another.

The first serious approach to the problem was made by Benson in 1935.[7] Professor Benson had the mind, if not the training, of a mathematician. Rather than seek ways around the Circular system, he adopted a formulation of it to serve as a fundamental guiding principle: "Each and every claimant is entitled to have applied to his claim the part of the fund remaining after an amount equal to the sum of the claims prior to his has been set aside."

In the ordinary case when the priorities are not circular, the above statement is an accurate description of the amounts to be paid to each creditor. If we note further that no creditor should ever have a negative entitlement or one which exceeds his own claim on the fund, then Benson's principle may be formalized by describing the ith creditor's entitlement as

$$E_i = \max(0, \min(X - \Sigma C_k, C_i))$$

where X is the size of the fund to be distributed and the sum is to be taken over all creditors having priority over the ith. We emphasize that the expression gives the usual entitlement in the ordinary case when the priorities are not circular.

In the case of circular priorities involving three creditors, each creditor has exactly one creditor prior to him; denote by $p(i)$ the creditor prior to the ith. Then

$$E_i = \max(0, \min(X - C_{p(i)}, C_i)).$$

Benson's proposed distribution of the fund depends upon the size of ΣE_i as compared with X; here and throughout the remainder of this paper, the sum is from $i=1$ to $i=3$.

If $\Sigma E_i \leqslant X$, then Benson proposed to pay the ith creditor

$$P_i = E_i + \tfrac{1}{3}(X - \Sigma E_k).$$

However, if $\Sigma E_i > X$, Benson would proceed as follows: noting that E_i is a function of X, find X_0 such that $\Sigma E_i(X_0) = X_0$. Then pay the ith creditor

$$P_i = E_i(X_0) + \tfrac{1}{3}(X - X_0).$$

It is easily deduced that there is a unique X_0 in the open interval $(0, \Sigma C_i)$ with the above property. A tedious but straightforward calculation shows that $X_0 = \frac{1}{2}\Sigma C_i$.

Benson gives little in the way of explanation as to why he settled upon this particular distribution of the fund. He was not a mathematician; his solution was described entirely in words. Gilmore[1] refers to the method as being "exceedingly complex." Nevertheless, as will be seen, Benson had a remarkably good intuitive grasp of the problem.

For a mathematician, the path to a solution beginning from the Benson principle is probably clear. We seek the distribution of funds which best satisfies Benson's principle; that calls for the definition of an objective function; the obvious choice is the sum of squared deviations, i.e.,

$$J = \Sigma (E_i - P_i)^2$$

where P_i is the amount of the fund to be paid to the ith creditor.

If we write "$K_i \to K_j$" for "K_i has priority over K_j," then the circular priorities may now be precisely stated:

Given a fund of size X and creditors K_i with claims C_i, $i = 1,2,3$, and a circular priority pattern $K_1 \to K_2 \to K_3 \to K_1$, find the amounts P_i which minimize J subject to the constraint $\Sigma P_i = X$.

The solution may readily be found by the Lagrange multiplier technique. For this reason, I will call it the Lagrange solution; it is

$$P_i = E_i + \tfrac{1}{3}(X - \Sigma E_k).$$

A simple variational argument shows that it is a true minimum of J.

Note that the distribution agrees with that of Benson when $\Sigma E_i \leqslant X$. We further note that the Lagrange solution is a true generalization of the usual case of noncircular priorities, for it is easily seen that $\Sigma E_k = X$ in such a case.

A simple numerical solution is valuable at this point to illustrate the payoffs under the two solutions; it will also serve to illustrate an interesting deficiency in the two solutions.

Example. Suppose that $C_1=5$, $C_2=4$, and $C_3=3$; suppose that $K_1 \to K_2 \to K_3 \to K_1$. Table 1 shows the distribution of funds for $0 \leqslant X \leqslant 12$. Note that the Benson and Lagrange solutions coincide until the size of fund exceeds 6.

TABLE 1

The Lagrange and Benson Solutions

				Lagrange				Benson			
X	E_1	E_2	E_3	P_1	P_2	P_3	J	P_1	P_2	P_3	J
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
2	0	0	0	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$1\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$1\frac{1}{3}$
3	0	0	0	1	1	1	3	1	1	1	3
4	1	0	0	2	1	1	3	2	1	1	3
5	2	0	1	$2\frac{2}{3}$	$\frac{2}{3}$	$1\frac{2}{3}$	$1\frac{1}{3}$	$2\frac{2}{3}$	$\frac{2}{3}$	$1\frac{2}{3}$	$1\frac{1}{3}$
6	3	1	2	3	1	2	0	3	1	2	0
7	4	2	3	$3\frac{1}{3}$	$1\frac{1}{3}$	$2\frac{1}{3}$	$1\frac{1}{3}$	$3\frac{1}{3}$	$1\frac{1}{3}$	$2\frac{1}{3}$	$1\frac{1}{3}$
8	5	3	3	4	2	2	3	$3\frac{2}{3}$	$1\frac{2}{3}$	$2\frac{2}{3}$	$3\frac{2}{3}$
9	5	4	3	4	3	2	3	4	2	3	5
10	5	4	3	$4\frac{1}{3}$	$3\frac{1}{3}$	$2\frac{1}{3}$	$1\frac{1}{3}$	$4\frac{1}{2}$	$2\frac{1}{2}$	3	$2\frac{1}{2}$
11	5	4	3	$4\frac{2}{3}$	$3\frac{2}{3}$	$2\frac{2}{3}$	$\frac{1}{3}$	5	3	3	1
12	5	4	3	5	4	3	0	5	4	3	0

If there were no more to the problem than this, it would scarcely be worth the time spent. But in 1938 a note appeared in the *Columbia Law Review* calling attention to an undesirable property of the Benson solution. It is one which is shared by the Lagrange solution.[8] Note that between fund sizes of 4 and 5 in the example the second creditor's share actually declines. This is not only anomalous, but also leads to the following "fraud."

We have been assuming that the fund size X is fixed independently of the creditors. In practice, X is likely to be the result of competitive bidding at open auction; in particular, the creditors themselves may bid. Assuming that K_1 and K_3 are mathematicians, they will perform the following "fraud" on K_2: when the bidding reaches $X=4$, each contributes $\frac{1}{2}$ to raise the bid to $X=5$. The result is that each receives a clear profit of $\frac{1}{6}$ from their collusion, a profit which comes directly from K_2's pocket.

The Columbia solution was to note that ΣE_i is a nondecreasing function of X. Consequently, there is an X_0 for which $\Sigma E_i(X_0)=X$. The Columbia solution is to pay each creditor

$$P_i = E_i(X_0).$$

Like the Benson solution, this *ad hoc* production of a formula is unsatisfying to a mathematician.

Once again the approach to take is relatively clear. The deficiency in the Lagrange solution is that $P_i(X)$ decreases with X over certain ranges, i.e., $P_i'(X) < 0$. If we wish to restrict ourselves to solutions for which $P_i'(X) \geqslant 0$, the natural formulation of the problem would seem to be within the framework of optimal control theory; we think of X increasing continuously, our task being to distribute the incoming funds among the three creditors.

As a matter of convenience, identify X with time and treat all functions as functions of time. The state equations of the system being controlled are

$$P_i'(t) = u_i(t) \qquad i = 1,2,3,$$

where we are to choose the functions u_i. The choice is not, of course, unrestricted. We require $u_i(t) \geqslant 0$ for all i and all t in order to satisfy the Columbia objection; we need $\Sigma u_i(t) = 1$ since all of the fund is to be distributed. And we wish to choose $u_i(t)$ in an "optimal" fashion, which means that we must have an objective function; the natural choice is

$$J = \int_0^{\Sigma C_i} \Sigma (E_i - P_i)^2 \, dt.$$

Note that the choice of the functions u_i determines the value of J.

The functions $u_i(t)$ are called controls. In accordance with the language of control theory, let us agree to call a triple of functions *admissible* (for our problem) if they are piecewise continuous, satisfy the constraints mentioned above, and transfer the system from $(P_1, P_2, P_3) = (0,0,0)$ to $(P_1, P_2, P_3) = (C_1, C_2, C_3)$.

We may now state the circular priority problem for variable fund size: choose from the set of admissible controls one which makes J a minimum, in the sense that no other admissible control gives a smaller value of J.

It should be noted that the problem may be as easily phrased in a discrete form as in the continuous form used here; one merely must suffer more subscripts.

The solutions of problems such as this are typically more difficult to find than the simple optimization problems of the Lagrange type. There are two general approaches, the dynamic programming theory of Bellman and the maximum principle of Pontryagin. It is unusual to find closed form solutions; but for problems of small dimension such as this one, numerical solutions may usually be generated. It is interesting to return to our previous numerical example to see how the Columbia solution compares with the "Optimal" solution. Table 2 shows these results.

TABLE 2

The Optimal and Columbia Solutions

				Optimal			Columbia		
X	E_1	E_2	E_3	$P_1(X)$	$P_2(X)$	$P_3(X)$	$P_1(X)$	$P_2(X)$	$P_3(X)$
1	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1	0	0
2	0	0	0	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$1\frac{1}{2}$	0	$\frac{1}{2}$
3	0	0	0	$1\frac{1}{18}$	$\frac{8}{9}$	$1\frac{1}{18}$	2	0	1
4	1	0	0	$2\frac{1}{18}$	$\frac{8}{9}$	$1\frac{1}{18}$	$2\frac{1}{3}$	$\frac{1}{3}$	$1\frac{1}{3}$
5	2	0	1	$2\frac{5}{9}$	$\frac{8}{9}$	$1\frac{5}{9}$	$2\frac{2}{3}$	$\frac{2}{3}$	$1\frac{2}{3}$
6	3	1	2	3	1	2	3	1	2
7	4	2	3	$3\frac{4}{9}$	$1\frac{4}{9}$	$2\frac{1}{9}$	$3\frac{1}{3}$	$1\frac{1}{3}$	$2\frac{1}{3}$
8	5	3	3	$3\frac{17}{18}$	$1\frac{17}{18}$	$2\frac{1}{9}$	$3\frac{2}{3}$	$1\frac{2}{3}$	$2\frac{2}{3}$
9	5	4	3	$3\frac{17}{18}$	$2\frac{17}{18}$	$2\frac{1}{9}$	4	2	3
10	5	4	3	$4\frac{1}{3}$	$3\frac{1}{3}$	$2\frac{1}{3}$	$4\frac{1}{2}$	$2\frac{1}{2}$	3
11	5	4	3	$4\frac{2}{3}$	$3\frac{2}{3}$	$2\frac{2}{3}$	5	3	3
12	5	4	3	5	4	3	5	4	3

If the Pontryagin maximum principle is used, it is possible to deduce that $u_i(t) \in \{0, \frac{1}{3}, \frac{1}{2}, 1\}$ for each i and for all t.

The following graph illustrates the continuous solution for the numerical example; it shows how the continuous version "switches" at noninteger points so as to confirm with the discrete version.

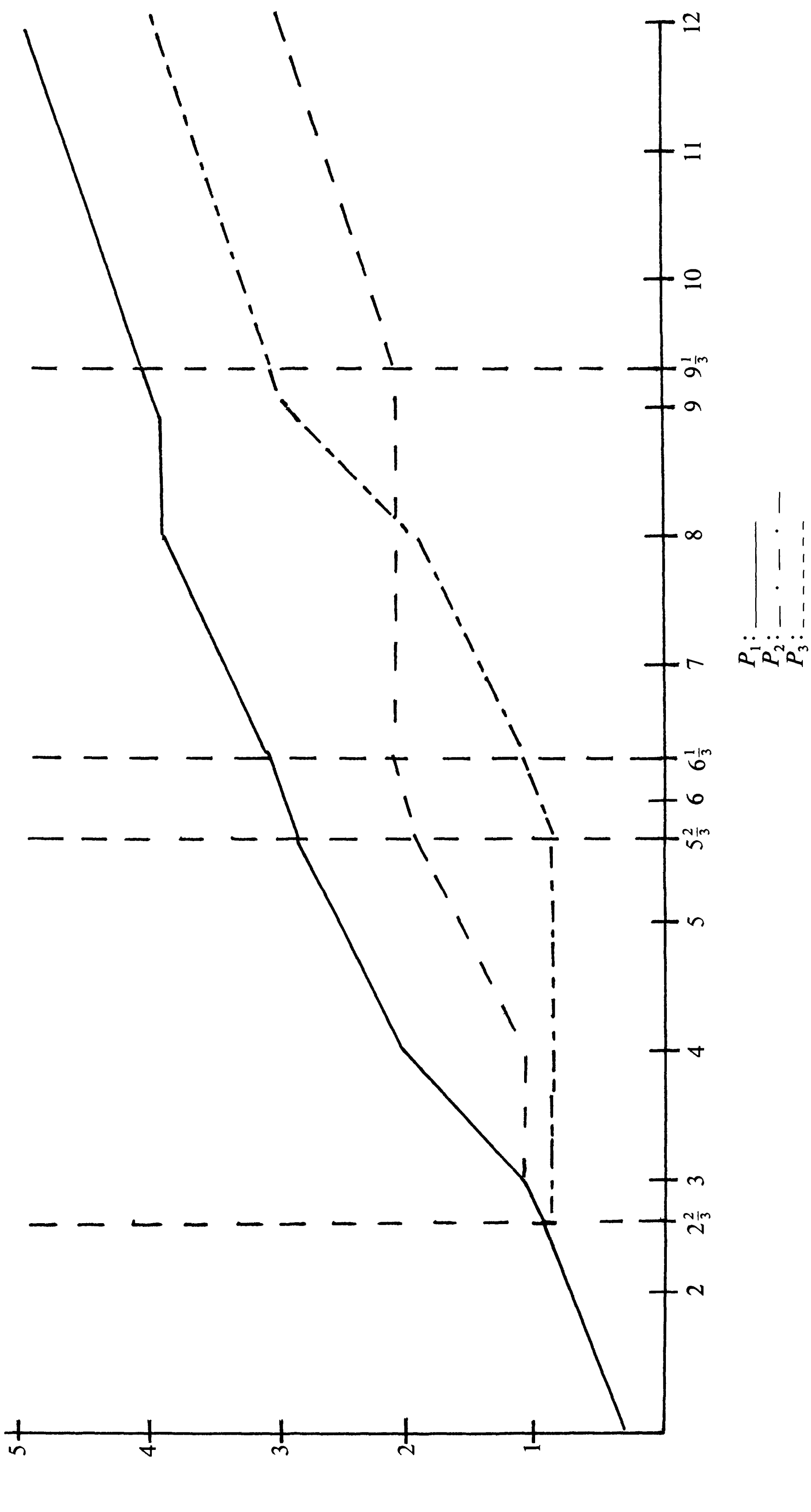

FIG. 1. Optimum Distribution of Funds

A final comment: There is little hope that the Courts will use the solutions offered here or, indeed, any better solutions which may be found. Although both the Benson and the Columbia solutions have been praised in influential textbooks, neither has ever been applied in the Courts. It is a sad truth that mathematics, so successful in the other social sciences, has had a minimal impact upon legal thought.

Notes

1. This is a very simplified explanation of the rights of the parties. Most jurisdictions have detailed and complex modifications of the simple model presented here. See G. Gilmore, *Security Interests in Personal Property*, Toronto, 1965.
2. The use of the word "priority" is perhaps unfortunate; as will be seen below, the relation is not transitive.
3. This prohibition also applied to the sale of goods which the seller retained in his possession.
4. *Twyne's Case* (1601) 3 Co. Rep. 806, 76 E.R. 809.
5. As late as 1851, it was still possible for a New York Court to adopt this point of view; the Court did not even feel the need to refer to supporting authority; *Griswold* v. *Sheldon* 4 N.Y. 581 (1851).
6. For a full discussion of judicial and academic discussions, see G. E. Osborne, *Handbook on the Law of Mortgages*, 2nd ed., St. Paul, 1970.
7. Benson, "Circularity of Lien—A Problem in Priorities," 19 *Minnesota Law Review*, 139 (1935).
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MATHEMATICAL NOTES

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PLANCHEREL'S THEOREM AND THE SHANNON SERIES DERIVED SIMULTANEOUSLY

Antonio Steiner

Introduction. Let I be any interval, finite or infinite, on the real line $\mathbb{R}$. Denote by $L^2(I)$ the usual complete, normed linear space of all complex-valued Lebesgue square integrable functions on I. Recall that two such functions are identified if they are equal a.e. on I and that complete means that Cauchy sequences converge. For $f,g\in L^2(I)$, define the inner product

$$(f,g)=\int_I f(t)\,\overline{g(t)}\,dt.$$

Hölder's inequality implies that this is well defined and moreover yields $|(f,g)|^2\leqslant (f,f)(g,g)$, which is known as Schwarz's inequality. The norm on $L^2(I)$, which we denote by $\|\cdot\|_{2,I}$, can be expressed as $\|f\|_{2,I}=\sqrt{(f,f)}$. Under this inner product $L^2(I)$ is a separable Hilbert space and therefore enjoys a very pleasant structure whose foundations lie in the work of Fourier and his contemporaries over a century and a half ago. The aspects of this that we use here can be developed as follows.

An orthonormal system in $L^2(I)$ is a sequence $\{\psi_k\}_{-\infty}^{\infty}\subset L^2(I)$ satisfying $(\psi_n,\psi_m)=\delta_{nm}$(1 if $n=m$ and 0 otherwise). Given such a system and an $f\in L^2(I)$, set $s_n(f)=\sum_{|k|\leqslant n}(f,\psi_k)\psi_k$. Then

one can easily compute:

$$0 \leqslant (f-s_n(f), f-s_n(f)) = (f,f) - 2\Sigma_{|k|\leqslant n}|(f,\psi_k)|^2 + \Sigma_{|k|\leqslant n}|(f,\psi_k)|^2;$$

hence $(f,f) \geqslant \Sigma_{|k|\leqslant n}|(f,\psi_k)|^2 = (s_n(f), s_n(f))$, which is called Bessel's inequality. It follows that $\{s_n(f)\}$ is Cauchy in $L^2(I)$ and therefore converges to some $s(f) \in L^2(I)$, where $(s(f)-f, \psi_k) = 0$ for $|k| = 0, 1, 2, \ldots$. The orthonormal system $\{\psi_k\}$ is complete if $(h, \psi_k) = 0$ for all k implies $h = 0$ (a.e.). In particular, if $\{\psi_k\}$ is a complete orthonormal system then $s(f) = f$ and one writes $f = \Sigma_{-\infty}^{\infty}(f,\psi_k)\psi_k$, meaning that the series converges to f in $L^2(I)$. The coefficients (f,ψ_k) are called the Fourier coefficients of f and the series $\Sigma(f,\psi_k)\psi_k$ is called the Fourier series of f, both relative to $\{\psi_k\}$. The identity $(f,f) = (s(f), s(f)) = \Sigma|(f,\psi_k)|^2$ is called Parseval's equation. One obtains the classical Fourier series by setting $I = [0, 2\pi]$ and $\psi_k(t) = e^{ikt}/2\pi$. To see that $\{e^{ik\cdot}/2\pi\}$ is complete, observe that finite linear combinations of these functions span a subalgebra of the continuous functions on $[0, 2\pi]$ which is supremum norm dense via the Stone-Weierstrass theorem.

When $I = \mathbb{R}$, the complex exponentials are no longer elements of $L^2(I)$. Hence it can be quite surprising to discover that the classical Fourier expansion results, suitably modified, are still valid, as the following theorem indicates.

PLANCHEREL'S THEOREM. *The formula*

$$f(x) = \hat{g}(x) = \lim_{a\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} g(t) e^{ixt}\, dt$$

defines a linear, norm-preserving transformation of $L^2(\mathbb{R})$ *onto itself. The inverse is given by*

$$g(t) = \check{f}(t) = \lim_{a\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(x) e^{-ixt}\, dx.$$

This result is fundamental to both theoretical and applied Fourier analysis. The standard methods of proof employ either the convolution kernels $\{e^{-\varepsilon t^2}\}_{\varepsilon>0}$ or the Hermite functions $\psi_n(t) = (-1)^n e^{t^2/2} d^n e^{-t^2}/dt^n$, which form a complete orthonormal system when properly normalized (e.g., [4]). Unfortunately these methods quite often obscure the connection with the more elementary classical Fourier series expansion. A related theorem is the following interesting result.

THE SAMPLING THEOREM. *Suppose* $g \in L^2(\mathbb{R})$ *and that* $g \equiv 0$ *off* $[-a,a]$. *Let* $f = \hat{g}$. *Then the values of* f *are completely determined by the discrete set of sampling points* $n\pi/a$, $n = 0, \pm 1, \ldots$, *and moreover*

$$f(x) = \lim_{x\to\infty} \sum_{|n|\leqslant k} f(n\pi/a) \frac{\sin a(x - n\pi/a)}{a(x - n\pi/a)},$$

where the series converges both uniformly on $\mathbb{R}$ *and in* $L^2(\mathbb{R})$.

In communication theory this series expansion for f is called the Shannon series. Shannon himself did not prove it; his derivation is purely heuristic and does not specify the sense of convergence (e.g., [7]; see also Section 2.13 of [4]). In point of fact it had been known under the name of cardinal series at least since 1915 ([8], and is discussed in [9], for example). There are many rigorous derivations of the sampling theorem (e.g., [1], [2], [6]).

The purpose of this note is to present an alternate proof of Plancherel's theorem which simultaneously yields the sampling theorem and clearly exhibits the connection with classical Fourier series. Indeed we shall only use the series results already established and the usual, necessary Lebesgue theory, as found, for example, in [1].

Proof of the two theorems. We will approximate the results on $L^2(-\infty,\infty)$ via the classical results on $L^2[-a,a]$. Denote the norms on these two spaces by $\|\cdot\|_2$ and $\|\cdot\|_{2,a}$, respectively. Given $g \in L^2(-\infty,\infty)$, let g_a denote the product of g with the characteristic function of $[-a,a]$. We apply the classical Fourier series theory to the transform $\hat{g}_a(x) = \sqrt{1/2\pi}\ \int_{-a}^{a} e^{ixt} g(t)\, dt$ with

the aid of the complete orthonormal sequence ψ_n for $L^2[-a,a]$ where

$$\psi_n(t,\alpha)=\frac{1}{\sqrt{2a}}e^{-it(n\tau+\alpha)}\qquad (n=0,\pm 1,\pm 2,\dots)$$

with $\tau=\pi/a$ and with $\alpha\in\mathbb{R}$ being any fixed number. Notice that for $a=\pi$ and $\alpha=0$ the sequence becomes the familiar $\{e^{-int}/\sqrt{2\pi}\}$. Set $c_n(\alpha)=\int_{-a}^{a}g_a(t)\overline{\psi_n(t,\alpha)}dt$ and recall that the classical theory yields Parseval's equation $\Sigma_n|c_n(\alpha)|^2=(\|g_a\|_{2,a})^2$ and that the sequence of partial sums $s_k(t,\alpha)\equiv\Sigma_{|n|\leqslant k}c_n(\alpha)\psi_n(t,\alpha)$ converges to g_a in $L^2[-a,a]$. But notice that $c_n(\alpha)=\sqrt{\tau}\,\hat{g}_a(n\tau+\alpha)$. Hence one can average over α to obtain, approximately, the desired Plancherel theorem. We do this in two stages. First, using Beppo Levi's theorem, one obtains from Parseval's equation

$$\begin{aligned}\|\hat{g}_a\|_2^2&=\int_{-\infty}^{\infty}|\hat{g}_a(x)|^2dx=\sum_{-\infty}^{\infty}\int_{n\tau}^{(n+1)\tau}|\hat{g}_a(x)|^2dx\\&=\sum_{-\infty}^{\infty}\int_0^{\tau}|\hat{g}_a(n\tau+\alpha)|^2d\alpha=\sum_{-\infty}^{\infty}\frac{1}{\tau}\int_0^{\tau}|c_n(\alpha)|^2d\alpha\\&=\frac{1}{\tau}\int_0^{\tau}\sum_{-\infty}^{\infty}|c_n(\alpha)|^2d\alpha=\frac{1}{\tau}\int_0^{\tau}(\|g_a\|_{2,a})^2d\alpha\\&=\|g_a\|_{2,a}^2.\end{aligned}\tag{1}$$

Next recall that $s_k(t,\alpha)$ is the kth partial sum of the Fourier series relative to $\{\psi_n(\cdot,\alpha)\}$. In particular, $\|s_k(\cdot,\alpha)\|_{2,a}\leqslant\|g_a\|_2$ and $\|g_a-s_k(\cdot,\alpha)\|_{2,a}\to 0$ as $k\to\infty$. It follows that $D_k(\alpha)\equiv(\|g_a-s_k(\cdot,\alpha)\|_{2,a})^2$ is bounded by $4\|g_a\|_{2,a}^2$ and that $D_k(\alpha)$ converges pointwise to 0 for each α. Hence Lebesgue's bounded convergence theorem yields $\int_0^{\tau}|D_k(\alpha)|d\alpha\to 0$ as $k\to\infty$. Coupled with Fubini's theorem and Schwarz's inequality, this yields:

$$\begin{aligned}\frac{1}{\tau}\int_0^{\tau}|D_k(\alpha)|d\alpha&=\frac{1}{\tau}\int_0^{\tau}\int_{-a}^{a}|g_a(t)-s_k(t,\alpha)|^2dt\,d\alpha\\&=\frac{1}{\tau^2}\int_{-a}^{a}\int_0^{\tau}|g_a(t)-s_k(t,\alpha)|^2d\alpha\int_0^{\tau}1\,d\alpha\,dt\\&\geqslant\frac{1}{\tau^2}\int_{-a}^{a}\left|\int_0^{\tau}g_a(t)-s_k(t,\alpha)\,d\alpha\right|^2dt\\&=\int_{-a}^{a}\left|g_a(t)-\int_0^{\tau}\frac{1}{\tau}\sum_{-k}^{k}c_n(\alpha)\psi_n(t,\alpha)\,d\alpha\right|^2dt\\&=\int_{-a}^{a}\left|g_a(t)-\sum_{-k}^{k}\int_0^{\tau}\frac{1}{\sqrt{\tau}}\hat{g}_a(n\tau+\alpha)\psi_n(t,\alpha)\,d\alpha\right|^2dt\\&=\int_{-a}^{a}\left|g_a(t)-\sum_{-k}^{k}\frac{1}{\sqrt{2\pi}}\int_0^{\tau}\hat{g}_a(n\tau+\alpha)e^{-it(n\tau+\alpha)}d\alpha\right|^2dt\\&=\int_{-a}^{a}\left|g_a(t)-\frac{1}{\sqrt{2\pi}}\int_{-k\tau}^{(k+1)\tau}\hat{g}_a(x)e^{-itx}dx\right|^2dt.\end{aligned}$$

Since the left-hand side tends to 0 as $k\to\infty$, one concludes that

$$\frac{1}{\sqrt{2\pi}}\int_{-k\tau}^{(k+1)\tau}\hat{g}_a(x)e^{-itx}dx\to g_a(t)\quad\text{in } L^2[-a,a].$$

Hence

$$\frac{1}{\sqrt{2\pi}}\int_{-k\tau}^{k\tau}\hat{g}_a(x)e^{-itx}dx\to g_a(t)\quad\text{in } L^2[-a,a].\tag{2}$$

To extend these results to $L^2(-\infty,\infty)$, let $g\in L^2(-\infty,\infty)$ and let $a\to\infty$ through any sequence. Then $\{g_a\}$ is Cauchy in $L^2(-\infty,\infty)$. Hence by (1) $\{\hat{g}_a\}$ is Cauchy in $L^2(-\infty,\infty)$ and therefore converges to the $\hat{g}$ of the theorem. Also by (1)

$$\|g\|_2=\lim\|g_a\|_{2,a}=\lim\|\hat{g}_a\|_{2,a}=\|\hat{g}\|_2,$$

which gives Parseval's equation, i.e., the norm-preserving property and, most certainly, the continuity of the Fourier-Plancherel transform $\hat{}$ in $L^2(-\infty,\infty)$. Now from (2) one concludes that

$$(\hat{g}_a)^{\check{}}(t)=\lim_{k\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-k\tau}^{k\tau}e^{-itx}\hat{g}_a(x)\,dx=g_a(t)\quad \text{in } L^2[-a,a].$$

Moreover $g\to\check{g}$ is continuous in $L^2(-\infty,\infty)$ for the same reason that $g\to\hat{g}$ is continuous. Thus in $L^2(-\infty,\infty)$

$$g=\lim_{a\to\infty}g_a=\lim_{a\to\infty}(\hat{g}_a)^{\check{}}=\left(\lim_{a\to\infty}\hat{g}_a\right)^{\check{}}=(\hat{g})^{\check{}},$$

which is Plancherel's inversion formula. Finally, we once again use the continuity of the Fourier-Plancherel transform together with the direct computation of $(\psi_n)\hat{}_a$ to conclude that

$$g_a(t)=\lim_{k\to\infty}\sum_{-k}^{k}c_n(\alpha)\psi_n(t,\alpha)\qquad(\text{in } L^2[-a,a])$$

implies

$$\begin{aligned}\hat{g}_a(x)&=\lim_{k\to\infty}\sum_{-k}^{k}c_n(\alpha)(\psi_n)\hat{}_a(x,\alpha)\qquad(\text{in } L^2(-\infty,\infty))\\&=\lim_{k\to\infty}\sum_{-k}^{k}\hat{g}_a(n\tau+\alpha)\frac{\sin a[x-(n\tau+\alpha)]}{a[x-(n\tau+\alpha)]}\\&=\lim_{k\to\infty}\sum_{-k}^{k}\hat{g}_a(n\tau)\frac{\sin a(x-n\tau)}{a(x-n\tau)},\qquad \alpha=0,\end{aligned}$$

which is the Shannon series. Actually this last limit exists in the supremum norm also. To see this, observe that

$$\begin{aligned}&\sup_x\left|\hat{g}_a(x)-\sum_{-k}^{k}\hat{g}_a(n\tau)\frac{\sin a(x-n\tau)}{a(x-n\tau)}\right|\\&=\sup_x\left|\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(g_a(t)-s_k(t,0))e^{itx}\,dt\right|\\&\leqslant\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}|g_a(t)-s_k(t,0)|\,dt\\&\leqslant\sqrt{\frac{a}{\pi}}\;\|g_a-s_k(\cdot,0)\|_{2,a}\to 0,\quad \text{as } k\to\infty.\end{aligned}$$

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QUADRANGLES, BUTTERFLIES, PASCAL'S HEXAGON, AND PROJECTIVE FIXED POINTS

DIXON JONES

It is a consequence of an important result in projective geometry, the so-called Fundamental Theorem on Quadrangular Sets, that if five sides of a complete quadrangle pass through fixed points on a given line, then the sixth side also meets that line at a fixed point; further, it may be shown that this theorem is equivalent to Desargues' Theorem for perspective triangles. In this article we exhibit results of a similar flavor but involving only simple quadrangles and the Theorem of Pascal and carrying implications for the classical Butterfly Theorem and one of its recent extensions.

(We employ the notion of lines of a configuration which pass through "fixed points" on a given line and, dually, points of a configuration which lie on "fixed lines" through a given point. This is used as a shorthand for more rigorous but less economical statements about configurations with sides and vertices placed in one-to-one correspondence such that corresponding sides meet at points of a given line, and dually; thus a phrase such as "the sides of a quadrangle meet a given line at fixed points" should be understood to mean, equivalently, "the sides of a second quadrangle, distinct from the first, whose vertices have been placed in one-to-one correspondence with those of the first meet the corresponding sides of the first at points on the given line." The "fixedness" of these points can be made more tangible by imagining that the quadrangle moves about continuously, as in the real projective plane, but that its sides always pass through collinear points which do not move.)

Figure 1 illustrates our first and primary result.

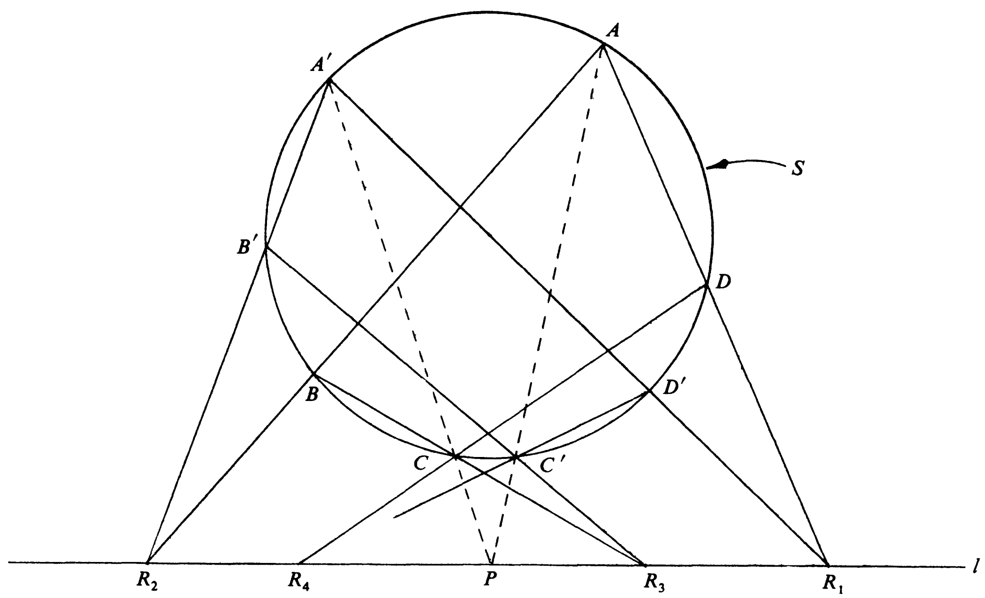


FIG. 1

THEOREM 1. *If three sides of a simple quadrangle inscribed in a point conic pass through fixed points on a given line, then the fourth side also passes through a fixed point on that line. Dually, if three vertices of a simple quadrilateral escribed about a line conic lie on fixed lines through a given point, then the fourth vertex also lies on a fixed line through that point.*

Proof. Referring to Figure 1, let quadrangles $ABCD$ and $A'B'C'D'$ be inscribed in a conic S, with three pairs of corresponding sides intersecting at points R_1, R_2, and R_3 of a given line l. CD meets l at R_4; it must be shown that $C'D'$ also meets l at R_4. Consider the hexagon $ABCA'B'C'$. By the Theorem of Pascal, since opposite sides AB and $A'B'$ meet on l at R_2, and BC and $B'C'$ meet on l at R_3, it follows that $A'C$ and AC' also meet on l, say at P. Now consider hexagon $ADCA'D'C'$. Again by Pascal's Theorem, since opposite sides AD and $A'D'$ meet on l at R_1, and $A'C$ and AC' meet on l at P, it follows that CD and $C'D'$ meet on l, in fact at the point R_4, which completes the proof. The dual follows by the principle of duality in the plane.

Theorem 1 represents the simplest case of the following more general theorem.

THEOREM 2. *Let an n-gon, $n=2k$, be inscribed in a point conic and let $n-1$ of its sides meet a given line at fixed points. Then the nth side also meets that line at a fixed point. Dually, let an n-lateral, $n=2k$, be escribed about a line conic and let $n-1$ of its vertices lie on fixed lines through a given point. Then the nth vertex also lies on a fixed line through that point.*

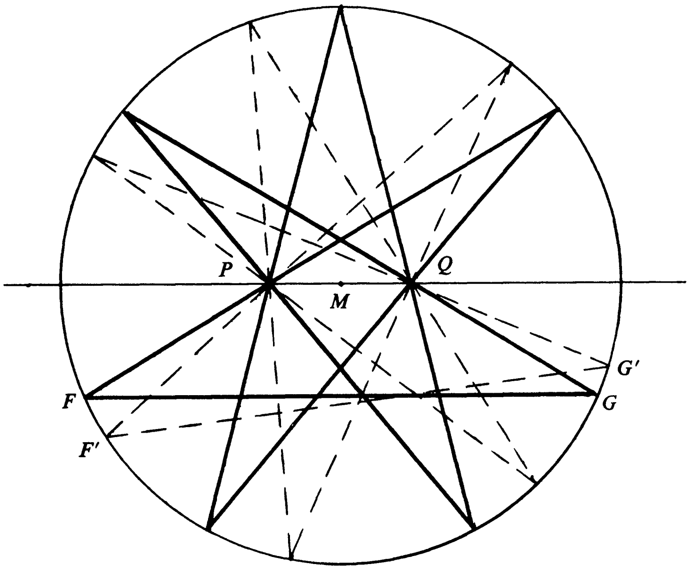
Proof. Our proof will be by induction on k. Clearly, the simplest polygon of $2k$ vertices occurs when $k=2$, and in this initial case Theorem 2 reduces to Theorem 1, which has been proved. Suppose now that Theorem 2 holds for some integer k; it must be shown to hold for $k+1$. Let an inscribed polygon P have consecutive vertices $P_1, P_2, \ldots, P_n, P_{m-1}, P_m$, where $m=2(k+1)$. It is given that $m-1$ sides of P meet a given line l at fixed points; without loss of generality, let P_1P_2 through $P_{m-1}P_m$ be the sides so given. By supposition, the theorem holds for a polygon of $n=2k$ sides; specifically, since the sides P_1P_2 through $P_{n-1}P_n$ of the polygon $P_1P_2 \cdots P_n$ pass through fixed points on l, the nth side P_1P_n also meets l at a fixed point. Now, P_1P_n, P_nP_{m-1}, $P_{m-1}P_m$, and P_mP_1 are the sides of an inscribed quadrangle, of which sides three are given or have been shown to meet l at fixed points; hence P_mP_1, the fourth side of the quadrangle and the mth side of P, also meets l at a fixed point, by Theorem 1. We have thus shown, by assuming its validity for $n=2k$, that Theorem 2 holds for $m=2(k+1)$, which completes the proof. The dual, of course, follows automatically.

One would naturally wonder if Theorem 2 holds for odd n. Unfortunately, the following construction disposes of any hopeful speculation on the matter. Given a circle, a chord therein, and points P and Q on the chord equidistant from its midpoint, construct for n an odd integer an inscribed n-gon with one vertex on the chord's perpendicular bisector, such that $n-1$ sides alternately pass through P and Q. Due to the bilateral symmetry imposed on the n-gon in this way, the nth side will be parallel to PQ. By perturbing any vertex to yield a second n-gon also passing alternately through P and Q $n-1$ times, it will be seen that the nth side cannot be parallel to PQ. Thus, while the two n-gons meet the conditions of Theorem 2, the result does not follow. Figure 2 illustrates this construction for the case $n=7$.

We mentioned in an earlier comment that the Fundamental Theorem on Quadrangular Sets is equivalent to Desargues' Theorem for perspective triangles. An equally intimate relationship is shared by Theorem 1 and another celebrated projective theorem.

THEOREM 3. *Theorem 1 is equivalent to the Theorem of Pascal.*

Proof. It was shown in our first proof that Pascal's Theorem implies Theorem 1; the reverse implication remains to be shown. Let hexagon $ABCDEF$ be inscribed in a conic, let AB meet DE at P, BC meet EF at Q, CD meet PQ at R, and let AD meet PQ at T. We discern that the



FG is parallel to *PQ*, but *F'G'* is not.

FIG. 2

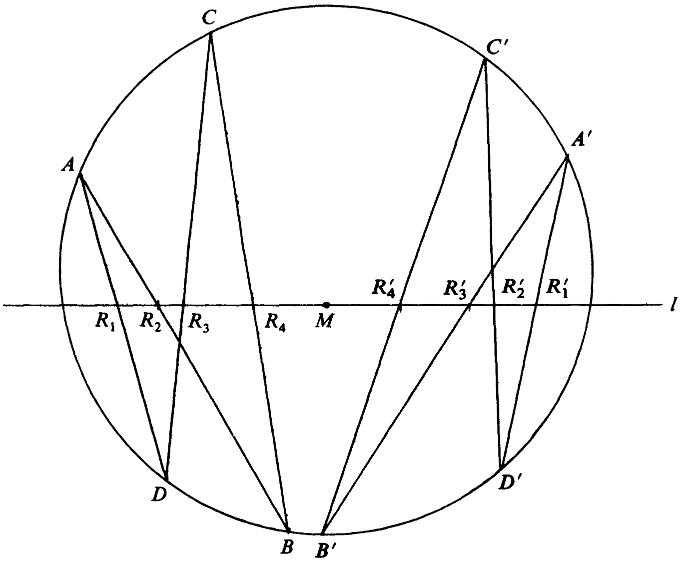


FIG. 3

quadrangles *ABCD* and *DEFA* possess three pairs of corresponding sides, those which pass through the points *P*, *Q*, and *T* on *PQ*, and hence the fourth pair of corresponding sides, *CD* and *AF*, meet *PQ* at the point *R*, which completes the proof.

Our final result is presented as a matter of incidental interest, to indicate a further connection between Pascal's Theorem and the notion of fixed points on a line.

THEOREM 4. *If five sides of a hexagon inscribed in a point conic meet a given line at fixed points, then its Pascal line meets that line at a fixed point.*

Sketch of proof. With reference to the construction and notation used in the previous proof, let AB, CD, DE, EF, and AF be the sides meeting a given line l at fixed points. Application of Theorem 1 to the simple quadrangle $ADEF$ yields that AD meets l at a fixed point, and application of the Fundamental Theorem on Quadrangular Sets to the complete quadrangle $PADR$ yields that PR, the Pascal line of $ABCDEF$, meets l at a fixed point. Details of the proof are left to the reader.

As a concluding remark, we observe that Theorem 1 is a generalization of a recent result called the Double Butterfly Theorem [1], which states that if two "butterflies" (re-entrant quadrangles $ABCD$ and $A'B'C'D'$) inscribed in a circle have their respective sides meeting a chord of the circle at R_1, R_2, R_3, R_4 and R_1', R_2', R_3', R_4', and if the chord's midpoint M bisects the segments R_1R_1', R_2R_2', and R_3R_3', then M bisects R_4R_4' (Fig. 3). Theorem 1 generalizes this by removing the midpoint and its metric aspects, by allowing all simple quadrangles, by removing the restriction that the given line intersect the circle, and by including conics other than the circle. Indeed, the proof of the Double Butterfly Theorem becomes a simple matter using Theorem 1: referring to Figure 3, map $ABCD$ to $A''B''C''D''$ by reflection in the midpoint M. By hypothesis, therefore, three corresponding sides of $A'B'C'D'$ and $A''B''C''D''$ pass through R_1', R_2', and R_3'. By Theorem 1, the fourth sides $B'C'$ and $B''C''$ pass through R_4', and since $B''C''$ is a reflection of BC in M, we have $R_4M = R_4'M$. That the Double Butterfly Theorem implies the classical Butterfly Theorem was pointed out in [1]; to prove the latter result directly using Theorem 1, align $ABCD$ and $A'B'C'D'$ so that $R_1 = R_4'$ and $R_2 = R_2' = R_3 = R_3' = M$, assume that R_4 is distinct from R_1', and apply the above argument.

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THE RETRIAL OF THE LOWER SLOBBOVIAN COUNTERFEITERS

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Lower Slobbovia is a poor country, unable to mint its local currency, the Rasbucknik [1]. Hence nine (N) coiners, C_1, $C_2, \ldots, C_9$ were engaged to produce coins to government specifications. However, it was suspected that some of the coiners were counterfeiting by introducing some base metal into the alloy. Any pair of counterfeit coins weighed the same, but differed slightly in weight from good coins. Each coiner produced either all good coins or all counterfeits. A procedure was called for to determine in three weighings which (if any) of the coiners were dishonest, using a beam balance with a set of infinitely refinable weights, as many coins from each coiner as may be needed, and one good coin.

The Court of Lower Slobbovia was supplied with the following solution [2]. In the first weighing, the weight Wg of the good coin is determined. In the second weighing, a single coin from each coiner is selected and these nine (N) coins are weighed, giving a total weight T. If the discrepancy $D = T - NWg$ is zero then all the coiners are honest. In the third weighing a sample of 2^{i-1} coins are selected from C_i, $i = 1, \ldots, 9$, and weighed, giving a total weight T'. The discrepancy here is $D' = T' - (2^N - 1)Wg$.

Now the integer S such that $D'/D = S/\beta(S)$ is determined where $\beta(S)$ is the number of ones in the binary representation of S, i.e., $\beta(S) = \sum_{i=1}^{N} B_i$ where $S = \sum_{i=1}^{N} B_i 2^{i-1}$. For such an S, coiner C_i is a counterfeiter if and only if $B_i = 1$, and $\beta(S)$ is the number of counterfeiters.

This procedure was followed at the trial of the coiners in 1954 and the fateful ratio $D'/D = 23$ was determined. After further computation it was discovered that $S = 69 = 2^0 + 2^2 +$

2^6, with $\beta(69)=3$, satisfied the ratio and accordingly coiners C_1, C_3, and C_7 were arrested, to be detained indefinitely at the Archduke's pleasure. They were dragged away pleading innocence, and the counsel for C_1 immediately lodged an appeal.

The wheels of Lower Slobbovian justice moved steadily but slowly, and the retrial was held before the Archduchess only 24 years later. In presenting his case C_1's counsel argued that $S=92=2^2+2^3+2^4+2^6$ with $\beta(92)=4$ satisfied the ratio $S/\beta(S)=23$ and hence the evidence convicting his client was faulty, while coiners C_4 and C_5 were probably counterfeiters and had been left unpunished.

Lower Slobbovia's very embarrassed Mathematician Laureate was called to explain. In the light of such a concrete counterexample he had to admit that the mapping $S \to S/\beta(S)$ was not, in fact, always 1-1 and therefore did not always admit a unique inverse. Hence in the light of the above evidence he could only conclude C_3 and C_7 were obviously guilty, C_2, C_6, C_8, and C_9 were clearly innocent, but which of the others, C_1, C_4, and C_5, were guilty was not clear. Unable to give a decisive conclusion he was invited to share the counterfeiters' fate and was led away muttering things like, "I forgot about carry digits."

The apprentice mathematician was instantly promoted and told the same fate awaited him should he be unable to produce a foolproof technique. After a little thought he stated that, although the restriction to only three weighings appeared to make the problem difficult, it could still be resolved unambiguously, altering only the third weighing. Instead of calling for 2^{i-1} coins from C_i, you should in fact select b^{i-1}, for any integer $b>N$. Let $\beta_b(S)$ be the sum of the digits of S expressed to the base b. Although $S \to S/\beta_b(S)$ would still not be 1-1 for all $S \in Z^+$, the only integers S generated in this process would be those whose digits to the base b are 0 or 1. In this case should $S/\beta_b(S)=T/\beta_b(T)$ *then* $\beta_b(T)\cdot S=\beta_b(S)\cdot T$. The digits of $\beta_b(S)\cdot T$ expressed to base b will be either 0 or $\beta_b(S)\leqslant N<b$, and similarly those for $\beta_b(T)\cdot S$ will be 0 or $\beta_b(T)$. As these integers are equal we must find $\beta_b(S)=\beta_b(T)$ and consequently $S=T$.

The new Mathematician Laureate was applauded and as $N=9$, the court officials set $b=10$, and selected one coin from C_1, 10 from C_2, etc. Reweighing T', and calculating $D'=T'-(10^N-1)Wg/9$ the ratio

$$S/\beta_{10}(S)=222002.2$$

was determined. "The only integer," concluded the young mathematician, "whose digits base 10 are zeros and ones, that satisfies this ratio is $S=1110011$, so in fact there are *five* counterfeiters, C_1, C_2, C_5, C_6 and C_7!"

There were howls of protest and disbelief. How could this be compatible with the earlier calculation with $S=69$ or 92? The young mathematician then pointed out that 1110011 as an integer (base 2) was $2^6+2^5+2^4+2^1+2^0=115$, and $115/\beta_2(115)=23$, a value not previously recognized. These three integers, 69, 92, and 115, are the only three giving the ratio 23, however. We note that if $\beta_2(23r)=r$, then $23r \geqslant 2^{r-1}$, which is true only for $r<9$. Testing for $r=1$, 2, 6, 7, and 8 we find these do not satisfy the ratio and hence we have uncovered all solutions to $S/\beta_2(S)=23$.

Also 23 is not the only nonunique ratio of $S/\beta_2(S)$. We can show that 26.5, 27, 37, 38.5, 41, 41.5, 46 and many more each are satisfied by multiple values of S. (For related problems, see [3].)

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CAYLEY'S THEOREM FOR TOPOLOGICAL GROUPS

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The following theorem is almost universally referred to as Cayley's Theorem:

(*) *Every group of order n is isomorphic to a subgroup of the symmetric group* S_n.

This result appears in every presentation of the elements of group theory (see, e.g., [**10**], [**2**], [**7**], and [**6**]). It has been generalized to infinite groups by replacing S_n by the group of transformations on a set of appropriate cardinality, and analogs of (*) for monoids, rings, and algebras appear in [**8**]. The qualifier "almost" appears in the first sentence above because of Burnside's attribution (in [**4**]) of (*) to Jordan.

The collected works of Cayley are readily accessible. They indicate that in 1854 Cayley defined the notion of (finite) group. He emphasized the abstract nature of a group by asserting that "a group is defined by means of the laws of combination of its symbols." He also represented a group by its multiplication table (sometimes called a Cayley table), and remarked that each row (column) of the table contains all the group elements, each one appearing only once [**5**]. Thus he implicitly makes the connection between group elements and regular permutations, but he does *not* explicitly prove (*). Moreover, as Burnside quite correctly notes, Jordan did indeed prove (*) in 1870 [**9**, pp. 60–61].

Nevertheless, to call (*) Cayley's Theorem is not an inappropriate attribution. Cayley essentially showed that the correspondence which takes a group element a to the permutation τ_a (where $\tau_a(x)=ax$ for any group element x) is one-to-one, even though he failed to demonstrate explicitly that $\tau_a\tau_b=\tau_{ab}$. More important, he communicated his awareness of this correspondence to the mathematical community at large.

Having introduced Cayley's Theorem, we wish to observe that it has a natural generalization to topological groups. In this case, sets are replaced by uniform spaces and the group of permutations of a set by the topological automorphism group of a uniform space. Before stating (and proving) the topological Cayley Theorem, we record the relevant definitions (as found in, e.g., [**3**]).

If X is a uniform space, then its automorphism group Aut(X) consists of all uniformly continuous functions $X \to X$ with uniformly continuous inverse, the topology being that of uniform convergence. For this topology, a fundamental system of neighborhoods of the identity automorphism is given by the sets $\mathcal{V}=\{\sigma|(x,\sigma(x))\in V$ for all $x\in X\}$, where V ranges over the entourages of the uniformity of X. It is routine to verify that this fundamental system of neighborhoods of the identity is compatible with the group structure of Aut(X), and thus that Aut(X) is a topological group. Moreover, if X is Hausdorff (respectively, discrete), then Aut(X) is Hausdorff (respectively, discrete). These last two observations emphasize that our topological discussion is not limited to Hausdorff topological groups only, and that it is really a generalization of the algebraic situation.

Now if V is a neighborhood of the identity e of the topological group G, we set $V_d=\{(x,y)\in G\times G|yx^{-1}\in V\}$. Then as V runs through a fundamental system of neighborhoods of e, the sets V_d describe a fundamental system of entourages of μ, the right uniformity of G. (The subscript d stands for "dextra," the Latin word for "right." If we had multiplied y on the left by x^{-1}, we would have obtained the left uniformity on G, which is in general distinct from the right uniformity. In this case we would have used a subscript s standing for "sinistra.")

We can now state the analog of Cayley's Theorem for topological groups:

(**) *Every topological group G is isomorphic to a subgroup of* Aut(μG), *the topological group of automorphisms of the right uniform structure of G.*

To the best of our knowledge, (**) was first proved in [**1**]; there seems to be no other proof in

the literature. To prove (**), we consider a topological group G. For $s \in G$, define $\tau_s : \mu G \to \mu G$ by setting $\tau_s(x) = sx$ for all $x \in G$. Then $\tau_s \in \mathrm{Aut}(\mu G)$, and the function $\tau : G \to \mathrm{Aut}(\mu G)$ which takes s to τ_s is an injective group homomorphism. Finally, the inequalities

$$\begin{aligned} \mathcal{V} \cap \tau(G) &= \{\tau_s \mid (x, \tau_s(x)) \in V_d \quad \text{for all} \quad x \in G\} \\ &= \{\tau_s \mid sx \cdot x^{-1} \in V \quad \text{for all} \quad x \in G\} \\ &= \{\tau_s \mid s \in V\} = \tau(V) \end{aligned}$$

imply that τ is an embedding. Thus (**) is proved.

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REGIONS OF HARMONICITY

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It is known that an arbitrary open subset of the complex plane is a **region of holomorphy**. This means that there always exists a holomorphic function having no analytic continuation across any point of the boundary. In other words, the topological boundary of each open subset is the **natural boundary** for some holomorphic function. Various constructions of functions possessing this property are available in the literature (see, e.g., [**3**, pp. 252, 294]). Let us also note that the set $A(D)$ of all functions holomorphic in a fixed open set D can be regarded as a complete metric space, and it is not difficult to see that a typical function of $A(D)$ (that is, one not belonging to a certain first-category subset of $A(D)$) has the boundary of D as its natural boundary (cf. [**3**, p. 253]).

Recall that both the real and the imaginary parts of a holomorphic function are **harmonic functions**. Conversely, any harmonic function is locally the real part of a holomorphic function. It is not too surprising that each open subset of R^2 is, in an obvious sense, a **region of harmonicity**. The same is easily seen to be true also in higher-dimensional Euclidean spaces. Indeed, define for $x \in R^m$, $m > 2$, the harmonic function p_x by setting $p_x(y) = |x-y|^{2-m}$, $y \in R^m \setminus \{x\}$. Now fix an open set $D \subset R^m$ with $\partial D \neq \varnothing$. We may assume that D is connected. Choose a countable set $\{x_1, x_2, \ldots\}$ that is dense in ∂D. If the positive numbers $\alpha_1, \alpha_2, \ldots$ are chosen in such a way that the series $\Sigma \alpha_i p_{x_i}$ is convergent at a point of D, then its sum defines a harmonic function in D whose set of points of continuability is empty. (A boundary point x of D is said to be a **point of continuability** of a harmonic function h defined on D if there are a neighborhood V of x and a function h_1 harmonic on V such that $h = h_1$ on $V \cap D$. The set of all points of continuability of h will be denoted by $C(h)$. Clearly, $C(h)$ is an open subset of ∂D.)

In summary, for each such $D \subset R^m$ $(m > 2)$ there is a harmonic function h on D with $C(h) = \varnothing$. (Note that even typical harmonic functions on D, in the sense used above, have this property. We shall omit the proof of this assertion here.)

One may find the foregoing construction a bit unfair in the sense that our function does not have a chance to be continued over ∂D because it is unbounded near any point of ∂D. Could we, in addition, require the continuous extendability of such a function to the closure $\bar{D}$ of D? Simple examples like a ball with a point deleted indicate that some troubles may arise with "small" parts of ∂D contained in the interior of $\bar{D}$. For the sake of simplicity, consider now a fixed bounded nonempty open set $D \subset R^m$ $(m \geqslant 2)$ such that $\partial D = \partial \bar{D}$ and denote by $H = H(D)$ the set of all functions continuous on $\bar{D}$ and harmonic on D. Note that H, equipped with the supremum norm, is a Banach space.

The existence of noncontinuable functions belonging to H follows now (by the Baire Category Theorem) from the next assertion.

THEOREM. *The set of all functions $h \in H$ with $C(h) = \varnothing$ is a dense G_δ in H.*

The proof of the theorem will be divided into several steps and will depend on two lemmas proved below. The proof of Lemma 1 depends on a simple geometrical argument, while the proof of Lemma 2 uses the fact that a uniformly bounded family of harmonic functions is equicontinuous.

We shall say that a point $x \in \partial D$ has the **ball property** if there is a closed (nondegenerate) ball K such that $K \cap D = \varnothing$ and $x \in K \cap \bar{D}$.

LEMMA 1. *Suppose that $x \in \partial D$ has the ball property and U is a neighborhood of x. Then there are a function $g \in H$ and a point $y_1 \in D \cap U$ such that $0 \leqslant g \leqslant 1$, $g(x) = 1$ and $g(y_1) \leqslant 1/3$.*

Proof. Without loss of generality we shall suppose that U is bounded. Let d be the diameter of $D \cup U$. For $z \in R^m$ we have defined $p_z(y) = |z-y|^{2-m}$ if $m > 2$; if $m = 2$, put $p_z(y) = \log(d/|z-y|)$. For each $z \in U \setminus \bar{D}$ the function p_z is a continuous nonnegative function on $\bar{D}$ that is harmonic on D. Given $y \in R^m$ and $\delta > 0$, set $K_\delta(y) = \{z \in R^m\colon |z-y| \leqslant \delta\}$. Since x has the ball property, one easily finds $y \in R^m$ and $\delta > 0$ such that $K_\delta(y) \subset U$ and $K_\delta(y) \cap \bar{D} = \{x\}$. For $0 < t \leqslant 1$ define $z_t = x + t(y-x)$, $\alpha_t = [p_{z_t}(x)]^{-1}$, and $f_t = \alpha_t p_{z_t}$. It is easily seen from the definition of f_t that $f_t(w) \to 0$ as $t \to 0_+$ whenever $w \in D \cap U$. Thus, choosing a small enough t, we can get a g with the required properties by restricting the corresponding f_t to $\bar{D}$, and the lemma is proved.

LEMMA 2. *Let $x \in \partial D$ have the ball property. If $M(x)$ is the set of all $h \in H$ for which $x \notin C(h)$, then $M(x)$ is a dense G_δ in M.*

Proof. Choose a decreasing sequence $\{U_n\}$ of open bounded sets such that $\cap_n U_n = \{x\}$. Denote now by P_n the set of all functions f continuous on $\bar{D} \cup U_n$, harmonic on $D \cup U_n$, and satisfying $|f| \leqslant n$ on $D \cup U_n$. The set of all restrictions of functions f in P_n to $\bar{D}$ will be denoted by M_n. Clearly, $M_n \subset H$, and for $h \in H$ we have $x \in C(h)$ if and only if $h \in M_n$ for a suitable n. Consequently, $M(x) = H \setminus \cup_n M_n$, and it is enough to prove that the sets M_n are closed and nowhere dense in H.

Let us fix n and choose an arbitrary sequence $\{h_k\}$ of functions from M_n, $h_k \to h$ in H. Now for any k there is an $f_k \in P_n$ whose restriction to $\bar{D}$ is equal to h_k. Since $\{f_k\}$ is a uniformly bounded family of harmonic functions on $D \cup U_n$, there is a subsequence $\{f_{k_m}\}$ converging uniformly on compact subsets of $D \cup U_n$ to a function f (see **[1**, p. 32, Th. 2.18]). It is easily seen that h is the restriction of f to $\bar{D}$ and that it belongs to M_n. But this implies that M_n is closed.

Choose $f \in M_n$ and $\varepsilon > 0$; we shall find a function $f' \in H \setminus M_n$ such that $|f - f'| \leqslant \varepsilon$ on $\bar{D}$. The equicontinuity of the functions in P_n (cf. the first part of the proof of Th. 2.18 in **[1]**) can be used to find a neighborhood V of x such that for every $h \in M_n$ and every $y \in V \cap \bar{D}$ we have $|h(x) - h(y)| < \varepsilon/4$. Now we apply Lemma 1 with $U = V$ to find $g \in H$ and $y_1 \in V \cap D$ with $g(x) - g(y_1) \geqslant 2/3$.

For $f'=f+\varepsilon g$ we obviously have $|f-f'|\leqslant\varepsilon$, but, on the other hand,

$$\begin{aligned}|f'(x)-f'(y_1)| &\geqslant \varepsilon|g(x)-g(y_1)|-|f(x)-f(y_1)|\\ &\geqslant \frac{2\varepsilon}{3}-\frac{\varepsilon}{4}>\frac{\varepsilon}{3},\end{aligned}$$

and so f' cannot belong to M_n. Thus the lemma is proved.

Now we are in position to complete the proof of the theorem. Observe first that the set of all points having the ball property is dense in ∂D. To see this, fix $x\in\partial D$ and $\delta>0$. If y is an arbitrary point of $K_\delta(x)\setminus\overline{D}$ and x_1 is the nearest point to y in the compact set $\overline{D}$, then $x_1\in\partial D\cap K_{2\delta}(x)$ and x_1 obviously has the ball property.

Consequently, there is a countable set $B\subset\partial D$ that is dense in ∂D and such that each point of B has the ball property. Now the Baire theorem and Lemma 2 imply that $M=\cap\{M(x): x\in B\}$ is a dense G_δ in H. To finish the proof, observe the simple fact that M consists exactly of those $h\in H$ for which $C(h)=\varnothing$.

In conclusion we add several remarks. The proof of the fact that all the sets M_n in the proof of Lemma 2 are nowhere dense can be considerably simplified by using the analyticity property of harmonic functions (in R^m). We avoided this shorter but less elementary argument (which should be used without hesitation, for example, in the proof of the analogous theorem for holomorphic functions).

The linear functions are naturally considered as harmonic functions on the real line R^1. In this simple case, however, the continuability has quite different properties (every linear function can be extended from any interval onto the whole line).

We have a less trivial example showing a new phenomenon if we define "harmonic functions" as the solutions of the heat equation. If we consider, say, a two-dimensional open interval I, then any function continuous on $\bar{I}$ and satisfying the heat equation on I can be "harmonically" continued across the upper part of ∂I. This contrasts strongly with the case of the Laplace equation.

In the framework of an axiomatic potential theory which covers not only harmonicity in R^m, $m\geqslant 2$, but also the two cases just mentioned as basic examples, the question of harmonic continuation is investigated in [2]. In particular, the results of [2] also show what happens when the hypothesis $\partial D=\partial\overline{D}$ in the above theorem is omitted. Let us agree to call a point x of ∂D "inessential" provided x has a neighborhood V such that $V\cap\partial D$ is negligible in the potential-theoretic sense (which means being polar for the case studied; see [1]). It turns out that in our case the set Q of inessential points is contained in the interior of $\overline{D}$ and $Q\subset C(h)$ for every $h\in H$, while the set of all $h\in H$ with $C(h)\cap(\partial D\setminus Q)=\varnothing$ is again a dense G_δ in H. A similar situation occurs if solutions of the Laplace equation are replaced by those of a more general partial differential equation of elliptic type. As already indicated, the "harmonic functions" defined by parabolic equations behave in a quite different manner.

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RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada T2N 1N4.

A POSTAGE STAMP PROBLEM

RONALD ALTER AND JEFFREY A. BARNETT

The postage stamp problem is the following: An envelope may carry no more than h stamps, and one has available k integer-valued stamp denominations. Given h and k, find the maximal integer $n=n(h,k)$ such that all integer postage values from 1 to n can be made up. In addition, find all sets of k stamp denominations satisfying this condition.

The problem statement is usually modified by augmenting the solution sets with a stamp of value zero, and requiring that a letter carry exactly h stamps. For example, if $h=2$ and $k=3$, then $n(h,k)=8$. The unique solution set is $\{0,1,3,4\}$. A construction of the integers $1,\dots,8$ is

$$\begin{array}{llll} 1=0+1 & 3=0+3 & 5=1+4 & 7=3+4 \\ 2=1+1 & 4=0+4 & 6=3+3 & 8=4+4. \end{array}$$

Many solution sets may exist. For example, $n(2,6)=20$, and the five solution sets are $\{0,1,2,5,8,9,10\}$, $\{0,1,3,4,8,9,11\}$, $\{0,1,3,4,9,11,16\}$, $\{0,1,3,5,6,13,14\}$, and $\{0,1,3,5,7,9,10\}$.

Clearly, $n(1,k)=k$ with the solution set $\{0,1,\dots,k\}$, and $n(h,1)=h$ with the solution set $\{0,1\}$. Stöhr **[44]**, Henrici **[12]**, and Stanton et al. **[43]**, independently, show that

$$n(h,2)=\lfloor (h^2+6h+1)/4 \rfloor,$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. If h is odd, then the unique solution set is $\{0,1,(h+3)/2\}$. If h is even, then there are two solution sets: $\{0,1,(h+2)/2\}$ and $\{0,1,(h+4)/2\}$.

The only other case of a near closed-form solution is $k=3$. Hofmeister **[17]** shows that

$$\frac{4}{81}h^3+\frac{2}{3}h^2+\frac{66}{27}h \leqslant n(h,3) \leqslant \frac{4}{81}h^3+\frac{2}{3}h^2+\frac{71}{27}h-\frac{1}{81}, \quad h \geqslant 34.$$

The lower bound is achieved whenever $h\equiv 0 \pmod 9$. Klotz **[20]**, **[21]** and Henrici **[12]** independently report similar but weaker results for the lower bound.

In 1936, Rohrbach **[37]** developed asymptotic bounds for n with h fixed and k large. He showed that

$$\left(\frac{k}{h}\right)^h \leqslant n(h,k) \leqslant \frac{k^h}{h!}+O(k^{h-1}).$$

The lower bound is developed constructively. The upper bound is obtained by noting that $\binom{k+h}{h}-1$ is sufficient because $\binom{k+h}{h}$ is the number of combinations of $k+1$ things (recall the zero stamp) taken h at a time with replacement. The -1 appears because 0 is a potential sum but illegal postage.

Hofmeister **[19**, p. 112**]** derives the best-known lower bound by using an unpublished result of R. Windecker:

$$n(h,k) \geqslant (4/3)^{\lfloor h/3 \rfloor}(8/7)^{\lfloor (h-3\lfloor h/3 \rfloor)/2 \rfloor}(k/h)^h - O(k^{h-1}).$$

Hofmeister [19, p. 104] also gives bounds for the case $k \geqslant 3$ fixed and h large, as

$$2^{\lfloor k/4 \rfloor}(4/3)^{\lfloor (k-4\lfloor k/4 \rfloor)/3 \rfloor}(h/k)^k + O(h^{k-1}) \leqslant n(h,k) \leqslant \frac{h^k}{k!} + O(h^{k-1}).$$

A nontrivial upper bound for $h=2$ appears in Rohrbach [37] as

$$n(2,k) \leqslant \tfrac{1}{2}(1-0.0016)k^2 + O(k).$$

This bound is improved in Klotz [20], [21], [22], who replaces 0.0016 by 0.0369.

Additional work on upper bounds for n is reported in Moser [27], Riddell [36], Salié [38], and Moser and Riddell [28]. For the case of large k, the best results to date are given by Moser et al. [29] as

$$n(h,k) < (1-b_h)\frac{k^h}{h!},$$

where $b_3 = 0.0221$ and $b_4 = 0.0115$. Further, $b_h = (1.02f(h))^h$ when $h \geqslant 5$ and $b_h = (1.1f(h))^h$ when $h \geqslant 8$, where $f(h) = \cos(\pi/h)/(2+\cos(\pi/h))$.

Richard K. Guy suggests that, for h large enough, $n(h,k)$ is given by a finite set of polynomials in h of degree k. For example, Stöhr's solution for $k=2$ may be written $n(h,2) = (h^2+(3+3c)h+d)/4$, where $c=d\equiv h$, mod 2. Guy's conjecture for $k=3$ is that, for $h \geqslant 20$,

$$n(h,3) = (4h^3 + 54h^2 + (204+3c_r)h + d_r)/81,$$

where c_r, d_r are given, for $h \equiv r$, mod 9, by:

$r=$	-4	-3	-2	-1	0	1	2	3	4
$c_r=$	0	1	3	0	-2	0	3	1	0
$d_r=$	46	-81	-1	-170	0	62	-26	0	-154.

This problem has been around for a long time; however, the earliest reference we could find is Rohrbach [37]. Several special cases of the postage stamp problem appear in the recreational literature. See, for example, Sprague [42, Prob. 18], Gardner [3, Prob. 4], and Legard [24].

A problem closely related to solving $n(2,k)$ is the representation of the integers $1,\ldots,n$ by differences of the members of a solution set. Miller [26] and Leech [23] describe this problem.

Alter and Barnett [1] describe an application of the $n(2,k)$ problem to the optimal allocation of index registers on computers. Hargraves [6] describes another application. He uses solution sets for $n(h,k)$ to design optimal wiring patterns for an associative cache memory.

Several thousand hours of computer time have been dedicated to obtaining values of $n(h,k)$. All reported algorithms are exponential in h and k. Table 1 summarizes the known values of n except those given by simple expressions (i.e., $h=1$ or $k=1,2$). The first publications of the included values are found in Stöhr [44], Henrici [12], Lunnon [25], Seldon [39], [40], Phillips [35], and Alter and Barnett [1]. Later, confirmatory results are given by Stanton et al. [43] and Heimer and Langenbach [11]. Henrici reports additional values for $n(2,k)$ where $k=14,\ldots,18$, as, respectively, 80, 92, 104, 116, and 128. He obtains these values using an unproved pruning heuristic. Thus they should be viewed as lower bounds until more reliable methods are employed. The values for $n(3,k), k \leqslant 47$, were calculated by John A. Bate; we are grateful to him for allowing us to publish them.

Several special-case investigations are worthy of note. Wegner and Doig [45] examine symmetric denomination sets. Let $\nu = \{a_0 = 0 < a_1 < \cdots < a_k\}$ be a denomination set. Then ν is symmetric if the sequence of differences of consecutive elements is palindromic. Symmetric solution sets exist for all values of $n(2,k)$ that are known except $k=10$. Rohrbach [37] investigates a restricted class of symmetric sets to derive his asymptotic bounds.

Henrici [12] drops the restriction that a_i be positive. He finds the solution set $\{-1,2,3,4,10,11,12,15\}$ for $n(2,7)$ and claims $n=27$. Table 1 shows $n(2,7)=26$. Note, $k=7$ by

TABLE 1
Known Values of $n(h,k)$

h \ $k=$	3	4	5	6	7	8	9	10	11	12	13
$h=$ 2	8	12	16	20	26	32	40	46	54	64	72
3	15	24	36	52	70	93	121	154			
4	26	44	70	108	162	220					
5	35	71	126	211							
6	52	114	216	388							
7	69	165	345								
8	89	234	512				$=n(h,k)$				
9	112	326	797								
10	146	427									
11	172	547									
12	212	708									
13	259	873									
14	302	1094									

$k=$	15	16	17	18	19	20	21	22	23	24	25
$n(3,k)=$	354	418	476	548	633	714	805	902	1012	1127	1254
$k=$	26	27	28	29	30	31	32	33	34	35	36
$n(3,k)=$	1382	1524	1678	1841	2010	2188	2382	2584	2801	3020	3256
$k=$	37	38	39	40	41	42	43	44	45	46	47
$n(3,k)=$	3508	3772	4043	4326	4628	4941	5272	5606	5960	6334	6723

Henrici's definition. The claim is justified because the normal problem statement allows an uncounted zero element. On the other hand, this result has the range $1,\dots,27$, whereas the normal result has the range $0,\dots,26$.

Henrici finds the *symmetric* (and unique) solution set $\{-1,1,2,4,8,12,16,20,22,23,25\}$ with range $0,\dots,48$ for $n(2,10)$. The value -1 is not included because sums must be formed from exactly two elements.

Alter and Barnett [1] derive an interesting bound for the case $h=k$. Namely, $n(h,h)\geqslant f_{2h}-1$, where f_i is the ith Fibonacci number.

Since the initial statement of the postage stamp problem by Rohrbach, significant progress toward a solution has been made. However, many issues remain open.

Problem 1. Can the bounds on n be improved? The distance between the best known upper and lower bounds is large. Clearly, there is room for progress short of finding a simple formula for n.

Problem 2. Is there a simple relation between $n(h,k)$ and $n(k,h)$?

Problem 3. What is the multiplicity of solution sets as a function of h and k?

Problem 4. Let $\nu=\{a_1,\dots,a_k\}$ and define $n(h,\nu)$ as the maximum integer, n, such that all integers $1,\dots,n$ can be made up as sums of no more than h of the a_i. Can $n(h,\nu)$ be expressed by a simple formula? Note that

$$n(h,k)=\max_{\nu\in U_k} n(h,\nu),$$

where U_k is the set of all k-element denomination sets.

Knowledge of $n(h,\nu)$ would be a tremendous aid to improving estimates of $n(h,k)$. The known lower bounds are generated by restricting U_k so that $n(h,\nu)$ is easily represented.

Problem 5. Let $\{a_1,\dots,a_k\}$ be a solution set for $n(h,k)$. What are bounds for a_i as a function of i, h, and k? Also, what is the magnitude of a_{i+1} relative to a_i?

Problem 6. What is the behavior of $n(h,k)$ if negative and rational stamp denominations are permitted?

Problem 7. For what values of h and k do symmetric solutions exist?

Problem 8. Do polynomial-time computational algorithms exist for $n(h,k)$ and the corresponding solution sets?

The bibliography includes several papers not cited in the text. Our literature search was more difficult than usual because the postage stamp problem seems to have been reinvented many times. Stöhr [44] summarizes work prior to 1955.

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CLASSROOM NOTES

EDITED BY DEBORAH TEPPER HAIMO AND FRANKLIN TEPPER HAIMO

Material for this department should be sent to Professor Deborah Tepper Haimo, Department of Mathematical Sciences, University of Missouri, St. Louis, MO 63121.

DERIVATION OF THE GIBBS PHENOMENON

DAVID SHELUPSKY

The Gibbs phenomenon in the theory of Fourier series is an example of what Carathéodory [**1**, p. 3] calls non-continuous convergence: a sequence of functions f_n, $n \geqslant 1$, converges to a function f at each point of some interval; but at some point x, and for some real sequence x_n converging to x, $f_n(x_n)$ does not converge to $f(x)$. In our discussion, f_n is the nth partial sum of the Fourier series of the 2π-periodic function f, and x is a point of discontinuity of f. The Gibbs phenomenon is essentially the following: we do not have $f_n(x_n) \to f(x+0)$ generally if $x_n \downarrow x$ or $f_n(x_n) \to f(x-0)$ if $x_n \uparrow x$. In this note the phenomenon is obtained by a process different from that used in most of the standard works on trigonometric and Fourier series (for example, [**2**, p. 179]); the treatment is simpler since it uses no integrals with Dirichlet kernels.

Recall that if f is assumed to be piecewise continuous, continuously differentiable, and normalized so that $f(x)=\frac{1}{2}[f(x+0)+f(x-0)]$, then the Fourier series of f converges at each point, and uniformly on any closed set not containing a discontinuity of f. This is the representation theorem that goes back to Dirichlet. (In fact, by Jordan's theorem it suffices to take f piecewise continuous and of bounded variation.)

If f is continuous, it follows that the convergence is continuous everywhere because the partial sums of the Fourier series of f converge uniformly to f. Let us denote by ϕ the 2π-periodic function given by $\phi(0)=0$, and $\phi(x)=\frac{1}{2}(\pi-x)$ for $0<x<2\pi$. Then, in the general case, when f is as above but not continuous, it can be represented as a sum of a continuous function and of a linear combination of a finite number of translates of ϕ. In this way the problem of the continuity of the convergence is reduced to that of the Fourier series of ϕ, which is

$$\sum_{k=1}^{\infty} k^{-1}\sin kx \tag{1}$$

and at the single point $x=0$. (Indeed, this reduction is standard in proving Dirichlet's theorem; one proves it separately for f continuous and piecewise differentiable, and for the function [3, p. 70].)

To exhibit the Gibbs phenomenon that occurs for (1) at $x=0$, write

$$s_n(x)=\sum_{k=1}^{n} k^{-1}\sin kx$$

and let $c>0$. Then

$$\begin{aligned} s_n(c/n) &= \sum_{k=1}^{n}\frac{\sin(kc/n)}{k} \\ &= \sum_{k=1}^{n}\frac{\sin(kc/n)}{(kc/n)}\left[\frac{kc}{n}-\frac{(k-1)c}{n}\right]; \end{aligned}$$

hence, by definition of the Riemann integral, as n increases

$$s_n(c/n)\to\int_0^c\frac{\sin t}{t}\,dt. \tag{2}$$

Denote the integral on the right-hand side by $I(c)$. As a function of c, $c\geqslant 0$, $I(c)$ is easily seen to have extrema at $c=m\pi$, $m=1,2,3,\ldots$ and a single absolute maximum at $c=\pi$. The value of $I(\pi)$ is approximately 1.18 times $\frac{1}{2}\pi$. Thus the limiting value of $s_n(\pi/n)$ is about 18 percent larger than $\phi(0+0)=\frac{1}{2}\pi$. From (2) we see that for each real u such that $0\leqslant u\leqslant I(n)$, a sequence x_i exists such that $x_i\downarrow 0$ and $s_n(x_n)\to u$. This is, perhaps, the basis for the statement found in some of the older books on the subject, that at $x=0$ the Fourier series of ϕ converges to the vertical line segment with endpoints $(0,I(\pi))$ and $(0,-I(\pi))$ or to infinitely many such lines [4, p. 294].

We can consider the matter alternatively in terms of maxima of the remainders, $r_n=\phi-s_n$. From

$$s_n'(x)=\sum_{k=1}^{n}\cos kx$$

we get

$$\begin{aligned} (2\sin\tfrac{1}{2}x)s_n'(x) &= \sum_{k=1}^{n}\left[\sin\left(k+\tfrac{1}{2}\right)x-\sin\left(k-\tfrac{1}{2}\right)x\right] \\ &= \sin\left(n+\tfrac{1}{2}\right)x-\sin\tfrac{1}{2}x, \end{aligned}$$

so that

$$\left(2\sin\tfrac{1}{2}x\right)\frac{d}{dx}\left[\tfrac{1}{2}(\pi-x)-s_n(x)\right]=\left(2\sin\tfrac{1}{2}x\right)r_n'(x)=-\sin\left(n+\tfrac{1}{2}\right)x.$$

It follows that $r_n'(x)=0$ in $0\leqslant x\leqslant 2\pi$ at the points $x=\pi m/(n+\frac{1}{2})$, $m=1,2,\ldots,2n$. At $\pi m/(n+\frac{1}{2})$, the mth positive extremum of r_n, the value of s_n is $s_n(\pi m/n+\frac{1}{2})$, and as above we show that as n increases

$$s_n\left(\pi m/\left(n+\tfrac{1}{2}\right)\right)=\sum_{k=1}^{n}\frac{\sin\left(\pi mk/\left(n+\frac{1}{2}\right)\right)}{k}$$
$$=\sum_{k=1}^{n}\frac{\sin\left(\pi mk/\left(n+\frac{1}{2}\right)\right)}{mk/\left(n+\frac{1}{2}\right)}\left[\frac{m(k+1)}{n+\frac{1}{2}}-\frac{mk}{n+\frac{1}{2}}\right]\to\int_0^{m\pi}\frac{\sin t}{t}\,dt.$$

The Gibbs phenomenon is usually stated in terms of this limit for $m=1$.

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AN EXAMPLE OF THE USE OF A COMPUTER IN NUMBER THEORY

J. C. HINTZ

1. Introduction. Many number theory problems are solvable by proofs which proceed in the following manner: If some proposition is true for a large number of cases, then it can be shown to be true for all natural numbers. Computers are being used more and more frequently to analyze the "large number of cases."

In this paper, we consider fixed-point iteration with a class of functions from the natural numbers, $\mathbb{N}$ into $\mathbb{N}$. We will show, for a fixed function of that class and any $n\in\mathbb{N}$ as an initial point, that fixed-point iteration always results in a finite number of finite cycles. In Section 2, we define our terms and state general results. In Section 3, we consider specific examples and consider some of the computational problems. We conclude in Section 4 with some remarks.

2. General Results.

DEFINITION. Let $F:\mathbb{N}\to\mathbb{N}$. A subset A of $\mathbb{N}$ is called a *target* set for F, provided

(a) card $A<\infty$, and

(b) for every $n\in\mathbb{N},\exists l\in\mathbb{N}$ such that

$$F^l(n)\in A,$$

where superscript denotes function composition. A subset A of $\mathbb{N}$ is a *minimal target* set for F if A is a target set and if, for all $a\in A, A-\{a\}$ is not a target set.

We now show, using a fairly simple condition on F, that target sets for F exist. In the next section, we will consider specific functions and consider their minimal target sets.

DEFINITION. Let $F:\mathbb{N}\to\mathbb{N}$. F is called a *diminishing* function provided $\exists N\in\mathbb{N}$ such that for all $n\in\mathbb{N}, n\geqslant N$

$$F(n)<n.$$

PROPOSITION. *If F is a diminishing function, then there exist target sets for F.*

A CONVERSE TO A COMPLETENESS THEOREM

W. E. Pfaffenberger

The following theorem [1, p. 211] is well known.

THEOREM 1. *Let X and Y be normed linear spaces over the field F (F is either the real or complex numbers) and let $B(X, Y)$ denote the set of all bounded linear transformations from X into Y. Then, with pointwise linear operations and the operator norm, $B(X, Y)$ is a normed linear space. Moreover, $B(X, Y)$ is a Banach space (i.e., is complete) if Y is a Banach space.*

We are interested in the converse of this last statement. That is, if $B(X, Y)$ is complete, does this imply Y must be complete? In general the answer is negative, as the next trivial example shows.

Example. If $X=\{0\}$, then $B(X, Y)=\{0\}$ is complete for any normed linear space Y.

It is interesting, however, that the following theorem shows our example to be the only case where $B(X, Y)$ is complete and Y may not be complete.

THEOREM 2. *Let $X\neq\{0\}$ and Y be normed linear spaces over F. Then $B(X, Y)$ is complete if and only if Y is complete.*

Proof. ($\Leftarrow$) Theorem 1.

($\Rightarrow$) Suppose $B(X, Y)$ is complete. Take $x\neq 0$, $x\in X$. Look at the one-dimensional subspace $M=\{\lambda x:\lambda\in F\}$ in X.

We show the existence of a continuous projection from X onto M, that is, of a bounded linear transformation P from X onto M, such that $P^2=P$. We construct P as follows:

By the Hahn-Banach theorem [1] there exists an $f\in B(X,F)$ such that $f(x)=1$. Define $P(z)=f(z)x$ for all $z\in X$. P is bounded since $\|P(z)\|=\|f(z)x\|=|f(z)|\,\|x\|\leqslant\|f\|\;\|z\|\;\|x\|$ implies $\|P\|\leqslant\|f\|\;\|x\|$. Also, $P^2=P$ since $P^2(z)=P(P(z))=P(f(z)x)=f(z)P(x)=f(z)f(x)x=f(z)x=P(z)$ for all $z\in X$.

Let $\{y_n\}$ be any Cauchy sequence in Y. We construct a sequence of bounded operators in $B(M, Y)$ as follows: $P_n(\lambda x)=f(\lambda x)y_n=\lambda f(x)y_n=\lambda y_n$ for all $\lambda\in F$ and each $n=1,2,\ldots$.

Construct the sequence $\{S_n\}$ in $B(X, Y)$ by $S_n=P_nP$ for $n=1,2,\ldots$.

We show that $\{S_n\}$ is a Cauchy sequence in $B(X, Y)$. $\|S_n-S_m\|=\|P_nP-P_mP\|=\|(P_n-P_m)P\|\leqslant\|P_n-P_m\|\;\|P\|$, so it suffices to show that $\{P_n\}$ is Cauchy in $B(M, Y)$.

$$\begin{aligned}\|P_n-P_m\|&=\sup\{\|(P_n-P_m)\|(\lambda x):\|\lambda x\|=1,\lambda x\in M\}\\&=\sup\left\{\|\lambda y_n-\lambda y_m\|:|\lambda|=\frac{1}{\|x\|},\lambda\in F\right\}\\&=\sup\left\{|\lambda|\,\|y_n-y_m\|:|\lambda|=\frac{1}{\|x\|},\lambda\in F\right\}=\frac{1}{\|x\|}\|y_n-y_m\|.\end{aligned}$$

Therefore since $\{y_n\}$ is Cauchy in Y, this implies $\{P_n\}$ is Cauchy, which implies $\{S_n\}$ is Cauchy.

Since $B(X, Y)$ is complete, we have $\lim_{n\to\infty}S_n=S\in B(X, Y)$. This implies $\lim_{n\to\infty}\|S_n(z)-S(z)\|=0$ for all $z\in X$, and therefore $\lim_{n\to\infty}S_n(z)=S(z)$ for all $z\in X$.

Taking $z=x$, this implies $\lim_{n\to\infty}S_n(x)=\lim_{n\to\infty}P_nP(x)=\lim_{n\to\infty}P_n(x)=\lim_{n\to\infty}y_n=S(x)\in Y$.

Therefore, we have shown that $\{y_n\}$ converges in Y and, since $\{y_n\}$ was an arbitrary Cauchy sequence, Y must be complete.

Reference

1. E. Hewitt and K. Stromberg, Real and Abstract Analysis, Springer-Verlag, New York, 1965.

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A NONTOPOLOGICAL PROOF OF THE UNIFORM BOUNDEDNESS THEOREM

JULIEN HENNEFELD

The purpose of this article is to present a proof of the uniform boundedness theorem which can be comprehended by observing a single equation. The proof is elementary in Halmos's sense of the word in that it does not use the Baire Category theorem or any related lemmas. In fact, this proof was prompted by Halmos's [**1**, Problem 20], which asks for an elementary proof in the special case of Hilbert space.

THEOREM. *Let* $\{A_\alpha\}_{\alpha\in J}$ *be a family of bounded linear operators from a Banach space X to a normed space Y. If* $\sup_\alpha \|A_\alpha x\| < \infty$ *for each x, then* $\sup_\alpha \|A_\alpha\| < \infty$.

Proof. Suppose the contrary, that $\|A_\alpha\|$ is unbounded. We will construct a sequence of elements $\{x_n\}$ and operators $\{A_n\}$ from $\{A_\alpha\}$ so that $\|x_n\| = 4^{-n}$ and so that for $x = \sum_{n=1}^{\infty} x_n$, $\|A_n x\| > n$ for each n.

Suppose $x_1,\dots,x_{n-1},A_1,\dots,A_{n-1}$ have already been selected. Consider

$$\boxed{\underbrace{A_n x = A_n(x_1 + \cdots + x_{n-1})}_{\text{Past}} + \underbrace{A_n x_n}_{\text{Present}} + \underbrace{A_n(x_{n+1} + \cdots)}_{\text{Future}}}$$

We will show that the *Present* term has large norm and in fact dominates the right side of the equation. The *Past* term has norm bounded by $\sup_\alpha \|A_\alpha(x_1 + \cdots + x_{n-1})\|$. The *Present* term can be chosen with norm large in relation to the norm of the *Past* term and n, and also so that $\|A_n x_n\| \approx \|A_n\| \|x_n\|$.

The size of the *Future* term is controlled by the inequality

$$\left\| \sum_{n+1}^{\infty} x_k \right\| \leqslant \tfrac{1}{3}\|x_n\|$$

and its consequence

$$\left\| A_n \sum_{n+1}^{\infty} x_k \right\| \leqslant \tfrac{1}{3}\|A_n\| \|x_n\|.$$

Here are the formal details: Suppose $x_1,\dots,x_{n-1},A_1,\dots,A_{n-1}$ have already been selected. Let $M_{n-1} = \sup_\alpha \|A_\alpha(x_1 + \cdots + x_{n-1})\|$. First select A_n so that $\|A_n\| > 3\cdot 4^n[M_{n-1} + n]$. Then select x_n of norm 4^{-n} so that $\|A_n x_n\| > \frac{2}{3}\|A_n\| \|x_n\|$. It follows that

(1) $\|A_n x_n\| > 2[M_{n-1} + n]$, (2) $\|A_n(x_{n+1} + \cdots)\| \leqslant \frac{1}{2}\|A_n x_n\|$.

Hence, by the triangle inequality and (1) and (2) we have

$$\|A_n x\| \geqslant [\|\text{Present}\| - \|\text{Future}\|] - \|\text{Past}\| > [M_{n-1} + n] - M_{n-1} = n.$$

This concludes the proof.

It may seem surprising that the elementary proof given here, which is more direct and illuminating than the standard Baire Category Theorem proofs, should not have been discovered before. In fact, it has; it is essentially the same proof given by Hausdorff in 1932 [**2**; item VII] as was pointed out by a referee. A remaining question is why has Hausdorff's proof fared so poorly in comparison to rival proofs—textbooks in Analysis and Functional Analysis give only the Baire Category proofs without any mention of an elementary proof in the text or even as an exercise.

References

1. P. Halmos, A Hilbert Space Problem Book, Van Nostrand, Princeton, N.J., 1967.
2. F. Hausdorff, "Zur Theorie der linearen metrischen Räume," J. Reine Angew. Math., 167 (1932) 294–311.

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PROBLEMS AND SOLUTIONS

EDITED BY VLADIMIR DROBOT

Send all proposed problems, in duplicate if possible, to Prof. Vladimir Drobot, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053. Please include solutions, relevant references, etc.

An asterisk (*) indicates that neither the proposer nor the editors supplied a solution.

Solutions should be sent to the addresses given at the head of each problem set.

A publishable solution must, above all, be correct. Given correctness, elegance and conciseness are preferred. The answer to the problem should appear right at the beginning. If your method yields a more general result, so much the better. If you discover that a MONTHLY *problem has already been solved in the literature, you should of course tell the editors; include a copy of the solution if you can. Before taking the trouble to write up a very lengthy solution, you may ask the appropriate editor whether any solutions have already been received. Enclose a self-addressed card or envelope.*

PROBLEMS DEDICATED TO EMORY P. STARKE

Solutions of these problems dedicated to E. P. Starke should be mailed to Prof. A. P. Hillman, Department of Mathematics & Statistics, University of New Mexico, Albuquerque, NM 87131 by July 31, 1980. To facilitate consideration, type with double spacing in the format used below. If acknowledgment is desired, include a self-addressed card, label, or envelope.

S 27. *Proposed by Emma Lehmer, Berkeley, Ca.*

Let p be an odd prime and (n/p) be the Legendre symbol. Set $\xi=\exp(2\pi i/p)$. Let Σ denote the sum taken over the range $1\leqslant n\leqslant(p-1)/2$.

It is well known that the Gauss half-sum $\Sigma(n/p)\xi^n$ has the value

$$\tfrac{1}{2}p^{1/2}+i\Sigma(n/p)\sin(2\pi n/p)\quad \text{if } p\equiv 1 \pmod 4$$

and has the value

$$\Sigma(n/p)\cos(2\pi n/p)+\tfrac{1}{2}ip^{1/2}\quad \text{if } p\equiv 3 \pmod 4.$$

Show that

$$(2/p)\Sigma(n/p)\sin(2\pi n/p)<0\quad \text{if } p\equiv 1 \pmod 4,$$

$$(2/p)\Sigma(n/p)\cos(2\pi n/p)>0\quad \text{if } p\equiv 3 \pmod 4.$$

S 28. *Propsed by David A. Sánchez, University of New Mexico.*

Find the general solution of the differential equation

$$x^3y''+2x^2=(xy'-y)^2.$$

SOLUTIONS OF PROBLEMS DEDICATED TO EMORY P. STARKE

Approaching Equal Folds

S 4 [1979, 127]. *Proposed by Richard K. Guy, University of Calgary.*

For Emory Starke, who welcomed and inspired a wandering amateur mathematician 19 years ago and gave him a copy of the Otto Dunkel Memorial Problem Book.

In order to store a given length L of paper tape in an accessible way, I choose a length, λ, and an even integer, $2n$, so that $2n\lambda = L$. I then screenfold the tape with n "odd" folds in one sense at distances $f_1, f_3, \ldots, f_{2n-1}$ along the tape, and $n-1$ "even" folds, in the other sense, at distances $f_2, f_4, \ldots, f_{2n-2}$. The ends of the tape are $f_0 = 0$ and $f_{2n} = L = 2n\lambda$. I try to arrange that the quantities $f_{i+1} - f_i = \lambda_i, 0 \leqslant i \leqslant 2n-1$ are each equal to λ, but in practice this rarely happens, so I then endeavor to improve the situation by lining up the ends and the even folds, $f_0, f_2, \ldots, f_{2n}$ and recreasing the odd folds at $f_1', f_3', \ldots, f_{2n-1}'$, so that hopefully better approximations, λ_i' to λ_i are produced, namely, $\lambda_i' = \lambda_{i+1}'[=(\lambda_i + \lambda_{i+1})/2]$ for $i = 0, 2, \ldots, 2n-2$. I then line up the odd folds and recrease the even ones, giving $\lambda_{i-1}'' = \lambda_i''[=(\lambda_{i-1}' + \lambda_i')/2]$ for $i = 2, 4, \ldots, 2n-2$. I then repeat the process. Does it terminate, or even converge?

Solution by J. G. Wendel, California Institute of Technology. The process may not terminate but must converge to equifoldedness.

To show convergence we observe that the matrix of the mapping $(\lambda_i) \rightarrow (\lambda_i'')$ is doubly stochastic; its top row is (1/2, 1/2, 0,...,0), its bottom row is (0,...,0, 1/2, 1/2), and the intermediate rows come in equal pairs, the first two being (1/4, 1/4, 1/4, 1/4, 0,...,0), then (0, 0, 1/4, 1/4, 1/4, 1/4, 0,...,0), and so on. In the language of Markov chain theory, the system is aperiodic and all states communicate. Then the powers converge to the projection with all equal elements, i.e., the matrix with all elements $1/2n$.

To see an example of nontermination, take $n = 2$ and observe that the column vector $(1, 0, 0, -1)^{\text{tr}}$ is an eigenvector for the eigenvalue $1/2$. Then powers of the matrix applied to the column vector $(1+c, 1, 1, 1-c)^{\text{tr}}$ have the form $(1+2^{-n}c, 1, 1, 1-2^{-n}c)^{\text{tr}}$; it suffices to take c positive and less than unity to get an example with $L = 4$.

Also solved by O. P. Lossers (Netherlands), L. E. Mattics, Walter Taylor, Paul A. Vojta, an anonymous solver, and the proposer.

Intersecting Sets of Differences

S 5 [1979, 127]. *Proposed by R. L. Graham, Bell Laboratories, Murray Hill, N.J.*

For a finite set X of integers, let $|X|$ denote the cardinality of X and let $X - X$ denote $\{x - x' : x, x' \in X\}$. Show that if $A, B \subseteq \{1, 2, \ldots, n\}$ with $|A||B| \geqslant 2n-1$ then $(A-A) \cap (B-B)$ contains a positive element. Here $n > 1$.

Solution by Thomas Jager, Calvin College. Consider the mapping $T: A \times B \rightarrow \{2, 3, 4, \ldots, 2n\}$ defined by $T(a,b) = a+b$. If T is onto, $(1,1), (n,n) \in A \times B$, so that $n-1 \in A - A \cap B - B$. If T is not onto, since $|A||B| \geqslant 2n-1, T$ cannot be one-one. Hence, there exist $(a,b) \neq (a',b')$ such that $a+b = a'+b'$. Assuming $a > a'$, we have that $a - a' = b' - b$ is a positive element in $A - A \cap B - B$.

Also solved by Carl E. Bredlau, Duane M. Broline, Boris Datskovsky, Bob Dickinson & Grant Guenther, Curtis D. Herink, R. Hill (England), Steve Hubbard, Lillian E. Peters Hupert, O. P. Lossers (Netherlands), L. E. Mattics, Mark Merriman, R. G. E. Pinch (England), Kenneth Rogers, St. Olaf College Problems Group, Dan Sokolowsky, J. M. Stark, William Staton, Charles Vanden Eynden, David Van Leeuwen, Paul A. Vojta, Hann Tzong Wang (Taiwan), N. Williams, an anonymous solver, and the proposer.

ELEMENTARY PROBLEMS

Solutions of these Elementary Problems should be mailed in duplicate to Dr. J. L. Brenner, 10 Phillips Road, Palo Alto, CA 94303 (USA), by July 31, 1980. Please place the solver's name and mailing address on each (double-spaced) sheet. Include a self-addressed card or label (for acknowledgment).

E 2821. *Proposed by Jeffrey Shallit, undergraduate, Princeton University.*

Express $S(n)=\sum_{k=1}^{n}\gcd(k,n)$ in terms of the factorization of n as a product of powers of distinct primes.

E 2822. *Proposed by R. S. Lehman, University of California, Berkeley.*

Evaluate the limits

(a) $$\lim_{n\to\infty}\int_0^{(2n+1)\pi/2}(\sin t)\ln(1/t)\,dt,$$

(b) $$\lim_{n\to\infty}\int_0^{n\pi}(\cos t)\ln(1/t)\,dt,$$

where n takes on integer values and the integrals are improper Riemann integrals.

E 2823. *Proposed by Herbert Carus, Lanham, Md.*

In Honsberger's *Mathematical Morsels* (see also E 1366 [1959, 432; 1960, 82]) it is shown that the function $f(x)=x^p$, $p=1/2,1/3,\ldots$, is triangle-preserving, i.e., if x_1,x_2,x_3 are the lengths of the sides of a triangle, then $f(x_1),f(x_2),f(x_3)$ are also the lengths of the sides of a triangle. Show that any (positive) increasing function $f(x)$ for which $f''(x)<0$ is triangle-preserving.

E 2824. *Proposed by Barry J. Powell, Kirkland, Wash.*

Prove that for p any odd prime with $p\not\equiv 1 \pmod 4$, the equation

$$x^{2p}+y^{2p}+z^{2p}=w^{2p}$$

has no solution in positive integers x,y,z,w with $xyzw\not\equiv 0 \pmod p$.

(Compare this problem with E 2771 [1979, 308].)

E 2825. *Proposed by R. A. Melter, Southampton College.*

In which rings is the following proposition valid: $x=y$ if and only if $(1-x+xy)(1-y+yx)=1$?

SOLUTIONS OF ELEMENTARY PROBLEMS

Double Series

E 2743 [1978, 823]. *Proposed by Peter Ungar, Courant Institute, New York University.*

Find

$$\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{(-1)^{i+j}}{i+j}.$$

I. *Solution by L. Van Hamme, Vrije Universiteit, Brussels, Belgium.*

$$S_{m,n}=\Sigma_{i=1}^{m}\Sigma_{j=1}^{n}(-1)^{i+j}/(i+j)=\int_0^1\Sigma_i\Sigma_j(-1)^{i+j}x^{i+j-1}dx.$$

$$=\int_0^1 x\{\Sigma_{i=0}^{m-1}(-x)^i\}\{\Sigma_{j=0}^{n-1}(-x)^j\}dx$$

$$=\int_0^1 x[1-(-x)^m][1-(-x)^n]/(1+x)^2dx.$$

Now

$$\int_0^1 x^n/(1+x)^2dx<\int_0^1 x^n dx=1/(n+1)$$

and hence $\lim_{n\to\infty}\int_0^1 x^n/(1+x)^2dx=0$. Therefore $\lim_{m,n\to\infty}S_{m,n}=\int_0^1 x/(1+x)^2dx=\ln 2-1/2$.

II. *Solution by Harley Flanders, Florida Atlantic University.*

$$|S_{m,n}-S_{m-1,n}|<1/(m+1)\quad\text{and}\quad|S_{m,n}-S_{m,n-1}|<1/(n+1).$$

Therefore $\lim_{m,n\to\infty}S_{m,n}=\lim_{m,n\to\infty}S_{2m,2n}$. Summing "by squares"

$$S_{2m,2n}=\Sigma_{i=1}^{m}\Sigma_{j=1}^{n}\left[\frac{1}{2i+2j-2}-\frac{2}{2i+2j-1}+\frac{1}{2i+2j}\right]$$

$$=\Sigma_{i=1}^{m}\Sigma_{j=1}^{n}\frac{2}{(2i+2j-1)(2i+2j-2)(2i+2j)}$$

We can sum this series of positive terms in any order we please. There are $k-1$ terms on the diagonals where $i+j=k$, so

$$\lim S_{2m,2n}=\Sigma_{k=2}^{\infty}\frac{2(k-1)}{(2k-1)(2k-2)(2k)}=\Sigma_{k=2}^{\infty}\frac{1}{(2k-1)(2k)}$$

$$=\Sigma_{k=2}^{\infty}\left(\frac{1}{2k-1}-\frac{1}{2k}\right)=\ln 2-1/2.$$

Comment: By the same method and using the backward difference ∇, if $(i)=(i_1,\dots,i_p)$

$$S_p=\lim_{(n)\to(\infty)}\Sigma_{(i)=(1)}^{(n)}\frac{(-1)^{\Sigma i}}{\Sigma i}=\Sigma_{(i)=(1)}^{(\infty)}\frac{(-1)^p p!}{(2x)(2x-1)\cdots(2x-p)}$$

where $x=\Sigma i$. Summed by the diagonal,

$$S_p=\Sigma_{x=p}^{\infty}\frac{(-1)^p p!}{(2x)(2x-1)\cdots(2x-p)}\binom{x+p-3}{p-1}.$$

The first three values are

$$S_1=-\Sigma_{x=1}^{\infty}\frac{1}{(2x)(2x-1)}=-\ln 2$$

$$S_2=\ln 2-1/2$$

$$S_3=-\tfrac{3}{4}\Sigma_{x=3}^{\infty}1/[(2x-1)(2x-3)]=-1/8.$$

In fact, for $p>3$ it can be shown that $S_p=A_p-B_p\ln 2$ where A_p and B_p are rational and $B_p>0$. For instance

$$S_4=(1-1/2+1/3-1/4+1/5)-\ln 2.$$

Comment by H. Schmidt, Würzburg-Heidingsfeld. For some double sums of a related type, but with iterated summation only, see Kiyek and Schmidt, Archiv. der Mathematik, 18 (1967) 242–243.

Also solved by H. L. Abbott, K. F. Andersen, J. M. Ash, G. Bach (Germany), D. M. Bloom, T. S. Bolis, A. Bondesen (Denmark), P. Bracken (Canada), Case Western Reserve Problem Solving Team, P. Chauveheid

(Belgium), L. L. Foster & G. Biriuk, D. Gallin, J. A. Gillespie, R. M. Giuli, M. L. Glasser, J. W. Grossman, J. R. Hatcher, P. Henrici, G. A. Heuer, E. L. Isaacson, A. A. Jagers (Netherlands), M. S. Klamkin (Canada), O. P. Lossers (Netherlands), W. F. Martens, R. M. McLeod, V. N. Murty, O. G. Ruehr, A. A. Sardinas, P. H. Savet, J. Silverman, M. Skalsky, University of South Alabama Problem Group, A. Stenger, R. Van Gaalen (Canada), M. Vowe (Switzerland), D. Zeitlin, P. Zwier, and the proposer.

Nonoverlapping Pennies

E 2745* [1978, 824]. *Proposed by David Hammer, Santa Cruz, California.*

Can every collection of non-overlapping pennies in the plane be colored with three colors so that no penny touches more than one penny with the same color?

Solution by G. L. Watson, University College, London, England. The answer is yes. Consider the lattice H of points (x,y) (x,y integers) in a coordinate system in which the six points $(\pm 1,0)$, $(0,\pm 1)$, $(\pm 1,\mp 1)$ are each at distance 1 (the diameter of a penny) from $(0,0)$. The distance between two different points (x,y), (x',y') of H is greater than 1 if $x+x'\equiv y+y' \pmod 3$. This disposes of the special case in which every penny has its center at a point of H; we assign the three colors to the pennies with centers satisfying $x-y\equiv 0,1,2 \pmod 3$, respectively.

In general, if any penny, say, p_1, touches at least six others, say, $p_2,\ldots,p_7$, it touches exactly six, and their centers all lie in a grid G derived from H by rotation and translation. For each such grid G_α, let S_α be the set of all the pennies with centers in G_α that have six neighbors. Any given set R of pennies contains at most countably many nonempty sets S_α. Set $T=\cup S_\alpha$. Then R may be expressed as the union of T and $W=\{q_1,q_2,\ldots\}$, where each q_i is a penny that touches at most five others. Clearly each S_α can be colored.

Next note that any two sets S_1,S_2 are separated; so T can be colored. In fact, if p_1,p_2 are respectively in S_1,S_2 and touch each other, then each of p_1,p_2 touches at least seven other pennies, which is impossible. We set $Q_1=T, Q_i=T\cup\{q_1,\ldots,q_{i-1}\}$ $(i>1)$. Having colored T, we color the q_i in order. Since q_i touches at most five of the pennies in Q_i, there must be at least one color that has been assigned to at most one of these five (or fewer) pennies. This color will do for q_i. This (transfinite or finite) induction completes the proof, which clearly uses the axiom of choice.

Almost the same argument shows that any number of nonoverlapping pennies in the plane can be colored with two colors so that no penny touches more than two other pennies of the same color.

Zeros of Derivatives of a Fading Function

E 2755 [1979, 127]. *Proposed by Michael Slater, University of Bristol, England.*

Let $f\in C^\infty(\mathbb{R})$ and suppose that $f(x)=o(x^n)$ as $x\to\pm\infty$. Show that $f^{(r)}$ has a zero for every $r\geqslant n+1$.

Is this conclusion the best possible?

Solution by Roger L. Cooke, University of Vermont. It is sufficient to show that $f^{(n+1)}(x)$ has a zero, since the hypothesis implies that $f(x)=o(x^k)$ for any $k\geqslant n$.

Suppose that $f^{(n+1)}(x)$ has no zeros. The $f^{(n)}(x)$ is a strictly monotonic function, and hence has at most one zero. Let $x_0=0$ if $f^{(n)}(x)$ has no zeros, and $x_0=z_0+1$ if $f^{(n)}(z_0)=0$. By Taylor's theorem with remainder, for any $x>x_0$ there exists $c=c(x)>x_0$ such that

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\cdots+f^{(n-1)}(x_0)(x-x_0)^{n-1}/(n-1)!+f^{(n)}(c)(x-x_0)^n/n!.$$

Since $f(x)=o(x^n)$, it follows that $f^{(n)}(c(x))\to 0$ as $x\to+\infty$. Since $f^{(n)}(x)$ is strictly monotonic and $c(x)>z_0+1$, this implies that $c(x)\to+\infty$. Hence $f^{(n)}(x)\to 0$ as $x\to+\infty$. A similar argument shows that $f^{(n)}(x)\to 0$ as $x\to-\infty$. But this is impossible for a strictly monotonic function.

Therefore $f^{(n+1)}(x)$ must have a zero. The example

$$f(x)=\int_{-\infty}^{x}\int_{-\infty}^{t_n}\cdots\int_{-\infty}^{t_2}\exp(-t_1^2)\,dt_1dt_2\cdots dt_n$$

shows that $f^{(k)}(x)$ may fail to have zeros for $0\leqslant k\leqslant n$.

Also solved by I. Bivens, R. Breusch, F. S. Cater, S. Hubbard, L. Kuipers (Switzerland), O. P. Lossers (Netherlands), A. Riese, University of South Alabama Problem Group, J. Suck (Germany), H. T. Wang (Taiwan), and the proposer.

Zeros of Successive Derivatives

E 2756 [1979, 128]. *Proposed by Michael Slater, University of Bristol, England.*

Let $f\in C^{\infty}(R)$, $f(0)f'(0)\geqslant 0$, and $f(x)\to 0$ as $x\to\infty$. Show that there exists a sequence $\{x_n\}$ with $0\leqslant x_1<x_2<\cdots$ such that $f^{(n)}(x_n)=0$ for $n=1,2,\ldots$.

Solution by the University of Santa Clara Problem Solving Ring. The solution of the problem follows from the two results: (A) $f'(x_1)=0$ for some $x_1\geqslant 0$; (B) If $f^{(k)}(t)=0$ then $f^{(k+1)}(s)=0$ for some $s>t$.

Re (A). If $\forall_{x\geqslant 0}[f'(x)\neq 0]$ then either $f'(x)>0$ or $f'(x)<0$. In the first case $f(0)\geqslant 0$ (since $f(0)f'(0)\geqslant 0$) and $f(x)$ is increasing, in the second $f(0)\leqslant 0$ and $f(x)$ is decreasing. In either case it is impossible that $f(x)\to 0$ holds.

Re (B). Suppose $f^{(k+1)}(s)\neq 0$ for $s>t$ so that $f^{(k+1)}(s)$ is either positive or negative. If it is positive then $f^{(k)}(s)$ is strictly increasing for $s>t$ and so $f^{(k)}(s)\geqslant\delta>0$ for $s\geqslant t_1>t$ (since $f^{(k)}(t)=0$). Integrating k times we obtain that $f(s)\geqslant(\delta/k!)s^k+p(s)$ for $s\geqslant t_1$ where $p(s)$ is a polynomial of degree $k-1$. This contradicts the fact that $f(x)\to 0$. Similarly, if $f^{(k+1)}(s)<0$ for $s>t$ we obtain $f(s)<-cs^k+p(s)$ for some $c>0$ and $p(s)$ as before; again contradicting $f(x)\to 0$.

Also solved by I. Bivens, D. M. Bloom, R. Breusch, F. S. Cater, R. L. Cooke, G. Gripenberg (Finland), F. D. Hammer, J. Hook, L. C. House, S. Hubbard, L. Kuipers (Switzerland), S. V. Noltie, V. Pambuccian (Rumania), A. Riese, D. Ross, University of South Alabama Problem Solving Group, J. Suck (Germany), H. T. Wang (Taiwan), and the proposer.

A Sum of Legendre Symbols

E 2760 [1979, 128]. *Proposed by Kenneth S. Williams, Carleton U., Canada.*

Let p be a prime. If $p\equiv 1\pmod 4$ let a be the unique integer such that

$$p=a^2+b^2,\quad a\equiv -1\pmod 4,\ b\text{ even}.$$

Prove that

$$\sum_{i=0}^{p-1}\left(\frac{i^3+6i^2+i}{p}\right)=\begin{cases}2\left(\dfrac{2}{p}\right)a, & \text{if } p\equiv 1\pmod 4\\ 0, & \text{if } p\equiv 3\pmod 4\end{cases}$$

where $\left(\dfrac{n}{p}\right)$ is the familiar Legendre symbol.

Solution by Ronald J. Evans, UCSD, La Jolla, Ca. Let S be the given sum and write $\chi(n)$ for the Legendre symbol (n/p). Make the 1-1 transformation $i\to(1-i)/(1+i)\pmod p$ below to get

$$S-1=\sum_{i=0}^{p-2}\chi(i)\chi(i^2+6i+1)=\sum_{i=0}^{p-1}\chi((i^2-1)(i^2-2)).$$

Since i^2 runs through the quadratic residues $\pmod p$ twice as i runs from 1 to $p-1$,

$$S-1=\sum_{0}^{p-1}\chi((i-1)(i-2))\{1+\chi(i)\}.$$

But

$$\sum_{0}^{p-1}\chi((i-1)(i-2))=\sum_{1}^{p-1}\chi(i)\chi(i+1)=\sum_{1}^{p-1}\chi(i^{-1})\chi(i+1)$$
$$=\sum_{1}^{p-1}\chi(1+i^{-1})=\sum_{1}^{p-1}\chi(1+i)=-1.$$

Thus,

$$S=\sum_{0}^{p-1}\chi(i(i-1)(i-2))=\sum_{0}^{p-1}\chi(i)\chi(i^2-1).$$

Replace i by $-i$ to see that $S=\chi(-1)S$. Thus $S=0$ if $p\equiv 3 \pmod 4$. If $p\equiv 1 \pmod 4$, the required result follows from Jacobsthal's theorem [**1**, p. 45].

REMARK. The sum S corresponds to curve 10 in Hadano's table [**2**] of thirteen elliptic curves with complex multiplication. The sums for curves 1, 2, 3, and 11 also admit elementary evaluations. The sum for curve 4, which is elementarily equivalent to that for curve 12, has been evaluated by Rajwade [**3**]. The sums for the remaining curves 5, 6, 7, 8, 9, 13 have been elegantly evaluated via complex multiplication by Stark [**4**].

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1. S. Chowla, The Riemann Hypothesis and Hilbert's Tenth Problem, Gordon and Breach, New York, 1965.
2. T. Hadano, Conductor of elliptic curves with complex multiplication and elliptic curves of prime conductor, Proc. Japan Acad. 51 (1975) 92–95.
3. A Rajwade, The Diophantine equation $y^2=x(x^2+21Dx+112D^2)$ and the conjectures of Birch and Swinnerton-Dyer, J. Austral. Math. Soc. 24 (1977) 286–295.
4. H. Stark, personal communication, 1979.

Also solved by L. E. Mattics, Claudia Spiro, and the proposer.

Polynomial with Zeros in Upper and Lower Half Planes

E 2761 [1979, 222]. *Proposed by Ron Adin, undergraduate, Technion, Haifa, Israel.*

Let $P(z)$ be a polynomial of degree at least 2 with complex coefficients, not all of them real. Prove that the equation $P(z)P(-z)=P(z)$ has roots in both the upper and lower open half-planes, $\operatorname{Im} z>0$ and $\operatorname{Im} z<0$.

Solution by Benjamin Klein, Davidson College. Let w be an arbitrary complex number, and set $Q_w(z)=P(z)P(-z)-wP(z)$. Since $P(z)P(-z)$ is an even polynomial and since the degree of $P(z)P(-z)$ is at least 2 more than the degree of $wP(z)$, we see that the sum of the zeros of $Q_w(z)$ is 0. Hence, if $P(z)$ has a zero with $\operatorname{Im}(z)\neq 0$, then $Q_w(z)$ has zeros in $\operatorname{Im}(z)>0$ and $\operatorname{Im}(z)<0$. Therefore, $P(z)P(-z)=wP(z)$ has roots in both $\operatorname{Im}(z)>0$ and $\operatorname{Im}(z)<0$ with w an arbitrary complex number provided $P(z)$ has a zero which is not real. Next, suppose that $P(z)$ has only real zeros. In this case, $P(z)=b\cdot[\Pi(z-a_j)]=b\cdot P_1(z)$ where the a_j are real and $\operatorname{Im}(b)\neq 0$. With r an arbitrary, non-zero real number, let $R_r(z)=P_1(-z)-(r/b)=\Pi(-z-a_j)-(r/b)$. We see that the sum of the zeros of $R_r(z)$ has imaginary part 0 since the a_j are real. However, since $P_1(z)$ is a polynomial with real coefficients and since $\operatorname{Im}(r/h)\neq 0$, $R_r(z)$ has no real zeros. Thus, $R_r(z)$ has zeros in both $\operatorname{Im}(z)>0$ and $\operatorname{Im}(z)<0$. Now, $Q_r(z)=P(z)[P(-z)-r]=b\cdot P(z)\cdot R_r(z)$ so that $Q_r(z)$ has zeros in both $\operatorname{Im}(z)>0$ and $\operatorname{Im}(z)<0$, whence $P(z)P(-z)=rP(z)$ has zeros in both half-planes for all non-zero real numbers r.

Also solved by Donald D. Eliott & William Bosch, Thomas Jager, Lauwerens Kuipers (Switzerland), O. P. Lossers (Netherlands), Martin Markl (Czechoslovakia), William Myers, Michael Skalsky, University of South Alabama Problem Group, Gerard Vinel (France), Joseph Wiener & John Spellman, Ye Yangbo (China), Paul Zwier, and the proposer.

ADVANCED PROBLEMS

Solutions of these Advanced Problems should be mailed in duplicate to Professor Roger C. Lyndon, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109 (USA), by July 31, 1980. The solver's full post-office address should be on each sheet.

6291. *Proposed by Joshua Barlaz, Rutgers University.*

Let X_i $(i=1,\dots,n)$ be independent, identically distributed random variables; $E[X_i]$ finite, with continuous density $f(x)$ with respect to Lebesgue measure on $(-\infty,\infty)$. Let $X^{(n)}=\max\{X_i\}_1^n$; $M_n=E[X^{(n)}]$, $p_n=\Pr(X_i\leqslant M_n)$.

Find necessary and sufficient conditions on f such that, for some $\lambda>1$, $p_n-n/(n+1)=O(n^{-\lambda})$, $n\to\infty$.

6292. *Proposed by L. Van Hamme, Vrije Universiteit Brussel, Brussels.*

Prove that in any p-adic field, for $p\neq 2$,

$$\sum_{n=1}^{\infty}\frac{n^2\cdot(n+1)!}{4^{n+1}}=-1.$$

SOLUTIONS OF ADVANCED PROBLEMS

Polynomials in Two Variables

6136 [1977, 141; 1978, 772]. *Proposed by H. L. Montgomery, University of Michigan.*

Let $P(z,w)=\sum c_{mn}z^m w^n$ be a polynomial in $\mathbf{C}[z,w]$. Suppose that $Q(z,w)=P(z,w/z)$ is also a polynomial: that is $c_{mn}=0$ whenever $n>m$. Show that

$$P=\{P(z,w):|z|<1,|w|<1\}=\{Q(z,w):|z|<1,|w|<1\}.$$

The published solution is erroneous, as pointed out by M. J. Pelling.

Solution by David Wigner, University of Dijon, France. Since the constant term of P is arbitrary, it suffices to show that $P(z,w)$ assumes the value 0 in the open unit polydisc $U^2=\{(z,w):|z|<1,|w|<1\}$ if and only if Q does. If $P(z_0,w_0)=0$, $(z_0,w_0)\in U^2$, then $Q(z_0,z_0w_0)=0$ and $(z_0,z_0w_0)\in U^2$. If $Q(z_0,w_0)=0,(z_0,w_0)\in U^2$, we need to show that there is a zero of P in U^2. To this end we prove the following:

THEOREM. *Let ϕ and ψ be continuous increasing bijections of* $[0,1]$ *to itself. If Q is analytic in the open unit polydisc U^2 and has a zero in U^2, then there is a $t_1, 0\leqslant t_1<1$, and a point (z_1,w_1) such that $Q(z_1,w_1)=0$ and $|z_1|=\phi(t_1),|w_1|=\psi(t_1)$.*

This solves the problem, since if $\phi(t)=t,\psi(t)=t^2$ we obtain a zero (z_1,w_1) of Q such that $|z_1|=t_1<1,|w_1|=t_1^2$. Hence $(z_1,w_1/z_1)$ is a zero of P in U^2.

Proof of theorem. For $0\leqslant t<1$ let D_t be the polydisc $D_t=\{(z,w):|z|\leqslant\phi(t),|w|\leqslant\psi(t)\}$; thus $U^2=\bigcup_{t<1}D_t$. Put $m(t)=\min_{D_t}|Q(z,w)|$. Then $m(t)$ is decreasing, continuous, and $m(1-\varepsilon)=0$ for all small $\varepsilon>0$. If $m(0)=0$ then we are done, for then $Q(0,0)=0$. Otherwise let t_1 be the

minimum t for which $m(t)=0$. For $0\leqslant t<t_1$, let $(z(t),w(t))$ be a point in D_t at which $|Q(z,w)|$ is minimal. By two applications of the minimum modulus theorem in one variable we see that $|z(t)|=\phi(t), |w(t)|=\psi(t)$. A sequence of t tending to t_1 from below gives rise to a sequence $(z(t),w(t))$ of points in D_{t_1}, some subsequence of which converges, say to (z_1,w_1). By continuity it is clear that (z_1,w_1) has the desired properties.

Permuted Residue Classes Under a Parabola

6199 [1978, 203]. *Proposed by Hugh L. Montgomery, University of Michigan.*

Suppose that q is a positive integer, and that $(a,q)=1$. Put

$$\mathbb{S}=\{n:1\leqslant n\leqslant q,\{an/q\}\leqslant(n/q)^2\},$$

where $\{\theta\}=\theta-[\theta]$ is the fractional part of θ. Show that

$$\sum_{n\in\mathbb{S}} n^{-2}\leqslant 9/q.$$

Solution by the proposer. Let $\mathcal{P}=\{(n/q,\{an/q\}):n\in\mathbb{S}\}$, and for $0\leqslant X\leqslant 1$ let $\mathcal{R}(X)=\{x,y):0\leqslant x\leqslant X, 0\leqslant y\leqslant X^2\}$. Let X_0 be the least X for which $\mathcal{R}(X)$ contains a point of $\mathcal{P}$; write that point as $(r/q,s/q)=(dt/q,du/q)$, where $(t,u)=1$. Let X_1 be the least X such that $\mathcal{R}(X)$ contains a point of $\mathcal{P}$ not of the form $(kt/q,ku/q)$. We let $N(X)$ denote the number of points of $\mathcal{P}$ which lie in $\mathcal{R}(X)$, and consider various ranges of X.

Suppose $X_0\leqslant X<X_1$. If $(kt/q,ku/q)\in\mathcal{R}(X)$, then $kt/q\leqslant X$; hence $k\leqslant qX/t$, and so $N(X)\leqslant qX/t$. Moreover, $(dt/q,du/q)\in\mathcal{R}(X)$, so $du/q\leqslant(dt/q)^2$, and hence $d\geqslant qu/t^2\geqslant q/t^2$. Consequently $X_0=dt/q\geqslant 1/t$.

Suppose $X_1\leqslant X\leqslant 1$. Consider $O=(0,0)$ and the points common to $\mathcal{P}$ and $\mathcal{R}(X)$. These points are not collinear, so their convex hull is a nondegenerate polygon. We may triangulate this polygon, say into T triangles, in such a way that each of our points is a vertex of at least one triangle, and each vertex is one of our points. But $\Lambda=\{(n/q,m+an/q):m,n\in\mathbb{Z}\}$ is a lattice of determinant $1/q$; hence a nontrivial triangle whose vertices lie in Λ has area $\geqslant(2q)^{-1}$. The large triangle with vertices $(0,0),(X,0),(X,X^2)$ contains the triangulated polygon; by comparing areas we see that $T/(2q)\leqslant\frac{1}{2}X^3$. But $N(X)\leqslant 3T$, so $N(X)\leqslant 3qX^3$.

Observing that $\int_{n/q}^{\infty}X^{-3}dX=\frac{1}{2}q^2n^{-2}$, we now bound the quantity

$$\frac{1}{2}q^2\sum_{n\in\mathbb{S}}n^{-2}=\int_{X_0}^{\infty}N(X)X^{-3}dX.$$

From the above remarks we see that

$$\int_{X_0}^{X_1}\leqslant\int_{1/t}^{\infty}qXt^{-1}X^{-3}dX=q,$$

$$\int_{X_1}^{1}\leqslant\int_{0}^{1}3qX^3X^{-3}dX=3q,$$

$$\int_{1}^{\infty}=N(1)\int_{1}^{\infty}X^{-3}dX\leqslant\frac{1}{2}q.$$

Hence the result.

Also solved by L. E. Mattics and by Fred Roush & Dave Joyner.

Torsion-Free Finite Extensions of Cyclic Groups

6205 [1978, 282; correction 1979, 828]. *Proposed by Alan McConnell and Louis Shapiro, Howard University.*

Let G be a group with no nontrivial elements of finite order, and let H be a cyclic subgroup

of finite index in G. Show that G is itself cyclic.

All correct solutions except one appealed to some at least mildly advanced, known theorem. (Two used the Feit-Thompson Odd Order Theorem.) We give one solution of this sort, which is especially simple. We give also a somewhat modified form of the only self-contained solution, submitted by T. P. McDonough, University College of Wales, Aberystwyth.

Solution I. If $H=1$, then G is a finite torsion-free group, whence $G=1$. We may assume then that H is infinite cyclic. Replacing H by the intersection of its (finitely many) conjugates, we may suppose H is normal in G. Now every automorphism of H either fixes all elements or replaces each element by its inverse. If $g \in G$ and $g \neq 1$, then some power $g^k \neq 1$ of g lies in H; since conjugation by g fixes g^k, it must fix all elements of H. Thus H is contained in the center of G.

The center of G therefore has finite index in G. A theorem of Schur and Baer [see D. J. S. Robinson, Finiteness Conditions and Generalized Soluble Groups, vol 1, Springer, 1972, p. 102] now says that the derived group G' of G is finite. Since G is torsion-free, $G'=1$, and G is abelian. As a finitely generated torsion-free abelian group, G is free abelian, and, indeed, infinite cyclic.

Solution II (after T. P. McDonough). As before, we may suppose that H is central. The remainder of the proof is contained in that of the following more general theorem [R. C. Lyndon and P. E. Schupp, Combinatorial Group Theory, Springer, 1977, p. 145], proved for H of rank 2 by H. Zieschang [see preceding reference].

THEOREM 1. *If a torsion-free group G contains a central free abelian group H of finite index, then G is free abelian.*

We prove Theorem 1 by induction on the order n of $Q=G/H$. If $n=1$, the assertion is trivial. If G is abelian, the conclusion follows easily as before. We assume then that G is not abelian, and derive a contradiction.

Let A be a nontrivial proper subgroup of Q and $\tilde{A}$ its preimage in G; by the induction hypothesis, $\tilde{A}$ is free abelian. Let N be the normalizer of A in Q and $\tilde{N}$ its preimage in G. Since conjugation by any element of $\tilde{N}$ fixes all elements in the subgroup H, of finite index in $\tilde{A}$, it fixes all elements of $\tilde{A}$; thus $\tilde{A}$ is central in $\tilde{N}$, and, by the induction hypothesis, $\tilde{N}$ is free abelian. Since G is not abelian, $\tilde{N}<G$ and hence $N<Q$.

Now Q is the union of its maximal proper (abelian) subgroups A, and they are self-normalizing. Moreover, if A_1 and A_2 are distinct maximal proper subgroups, and $B=A_1 \cap A_2 \neq 1$, then both A_1 and A_2 are contained in the normalizer $N<G$ of B, a contradiction. Thus the maximal proper subgroups A have pairwise trivial intersections. If a maximal proper subgroup A has order a, then it has n/a distinct conjugates, which contain in all $1+(n/a)(a-1)>((a-1)/a)n \geqslant \frac{1}{2}n$ elements. Thus all maximal proper subgroups are conjugate to A, and $n=1+(n/a)(a-1)$, $(n-1)/n=(a-1)/a$, $n=a$, and hence $Q=A$, a contradiction.

Also solved by J. M. Ash, Steven Assa, Duane Broline, John Bryant & Robert Gilmer, P. L. Chabot, Jeffrey Mitchell Cohen, Ömer Egecioglu, Peter Flusser, Stephen M. Gagola, Tom Jager, Michael Josephy (Costa Rica), Ernst Kani (West Germany), Luise-Charlotte Kappe & Wolfgang P. Kappe (2 solutions), Seymour Kass, Stephen C. King, Antonio Machì (Italy), John Petro, Veril Phillips & Hugh Edgar, Joseph Rotman, Herman Simon, Blair Spearman, Don Taylor (Netherlands), Gary L. Walls, Daniel Weisser, Edward T. Wong, and the proposers.

Comments. (1) Robinson (p. 102) says that the theorem cited in Solution I is implicit in the work of Schur, and gives references to later proofs, of which the first is that of Baer.

(2) McDonough and Rotman point out the assertion of the problem is a special case of a theorem of J. Stallings [Groups of dimension 1 are locally free, Bull. Amer. Math. Soc. 74 (1968) 361–364; On torsion-free groups with infinitely many ends, Annals of Math, 88 (1968) 312–334], in which he establishes a conjecture of J.-P. Serre [Sur la dimension cohomologique des groupes profinis, Topology, 3 (1965) 413–420], as follows:

THEOREM 2. *If a torsion free group G contains a free group of finite index, then G is free.*

Condition for a Composite of Polynomials

6208 [1978, 283]. *Proposed by Gary Gundersen, University of New Orleans.*

Let $p(z)$ and $q(z)$ be two polynomials with $\deg(q)\geqslant\deg(p)$, and suppose there is a discrete real sequence $\{x_j\}_{j=1}^{\infty}$ with cluster points at $\pm\infty$. Prove: if $q(z)\in\mathbb{R}$ whenever $p(z)\in\{x_j\}_{j=1}^{\infty}$, then $q(z)=\sum_{i=0}^{n}c_i(p(z))^i$ where $c_i\in\mathbb{R}$ $(0\leqslant i\leqslant n)$.

*Can the condition $\deg(p)\leqslant\deg(q)$ be dropped?

Solution by I. N. Baker, Imperial College, London. The condition on the degrees can be omitted—it is necessary to assume only that p is non-constant.

Suppose that $w=p(z)=a_m z^m+\cdots a_0, a_m\neq 0$. Then the inverse $z=p^{-1}(w)$ has a branch point of order $m-1$ at ∞ where all the branches join and in a neighborhood of ∞ we have

$$z=a_m^{1/m}w^{1/m}+\lambda+\mu w^{-1/m}+\cdots.$$

Then if q has degree n

$$\psi(w)=q\{p^{-1}(w)\}=Aw^{n/m}+\text{lower powers of } w^{1/m},\quad A\neq 0, \tag{1}$$

is valid in some neighborhood of ∞.

Now $\psi(x_j)$ is real for each x_j and so for large positive and negative x_j and every determination of $w^{n/m}=x_j^{n/m}$ in (1) we have $\psi(x_j)\simeq Ax_j^{n/m}$ real. Taking a positive and a negative x_j leads to values of $Ax_j^{n/m}$ whose arguments differ by $(n\pi/m)$, and the reality of $\psi(x_j)$ in both cases shows that m must divide n. Thus $w^{n/m}$ is real for real $w=x_j$ and so A must also be real. Applying the same arguments to $\psi(w)-Aw^{n/m}$ now shows that the second term in the expansion (1) must also have the form cw^i where i is an integer and c real. Inductively we see that

$$\psi(w)=\sum_{i=-\infty}^{k}c_iw^i,\qquad c_i\in R.$$

The function $\psi(w)$ is algebraic and all its branches at ∞ are included in the expansion (1). Thus ψ must be single-valued and therefore rational. Further $\psi(w)=\infty$ implies $p^{-1}(w)=\infty$ and so $w=\infty$. Thus ψ is a polynomial $\sum_{i=0}^{k}c_iw^i, c_i\in R$.

Also solved by L. E. Clarke (England), Gustaf Gripenberg (Finland), Nicholas Passell, and the proposer.

Integral Matrices Congruent to I

6210 [1978, 389]. *Proposed by Olga Taussky, California Institute of Technology.*

A theorem of Minkowski (*Gesammelte Werke* I, p. 212) states: Let A be an integral square matrix which is congruent to the unit matrix I modulo an odd prime number. Then A is either equal to I or it is of infinite order. Give a proof based on the eigenvalues of A.

Solution by Robert M. Guralnick, California Institute of Technology. Say $A=I+pB$, where p is an odd prime and B is an integral matrix. Then

$$\det(xI-A)=\det((x-1)I-pB).$$

Hence β is an eigenvalue of B if and only if $\beta=(\alpha-1)/p$, where α is an eigenvalue of A.

Suppose A has finite order. Then all eigenvalues α of A are roots of unity and in particular satisfy $|\alpha|=1$. Thus if β is an eigenvalue of B,

$$|\beta|=\frac{|\alpha-1|}{p}\leqslant\frac{2}{p}<1.$$

Hence β is an algebraic integer all of whose conjugates have absolute value less than 1. This implies $\beta=0$, and hence A is unipotent. As the only unipotent complex matrix of finite order is I, this completes the proof.

Actually what is shown above is the standard result that if $1\neq\alpha$ is a root of unity and $p>2$,

then $(\alpha-1)/p$ is not an algebraic integer. If $p=2$, this proof shows that A has finite order if and only if $A^2=I$.

Also solved by Manfred Leitz (West Germany), O. P. Lossers (Netherlands), and the proposer. Lossers draws the same conclusion for $A=I+\alpha B$, B an integral matrix and $\alpha>2$.

Coloring a Chessboard

6211* [1978, 389]. *Proposed by Alvin J. Paullay and Sidney Penner, Bronx Community College of CUNY.*

Suppose that each square of an $n\times n$ chessboard is colored either black or white. A square, formed by the horizontal and vertical lines of the board, will be called *chromatic* if its four distinct corner squares are all of the same color.

(a) Exhibit a black-white coloring of a 9×9 board in which every such square, as described above (there are 204), is not chromatic.

(b) Find the smallest n, say s, such that, with any such coloring, every $s\times s$ board must contain a chromatic square.

Solution to part (*a*) *by Ingrid de Buda, undergraduate, University of Toronto.* (a) The 10×10 chessboard below has no chromatic square. Therefore, the four 9×9 sub-chessboards have no chromatic square. The symbols "1" and "0" denote a white square and black square, respectively.

0	0	0	0	0	0	1	1	0	1
1	0	1	0	1	0	0	1	1	1
1	0	0	1	1	0	1	0	0	1
1	1	0	0	1	1	0	0	1	0
1	0	1	0	0	1	1	0	1	1
0	0	0	1	0	1	0	1	1	0
1	1	0	1	0	0	0	0	1	1
0	1	1	1	1	0	1	0	0	0
0	1	0	0	1	0	0	1	1	1
1	1	1	0	0	0	1	1	0	1

FIG. 1

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR., AND LYNN A. STEEN
with the assistance of the mathematics departments of St. Olaf, Carleton, and Macalester Colleges
COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

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Introduction to Operations Research Techniques. By Hans G. Daellenbach and John A. George. Allyn and Bacon, Boston, Mass., 1978. xvii+603 pp. \$17.95. (Telegraphic Review. December 1978.)

Operations Research, An Introduction, Second Edition. By Hamdy A. Taha. Macmillan, Riverside, N.J., 1976. xiv+648 pp. $19.50.

Textbooks in operations research are becoming more plentiful and available at several levels of difficulty. These two texts are appropriate for a homogeneous group of students, namely, for undergraduates with some mathematical background at the college level without any previous exposure to the subject. They are suitable for various courses which may be taught at the junior or senior level. This past semester I used the text by Taha in a course in operations research at a liberal arts college with a class consisting of students with interests in business, economics, mathematics, and computer science. Both texts are well written, treat essentially the same topics, but differ considerably in their style and presentation: Taha stresses operations research techniques and their mathematical foundations, while Daellenbach and George put more emphasis on the methodology and discipline of the subject.

Taha's approach is concise and analytical. He concentrates on the mathematical techniques used in a particular model as well as on the abstract formulation of the model. In Daellenbach and George there is more emphasis put on the processes one goes through in doing operations research. There are many large charts, diagrams, graphs—some rather complex; and there are boxes surrounding major formulas, rules, and algorithms. To present ideas and concepts, Daellenbach and George seem to depend on detailed examples, while Taha presents major concepts in terms of relatively simple models with numbers that are easy to work with.

Many of the chapters in the book by Daellenbach and George begin with a statement of the topic to be covered with an indication, if needed, of necessary preliminary material either in previous chapters or in one of the two appendices on linear algebra and probability. Then a specific example is covered in detail. Some of the examples are rather involved and computationally complex. This might obscure the principles being demonstrated. On the other hand, real problems are rarely simple. The authors do take the reader through all the steps of this example and then present a general model. The topics covered this way are standard: linear programming, dynamic programming, integer programming, nonlinear programming, networks, probabilistic models, simulation, and waiting-line models. There is no chapter on decision theory, but there is a chapter on heuristics, written by John L. Rogers. I would have liked to have seen the chapter on integer programming placed nearer those on linear programming rather than between chapters on classical optimization methods with applications to inventory control and on nonsimplex-based nonlinear programming.

There are some rather nice features of Daellenbach and George. Each chapter is followed by many problems and a list of references. The problems require the student to do a bit of thinking before applying the appropriate methods to obtain a solution. I think doing these problems would be satisfying to students; they would get a sense of having solved a complete problem and not of just having practiced a particular technique. There are several references given at the end of each chapter, including appropriate comments on the works cited. The bibliography includes a list of technical and professional journals in operations research.

The text by Taha is in its second edition. It is divided into three parts: linear, dynamic, and integer programming; probabilistic models; and nonlinear programming. Four appendices are included: a review of vectors and matrices, a review of basic theorems in differential calculus, a general computer program for computing Poisson queueing formulas, and answers to selected problems. Models are presented by straightforward examples in which the numbers or words do not get in the way. This is not a text for persons shy of mathematics, since Taha discusses most topics in mathematical terms, expects the reader to be somewhat mathematically sophisticated, and uses subtle arguments in deriving some results. This may be what one expects of a text at this level, and it is his emphasis on mathematics that sets this text apart from the other.

Taha presents mostly the same topics as Daellenbach and George. He does add a chapter titled "Review of Probability," which is needed for Part II, and a chapter on decision theory.

There are ample problems at the end of each chapter, but they do not contain as much detail describing the setting of the problem as do Daellenbach and George. My students found the problems interesting and challenging. Some problems of a theoretical nature are included. References are given in most chapters but are not plentiful.

I close with a summary comparison of the books. To paraphrase Daellenbach and George, they wish to highlight important aspects of the methodology of operations research, to give some illustrations, and to present the nature of operations research rather than just a collection of techniques. I feel they do this well. The emphasis in Taha's book is on the mathematical techniques of operations research and on construction of appropriate models. His text presents the option of considering some of the theoretical aspects of the subject, yet one may also use the text for a more elementary approach. I would like to see a text that has the best features of both, and one that contains more case studies. I enjoyed using the text by Taha, will probably use it again, and I also recommend the text by Daellenbach and George.

ERNEST C. ACKERMANN, Gustavus Adolphus College

Numerical Analysis. By Richard L. Burden, J. Douglas Faires, Albert C. Reynolds. Prindle, Weber & Schmidt, Boston, Mass., 1978. ix+579 pp. (Telegraphic Review, November 1978.)

Modern numerical analysis texts (those published in the past five years) are in great supply, so selecting the best one for a class is quite difficult, especially because so many of them are good. I chose Burden's book because it covered the material I thought would do the intended audience the most good; the parts I read prior to selection seemed clearly written; the notation was mostly standard and the algorithms (without programs) were neatly laid out, and there seemed to be an abundance of problems. My expectations were, I think, realized.

The class consisted mainly of engineers, a few science students, and a mathematics major. Our course in Numerical Analysis is only one semester (14 weeks), while the text is obviously intended as a year course. Forty-eight algorithms are explicitly developed and indexed on the front flyleaf. Twenty-two algorithms were covered in this course. As my students had been exposed to numerical solutions of differential equations in other courses, we decided to omit those chapters (6, 10, 11). A term project was required of everyone, and several students chose to discuss Runge-Kutta methods in order to do some differential equations on their own.

The book lived up to my expectations. The explanations were in general clear, and students were able to read and learn by themselves from the text.

Error analysis is nicely handled throughout, and approximation theory (Chapter 4) is introduced with a careful introduction to norms. I think the students developed a good idea about what is meant by convergence of iterative processes in different norms. The course covered Chapter 2, Solutions of Equations in One Variable; Chapter 3, Interpolation and Polynomial Approximation—the last section in cubic spline interpolation made a nice term project for two people; Chapter 4, Approximation Theory; Chapter 5, Numerical Differentiation and Integration; Chapter 7, Direct Methods for Solving Linear Systems; part of Chapter 8, Iterative Techniques in Matrix Algebra.

The algorithms are quite carefully laid out and the class had little difficulty implementing them. It is an interesting feature of the book that programs are deliberately not included so that these must be done by each student. It is my feeling that the apparent ease with which programs were implemented can be attributed to the explanation and description accompanying each algorithm.

The exercises are fairly numerous. Some are the usual routine ones that must be included for such a project. My preference would be to have a few more theoretical problems and perhaps

some more challenging ones. Many problem sets include at the end two or three "industrial" problems—that is, problems that read as though they had arisen in a practical industrial situation. The engineers liked this contact with reality.

In general, therefore, we found this a very useful and usable book (we were able to cover about two thirds of the book) and I will use it again. We found very few misprints.

All recent Numerical Analysis texts try to adapt to the availability of the high-speed computer. However, none that I'm aware of have adapted to the technology of other available equipment such as graphics equipment. It is possible to solve many problems visually—for example, a 3-place accuracy solution for $f(x)=0$ is almost trivial to obtain on the Tektronix 4015. The usual numerical analysis algorithms of course are needed if greater accuracy is required. But solving a problem graphically gives instant insight into the behavior of the function. Other problems can also be solved visually to 2- or 3-place accuracy quite easily, such as finding extreme points of sixth degree polynomials and other complicated functions. One plots the curve and uses the cursor to approximate the extreme value; on demand, the machine gives the approximate value. A blow-up function or zoom function permits a more accurate zeroing in on the desired point.

Graphics technology should now be exploited to solve numerical problems.

DAVID ROSEN, Swarthmore College

MISCELLANEA

32.

THE CREDIBILITY OF DIFFERENTIAL TOPOLOGY

In early 1969 I was called as a defense witness in a trial in Oakland, California, the so-called "Oakland Seven" conspiracy trial. This involved being cross-examined by a prosecuting attorney who seemed to feel that the Berkeley Mathematics Department was a great hotbed of subversion.

PROSECUTOR: You are doing graduate work in mathematics at the University of California campus at Berkeley?
WITNESS: Yes.
PROSECUTOR: Do you know Professor Mo Hirsh?
WITNESS: Yes.
PROSECUTOR: How do you know Professor Mo Hirsh?
WITNESS: I'm taking a course from him.
PROSECUTOR: What course is that?
WITNESS: Differential Topology.
PROSECUTOR: Will you explain differential topology to the jury?
DEFENSE ATTORNEY: Objection. Is this relevant, your honor?
PROSECUTOR: I'm trying to impugn the witness's credibility.
JUDGE: Objection denied. Answer the question. Remember that you are under oath.

I rather doubt that when I got done, the jury understood differential topology. In fact—since I actually tried to tell the truth—I rather suspect that the prosecuting attorney succeeded in impugning the witness's credibility!

— Edward T. Ordman, New England College, Henniker, New Hampshire 03242

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

BASIC, T(13), *An Introduction to Technical Mathematics with Computing*. Gabriel Kousourou, Sidney N. Sonsky, Frank Scalzo. Petrocelli Book, 1979, xv + 510 pp, $24. [ISBN: 0-89433-038-1] A very elementary text introducing basic algebraic tools, some trigonometry, logarithms, exponents, unit conversions, and use of hand-held calculators. LLK

PRECALCULUS, T(13), *Precalculus, An Elementary Functions Approach*. Robert P. Webber. Brooks/Cole, 1980, xiii + 393 pp, $15.95. [ISBN: 0-8185-0292-4] A text for college freshmen; presupposes no trigonometry; includes exponential and logarithmic functions, the binomial theorem, and analytic geometry of conic sections. AWR

PRECALCULUS, T(13: 1), *Precalculus*. Michael D. Grady, Edwin F. Beckenbach, William Wooton. Wadsworth, 1980, xiii + 448 pp, $16.95. [ISBN: 0-534-00733-3] Handy survey for those not in a rush to begin calculus. Flexible, not always in depth, review of topics in algebra, functions and analytic geometry (in the plane and in space) which are encountered in calculus. Lots of examples. Plenty of exercises, none challenging. Appendices on sequences, series and limits. Tables. Pre-paid student response mailer provided. JK

PRECALCULUS, T(13: 1), *Analytic Trigonometry with Applications, Second Edition*. Raymond A. Barnett. Wadsworth, 1980, xii + 299 pp, $15.95. [ISBN: 0-534-00728-7] Many small changes were made in this new edition of this attractive text. FLW

PRECALCULUS, T(13: 1), *College Algebra*. Anne Pfenning, Clifton Gary. Wadsworth, 1980, xii + 367 pp, $16.95. [ISBN: 0-534-00707-4] Good text. Excellent features include many verbal exercises, introduced first as "reasoning things out," plus many examples, chapter summaries and tests, and cumulative review tests. LLK

PRECALCULUS, T(13: 1), *Trigonometry*. Vivian Shaw Groza, Gene Sellers. Saunders, 1980, vii + 494 pp, $13.95 (P). [ISBN: 0-7216-4325-6] A workbook style text. Almost all applications are in Chapter 2, rather than dispersed throughout. LLK

EDUCATION, T*(1), P, L, *Young Children Learning Mathematics*. Douglas E. Cruikshank, David L. Fitzgerald, Linda R. Jensen. Allyn, 1980, xi + 348 pp, $15.95. [ISBN: 0-205-06752-2] An excellent resource for pre-service and in-service teachers of children ages 3 to 9, but also valuable background for teachers of older children. A comprehensive blend of the psychology of learning mathematics with numerous activities to promote this learning. Extensive bibliography. JNC

EDUCATION, P, *Calculators*. Ed: Bruce C. Burt. NCTM, 1979, viii + 231 pp, $6.25 (P). [ISBN: 0-87353-144-2] A resource for teachers using or considering the use of calculators in the elementary or secondary classroom. Articles from the *Arithmetic Teacher*, *The Mathematics Teacher* and the 1977 N.C.T.M. *Yearbook* are organized under the headings: Policies, Uses and Selection; Calculators and the Curriculum; Activities; Research. JNC

EDUCATION, P*, *Organizing Data and Dealing with Uncertainty*. NCTM, 1979, viii + 135 pp, $6.25 (P). [ISBN: 0-87353-141-8] Two units taken from Volume 2 of *Experiences in Mathematical Ideas* (TR, February 1972), a set of teaching units designed for grades 5-8. Motivated by a recommendation from the National Advisory Committee on Mathematical Education that "ideas from probability and statistics should become part of the elementary and secondary school mathematics curricula." RSK

EDUCATION, S(13-14), P, *Taschenrechner*. Herbert Löthe, Kurt Peter Müller. Teubner Stuttgart, 1979, 168 pp, DM 18,80 (P). [ISBN: 3-519-02711-9] How and where to use hand calculators. Written for teachers and prospective teachers. A little about algorithms, but almost nothing on programming. JD-B

HISTORY, L, *A History of Mathematics, Third Edition*. Florian Cajori. Chelsea, 1980, x + 524 pp, $18.50. [ISBN: 0-8284-0303-1] A corrected reprinting of the 1919 *Second Edition*, supplemented with bibliographical notes calling attention to the publication in the intervening years of various collected works. The original edition of this classic was published in 1895. LAS

FOUNDATIONS, T(16-18: 1, 2), P, L*, *Topoi, The Categorial Analysis of Logic*. Robert Goldblatt. Stud. in Logic and Found. of Math., V. 98. North-Holland, 1979, xv + 486 pp, $83. [ISBN: 0-444-85207-7] An introduction to topos theory and its implications for logic and foundations. A carefully written presentation which makes a real effort to present the intuition behind these recent developments. Some exercises, good indexes of notation and definitions. JAS

FOUNDATIONS, P, *Logic Colloquium '78*. Ed: Maurice Boffa, Dirk van Dalen, Kenneth McAloon. Stud. in Logic and Found. of Math., V. 97. North-Holland, 1979, x + 434 pp, $58.50. [ISBN: 0-444-85378-2] Proceedings of a colloquium held in Mons, Belgium, in August, 1978 (10 of 16 invited lectures, 2 of 3 survey courses, and 5 of 57 contributed papers) centered on constructive mathematics, model theory and set theory. LAS

FOUNDATIONS, P. *Unsolvable Classes of Quantificational Formulas.* Harry R. Lewis. A-W, 1979, xv + 198 pp, $13.50 (P). [ISBN: 0-201-04069-7] Survey of recursive unsolvability results for a number of decision problems for combinatorial systems (tiling problems) and for first order quantification theory. Provides a uniform treatment of the strongest known results on unsolvable classes of formulas, based on combinatorial manipulations of Herbrand expansions of formulas (ideas going back to Wang and Büchi). More time is spent on interesting combinatorial problems than necessary for the logical applications, though no completeness is intended. Complements the book reviewed immediately below. GHM

FOUNDATIONS, P. *The Decision Problem, Solvable Classes of Quantificational Formulas.* Burton Dreben, Warren D. Goldfarb. A-W, 1979, xii + 271 pp, $27.50. [ISBN: 0-201-02540-X] Building on ideas of Skolem and Herbrand, the authors provide the first unified treatment of the positive results of the decision problem for restricted classes of quantificational formulas. Intended as a prolegomenon to a (future) abstract study of solvability aimed at finding informative general criteria distinguishing solvable from unsolvable classes. Complements the book reviewed above. Together they delimit the boundary between solvable and unsolvable, though each book stands on its own. GHM

FOUNDATIONS, L. *Logic for Mathematicians, Second Edition.* J. Barkley Rosser. Chelsea, 1978, xv + 574 pp, $19.95. [ISBN: 0-8284-0294-9] A corrected reprinting of the 1953 original edition, supplemented by four appendices containing a proof of the axiom of infinity, discussions of the axiom of counting and the axiom of choice, and an introduction to nonstandard analysis. LAS

FOUNDATIONS, T(15-17: 1), P, L. *Fundamentals of Contemporary Set Theory.* Keith J. Devlin. Springer-Verlag, 1979, vii + 182 pp, $9.50 (P). [ISBN: 0-387-90441-7] Introduction to the basic topics of pure set theory which every mathematician should know. Four rigorous chapters carry one up through ordinal and cardinal arithmetic, with a few selected topics from analysis and infinitary combinatorics. Two, more expository, chapters explain the bare fundamentals of constructibility and Boolean valued models used in independence proofs. The conversational style makes this a very accessible book for initial study, before tackling more advanced texts. Chapters I through III contain exercises. GHM

FOUNDATIONS, P. *Lecture Notes in Mathematics-718: The Computational Complexity of Logical Theories.* Jeanne Ferrante, Charles W. Rackoff. Springer-Verlag, 1979, x + 243 pp, $14.30 (P). [ISBN: 0-387-09501-2] The dichotomy between decidable and undecidable theories is a classical topic in mathematical logic. The study of degrees of undecidability is almost as old as Gödel's Theorem itself. Only recently, however, have studies been made of the relative efficiency (computational complexity) of decision procedures for decidable theories. The present treatise contains many original results which place both upper and lower bounds on the number of steps or size of memory required of an algorithm for deciding certain theories. Contains also a good summary of the current state-of-the-art in this recent and fertile marriage of classical logic and modern computer science. GHM

COMBINATORICS, P. *Lecture Notes in Mathematics-748: Combinatorial Mathematics VI.* Ed: A.F. Horadam, W.D. Wallis. Springer-Verlag, 1979, ix + 206 pp, $12.50 (P). [ISBN: 0-387-09555-1] Proceedings of the Sixth Australian Conference on Combinatorial Mathematics, Armidale, Australia, August 1978. JAS

COMBINATORICS, T*(17: 1, 2), S, P, L*. *Combinatorial Theory.* Martin Aigner. Grund. math. Wissenschaften, B. 234. Springer-Verlag, 1979, viii + 483 pp, $39. [ISBN: 0-387-90376-3] A substantial introduction to combinatorics at the beginning graduate level. The book covers topics which deal with enumeration and order theory, but not configurations. There are lots of good problems and an extensive bibliography. CEC

COMBINATORICS, T(16-17: 1), S, P, L*. *Permutation Groups and Combinatorial Structures.* N.L. Biggs, A.T. White. London Math. Soc. Lect. Note Ser., No. 33. Cambridge U Pr, 1979, viii + 140 pp, $13.95 (P). [ISBN: 0-521-22287-7] A complete rewrite of *Finite Groups of Automorphisms* (No. 6 of this series, TR, May 1972); Chapters: Permutation Groups, Finite Geometries, Designs, Groups and Graphs, Maps. Particularly appropriate for seminar study (e.g., weekly student-staff study group); includes thirteen "project" sections. LCL

NUMBER THEORY, S(18), P. *Analytic Arithmetic of Algebraic Function Fields.* John Knopfmacher. Lect. Notes in Pure and Appl. Math., V. 50. Dekker, 1979, iii + 130 pp, $16.50 (P). [ISBN: 0-8247-6907-4] Recent advances in the author's work. This book provides a parallel to the kind of analytic or asymptotic arithmetic (that has been applied to ordinary integers and algebraic number fields) to the context of polynomial rings, algebraic function fields over finite fields, and other related systems. CEC

NUMBER THEORY, T(14: 1), S. *Zahlentheorie: Eine Einführung.* Edmund Hlawka, Johannes Schoissengeier. Manzsche Verlag, 1979, 159 pp, DM 24 (P). [ISBN: 3-214-00005-5] A standard introduction to elementary number theory. Topics include divisibility, congruences, quadratic reciprocity, continued fractions, quadratic forms and a brief look at quadratic number fields. Written in a very formal style. Does not include exercises. CEC

NUMBER THEORY, P. *Lecture Notes in Mathematics-751: Number Theory, Carbondale 1979.* Ed: Melvyn B. Nathanson. Springer-Verlag, 1979, 342 pp, $16 (P). [ISBN: 0-387-09559-4] Proceedings of a conference on number theory held at Southern Illinois University at Carbondale on March 30 and 31, 1979. Includes twenty-one papers on diverse topics in analytic, algebraic, additive, and combinatorial number theory. CEC

NUMBER THEORY, T(13: 1), S*, L*. *Perfect Numbers.* Stanley Bezuszka. Boston College Pr, 1980, i + 169 pp, (P). An introduction to elementary multiplicative number theory through a self-motivating, problem solving approach. Perfect numbers serve as a unifying theme. More than half the book consists of appendices which, amongst other things, give a listing of all known perfect numbers in decimal notation. CEC

LINEAR ALGEBRA, T(16-17: 1), P, L. *Nonnegative Matrices in the Mathematical Sciences.* Abraham Berman, Robert J. Plemmons. Comp. Sci. and Appl. Math. Acad Pr, 1979, xviii + 316 pp, $32. [ISBN: 0-12-092250-9] Draws together widely scattered results on positive matrices. Attractive as a reference, not only for mathematicians, but also for economists, programmers, and statisticians. Last four chapters deal with applications to iterative solutions of linear systems, to finite Markov chains, to economics, to linear complementarity. AWR

LINEAR ALGEBRA, T(13-14: 1). *Modern Elementary Linear Algebra.* Noël Cortey. U Pr of America, 1978, vii + 129 pp, $7.50 (P). [ISBN: 0-8191-0524-4] A much too brief presentation of linear algebra. Poorly motivated; no applications; just definitions, theorems, and proofs. LLK

LINEAR ALGEBRA, P. *Quadratic Forms in Infinite Dimensional Vector Spaces.* Herbert Gross. Progress in Math., No. 1. Birkhäuser Boston, 1979, xii + 419 pp, $20 (P). [ISBN: 3-7643-1111-8] Systematic presentation of the author's investigations concerning the algebraic theory of quadratic forms in denumerably infinite dimensional vector spaces over division rings. "These results show that it is possible to generalize, without rarefying, classical results from finite dimensional orthogonal geometry." LCL

ALGEBRA, P. *Lecture Notes in Mathematics-752: *-Autonomous Categories.* Michael Barr. Springer-Verlag, 1979, 140 pp, $9 (P). [ISBN: 0-387-09563-2] A symmetric closed monoidal category is called autonomous. If such a category has an object K such that the internal Hom (-,K) induces an equivalence with its opposite category, the category is called *-autonomous. This monograph studies embeddings in such categories. JAS

ALGEBRA, P. *Lecture Notes in Mathematics-740: Séminaire d'Algèbre Paul Dubreil.* Ed: M.P. Malliavin. Springer-Verlag, 1979, v + 456 pp, $23.40 (P). [ISBN: 0-387-09537-3] Proceedings of the seminar's 31st year, 1977-78 in Paris, France. JAS

ALGEBRA, P. *Classification and Fourier Inversion for Parabolic Subgroups with Square Integrable Nilradical.* Joseph A. Wolf. Memoirs No. 225. AMS, 1979, iii + 166 pp, $7.60 (P). [ISBN: 0-8218-2225-X]

ALGEBRA, S(18), P. *Introduction to Affine Group Schemes.* William C. Waterhouse. Grad. Texts in Math., V. 66. Springer-Verlag, 1979, xi + 164 pp, $19.80. [ISBN: 0-387-90421-2] A fairly intuitive and accessible introduction to affine group schemes based on a background in algebra through tensor products and Galois theory. Numerous examples and applications. Exercises, index, bibliography, appendices. JS

ALGEBRA, S(18), P. *Lecture Notes in Mathematics-750: Moduln mit einem höchsten Gewicht.* Jens Carsten Jantzen. Springer-Verlag, 1979, 194 pp, $11.90 (P). [ISBN: 0-387-09558-6]

ALGEBRA, P. *Structure of Regular Semigroups.* K.S.S. Nambooripad. Memoirs No. 224. AMS, 1979, vii + 119 pp, $6.80 (P). [ISBN: 0-8218-2224-1] It is shown that the category of regular semigroups is equivalent to the category of inductive groupoids, and this is used to study the structure of regular semigroups in full generality. LCL

CALCULUS, T(13: 2). *Essential Calculus with Applications, Second Edition.* Margaret L. Lial, Charles D. Miller. Scott F, 1980, 493 pp, $16.95. [ISBN: 0-673-15248-0] A short course in calculus with several chapters reviewing precalculus topics. This quick and easy introduction includes exponential growth and decay problems and Lagrange multipliers, and has many good applications. Revisions from *First Edition* (TR, October 1975) include a new format of wide margins, more examples, applications, and actual cases. LLK

CALCULUS, T(13: 2), S, L. *Calculus and the Computer: An Approach to Problem Solving.* Timothy V. Fossum, Ronald W. Gatterdam. Scott F, 1980, 220 pp, $7.95 (P). [ISBN: 0-673-15158-7] For use in conjunction with a first year calculus course. Algorithms illustrate continuity, convergence, differentiation and integration. Each section includes motivation, theory, a flowchart of an algorithm, the annotated program in Basic, instructions for its use, sample runs and exercises. No programming ability is required. JNC

CALCULUS, T*(14-16: 2), S, L**. *Calculus in Vector Spaces.* Lawrence J. Corwin, Robert H. Szczarba. Pure and Appl. Math., V. 52. Dekker, 1979, x + 782 pp, $65. [ISBN: 0-8247-6832-9] This is not really the world's most expensive calculus text since there is "a special adoption price of $29.75 ...on orders of five or more copies...in the United States and Canada." A unique book based on the year course taught to sophomores (and some freshmen) at Yale University. Because the ideas of analysis and linear algebra are so interwoven this book could not be used for a one term course in either, but the result is a mathematically beautiful development of the material. The index is short but to the point. The small number of relatively sophisticated exercises won't fit weak students very well, but they definitely do a good job of developing important ideas. JAS

CALCULUS, T(16-17), S. *Einführung in die reelle Analysis II: Differentialrechnung der Funktionen Mehrerer Veränderlicher.* Georg Aumann, Otto Haupt. Walter de Gruyter, 1979, 313 pp, DM 128. [ISBN: 3-11-005720-4] Third, extensively revised edition. A careful, modern and sophisticated treatment of the differential calculus of functions of several variables. JD-B

REAL ANALYSIS, P. *Lecture Notes in Mathematics-739: Analyse Harmonique sur les Groupes de Lie II.* Ed: P. Eymard, *et al.* Springer-Verlag, 1979, vi + 646 pp, $29.70. [ISBN: 0-387-09536-5] Texts of a number of the presentations given at the Séminaire Nancy-Strasbourg 1976-1978. JAS

COMPLEX ANALYSIS, P. *Lecture Notes in Mathematics-747: Complex Analysis, Joensuu 1978.* Ed: Ilpo Laine, Olli Lehto, Tuomas Sorvali. Springer-Verlag, 1979, xv + 450 pp, $21.30 (P). [ISBN: 0-387-09553-5] Proceedings of the colloquium held in Joensuu, Finland, August 24-27, 1978. JAS

DIFFERENTIAL EQUATIONS, S(18), P. *Numerical Methods for Partial Differential Equations.* Ed: Seymour V. Parter. Acad Pr, 1979, ix + 332 pp, $14.50. [ISBN: 0-12-546050-3] A collection of ten papers representing the proceedings of an advanced seminar on numerical methods at the Mathematics Research Center in Madison, Wisconsin, October 1978. JS

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Mathematics-727: Spectral Representations for Schrödinger Operators With Long-Range Potentials.* Yoshimi Saitō. Springer-Verlag, 1979, 148 pp, $9.80 (P). [ISBN: 0-387-09514-4] A unified and self-contained account (modulo functional analysis and partial differential equations) of several of the author's recent papers on spectral theory for second order differential equations. LCL

NUMERICAL ANALYSIS, P. *Lecture Notes in Mathematics-757: Smoothing Techniques for Curve Estimation.* Ed: Th. Gasser, M. Rosenblatt. Springer-Verlag, 1979, 245 pp, $14 (P). [ISBN: 0-387-09706-6] Proceedings of the workshops held in Heidelberg, April 2-4, 1979. JAS

FUNCTIONAL ANALYSIS, T(18: 1), S, P. *Lecture Notes in Mathematics-736: M-Structure and the Banach-Stone Theorem.* Ehrhard Behrends. Springer-Verlag, 1979, x + 217 pp, $12.50 (P). [ISBN: 0-387-09533-0] Assuming a solid background in functional analysis, M-structure theory (an attempt to measure "to what extent a given Banach space X behaves like a CK-space") is developed in Part I and is applied in Part II to generalizations of the Banach-Stone Theorem. Indices, bibliography. JS

FUNCTIONAL ANALYSIS, P. *Lecture Notes in Mathematics-742: Seminormal Operators.* Kevin Clancey. Springer-Verlag, 1979, vii + 125 pp, $9 (P). [ISBN: 0-387-09547-0] A reasonably self-contained picture of some of the recent developments in the area of seminormal operators on a Hilbert space that have attracted much attention. LCL

OPTIMIZATION, S(16-17), P. *Lectures on Optimization--Theory and Algorithms.* Jean Cea. Tata Inst, 1978, v + 236 pp, $9.90 (P). Written in definition, theorem, proof style. Opening chapter on differential calculus in normed linear spaces (Gâteaux and Fréchet derivatives) sets the context. AWR

OPTIMIZATION, S(16-17), P, L. *Parametric Integer Programming.* Robert M. Nauss. U of Missouri Pr, 1979, 98 pp, $19. [ISBN: 0-8262-0250-0] Parametric linear programming is a familiar topic in management science, but parametric integer linear programming has become tractable only with relatively recent improvements in integer programming. Nauss compares recent approaches and suggests directions for future research. AWR

ANALYSIS, P. *Lecture Notes in Mathematics-765: Padé Approximation and its Applications.* Ed: L. Wuytack. Springer-Verlag, 1979, vi + 392 pp, $20 (P). [ISBN: 0-387-09717-1] Proceedings of the conference held in Antwerp, Belgium, April 4-6, 1979. Includes two current bibliographies on Padé approximations and algorithms for computing Padé approximants. JAS

ANALYSIS, P. *Lecture Notes in Mathematics-755: Global Analysis.* Ed: M. Grmela, J.E. Marsden. Springer-Verlag, 1979, vii + 377 pp, $20 (P). [ISBN: 0-387-09703-1] Eleven of the invited addresses from the biennial seminar of the Canadian Mathematical Congress, Calgary, Alberta, June 12-27, 1978. JAS

ANALYSIS, T(18: 2), P. *Abstract Harmonic Analysis, Volume I: Structure of Topological Groups, Integration Theory, Group Representations, Second Edition.* Edwin Hewitt, Kenneth A. Ross. Grund. math. Wissenschaften, B. 115. Spinger-Verlag, 1979, ix + 519 pp, $50.60. [ISBN: 0-387-02983-4] A corrected republication of the 1962 original edition. LAS

ALGEBRAIC GEOMETRY, P. *Lecture Notes in Mathematics-754: Formal Moduli of Algebraic Structures.* Olav Arnfinn Laudal. Springer-Verlag, 1979, 161 pp, $11.80 (P). [ISBN: 0-387-09702-3] Notes from a series of lectures given at the University of Oslo during 1974-75, dealing with deformation theory and particularly with the study of the hulls of some deformation functors encountered in algebraic geometry. The lectures were based upon the author's work in this area. CEC

ALGEBRAIC GEOMETRY, P. *Oscar Zariski, Collected Papers, Volume IV: Equisingularity on Algebraic Varieties.* Ed: J. Lipman, B. Teissier. MIT Pr, 1979, xvi + 651 pp, $50. [ISBN: 0-262-08049-4] 15 recent papers (1964-1979) seeking a natural way to stratify varieties so that the singularities on each stratum are "equivalent in some convincing sense." Includes a preface by Zariski and an introduction by the editors. LAS

ALGEBRAIC GEOMETRY, P. *Automorphic Forms, Representations and L-Functions.* Ed: A. Borel, W. Casselman. Proc. of Symp. in Pure Math., V. XXXIII. AMS, 1979. *Part 1,* x + 322 pp. [ISBN: 0-8218-1435-4]; *Part 2,* vii + 382 pp, $51.60 set. [ISBN: 0-8218-1437-0] The proceedings (Oregon State University, Summer 1977) appear in two parts. The first is devoted to the structure of reductive groups; their representation over local fields; automorphic forms and representations, with emphasis on the analytic theory. The second concerns automorphic representations, their L-functions, their relation to arithmetical algebraic geometry, and the base change problem for GL(2). LCL

GEOMETRY, T*(14: 1, 2), L. *Euclidean and Non-Euclidean Geometries: Development and History.* Marvin Jay Greenberg. Freeman, 1980, xv + 400 pp, $18. [ISBN: 0-7167-1103-6] An already strong text for prospective secondary teachers, liberal arts students and mathematics majors is strengthened by the addition of chapters on geometric transformations and further results in hyperbolic geometry and the addition of an appendix on geometry without continuity. (*First Edition,* TR, April 1975; ER, November 1977.) JNC

GEOMETRY, S(16), P**. *Problems in Discrete Geometry, Fourth Edition.* Ed: W. Moser. McGill U (Dept. of Math., 805 Sherbrooke St. W., Montreal, Canada H3A 2K6), 1979, 23 pp, (P). Twenty-eight succinctly-stated problems in discrete geometry, up from 14 in the first edition distributed to participants in the discrete geometry week, Oberwolfach, July 1977. Historical notes. References. Precursor of projected periodical on Problems in Discrete Geometry. Much nice mathematics, including solved, partially solved and unsolved problems. JK

PROBABILITY, T(15-16: 1), S*, P, L**. *Random Processes, A First Look.* R. Syski. Statistics, V. 29. Dekker, 1979, xvii + 290 pp, $14.50 (P). [ISBN: 0-8247-6893-0] An intuitive and light-hearted introduction to stochastic processes. Presupposes only elementary calculus. FLW

PROBABILITY, T(15-17: 2, 3), *Probabilistic Models in Engineering Sciences.* Harold J. Larson, Bruno O. Shubert. Wiley, 1979. *Volume I: Random Variables and Stochastic Processes.* x + 544 pp, $24.95 [ISBN: 0-471-01751-5]; *Volume II: Random Noise, Signals, and Dynamic Systems.* x + 737 pp, $26.95. [ISBN: 0-471-05179-9] *Volume I* is a comprehensive introduction to applied probability theory. Unique feature is its simultaneous treatment of random variables and stochastic processes. *Volume II* deals with further topics in the theory and applications of stochastic processes, but may be used independently of *Volume I* by students with a post-calculus probability background plus some additional analysis. Discusses both classical (spectral analysis, Markov chains) and recent topics (innovation processes, point processes). RSK

PROBABILITY, T(13: 1), *A Primer in Probability.* Kathleen Subrahmaniam. Statistics, V. 28. Dekker, 1979, viii + 326 pp, $12.50 (P). [ISBN: 0-8247-6836-1] Somewhat sophisticated presentation of elementary probability topics which provide a foundation for statistics. Makes good use of indicator random variables. Emphasizes problem solving. RSK

PROBABILITY, T(17-18: 1, 2), S, *Probability Theory.* R.G. Laha, V.K. Rohatgi. Wiley, 1979, xiii + 557 pp, $28.95. [ISBN: 0-471-03262-X] A measure theoretic treatment that presupposes advanced calculus and some measure theory. FLW

PROBABILITY, T(14-15: 1, 2), *An Introduction to Applied Probability.* Ian F. Blake. Wiley, 1979, 528 pp, $21.95. [ISBN: 0-471-03210-7] Intended primarily for students of engineering, science, and management, assuming one year of calculus. First half covers standard topics. Last half contains essentially independent chapters dealing with applications, both statistical (estimation, hypothesis testing, quality control and acceptance sampling, and linear models for data) and nonstatistical (queueing theory, reliability theory, and communication theory), from which a selection can be made for a one-semester course. RSK

PROBABILITY, P, *Classifying Infinitely Divisible Distributions by Functional Equations.* K. van Harn. Math. Centre Tracts, No. 103. Math Centrum, 1978, vii + 194 pp, Dfl. 24 (P). [ISBN: 90-6196-172-6] The probability distribution of a random variable X is infinitely divisible if, for each n, X is a sum of n independent and identically distributed random variables. This monograph examines classes of such distributions which are characterized by functional equations. LCL

PROBABILITY, T(16-17), P, *An Introduction to Stochastic Processes.* D. Kannan. North Holland, 1979, xiii + 296 pp, $24.50. [ISBN: 0-444-00301-0] "Intended for mathematicians requiring a rigorous introduction to major areas in stochastic processes." Topics include Markov chains and processes, stationary processes, Martingales and Brownian motion. Contains some exercises; more would be useful for a text. The writing is clear if a bit terse. TAV

STATISTICS, T(13-15), *General Applied Statistics, Third Edition.* Fadil H. Zuwaylif. A-W, 1979, xii + 435 pp, $13.95. [ISBN: 0-201-08994-7] First course in statistical inference without calculus. Characterized by simplicity of exposition (no proofs) and many illustrative examples. Covers a good range of the standard hypothesis tests (some quite briefly) with a new chapter on inferences in regression. Reasonable number of exercises, with answers in back. GHM

STATISTICS, T(13), *Probabilités et Statistique.* Jacqueline Fourastié, Benjamin Sahler. Dunod (US Distr: SMPF, 14 E. 60th St., NY 10022), 1978, x + 260 pp, 46 F (P). [ISBN: 2-04-010335-X] An elementary text devoted to descriptive statistics, probability calculation, estimation, hypothesis testing, and the χ^2-test. Chapter exercises. TRS

STATISTICS, T(16: 1), S, L, *Numerical Methods for Nonlinear Regression.* D. Royce Sadler. U of Queensland Pr, 1975, x + 89 pp, $12.95. [ISBN: 0-7022-0964-3] A rather non-mathematical presentation of several algorithms for fitting nonlinear functions to experimental data. FLW

STATISTICS, S(16-18), P*, L*, *Robustness in Statistics.* Ed: Robert L. Launer, Graham N. Wilkinson. Acad Pr, 1979, xvi + 296 pp, $18.50. [ISBN: 0-12-4381-50-2] Proceedings of a workshop held in April 1978 at Research Triangle Park, North Carolina. Surveys, applications, and new ideas in this important area. FLW

STATISTICS, T(16-18: 1, 2), S, P, L, *Introduction to the Theory of Nonparametric Statistics.* Ronald H. Randles, Douglas A. Wolfe. Wiley, 1979, xiii + 450 pp, $24.95. [ISBN: 0-471-04245-5] Presupposes a sound course in calculus-based statistics. Not a compendium of all nonparametric methods, but a study of the tools fundamental to the development of these methods. FLW

STATISTICS, S(18), P, *Some Basic Theory for Statistical Inference.* E.J.G. Pitman. Chapman and Hall (Distr: Halsted Pr), 1979, vii + 110 pp, $17.95. [ISBN: 0-470-26554-X] One of their Monographs on Applied Probability and Statistics. Develops various theoretical results. Topics include distance between probability measures, sensitivity, asymptotic relative efficiency, maximum likelihood estimation, and the sample distribution function. RSK

STATISTICS, P*, *Measures of Association for Cross Classifications.* Leo A. Goodman, William H. Kruskal. Series in Statistics, V. 1. Springer-Verlag, 1979, x + 146 pp, $12. [ISBN: 0-387-90443-3] Reprinting of four landmark, but still timely, papers from the *Journal of the American Statistical Association,* dated from 1954 to 1972. A comprehensive treatment of the subject. RSK

STATISTICS, T(15-16: 2), *Probability and Statistical Inference.* J.G. Kalbfleisch. Springer-Verlag, 1979, $15 each. *V. I,* 342 pp [ISBN: 0-387-90457-3]; *V. II,* 316 pp. [ISBN: 0-387-90458-1] *Volume I* deals with probability models, while *Volume II* treats data analysis and interpretation. Emphasis is on obtaining information from data (model testing and estimation) rather than decision making. Makes strong use of the likelihood function and procedures related to it. RSK

STATISTICS, T*(14-15: 1). *Regression Methods: A Tool for Data Analysis.* Rudolf J. Freund, Paul D. Minton. Statistics, V. 30. Dekker, 1979, xi + 261 pp, $13.50. [ISBN: 0-8247-6647-4] Clearly written introduction to linear model methodology, including analysis of variance and covariance, assuming a background of an elementary statistics course. Makes very good use of numerical examples to illustrate ideas and procedures. Provides some introduction to modern techniques such as ridge regression and C_p plots. RSK

STATISTICS, P*. *Tables of the F- and Related Distributions with Algorithms.* K.V. Mardia, P.J. Zemroch. Acad Pr, 1978, x + 256 pp, $28. [ISBN: 0-12-471140-5] Percentage points of the F, t and χ^2 distributions for *fractional* as well as integral degrees of freedom. Includes detailed procedures for interpolation or approximation, together with some applications involving fractional degrees of freedom. Also includes a comprehensive library of Algol 60 algorithms designed to calculate percentage points and probability integrals. RSK

STATISTICS, P. *Mathematik heute: Grundkurs Stochastik.* Hermann Athen, Heinz Griesel. Hermann Schroedel, 1979, 127 pp, (P). [ISBN: 3-507-83084-1] Probability and statistics for secondary schools. JD-B

COMPUTER PROGRAMMING, P. *Best of Interface Age, Volume 1: Software in BASIC.* Carl D. Warren. Dilithium Pr, 1979, viii + 314 pp, $12.95 (P). [ISBN: 0-918398-36-3] Four articles from *Interface Age* which describe and give source listings for four different versions of Basic for microprocessors: Lawrence Livermore's 8080 BASIC, Dr. Wang's Palo Alto TINY BASIC, Nationals TINY BASIC-NIBL and Uiterwyk's 6800 4K BASIC. CEC

COMPUTER SCIENCE, S(13-16), L*. *Computer Dictionary for Everyone.* Donald D. Spencer. Charles Scribner's, 1979, 191 pp, $9.95. [ISBN: 0-684-16305-5] A low-brow but nonetheless very useful guide to modern computer lingo, expressed in common terms that even mathematicians can understand. A specially useful feature is the unravelling of large numbers of acronyms, the common cryptographs of computer culture. LAS

COMPUTER SCIENCE, T(16-17: 1), S, P. *Fehlertolerante Systeme.* Mario Dal Cin. Teubner Stuttgart, 1979, 206 pp, DM 23,80 (P). [ISBN: 3-519-02352-0] On the theory of fault-tolerant systems. Treats reliability, availability, self-diagnosis and renewal processes. Problems, some solutions. JD-B

SYSTEMS THEORY, P. *Decision Information.* Ed: Chris P. Tsokos, Robert M. Thrall. Acad Pr, 1979, x + 528 pp, $34. [ISBN: 0-12-702250-3] Grew out of 1976 conference intended to identify and stimulate research on problems relevant to military decision information. Three parts: systems approach to large scale problems; statistical models for scheduling, combat strategy, inventory systems, etc.; specific military problems. AWR

SYSTEMS THEORY, T(17-18: 1, 2), P. *Adaptive Control: The Model Reference Approach.* Yoan D. Landau. Control and Systems Theory, V. 8. Dekker, 1979, xix + 406 pp, $45. [ISBN: 0-8247-6548-6] A comprehensive treatment of model reference adaptive systems. FLW

SYSTEMS THEORY, S(16-18), P, L. *Management Applications of System Theory.* Constantin Virgil Negoita. Birkhäuser Boston, 1979, 153 pp. [ISBN: 3-7643-1032-4] State spaces, control, trajectories, stability, realization, multistage linear programming, fuzzy programming, multicriteria optimization. Portions presuppose category theory and the theory of fuzzy sets. No index. FLW

SYSTEMS THEORY, P. *Lecture Notes in Control and Information Sciences-12: A Complex Variable Approach to the Analysis of Linear Multivariable Feedback Systems.* Ian Postlethwaite, Alistair G.J. MacFarlane. Springer-Verlag, 1979, iv + 177 pp, $9.80 (P). [ISBN: 0-387-09340-0] An extension, to multivariable systems, of the concepts underlying the classical techniques of linear feedback systems. It is shown that complex-variable ideas have an important role to play. LCL

APPLICATIONS (BIOLOGY), P, L. *Some Mathematical Questions in Biology, V. 12.* Ed: Simon A. Levin. Lect. on Math. in Life Sci., V. 12. AMS, 1979, ix + 218 pp, $12.40 (P). Five papers from the annual AMS-SIAM symposium held in conjunction with the 1979 Houston meetings of AAAS: two on mathematical models of epidemiology, and one each on allergic reactions, optimal harvesting, and respiration. LAS

APPLICATIONS (ECONOMICS), T(15-18: 1, 2), S, P, L*. *Econometrics, Second Edition.* Ronald J. Wonnacott, Thomas H. Wonnacott. Wiley, 1979, xxiii + 580 pp, $21.95. [ISBN: 0-471-95981-2] Part I presupposes only elementary statistics. Part II uses calculus and matrices. Regression, multiple regression, correlation, time series, the identification problem, systems estimation, and Bayesian inferences. Part II gives more general treatments of these same topics. This new edition contains improvements and new material. (First Edition, TR, October 1970.) FLW

APPLICATIONS (PHYSICS), T(18: 1), S, P, L. *Applied Semigroups and Evolution Equations.* Aldo Belleni-Morante. Clarendon Pr, 1979, xv + 387 pp, $27.50. [ISBN: 0-19-853529-5] Intended for applied mathematicians, physicists, and engineers with a "good" knowledge of classical calculus, enough of the theory of Banach-Hilbert space operators is developed to spend the last six chapters on applications in such areas as classical physics, quantum mechanics, and population theory. Examples, exercises, index, bibliography. JS

Reviewers Whose Initials Appear Above

Judith N. Cederberg, St. Olaf; Clifton E. Corzatt, St. Olaf; John Dyer-Bennet, Carleton; Lorraine Keller, St. Olaf; Richard S. Kleber, St. Olaf; Joseph Konhauser, Macalester; Loren C. Larson, St. Olaf; George H. Mills, St. Olaf; A. Wayne Roberts, Macalester; Thomas R. Savage, St. Olaf; John Schue, Macalester; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; T.A. Vessey, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY PAUL A. HAEDER, University of Nebraska at Omaha

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036

PERSONAL ITEMS

University of North Carolina, Wilmington: Dr. William Etheridge, Purdue University and Dr. Sandra C. McLaurin, University of South Alabama, have been appointed Assistant Professors.

Muskingum College, New Concord, Ohio: Assistant Professor Larry Zettel has been promoted to Associate Professor. Professor James L. Smith is serving as a Fullbright Fellow at Chancellor College, University of Malawi.

Lamar University, Beaumont, Texas: Assistant Professor Michael A. Laidacker has been promoted to Associate Professor. Philip W. Latimer has retired with the title of Professor and Regent Emeritus. Associate Professor Mary Katherine Bell has been awarded the Minnie Stevens Piper Award.

Associate Professors Gustave Efroymson and Cornelius Onneweer, University of New Mexico, have been promoted to Professors.

Charles M. Chambers, Staff Associate at the Council on Postsecondary Accreditation. Washington, D.C., has been promoted to Vice President and Legal Advisor.

Dr. Albert Y. Chi, formerly at Oklahoma State University, has been appointed Lecturer at Emporia State University, Emporia, Kansas.

Associate Professor Francis W. Kollett, Bard College, Annandale-on-the-Hudson, New York, has resigned to accept a position at Wheaton College, Wheaton, Illinois.

Dr. John Mooningham, Utica College, Utica, New York, has been appointed Assistant Professor at Saginaw Valley State College, University Center, Michigan.

Professor Marion B. Smith, California State-Bakersfield, has been awarded a Fellowship by the American Council of Learned Societies for research in the history of mathematics.

Professor Kenneth S. Williams, Carleton University, Ottawa, Canada, was awarded the degree Doctor of Science on December 14, 1979, by the University of Birmingham, England, for his research in the Theory of Numbers.

Lt. Col. Walter M. Patterson III, has been promoted from Chief of Operations Research to Deputy Director of Studies on Analysis, Military Airlift Command, U.S. Air Force.

Associate Professor James F. Hurley, University of Connecticut, has been awarded a Research Fellowship by the Japan Society for the Promotion of Science to support his visit to the University of Tsukuba during the period April-August, 1980.

Professor Alexander E. Clebowicz, Central Connecticut State College, New Britain, Connecticut, has retired with the title of Professor Emeritus.

Dr. Raymond F. Schnepp, San Antonio, Texas, died on August 8, 1979. He was a member of the Association for forty seven years.

Assistant Professor Francis J. Sholomskas, Temple University, died on April 7, 1979, at the age of 56. He was a member of the Association for twenty seven years.

Professor J. S. MacNerney, University of Houston, died on June 2, 1979, at the age of 57. He was a member of the Association for twenty six years.

Professor Gerson H. Robison, New Paltz, New York, died on May 23, 1979, at the age of 70. He was a member of the Association for thirty one years.

THIRD INTERNATIONAL TIME SERIES MEETING

The Third International Time Series Meeting (First American Conference) will be held August 14-15, 1980, at the Shamrock Hilton Hotel in Houston, Texas. Oliver D. Anderson and M. Ray Perryman are the program directors. The conference will convene immediately following the American Statistical Association meetings at the same location. Persons desiring to present a paper, serve on the program committee, serve as a session chairperson or discussant, or organize a session, should contact M. Ray Perryman, Director, Center for the Advancement of Economic Analysis, Hankamer School of Business, Baylor University, Waco, Texas 76706. Persons desiring to present a paper should send a two page abstract to Dr. Perryman by May 1, 1980 (papers will be published in a proceedings volume).

SIXTH ANNUAL RELIABILITY TESTING INSTITUTE

The Sixth Annual Reliability Testing Institute, sponsored by the University of Arizona College of Engineering and Hughes Aircraft Company, Tucson, Arizona Operations, will be held at the Ramada Inn, 404 North Freeway, Tucson, Arizona, on April 14-18, 1980.

The objectives are to provide Reliability Engineers, Product Assurance Engineers and Managers and all other engineers and teachers with a working knowledge of analyzing component, equipment, and system performance and failure data to determine the distributions of their times to failure, their failure rates, their reliabilities and their confidence limits; small sample size, short duration, low cost tests, and methods of analyzing their results; accelerated testing; test planning; electrical overstress and electrostatic failure protection; Bayesian testing; suspended items testing; sequential testing; and others.

For further information contact Dr. Dimitri Kececioglu, Institute Director, Aerospace and Mechanical Engineering Department, The University of Arizona, Building #16, Tucson, Arizona 85721.

TWO JUNE 1980 WORKSHOPS IN APPLICABLE MATH

The MD-DC-VA section of the MAA will sponsor two five-day workshops at Salisbury State College, Maryland. "Linear Algebra and Its Applications" will be led by Dr. M. Z. Nashed of the University of Delaware. This workshop will be held 2-6 June.

The second workshop, "Structured Programming in PASCAL" will be led by Dr. W. J. Collins of Salisbury State College. This workshop will be held 9-13 June.

These workshops are designed for teachers in two and four-year colleges. The total cost, including room and board, is $150 for each workshop. For more information, write Dr. B. A. Fusaro, Department of Math Sciences, SSC, Salisbury, MD 21801.

CONFERENCE ON NUMERICAL ANALYSIS OF SEMICONDUCTOR DEVICES AND INTEGRATED CIRCUITS

NASECODE II Conference: The second international conference on the Numerical Analysis of Semiconductor Devices and Integrated Circuits will be held in Dublin, Ireland, from 17th to 19th June 1981 under the auspices of the Numerical Analysis Group, Dublin. The conference is sponsored by IEEE (Electron Devices Society), IEE (Irish Brand), Royal Irish Academy and Irish Mathematical Society.

Contributed papers are solicited on any topic relevant to the numerical simulation, optimization and computer aided design of semiconductor devices or integrated circuits. The preliminary version of such a paper should be submitted not later than FRIDAY, 20th MARCH 1981. It must be accompanied by a separate one-page abstract. The final version of an accepted contribution should be delivered to the registration desk on the first day of the conference. It must be typed according to our instructions on special paper supplied by us and it should be at most five pages long.

The proceedings of the conference will be published in book form in August 1981. They will contain the full texts of both invited lectures and the contributed papers. Registered participants will receive one free copy.

The registration fee is £90 and all communications should be addressed to NASECODE II Conference 39 Trinity College, Dublin 2, Ireland, telephone No. (01) 772941 ext. 1889 or 1949; telex No. 5442 or 31166; telegraphic address TRINITY DUBLIN.

THE USE OF HISTORY IN TEACHING OF MATHEMATICS

A symposium on the use of history in the teaching of mathematics will be held on Friday, April 25, 1980, at Valparaiso University, in commemoration of Professor Arthur E. Hallerberg. The program is designed to accommodate 16 short papers (10-15 minutes) for which abstracts are due on March 1, 1980.

Major addresses will be given by Professor Phillip S. Jones of the University of Michigan, Professor Emeritus Karl Menger of Illinois Institute of Technology, and Professor Harry Pollard of Purdue University. Further information may be obtained from Professor Marvin G. Mundt, Department of Mathematics, Valparaiso University, Valparaiso, Indiana 46383.

WOODROW WILSON FELLOWS

The Woodrow Wilson National Fellowship Foundation on December 14, 1979, announced 12 winners in a competition among faculty for projects to expand their teaching ability. The $4,000 grants are being made to former Woodrow Wilson Fellows who have demonstrated imagination and flexibility in their teaching. In making the awards, Hans Rosenhaupt, President of the Foundation, said, "The best way to improve undergraduate education today is to provide teachers with a new challenge and support for creative activities."

Winners from the mathematical sciences are Harold Hastings, Hofstra University, *Mathematical modeling in biological and social sciences*, and Bernard Shiffman, The Johns Hopkins University, *Combining introductions to differential geometry and relativity.*

The grant money will be used for released time and costs for research necessary to develop new undergraduate courses. These outstanding young faculty members share a concern for the changes in our culture and the values inherent in such changes.

A new method for improving students' ability to write will be developed from the hypothesis that it is necessary to learn how to generate ideas and form concepts as a prelude to learning the means for expressing ideas. The resulting teaching model can be used to help teachers improve students' written expression in any field. All the work being supported by the awards will be translated directly into courses introduced into the curriculum. As a result, grants made to individual faculty members will eventually benefit a large number of undergraduate students.

More than one hundred applications were received for the 12 awards. Because of restrictions in funding, eligibility was limited to former Woodrow Wilson Fellows teaching in the northeastern states. More than 18,000 Woodrow Wilson Fellows, chosen for their commitment to college teaching and their academic excellence, were supported by the Foundation between 1945 and 1971. The Woodrow Wilson National Fellowship Foundation is an independent national agency which administers innovative programs for the improvement of education.

Members of the selection committee for the competition were Frederick deW. Bolman, former president of Franklin and Marshall College and former director of the Exxon Education Foundation; Mina Rees, President Emeritus, the Graduate Center, City University of New York; and Frank W. Wadsworth, former Woodrow Wilson Fellow and Professor of Literature, State University of New York College at Purchase.

AMERICAN JOURNAL OF MATHEMATICAL AND MANAGEMENT SCIENCES

The *American Journal of Mathematical and Management Sciences* will commence publication in January of 1981. *AJMMS* will seek to bring together the best new work in the various areas of the mathematical and management sciences (including, but not limited to, statistics (mathematical, applied, industrial, bio-, or educational), probability (pure or applied, decision theory, information theory, consulting, computer science, management science, and operations research). Publisher is the American Sciences Press, P.O. Box 5661, Columbus, Ohio 43221.

AJMMS will be inclusive rather than narrowly exclusive. Refereeing will usually be by Editorial Board members in their specialties, and manuscripts will be reproduced directly from carefully typed copy, in order to attain the highest levels of quality and simultaneously achieve extremely rapid publication of acceptable papers. The Editor is Edward J. Dudewicz and the Editorial Board includes thirty five well-known professionals in related fields.

AJMMS editorial policy is that a paper worth publishing, is worth publishing in a form in which it is readable and (if appropriate) usable. Hence: papers will not be shortened simply to save journal pages; detailed proofs, relevant tables, literature review, and worked examples and needed computer programs are encouraged. Manuscripts to be considered for publication should be submitted in 5 copies. No special format is required for submission; in order to save authors' time, detailed instructions and special typing paper (designed for easy use by secretaries) will be provided once an article is fully approved for publication. Correspondence and manuscripts should be directed to: Professor Edward J. Dudewicz, Editor, American Journal of Mathematical and Management Sciences, Department of Statistics, The Ohio State University, Columbus, Ohio 43210, U.S.A.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NOVEMBER MEETING OF THE NORTHEASTERN SECTION

The twenty-fifth annual meeting of the Northeastern Section of the MAA was held at the University of Hartford, Hartford, Connecticut, on November 17, 1979. Among the 141 people who attended the meeting was Anne F. O'Neil, the newly elected Sectional Governor. The following talks were given at the morning session:

The Poincare Model as a Barbilian Geometry, Howard Eves, University of Maine
The Christie Lecture: Is the Universe Simply Connected," John Milnor, Institute of Advanced Study

At the business meeting the following By-Law Amendments were presented and passed: A representative from the two-year colleges, elected to a two year term at the annual meeting in even numbered years, is to be added to the Executive Committee; the term of the Vice-Chairman is to be shortened to one year with election to this office occurring in even-numbered years. The Chairman will be elected in odd-numbered years to serve a two year term of office.

The following officers were elected:

Chairman: Roger L. Cook, University of Vermont
Secretary-Treasurer: George W. Best, Phillips Academy
Two-Year College Representative: Nancy Meyers, Bunker Hill Community College

The business meeting closed with the announcement that the section will hold a summer meeting in June 1980.

The afternoon program included a lecture and a panel discussion.

Panel on *Hand Calculators and Microprocessors*, Panelists: Louise Gould, Ethel Walker School
William Hudnall, The University of Hartford
John Morris, Pratt and Whitney Aircraft,Inc.
Recreational Mathematics in the Classroom, Richard S. Dolliver, Greater Hartford Community College

OCTOBER MEETING OF THE INDIANA SECTION

The fall meeting of the Indiana Section of the MAA was held at Wabash College in Crawfordsville on Saturday, October 27, 1979, with 65 people present. The chairman of the Section, Underwood Dudley of DePauw University, presided.

The following papers were presented:

Cancellation in Arithmetic and Topology, P. Hilton, Case Western Reserve University
Rearranging the Alternating Harmonic Series, C. Cowen, Purdue University
How to Paint a High-Dimensional Triangle, P. Halmos, Indiana University

A panel discussion *Can Remedial Students Be Saved?* was presented by M. Gemignani, J. Kuczkowski and I. Boodt, Indiana University-Purdue University Indianapolis.

P. T. Mielke read a statement of gratitude toward Professor J. Crawford Pauley on the anniversary of his fiftieth year at Wabash College. It was warmly supported by the entire section.

R. R. PATTERSON, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Sixtieth Summer Meeting, University of Michigan, Ann Arbor, August 18–20, 1980.

Sixty-fourth Annual Meeting, San Francisco, California, January 9–11, 1981.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, West Virginia Wesleyan College, Buckhannon, April 25–26, 1980.

EASTERN PENNSYLVANIA AND DELAWARE, Cedar Crest College, Allentown, Pennsylvania, April 26, 1980.

FLORIDA, Jacksonville University, Jacksonville, March 7–8, 1980.

ILLINOIS, John A. Logan College, Carterville, April 25–26, 1980.

INDIANA, Valparaiso University, Valparaiso, April 16, 1980.

INTERMOUNTAIN, Utah State University, Logan, late April or early May 1980.

IOWA, Simpson College, Indianola, April 18–19, 1980.

KANSAS, Kansas State University, Manhattan, April 12, 1980.

KENTUCKY, Western Kentucky University, Bowling Green, April 11–12, 1980.

LOUISIANA–MISSISSIPPI, Friday-Saturday before February 20. Deadline for papers three months before meeting.

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, University of Richmond, Virginia, April 12, 1980.

METROPOLITAN NEW YORK, Mercy College, Dobbs Ferry, May 4, 1980.

MICHIGAN, Hope College, Holland, May 2–3 1980.

MISSOURI, Westminster College, Fulton, April 25–26, 1980.

NEBRASKA, Doane College, Crete, April 18–19, 1980.

NEW JERSEY, Hyatt House, Cherry Hill, March 15, 1980.

NORTH CENTRAL, Gustavus Adolphus College, St. Peter, Minnesota, April 25–26, 1980.

NORTHEASTERN, Saturday after Thanksgiving, and third week in June in odd-numbered years.

NORTHERN CALIFORNIA, first or second Saturday in February.

OHIO, Wittenberg University, Springfield, April 25–26, 1980.

OKLAHOMA–ARKANSAS, Westark Community College, Fort Smith, Arkansas, March 28–29, 1980.

PACIFIC NORTHWEST, Central Washington University, Ellensburg, Washington, June 20–21, 1980.

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SEAWAY, Herkimer County Community College, Herkimer, New York, May 2–3, 1980.

SOUTHEASTERN, Appalachian State University, Boone, North Carolina, April 11–12, 1980.

SOUTHERN CALIFORNIA, California State University, Northridge, March 8, 1980.

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TEXAS, East Texas State University, Commerce, April 11–12, 1980.

WISCONSIN, University of Wisconsin, Milwaukee, March 28–29, 1980.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Toronto, January 3–8, 1981.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES

AMERICAN MATHEMATICAL SOCIETY, University of Michigan, Ann Arbor, August 19–22, 1980.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Amherst, Massachusetts, June 23–26, 1980.

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THE AMERICAN MATHEMATICAL MONTHLY

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Is a Quadratic an Iterate?

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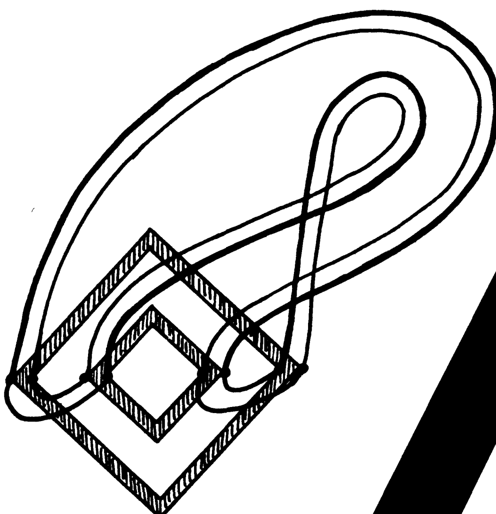

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CALCULATOR CALCULUS AND ROUNDOFF ERRORS

GEORGE MIEL

1. Introduction. There is considerable interest among educators in the use of hand-held electronic calculators as instructional aids in the classroom. Current trends seem to be promoting imaginative uses of these devices, not only to avoid the drudgery of pencil and paper calculations, but also to motivate and reinforce the learning of mathematical concepts; see, for example, [**A1**].

In particular, an elegant application of the calculator as a reinforcement aid in beginning calculus is in the demonstration of the limit process. A quantity carefully chosen by the instructor can be calculated rapidly and repeatedly and thus dramatically shown to get closer and closer to a certain value. However, indiscriminate use of the calculator and a lack of awareness of the effects of roundoff errors can lead the student to a mistaken interpretation of calculated results. Although roundoff errors may remain negligible or cancel one another, they usually propagate during the calculation and thus yield numbers which eventually differ significantly from the limit. In extreme cases, if the calculation is ill conditioned—that is, very sensitive to perturbations caused by roundoff—the calculated results may be totally erroneous. Such situations can confuse the uninitiated student and thus negate the benefits of the calculator as an instructional tool.

This is one of the reasons for a great deal of debate on the merits of student use of calculators in the classroom. Regardless of how one feels about the issue, an awareness of what is available and known is needed in making an informed decision on whether or not to supplement a calculus course with calculator work. For the reader's convenience, we give in Section 2 a basic list of references that may be helpful in making this decision.

A good part of the literature indicates that supervised usage of the calculator can indeed reinforce understanding of calculus notions, promote an appreciation of the difference between theory and practical calculation, and encourage independence and creativity in problem solving. However, the common belief that simplicity of classroom calculations does not allow excessive roundoff errors is ill founded. In calculator demonstration of the limit process, the arithmetic may involve operations which result in large losses of significance. Typically, such operations are additions of numbers which vary widely in size or subtractions of nearly equal numbers. If a calculus class is allowed or required to use calculators, the basic facts on roundoff errors, usually relegated to formal courses on numerical analysis, should be taught prior to turning students loose to their devices. The instructor is advised to keep in mind the cautionary attitude, endorsed by Scarborough [**36**, Section 14] and other numerical analysts, that fragmentary knowledge of approximate calculation can lead to faulty conclusions.

Most textbooks on computing pay attention to numerical errors; but the discussion is either very technical or quite cursory, and the emphasis is usually on theoretical errors of various approximation techniques. Moreover, there appears to be a paucity of examples which vividly show the effects of numerical errors, and which can be carried out on a calculator in the classroom. Section 3 presents an elementary analysis of roundoff errors in calculator demonstrations of the limit process. Section 4 gives simple numerical illustrations that can easily be shown in the classroom. These examples can be adapted for calculators of different makes and accuracies. The calculators need not be programmable, but they should have scientific notation and a basic mathematical package, in particular, the $\sqrt{\ }$ and ln functions. The following

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material is primarily geared to calculus instructors who wish to use calculators in order to help them design an introductory discussion on roundoff errors. The material can also be given in an elementary course in numerical analysis. References for advanced follow-up topics are given in Section 5.

2. Information Resources. There is at least one textbook [25] specifically aimed at supplemental use of the calculator in elementary calculus. The articles [20], [23], [24], [30] are directly pertinent. Texts on computers in the calculus [4], [17], [22], [44] can be modified for calculator use. The articles [6] and [29] deal with the last-mentioned text. Further information on computer calculus is given in recent notes in this MONTHLY [1], [2], [12], [19], [21], [26], [28], [31], [35], [45], [50].

The philosophy behind classroom use of calculators is discussed in [3]. Guidelines for classroom management can be found in various reports, for example, [49]. It must be remembered that the instructor cannot merely hold a pocket calculator up before a class, since the digits are too small to be seen from a distance. Moreover, considerable time can be spent speaking out numbers and writing them on the board, thus disrupting the continuity of calculator demonstrations. This difficulty can be resolved by using a special unit, available from [A2], which provides a display for the class with a viewing angle of 130° and a range of 60 feet. A less satisfactory, but more economical, device is available from [A3]; it consists of a calculator, with ceramic insulation, that is modified to let its display be projected on a screen or blackboard by any overhead projector.

Two filmstrips with records or cassettes, obtainable from [A4], may be used to introduce the electronic calculator to the class. The article [27] explains how a typical calculator works, by describing in layman's terms a simpler hypothetical four-function calculator. A user's guide and general lexicon is given in [38]. Guidelines for the selection of a calculator can be found in [10], [15], [37], [32]. The last survey is specifically oriented toward the mathematician. Although very informative, these surveys are somewhat obsolete, as fast-paced changes in prices and models make evaluations outdated by the time of publication. Purchase decisions should be delayed until the prospective buyer has communicated with manufacturers.

Although not needed in a beginning calculus class, a calculator with programming capabilities may be helpful later on for advanced applications. For this reason, the instructor and the students should consider the option of buying a programmable calculator from the onset. A guide is provided in [39]. Various programs and applications are available from [8], [18], [33], [41], [42]. There are also "users' groups" for specific programmable calculators, which enable members to exchange programs for their machines [A5], [A6].

The instructor should be aware that games, puzzles, and tricks are possible on the calculator, thus providing an occasional change of pace in the classroom (see [11] and [16]). There are numerous books on the recreational use of calculators; a prototype is [46].

We conclude our list of references by pointing out that [A7] and [A8] provide various information bulletins and updated bibliographies on calculators and computers.

3. Elementary Analysis. A discussion of numerical errors must necessarily begin with a description of scientific notation, truncation and rounding procedures, and floating-point arithmetic. Standard fares of these topics abound in the literature. Most scientific calculators in the market today have a display of 8 or 10 digits, with a floating-point mantissa that may contain one or more additional internal digits. The actual number of digits in the mantissa of a calculator, its truncation or rounding procedure, and the accuracy of its operations are stated in the owner's manual. Typically, the four basic arithmetic operations have a maximum error of ± 1 count in the least significant digit of the mantissa.

A certain phenomenon occurs when numbers widely varying in size are added. For example, consider a calculation of the form,

$$a+\sum_{i=1}^{m} b_i, \qquad 0<b_i\leqslant 0.4, \tag{1}$$

where a is extremely larger than each b_i. On a calculator with an 8-digit mantissa, if $a=10{,}000{,}000$ and $b_1=0.4$, we get

$$a+b_1=a, \quad b_1\neq 0, \tag{2}$$

regardless of whether the calculator rounds or truncates. The same effect occurs as the addition (1) proceeds with other b_i terms, yielding at the end again a. If $m=25$, we could thus lose two significant digits. If a computer with an 8-digit mantissa is programmed to evaluate (1) with $m=10{,}000$, as many as four digits could be lost. Observe that, if the summation $\sum_{i=1}^{m} b_i$ is calculated first and then added to a, no major loss of significance takes place. If possible, the terms of long sums should be ordered so that smaller terms are added first. Effects of type (2) are likely to occur in the evaluation of infinite series, when a partial sum is considerably larger than the corresponding next term in the series.

The greatest source of numerical error in many calculations is the loss of leading significant figures in the subtraction of nearly equal numbers. We can illustrate this serious type of error on a calculator with an 8-digit mantissa, by evaluating

$$\begin{gathered} a(b-1), \quad ab-a, \\ a=22{,}222{,}222, \quad b=1+10^{-n} \end{gathered} \tag{3}$$

for different values of n. Our calculator, which displays all 8-digits of its mantissa, generated the following values:

	$a(b-1)$	$ab-a$
0	2.222 2222+07	2.222 2222+07
1	2.222 2222+06	2.222 2220+06
2	2.222 2222+05	2.222 2200+05
3	2.222 2222+04	2.222 2000+04
4	2.222 2222+03	2.222 0000+03
5	2.222 2222+02	2.220 0000+02
6	2.222 2222+01	2.200 0000+01
7	2.222 2222+00	2.000 0000+00
8	2.222 2222−01	0.000 0000+00

These numbers are in scientific notation, with the decimal exponent indicated by the last two digits preceded by an algebraic sign. For each n, the calculated values $a(b-1)$ are correct to all 8 figures. On the other hand, the nth entry of $ab-a$ has lost n significant digits. These calculations show that

$$a(b-1)\neq ab-a, \tag{4}$$

that is, they violate the distributive law. As n increases, the number ab gets closer to a and the resulting subtraction error gets larger. Such subtraction errors occur in certain limit processes, for example, in the approximation of derivatives by difference quotients.

The two numerical examples dealing with (1) and (3) can be easily modified for calculators with different numbers of display and internal digits. As shown by (2) and (4), such examples indicate that actual calculations may violate ordinary laws of algebra. An excellent eye-opener for the student is to determine which of the axioms, defining the set of real numbers as a complete ordered field, are violated by floating-point arithmetic.

We are now ready to discuss the effects of roundoff in limit processes. Consider an infinite sequence of numbers,

$$\{t_n\}_{n=0}^{\infty}, \quad \lim_{n\to\infty} t_n=t<\infty, \tag{5}$$

generated by some well-defined algorithm and known to converge to a finite limit t. For a convergent infinite series, $\sum_{n=0}^{\infty} s_n = t$, we consider as usual the sequence (5) with $t_n = s_0 + \cdots + s_n$. For the limit of a function, $\lim_{x \to a} f(x) = t$, we first choose a sequence $\{x_n\}_{n=0}^{\infty}$ converging to a and then let $t_n = f(x_n)$. To demonstrate on a calculator the convergence in (5), we generate a finite sequence

$$\hat{t}_0, \hat{t}_1, \ldots, \hat{t}_N,$$

where $\hat{t}_n$ is the calculated value of t_n, with the hope that these values get increasingly close to t.

The total error $E_n = t - \hat{t}_n$ at the nth step is given by

$$E_n = \varepsilon_n + \delta_n,$$

$$\varepsilon_n = t - t_n, \quad \delta_n = t_n - \hat{t}_n,$$

where ε_n is the theoretical error and δ_n is the accumulated roundoff error. Whereas the magnitude of ε_n is known to tend to zero and can usually be estimated, the behavior of δ_n may be difficult to predict, as it depends not only on the algorithm but also on the calculator being used. It is possible to have ε_n and δ_n of the same order of magnitude but of opposite signs, thus nearly canceling each other, so that $\hat{t}_n$ actually approximates t better than t_n. On other occasions, δ_n can be considerably bigger in size than ε_n, thus making $\hat{t}_n$ completely erroneous. *A rule of thumb is that, as n increases, the magnitude of ε_n gets smaller while that of δ_n gets larger and that the behavior of E_n depends on the relative sizes and rates of change of ε_n and δ_n.* This rule will not apply for every specific case, but it serves well as a guiding principle for the novice. Typical situations are illustrated in the figures in which we graphed $(n, |E_n|)$ as points on the Cartesian plane.

If $|\varepsilon_n|$ is monotone decreasing and $|\delta_n|$ remains comparatively small for every term, we get a graph as shown in Figure 1, in which the points $(n, |E_n|)$ tend to lie on a monotone decreasing curve. In this desirable situation, since δ_n remains negligible, E_n is controlled by ε_n. In effect, this means that $\hat{t}_n$ is a good approximation to t_n. The calculated values may begin to repeat after a certain number of terms,

$$\hat{t}_{n_0} = \hat{t}_n, \quad n \geqslant n_0, \tag{6}$$

thus indicating that, relative to the accuracy of the calculator being used, the corresponding

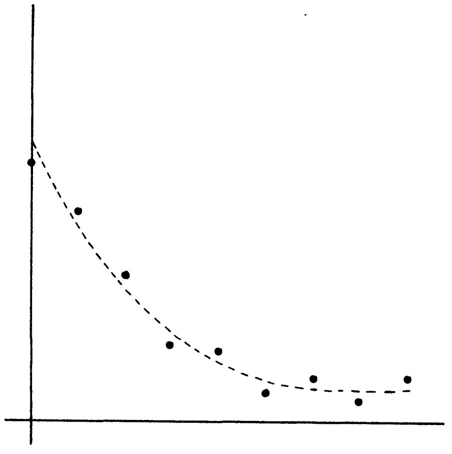

FIG. 1. Best Case—$|\varepsilon_n|$ decreases fast and $|\delta_n|$ stays small: the theoretical error dominates and the calculated values are relatively unaffected by roundoff.

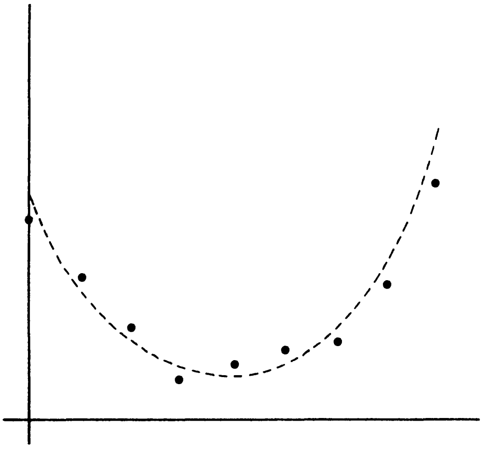

FIG. 2. Common Case—$|\varepsilon_n|$ decreases and initially dominates but $|\delta_n|$ increases and eventually takes over: the trade-off point occurs near the vertex of the curve.

exact values t_n, $n \geqslant n_0$, are indeed close to t. For a calculator with more significant digits, (6) is likely to occur after a larger number n_0 of terms, and the approximation t_{n_0} is then more accurate than before.

The most common type of behavior of the total error is indicated in Figure 2. Initially, $|\varepsilon_n|$ is much larger than $|\delta_n|$; but as n increases, the former gets smaller whereas the latter gets larger. This means that at first ε_n dominates and $|E_n|$ decreases; but after a while, δ_n takes over and $|E_n|$ gets larger. The points $(n,|E_n|)$ tend to lie on a concave upward curve, and the points close to the vertex correspond to calculated values with small errors. The best possible approximation of the limit is $\hat{t}_{n_0}$, where

$$n_0 = \min\{m:|E_m| \leqslant |E_n|, 0 \leqslant n \leqslant N\}.$$

If a calculator with a bigger number of significant digits is used, then the point $(n_0,|E_{n_0}|)$ gets closer to the abscissa and shifts to the right, since the roundoff errors are then smaller and take longer to contaminate the calculation.

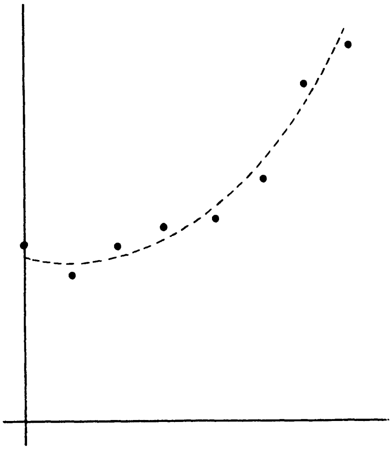

FIG. 3. Worst Case—$|\varepsilon_n|$ decreases slowly and $|\delta_n|$ grows large fast: the dominant roundoff error makes the calculated values meaningless.

The worst situation is shown in Figure 3, in which the points $(n,|E_n|)$ tend to lie on a monotone increasing curve. This pattern occurs when the rate of decrease of $|\varepsilon_n|$ is overwhelmed by a very fast growth of $|\delta_n|$. This propagation of roundoff errors renders the calculated values meaningless. Algorithms yielding such behavior are called ill conditioned or unstable. In such cases, an increase of the number of significant digits might reduce the rate of contamination; but, basically, the user has to find a better algorithm for generating either the same sequence or, possibly, a different sequence converging to the same limit.

4. Numerical Examples. We will illustrate numerically the situations described in the previous section. Our first example is based on a classical paradox described around 500 B.C. by Zeno of Elea. This paradox is sometimes used to introduce the notion of a limit in modern courses of calculus, e.g., Grossman [**14**, Sections 2.1 and 2.6]. Suppose that a tortoise starts crawling a certain distance ahead of Achilles, the swiftest of the Greeks. While Achilles runs to the starting point of the tortoise, the latter will have crawled ahead. Every time Achilles closes in on the tortoise's previous position, the latter will have moved ahead. Zeno concluded that, although Achilles gets closer and closer to the tortoise, he will never actually reach it. The flaw in this conclusion is explained by the convergence of a geometric series.

To put the paradox in a numerical context, assume that Achilles starts out 1000 paces behind the tortoise and that he runs at a rate of 10 paces a second, while the tortoise crawls 1 pace a second. The time needed by Achilles to reach the tortoise is given by

$$10t = 1000 + t, \quad t = 111\tfrac{1}{9}\,\text{sec}.$$

As pointed out by Vilenkin [47, Chapter 3], Zeno's argument consists of estimating t by successive approximations given by

$$t_0 = 0, \quad t_n = 100 + \frac{t_{n-1}}{10}, \quad n \geqslant 1. \tag{7}$$

In order to arrive at the place where the tortoise was after t_{n-1} seconds, Achilles needs 100 seconds to run the first 1000 paces and an additional $t_{n-1}/10$ seconds to run the t_{n-1} paces covered by the tortoise.

By using a calculator to evaluate the first few iterates (7), we will reach the same conclusion that Zeno did. This time, however, the conclusion will be erroneous because of roundoff errors. After $n=3$, each iterate (7) gains a new decimal place equal to 1. For a calculator with an 8-figure display, we find

$$\hat{t}_{n_0} = \hat{t}_n = 111.111\,11, \quad n \geqslant n_0 = 8. \tag{8}$$

The distance at time t separating Achilles and the tortoise is $D(t) = 1000 - 9t$. Since we have $D(\hat{t}_n) = 0.000\,01$ for $n \geqslant 8$, it again appears that Achilles never catches up with the tortoise. The reason is easily explained. From (7), t_n is the sum of the first n terms of a geometric series,

$$t_n = 100\left(1 + \frac{1}{10} + \cdots + \frac{1}{10^{n-1}}\right).$$

Since the calculator displays only 8 digits, every new term $100/10^{n-1}$ for $n > 8$ is too small compared to t_8, thus causing the effect (2) described earlier.

For a calculator with an N-digit display, (8) occurs with $n_0 = N$. In this case, simple calculations show that

$$\varepsilon_n = \frac{10^{3-n}}{9}, \quad \begin{cases} \delta_n = 0 & \text{and } E_n = \varepsilon_n, \quad n \leqslant N, \\ \delta_n = \dfrac{10^{3-N}}{9} - \varepsilon_n & \text{and } E_n = \dfrac{10^{3-N}}{9}, \quad n > N \end{cases}$$

For $n \leqslant N$, the roundoff error $\delta_n = 0$ is smaller than the decreasing theoretical error ε_n, and the resulting behavior of the total error E_n is described by Figure 1. The calculated approximation $\hat{t}_N$ agrees with the theoretical limit t to all N significant figures.

On some occasions, roundoff errors will work in the user's favor and yield an approximation $\hat{t}_n$ which agrees exactly with the limit t. For example, the sequence defined by

$$t_0 = 2, \quad t_n = \sqrt{t_{n-1}},$$

is known to converge to $t=1$ [5, p. 31]. On a calculator, the terms of this sequence are easily generated by entering 2 and then repeatedly pressing the $\sqrt{}$ key. We thus find

$$\hat{t}_{n_0} = \hat{t}_n = 1, \quad n \geqslant n_0,$$

where n_0 depends on the type of calculator used. For calculators with 8 and 10 display digits, n_0 is around 20 and 30, respectively. Although $t_n \neq \hat{t}_n$ for $n \geqslant n_0$, the calculated sequence indicates exactly the limit $t=1$. The behavior of the total error E_n is again described by Figure 1, with the condition $E_n = 0$ for $n \geqslant n_0$, since then the corresponding theoretical and roundoff errors, $\varepsilon_n = 1 - t_n$ and $\delta_n = t_n - 1$, cancel one another.

One of the simplest ways of illustrating the most common situation, shown in Figure 2, in which the size of the total error initially decreases and then eventually increases, is to take a sequence of difference quotients converging to a derivative. Consider a function $f(x)$ differentia-

ble at $x=a$ and let

$$t_n = 10^n[f(a+10^{-n})-f(a)], \quad t=f'(a). \tag{9}$$

By Taylor's theorem, if f'' exists and is continuous around $x=a$, the theoretical error is

$$\varepsilon_n = \frac{-10^{-n}}{2}f''(\xi_n), \quad a<\xi_n<a+10^{-n}. \tag{10}$$

The roundoff error generally starts out quite small, but gets rapidly large, usually not because of the arithmetic needed to evaluate the function, but mostly because of the increasing subtraction error as $f(a+10^{-n})$ approaches $f(a)$. This serious type of loss of significance was described earlier.

If we take $f(x)=\sqrt{x}$ and $a=\frac{9}{4}$, then (9) and (10) become

$$t_n = 10^n[\sqrt{2.25+10^{-n}} - 1.5], \quad t=1/3,$$

$$\varepsilon_n = \frac{10^{-n}}{8\sqrt{\xi_n^3}} < \frac{10^{-n}}{27}, \quad 2.25<\xi_n<2.25+10^{-n}. \tag{11}$$

The corresponding $\hat{t}_n$ values from calculators with 8, 10, and 13 digits in the mantissa are shown below:

n	8 digits	10 digits	13 digits
0	.302 775 60	.302 775 638 0	.302 775 637 7
1	.329 709 00	.329 709 720 0	.329 709 716 0
2	.332 960 00	.332 963 800 0	.332 963 783 6
3	.333 200 00	.333 296 000 0	.333 296 303 0
4	.333 000 00	.333 330 000 0	.333 329 610 0
5	.330 000 00	.333 300 000 0	.333 332 800 0
6	.300 000 00	.333 000 000 0	.333 331 000 0
7	.000 000 00	.330 000 000 0	.333 310 000 0
8	.000 000 00	.300 000 000 0	.333 100 000 0
9	.000 000 00	.000 000 000 0	.332 000 000 0

The values in the last column were generated with a calculator which displays 10 digits, but which actually uses a mantissa with 3 additional internal digits. The numbers vary slightly depending on the make of the calculator, but the pattern is the same.

For the case with an 8-digit mantissa, the calculated values $\hat{t}_n$ are increasingly more accurate for $0 \leqslant n \leqslant 3$, since then the rapidly decreasing theoretical error (11) is dominant; but for $3<n\leqslant 7$, the accuracy of $\hat{t}_n$ decreases due to the increasing seriousness of the subtraction error. Note that $\hat{t}_n=0$ for $n \geqslant 7$, since then the calculator is unable to distinguish between $\sqrt{2.25+10^{-n}}$ and $\sqrt{2.25}$. The behavior of the total error is described by Figure 2, with the least error occurring at $n_0=3$. The same situation occurs for the cases with 10 or 13 digits in the mantissa, except that the optimal numbers of terms is $n_0=4$ and $n_0=5$, respectively. These values agree with the numerical experiments of Morgan and Warnock [30], which indicate that the optimal approximation of derivatives by (9), for typical classroom functions and for current calculators, occurs in general with $n_0=3$, 4, or 5. Certain symmetric formulas for numerical differentiation minimize the anomalies due to subtraction errors. A derivation of such formulas and their error estimates, ideal for an elementary calculus class, is given by Smith [**40**].

Our last example deals with an ill-conditioned limit process. Let

$$t_n = \int_0^1 \frac{x^n}{a-x}\,dx, \quad n \geqslant 0, \tag{12}$$

where $a>1$ is constant. Since

$$at_{n-1}-t_n=\int_0^1 x^{n-1}dx=\frac{1}{n},$$

we obtain the recurrence relation,

$$t_0=\ln a-\ln(a-1),\quad t_n=at_{n-1}-\frac{1}{n},\quad n\geq 1. \tag{13}$$

The following calculated terms were obtained from (13) for different values of a and with the same calculators used in the previous example.

n	8 digits $a=100$	10 digits $a=700$	13 digits $a=1000$
0	1.005 000 0 −02	1.429 593 000 −03	1.000 500 334 −03
1	5.000 000 0 −03	7.151 000 000 −04	5.000 340 000 −04
2	0.000 000 0 +00	5.700 000 000 −04	3.340 000 000 −04
3	−3.333 333 3 −01	6.566 666 670 −02	6.666 666 667 −04
4	−3.358 333 3 +01	4.571 666 669 +01	4.166 666 667 −01
5	−3.358 533 3 +03	3.200 146 668 +04	4.164 666 667 +02

In each case, the calculated values $\hat{t}_n$ are absurd, since an inspection of (12) shows that the theoretical values t_n are monotone decreasing and convergent to $t=0$.

The algorithm (13) is severely unstable, especially for large values of a. As shown in Figure 3, since the roundoff error is dominant and very rapidly increasing, the total error grows large in size, causing completely false results. The reason can be explained as follows. For large a, the evaluation of t_0 yields a big subtraction error, since ln a and $\ln(a-1)$ are nearly equal. From (13), we have

$$t_n=a^n t_0-A_n,\quad A_n=\sum_{i=1}^{n}\frac{a^{n-i}}{i}.$$

Whereas the roundoff errors involved in calculating A_n are relatively unimportant, the initial subtraction error δ_0 gets magnified by a factor a^n in the calculation of t_n. Assuming exact arithmetic for $n\geq 1$, we get

$$\delta_n=a\delta_{n-1}=a^n\delta_0. \tag{14}$$

Since $E_n=\varepsilon_n+a^n\delta_0$, where $\varepsilon_n=-t_n$ converge monotonically to zero and $a^n\delta_0$ are unbounded if $\delta_0\neq 0$, we see that $\lim|E_n|=\infty$. The relation (14) shows that each iteration of the algorithm magnifies by a factor a the error inherited from previous cycles, and the resulting error propagation is disastrous. For small values of a, the rate of contamination is reduced, especially by taking $t_0=\ln(a/(a-1))$, but the algorithm (13) is basically not suited for numerical work.

5. Conclusion. Classroom use of calculators is a recent phenomenon, and measurements of the resulting long-range effects have yet to be firmly established. Regardless of how one feels about the issues on calculators, considerable literature and various information resources are available for keeping well informed. If instructors allow or require calculators, they are advised to warn their students of the effects of roundoff errors and to impress on them the need for thought when working with numerical methods. We have shown in this article that simple explanations and illustrations are accessible at the beginning calculus level. An intermediate treatment of roundoff errors can be found in some texts on numerical analysis; our favorite is Chapter 2 in Dahlquist and Björck [7]. The articles by Forsythe [**9**] and Stegun and Abramowitz [**43**] should be required reading. Advanced topics are covered by Gautschi [**13**], Rall [**34**], and Wilkinson [**48**].

Addresses and Notes

A1. The entire May 1978 issue of *Mathematics Teacher* deals with classroom use of the pocket calculator. The

magazine is published by the NCTM (see [**A8**] below), which distributes calculator and computer information for teachers.

A2. Educational Calculator Devices, Inc., P.O. Box 974, Laguna Beach, CA 92652. This outfit publishes a new journal, *Didactic Programming*, which deals with calculator-demonstrated instruction.

A3. Stokes Publishing Company, P.O. Box 415, Palo Alto, CA 94302.

A4. Two filmstrips with records or cassettes, "Introducing the Electronic Calculator," are available from: BFA Educational Media, P.O. Box 1795, Santa Monica, CA 90406.

A5. HP-65 User's Club, 2541 W. Camden Place, Santa Ana, CA 92704.

A6. SR-52 User's Club, 9459 Taylorsville Road, Dayton, OH 45424.

A7. Calculator Information Center, 1200 Chambers Road, Columbus, OH 43212.

A8. National Council of Teachers of Mathematics, 1906 Association Drive, Reston, VA 22091.

A9. The magazine *Calculators/Computers* often contains articles on instructional use of calculators. It is published by: Dymax, P.O. Box 310, Menlo Park, CA 94025.

A10. Hewlett Packard Advanced Products Division, HP-45 Applications Book, Hewlett Packard Co., Cupertino, CA, 1974. This book lists more than 200 keystroke sequences for a specific nonprogrammable calculator. With an understanding of the Reverse Polish Notation logic of HP-series calculators, most of these sequences can be readily converted for use on other scientific calculators. Applications include multiple linear regression, Gaussian quadrature, evaluation of Bessel functions, and others.

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WHEN IS $f(f(z)) = az^2 + bz + c$?

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1. Introduction. The surprising answer to the title question is: *never*. In this paper we prove this assertion and more: we prove that a quadratic polynomial defined on the entire complex

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plane has no iterative roots whatever.

To make this statement precise, we need a few definitions. Let E be a set and let f,g be functions mapping E into itself. The *composite* of f and g is the function $f\circ g$ defined by

$$(f\circ g)(x)=f(g(x)),\quad \text{for all } x \text{ in } E. \tag{1.1}$$

The *iterates* of f are the functions f^n defined recursively by:

$$f^0(x)=x,\quad \text{for all } x \text{ in } E, \tag{1.2}$$

$$f^{n+1}=f\circ f^n,\quad \text{for any nonnegative integer } n. \tag{1.3}$$

And, for any integer $r\geqslant 2$, f is an *iterative root of order r of g*, or an rth *iterative root of g*, if

$$f^r=g. \tag{1.4}$$

We can now formally state our principal result as:

THEOREM 1. *Let P be a polynomial of degree 2 defined on the entire complex plane C. Then P has no iterative roots of any order whatever; i.e., for any integer $r\geqslant 2$, there exists no function f whatever mapping C into itself such that $f^r=P$.*

It must be emphasized that the phrase "no function f whatever" is to be taken literally and does not mean merely "no entire function" or "no continuous function," etc. From this it is to be expected that none of the usual methods of analysis or topology play a role in what follows. This is the case. The proof of Theorem 1 is, in essence, purely combinatorial. The necessary tools for this proof are developed in Sections 2 and 3. Iterative square roots are discussed in Section 4, which culminates in the answer to the title question. The proof of Theorem 1 is completed in Section 5. Section 6 contains supplementary material, stated without proof; in particular, the conclusion of Theorem 1 is contrasted to the quite different situation for real quadratic polynomials.

Experience has shown that many mathematicians, analysts in particular, when confronted by the statement of Theorem 1, respond with the following instant "counterexample":

"Take $P(z)=z^2$, and let f be a branch, say, the principal one, of the $\sqrt{2}$th power, i.e., let $z\neq 0$ be expressed in the form

$$z=re^{i\theta},\quad r>0,\quad -\pi<\theta\leqslant\pi. \tag{1.5}$$

Define $f\colon C\to C$ by

$$f(0)=0,\quad f(z)=r^{\sqrt{2}}\,e^{i\theta\sqrt{2}} \text{ for } z\neq 0. \tag{1.6}$$

Then $f(f(0))=0=0^2$ and for any $z\neq 0$,

$$f(f(z))=f\big(r^{\sqrt{2}}\,e^{i\theta\sqrt{2}}\big)=\big(r^{\sqrt{2}}\big)^{\sqrt{2}}\,e^{i\theta\sqrt{2}\,\sqrt{2}}=r^2e^{2i\theta}=z^2. \tag{1.7}$$

Thus f is an iterative square root of P, contradicting the alleged Theorem 1."

The flaw in this argument lies in the fact that the second equality in (1.7) fails when $f(z)$, as defined in (1.6), is not in the same standard form as z in (1.5). Indeed, an explicit calculation, with proper attention paid to necessary details, yields:

$$f^2(z)=\begin{cases} z^2e^{i2\pi\sqrt{2}}, & -\pi<\theta\leqslant -\dfrac{\pi}{\sqrt{2}}, \\ z^2, & -\dfrac{\pi}{\sqrt{2}}<\theta\leqslant\dfrac{\pi}{\sqrt{2}} \text{ or } z=0, \\ z^2e^{-i2\pi\sqrt{2}}, & \dfrac{\pi}{\sqrt{2}}<\theta\leqslant\pi, \end{cases}$$

whence $f^2 \neq P$, since, e.g., $f^2(-1) = e^{-i2\pi\sqrt{2}} = -.858216 - i.513288 \neq 1 = P(-1)$. (The fact that f^2 and P coincide on part of their common domain is irrelevant: equality of functions means equality *everywhere*.)

(The same flaw has often affected the discussion of certain real quadratic polynomials, e.g., x^2-2. See this MONTHLY, problem E984 [1951, 564] and its treatment [1952, 252; 1976, 567; 1977, 739; 1980, 303].)

On the other hand, the validity of statements such as Theorem 1 is heavily domain-dependent. On the real line, for example, the function f_r defined by $f_r(x) = |x|^{2^{1/r}}$ for any $r \geqslant 2$ is an iterative rth root of the polynomial P given by $P(x) = x^2$.

The problem of finding iterative roots of functions dates back at least to Abel [**1**] and Babbage [**2**] (see also [**5**], especially Chapters 4 and 7). Since then it has attracted the attention of many authors. A comprehensive survey of the theory of iteration of continuous real functions, together with an extensive bibliography, is given in [**7**]. Further references may be found in the papers [**3**] and [**6**].

2. Orbits. Let E be a set, f a function from E into E, and $\sim_f$ the relation on E defined via:

$$x \sim_f y \text{ if and only if } f^m(x) = f^n(y) \tag{2.1}$$

for some nonnegative integers m, n. It is immediate that $\sim_f$ is an equivalence relation. Each equivalence class of $\sim_f$ determines a directed graph, called an *orbit of* f or *f-orbit*, which is constructed as follows: With each element x of an equivalence class, associate a point, called a *vertex*; and if $f(x) = y$, join the vertex representing x to the one representing y by an arc, called an *edge*, directed from x to y, thus:

In view of (2.1), an f-orbit is connected. It is also maximal in the sense that no further vertices or edges can be added. Since f is a function, and thus single-valued by definition, it is clear that while a finite, countable, or uncountable number of edges can enter a given vertex, exactly one edge exits. (Note that a point on the conventional graph of a real function, since it has coordinates of the form $(x, f(x))$, corresponds to an edge of an f-orbit.)

Orbits are divided into two types: cyclic (see Figs. 2 and 3) and acyclic (see Fig. 1). An f-orbit is *cyclic* if it contains a vertex x such that $f^n(x) = x$ for some $n > 0$. The vertices $x, f(x), \ldots, f^{n-1}(x), f^n(x) = x$ form a *cycle*. There is no "escape" from a cycle, whence an f-orbit cannot contain more than one cycle. The number of distinct vertices in the cycle of a cyclic orbit is the *order* of the cycle; a cycle of order n is called an *n-cycle*; and the orbit containing it, an *n-cyclic* orbit. Note that a 1-cycle corresponds to a fixed point of the function f and that the vertices of an n-cycle of f correspond to fixed points of its nth iterate f^n.

An orbit which is not cyclic is *acyclic*. An acyclic orbit must have at least countably many vertices whereas a cyclic orbit may have as few as one.

Iteration of a function generally splits its orbits. For the second iterate f^2, we have:

LEMMA 1. *For any function* f:

(a) *An acyclic f-orbit is the union of two acyclic f^2-orbits* (*see Fig. 1*).

(b) *A cyclic f-orbit of even order, say* $2m$, *is the union of two cyclic f^2-orbits of order* m (*see Fig. 2*).

(c) *A cyclic f-orbit of odd order, say* $2m+1$, *is a cyclic f^2-orbit of the same order. The graphs of these orbits are generally different* (*see Fig. 3*).

Proof. Suppose $y \sim_f x$. Then there are non-negative integers m, n such that $f^m(x) = f^n(y)$. If m and n are both even or both odd, then $y \sim_{f^2} x$; otherwise $y \sim_{f^2} f(x)$. On the other hand, it is

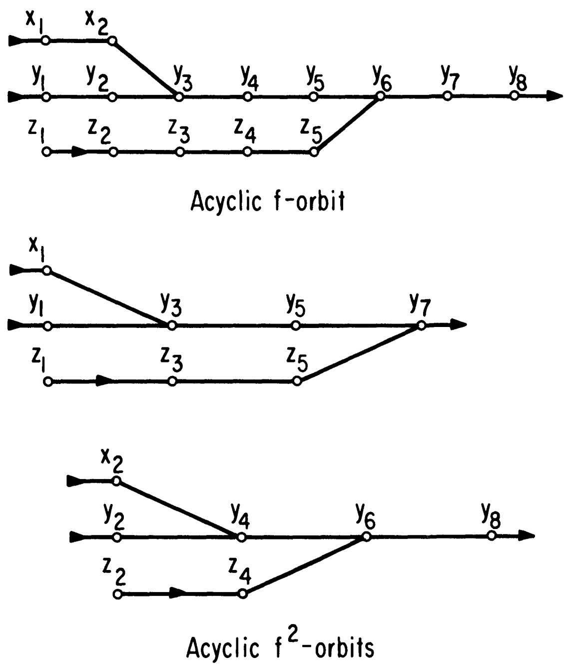
x1 x2
y1 y2 y3 y4 y5 y6 y7 y8
z1 z2 z3 z4 z5
Acyclic f-orbit
x1
y1 y3 y5 y7
z1 z3 z5
x2
y2 y4 y6 y8
z2 z4
Acyclic f2-orbits

FIG. 1

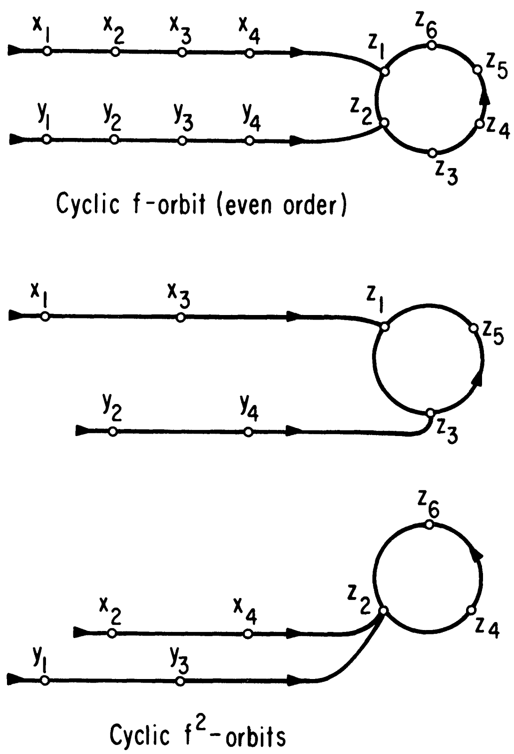
x1 x2 x3 x4
y1 y2 y3 y4
z1 z2 z3 z4 z5 z6
Cyclic f-orbit (even order)
x1 x3
y2 y4
z1 z3 z5
x2 x4
y1 y3
z2 z4 z6
Cyclic f2-orbits

FIG. 2

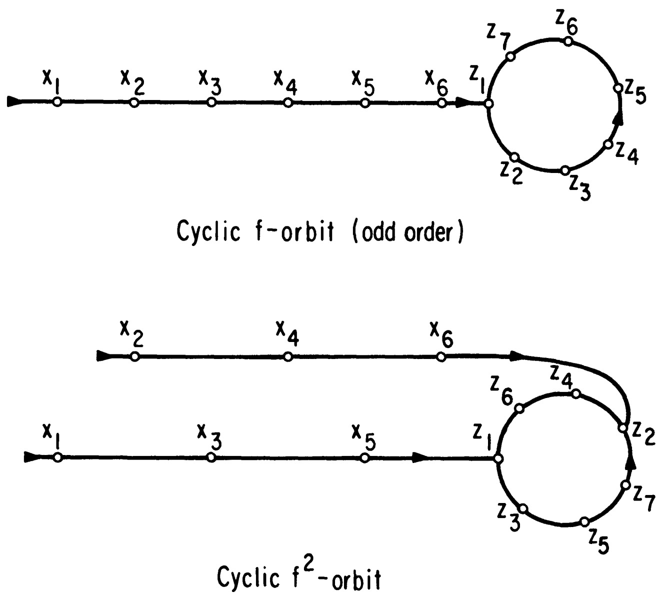

FIG. 3

immediate that either $y \sim_{f^2} x$ or $y \sim_{f^2} f(x)$ implies $y \sim_f x$. It follows that the f-orbit containing x is the (set-theoretical) union of the f^2-orbit containing x and the f^2-orbit containing $f(x)$. These two f^2-orbits coincide if and only if $x \sim_{f^2} f(x)$, i.e., if and only if there are nonnegative integers m, n such that

$$f^{2m}(x) = f^{2n+1}(x). \tag{2.2}$$

Let $p = \min(2m, 2n+1)$ and $q = |2n+1-2m|$. Note that q is odd, hence necessarily positive. Note also that $p + q = \max(2m, 2n+1)$, whence (2.2) is equivalent to

$$f^p(x) = f^{p+q}(x).$$

Hence upon setting $w = f^p(x)$, we have

$$f^q(w) = f^{p+q}(x) = f^p(x) = w,$$

which means that the f-orbit containing x is cyclic, and that its order, which must divide q, is odd. This proves (a), and, apart from the exact orders of the f^2-cycles, (b) and (c) as well.

To determine these orders, let x_0 be a vertex in the cycle of a cyclic f-orbit, and set $x_p = f^p(x_0)$ for every nonnegative integer p. If the order of the cyclic f-orbit is $2m$, then it is easily seen that $\{x_0, x_2, \ldots, x_{2m-2}\}$ is the cycle, of order m, of the f^2-orbit containing x_0, and $\{x_1, x_3, \ldots, x_{2m-1}\}$ is the cycle, of order m, of the f^2-orbit containing $f(x_0) = x_1$. If the order of the cyclic f-orbit is $2m+1$, then

$$x_{2m+2} = f^{2m+2}(x_0) = f(f^{2m+1}(x_0)) = f(x_0) = x_1,$$

whence it follows that $\{x_0, x_2, \ldots, x_{2m}, x_1, x_3, \ldots, x_{2m-1}\}$ is the cycle, of order $2m+1$, of the f^2-orbit containing x_0. This completes the proof.

Note that in cases (a) and (b) each f^2-orbit consists of "every other" vertex of the f-orbit. Note also that the property of being in or not in the cycle of a cyclic orbit is preserved under iteration.

Lemma 1 shows that a cyclic f^2-orbit of even order, say $2m$, can only arise from the splitting of a cyclic f-orbit of order $4m$. But since such an f-orbit splits into two f^2-orbits, it follows that the number, if finite, of $2m$-cyclic f^2-orbits must be even. Hence we have:

LEMMA 2. *Let g be a function. Then a necessary condition for g to have an iterative square root, i.e., for there to exist a function f such that* $f^2 = g$, *is that for any positive even integer* $2m$ *the number (if finite) of* $2m$*-cyclic g-orbits is even.*

The equivalence relation (2.1) was introduced by K. Kuratowski in a brief remark at the end of [**12**]. It was apparently G. T. Whyburn [**14**] (see also [**15**, Chapter 12, § 6]) who extended the term *orbit*, already used in related connections, to cover Kuratowski's definition. In [**6**], a basic and beautiful paper which deserves to be much better known, R. Isaacs obtained conditions on orbits that are both necessary and sufficient for the existence of iterative square roots of arbitrary functions. (In contrast, our Lemma 2 only yields a necessary condition; but this is adequate for our purpose.) Recently G. Zimmermann (née Riggert) has significantly extended Isaacs's results (see [**11**] and [**16**, § 1]).

3. Conjugacy. Let g be a function mapping a set E_1 into itself, and h a function mapping a set E_2 into itself. We say that g and h are *conjugate* if there exists a one-to-one function f mapping E_1 onto E_2 such that

$$f \circ g = h \circ f. \tag{3.1}$$

Equivalently, we could write either

$$g = f^{-1} \circ h \circ f \quad \text{or} \quad h = f \circ g \circ f^{-1}, \tag{3.2}$$

where f^{-1} is the inverse of f. Clearly, conjugacy is an equivalence relation among functions. Furthermore, we have:

THEOREM 2. *Let g be a function mapping a set* E_1 *into itself, and h a function mapping a set* E_2 *into itself. Then g and h are conjugate if and only if g and h are (orbit-)* isomorphic, *i.e., if and only if there exists a one-to-one function f mapping* E_1 *onto* E_2 *such that, for any x,y in* E_1, *and any nonnegative integers m,n we have*

$$g^m(x) = g^n(y) \quad \textit{if and only if } h^m(f(x)) = h^n(f(y)). \tag{3.3}$$

Proof. If g and h are conjugate, then an induction using (3.1) yields $f \circ g^m = h^m \circ f$ for any nonnegative integer m. Hence $f(g^m(x)) = f(g^n(y))$ if and only if $h^m(f(x)) = h^n(f(y))$. But since f is one-to-one, $f(g^m(x)) = f(g^n(y))$ if and only if $g^m(x) = g^n(y)$. Therefore g and h are (orbit-) isomorphic.

In the other direction, if g and h are isomorphic, let x be any element of E_1 and let $y = g(x)$. Then, using (3.3) with $m = 1$ and $n = 0$, we have

$$h(f(x)) = f(y) = f(g(x)),$$

which yields (3.1), since x is arbitrary. Note that Theorem 2 can be compactly summarized by the statement: "The diagram in Figure 4 commutes."

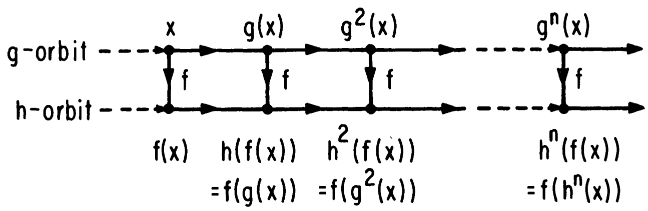

FIG. 4

In other words, two functions are conjugate if and only if their orbit structures are identical. This result goes back at least to 1960: it appears, e.g., in [**13**, Chapter 6, § 2], with orbits entering under the name "trees."

As an immediate consequence of Theorem 2 we have:

LEMMA 3. *If g and h are conjugate, and r is any integer $\geqslant 2$, then g has an iterative rth root if and only if h has an iterative rth root.*

Proof. Let g and h be conjugate via f and suppose that $\phi^r = g$. Let $\psi = f\circ\phi\circ f^{-1}$. Then $\psi^r = f\circ\phi^r\circ f^{-1} = f\circ g\circ f^{-1} = h$.

It follows from Theorem 2 and Lemma 3 that the existence or nonexistence of iterative roots of a function depends only on its orbit structure.

Two functions f and g defined on the complex plane C are *linearly conjugate* if there exists a nonconstant linear function L such that

$$L\circ f = g\circ L. \tag{3.4}$$

Since any such function L is a one-to-one mapping of C onto itself, it is clear that linear conjugacy is a special case of conjugacy.

We now turn our attention to quadratic polynomials P defined on C. Since we are interested in fixed points, i.e., roots of the equation $P(z) = z$, we use the standard form

$$P(z) = az^2 + (b+1)z + c, \tag{3.5}$$

where a, b, c are in C and $a \neq 0$. For any such polynomial P we define $\Delta(P)$, the *iterative discriminant* of P, by

$$\Delta(P) = b^2 - 4ac. \tag{3.6}$$

We then have the following basic:

LEMMA 4. *If P and Q are quadratic polynomials on C, then P and Q are linearly conjugate if and only if $\Delta(P) = \Delta(Q)$; i.e., Δ is a complete invariant for linear conjugacy of quadratic polynomials.*

Proof. Let $P(z) = az^2 + (b+1)z + c$ and $Q(z) = a'z^2 + (b'+1)z + c'$. Then P and Q are linearly conjugate if and only if there exists a linear function $L(z) = Az + B$, with $A \neq 0$, such that for all z in C,

$$\begin{aligned} L(P(z)) &= Aaz^2 + A(b+1)z + Ac + B = Q(L(z)) \\ &= A^2a'z^2 + [2ABa' + A(b'+1)]z + B^2a' + B(b'+1) + c'. \end{aligned} \tag{3.7}$$

Now (3.7) holds for all z in C if and only if

$$Aa = A^2a', \tag{3.8}$$

$$A(b+1) = A(2Ba' + b' + 1), \tag{3.9}$$

$$Ac + B = B^2a' + B(b'+1) + c'. \tag{3.10}$$

Solving (3.8) and (3.9) for A and B yields,

$$A = a/a' \quad \text{and} \quad B = (b-b')/2a', \tag{3.11}$$

whence the system (3.8), (3.9), (3.10) has a solution if and only if the substitution of A and B, as given by (3.11), into (3.10), yields an identity—that is, upon simplification, if and only if $b^2 - 4ac = b'^2 - 4a'c'$ or, equivalently, $\Delta(P) = \Delta(Q)$.

Now consider the family $\{P_\lambda\}$ of quadratic polynomials given by

$$P_\lambda(z) = z^2 + (1-\lambda)z = z(z-\lambda) + z, \tag{3.12}$$

where $\lambda = \mu + i\nu$ is such that either $\mu > 0$, or $\mu = 0$ and $\nu \geqslant 0$. We have $\Delta(P_\lambda) = \lambda^2$, whence the family $\{P_\lambda\}$ contains a single representative from each linear conjugacy class. Consequently, the problem of determining those quadratic polynomials which have iterative roots reduces at once to the problem of determining those values of λ for which P_λ has iterative roots.

4. Iterative square roots. To answer the question of the title, we begin with:

LEMMA 5. *A quadratic polynomial P has at most one 2-cyclic orbit.*

Proof. The two vertices in the 2-cycle of any 2-cyclic P-orbit are distinct solutions of the equation

$$P^2(z)=z. \tag{4.1}$$

But any fixed point of P is also a solution of (4.1), and by the Fundamental Theorem of Algebra, P has at least one fixed point. Thus if P had two or more 2-cyclic orbits then (4.1) would have 5 or more distinct solutions. But this is impossible since P^2 is a polynomial of degree 4.

If P has one 2-cyclic orbit, then Lemma 2 shows that P has no iterative square roots. Hence the only quadratic polynomials which can conceivably have iterative square roots are those which have no 2-cyclic orbits. To find such polynomials, we consider the family $\{P_\lambda\}$ defined in (3.12). A straightforward computation yields

$$P_\lambda^2(z)=z(z-\lambda)\left[z^2+(2-\lambda)z+(2-\lambda)\right]+z. \tag{4.2}$$

The roots of the equation $P_\lambda^2(z)=z$ are $0,\lambda$,

$$z_3=(\lambda-2+\sqrt{\lambda^2-4})/2 \quad \text{and} \quad z_4=(\lambda-2-\sqrt{\lambda^2-4})/2.$$

Now, 0 and λ are the fixed points of P_λ. Thus, since $P(z_3)=z_4$ and $P(z_4)=z_3$, P has no 2-cyclic orbit if and only if $z_3=z_4$; and this is the case if and only if $\lambda^2=4$. Since $\Delta(P_\lambda)=\lambda^2$, this yields:

LEMMA 6. *If P is a quadratic polynomial defined on C and if $\Delta(P)\neq 4$, then P has exactly one 2-cyclic orbit and hence no iterative square root.*

Lemma 6 shows that—up to linear conjugacy—there is exactly one complex quadratic polynomial which has no 2-cyclic orbit. This result is not new. Indeed, in **[3]** I. N. Baker has shown that, with the sole exception of P_2, which lacks only a 2-cyclic orbit, all complex polynomials have cyclic orbits of all orders. Also, it should be remarked that R. Isaacs noted in **[6]** that $P(z)=z^2$ has exactly one 2-cycle (the complex cube roots of unity) and hence no iterative square roots.

For $\Delta(P)=4$ we have $\lambda=2$, $P_2(z)=z^2-z$ and $z_3=z_4=0$. We now dispose of this one remaining case by proving:

LEMMA 7. *If P is a complex quadratic polynomial with $\Delta(P)=4$, then P has three 4-cyclic orbits. Hence by Lemma 2, P has no iterative square roots.*

Proof. If P is any complex quadratic polynomial, then P^4 is a polynomial of degree $2^4=16$. Since P has at least one fixed point, a counting argument similar to the one used in the proof of Lemma 5 shows that P has at most three 4-cyclic orbits. Turning specifically to P_2, we find after some computation that

$$P_2^4(z)=z^3(z-2)Q(z)+z,$$

where

$$Q(z)=\left[z^2(z-2)+1\right]^3(z^3+1)+1.$$

Thus the solutions of the equation $P_2^4(z)=z$, i.e., the fixed points of P_2^4, are 0, 2, and the roots of the equation $Q(z)=0$. Of these, 0 and 2 are the fixed points of P_2, and direct evaluation shows that neither is a root of $Q(z)=0$. Consequently, since P_2 has no 2-cycle, the roots of $Q(z)=0$ are precisely the vertices in the 4-cycles of P_2. It remains to show that these roots are all distinct. Suppose they are not. Then Q and its derivative Q' have a common factor. Now

$$Q'(z)=6z\left[z^2(z-2)+1\right]^2[2z^4-3z^3+2z-2].$$

It is immediate that the roots of $z^2(z-2)+1=0$ are not roots of $Q(z)=0$. Since we already know that 0 is not a root of $Q(z)=0$, it follows that any common factor of Q and Q' must be a common factor of Q and B, where

$$B(z)=2z^4-3z^3+2z-2.$$

But B is a polynomial of degree 4, and $Q(z)=0$ has either 4, 8, or 12 *distinct* roots (since these roots, being the vertices in the 4-cycles of P_2, come in bunches of 4). Thus if Q and Q' have one common factor, then Q and Q' must have four distinct common factors, whence B must divide Q. However, by direct calculation we find that

$$2^9Q(z)=B(z)D(z)+R(z), \tag{4.3}$$

where

$$D(z)=256z^8-1152z^7+1344z^6+736z^5-2096z^4$$
$$+504z^3+852z^2-498z-275,$$

and

$$R(z)=527z^3-372z^2-446z+474.$$

Thus B does not divide Q, the roots of $Q(z)=0$ are all distinct, and P_2 has three 4-cyclic orbits. Since the same conclusion holds for any P linearly conjugate to P_2, the proof of Lemma 7 is complete.

The long division indicated in (4.3) can be avoided by noting that Q and B are both primitive polynomials. If B divides Q, then $Q(z)=B(z)A(z)$, where A has rational coefficients. By a variant of Gauss's Lemma (see e.g., [**4**, pp. 168–169]), it follows that A has integer coefficients. Consequently, the leading coefficient of B, namely, 2, divides the leading coefficient of Q, namely, 1, which is false.

Combining Lemmas 6 and 7, we obtain:

THEOREM 3. *Let P be a quadratic polynomial defined on the complex plane* **C**. *Then P has no iterative roots of order* 2, *i.e., there exists no function f whatever such that*

$$f(f(z))=P(z) \quad \text{for all } z \text{ in } \mathbf{C}. \tag{4.4}$$

5. Proof of Theorem 1. For any vertex x of a cyclic f-orbit there is a smallest nonnegative integer m such that $f^m(x)$ is in the cycle of the orbit; this integer is the *f-height of x*, written $\mathrm{ht}(f;x)$. An n-cyclic f-orbit contains exactly n vertices of f-height 0 (the vertices in the n-cycle) but for an arbitrary function f the only restriction on the number of vertices of any given positive f-height is the obvious one: if there is a vertex of f-height $m\geqslant 2$, then for each positive integer $k\leqslant m$ there must be at least one vertex of f-height k.

The next lemma makes precise the fact that since f^r strides toward the cycle in r-league boots it takes roughly $1/r$ as many steps as f to get there.

LEMMA 8. *Let x be a vertex in a cyclic f-orbit and r an integer $\geqslant 2$. Then*

$$\mathrm{ht}(f^r;x)=\lceil(\mathrm{ht}(f;x)/r)\rceil \tag{5.1}$$

where, for any real number a, $\lceil a\rceil$ denotes the least integer $\geqslant a$.

Proof. If $\mathrm{ht}(f;x)=0$, then (5.1) is trivial. Otherwise, let $\mathrm{ht}(f;x)=p\geqslant 1$. Then $p/r\leqslant\lceil p/r\rceil<(p/r)+1$, whence $r\lceil p/r\rceil\geqslant p$. Thus $f^{r\lceil p/r\rceil}(x)$ is in the cycle of the f-orbit containing x, and so in the cycle of the f^r-orbit containing x. Consequently $\mathrm{ht}(f^r;x)\leqslant\lceil p/r\rceil$. In the other direction, if q is a nonnegative integer less than $\lceil p/r\rceil$, then

$$rq\leqslant r\lceil p/r\rceil-r<p.$$

Since p is the *least* integer such that $f^p(x)$ is in the cycle of the f-orbit containing x, it follows that $f^{rq}(x)$ is *not* in this cycle and hence not in the cycle of the f^r-orbit containing x.

LEMMA 9. *Let $r \geqslant 2$ and let x be a vertex of a 1-cyclic f^r-orbit. Then x is in a d-cyclic f-orbit where d is a divisor of r; and if $y \sim_f x$, then y is in a 1-cyclic f^r-orbit.*

Proof. Let z be the vertex in the 1-cycle of the f^r-orbit containing x. Then $f^r(z)=z$. Now let d be the least positive integer such that $f^d(z)=z$. Then $d \leqslant r$, and there are integers p, q with $p \geqslant 1$ and $0 \leqslant q \leqslant d-1$ such that $r=pd+q$. Consequently,

$$z=f^r(z)=f^{q+pd}(z)=f^q(f^{pd}(z))=f^q(z),$$

whence $q=0$ and d divides r. Next, if $y \sim_f x$, then there is a positive integer m such that $f^{mr}(y)=w$ is in the d-cycle of the f-orbit containing x and y. Thus $y \sim_{f^r} w$. Finally, w is a fixed point of f^r since $f^r(w)=f^{pd}(w)=w$.

LEMMA 10. *If for some integer $r \geqslant 2$, there is a 1-cyclic f^r-orbit containing a vertex x of f^r-height 2, then the number of vertices of f^r-height 1 in all the 1-cyclic f^r-orbits is at least r.*

Proof. From (5.1) we obtain $r<\mathrm{ht}(f;x) \leqslant 2r$, whence $\mathrm{ht}(f;x) \geqslant r+1$. Thus with $q=\mathrm{ht}(f;x)-r-1$ the r vertices $f^{q+1}(x), f^{q+2}(x), \ldots, f^{q+r}(x)$ are all distinct, have respective f-heights $r, r-1, \ldots, 1$, and consequently all have f^r-height 1. These vertices need not be in the same f^r-orbit; but by Lemma 9, the f^r-orbits containing them are all 1-cyclic.

A restatement of Lemma 10 yields:

LEMMA 11. *Let g be a function and r an integer $\geqslant 2$. Suppose g has a 1-cyclic orbit containing a vertex of g-height 2. Then a necessary condition for g to have an iterative rth root is that the number of vertices of g-height 1 in all the 1-cyclic g-orbits be at least r.*

We now apply these results to polynomials, beginning with:

THEOREM 4. *Let P be a polynomial of degree $d \geqslant 2$ defined on the complex plane C, and let r be an integer $\geqslant 2$. If P has an iterative rth root, then $r \leqslant d(d-1)$.*

Proof. We shall show that: (a) P has at least 1 and not more than d 1-cyclic orbits; (b) at least one of the 1-cyclic P-orbits contains a vertex of P-height 2; (c) the total number of vertices of P-height 1 in all the 1-cyclic P-orbits is $\leqslant d(d-1)$. The conclusion of the theorem then follows immediately from Lemma 11.

To prove (a), we need only observe that the number of 1-cyclic P-orbits is the same as the number of distinct solutions of the equation

$$P(z)=z. \tag{5.2}$$

By the Fundamental Theorem of Algebra, (5.2) has at least 1, and not more than d, distinct solutions. (Both extremes are attained: any polynomial linearly conjugate to z^d+z has precisely one 1-cyclic orbit, while any polynomial linearly conjugate to z^d has d 1-cyclic orbits.)

Turning to (b), we first note that if a 1-cyclic P-orbit contains a vertex z_0 of P-height 1, then it contains a vertex of P-height 2. For the equation $P(z)=z_0$ has at least one solution (Fundamental Theorem of Algebra again!) which cannot be equal to z_0, since z_0 is not a fixed point of P. Hence any such solution has P-height 2. Thus it remains to show that there is at least one 1-cyclic P-orbit which contains a vertex of P-height 1. Suppose the contrary, i.e., that there are no such vertices.

Let z_1 be a fixed point of P. Then z_1 is the only solution of the equation $P(z)=z_1$. And this is the case if and only if

$$P(z)=a(z-z_1)^d+z_1, \tag{5.3}$$

for some $a \neq 0$. But then

$$P(z)-z=(z-z_1)Q(z), \tag{5.4}$$

where

$$Q(z)=a(z-z_1)^{d-1}-1. \tag{5.5}$$

Since $d \geqslant 2$, the degree of Q is greater than 0. Let z_2 be a solution of the equation $Q(z)=0$. By (5.4), $P(z_2)=z_2$, whence z_2 belongs to a 1-cycle of P; and by (5.5), z_2 is distinct from z_1. Thus using the same argument as for z_1,

$$P(z)=a'(z-z_2)^d+z_2, \tag{5.6}$$

for some $a' \neq 0$. Equating coefficients of z^d in (5.3) and (5.6) yields $a=a'$; and then equating coefficients of z^{d-1} yields $z_1=z_2$, which is a contradiction. This proves (b).

As for (c), let z_0 be a fixed point of P. Then the 1-cyclic orbit containing z_0 cannot contain more than $d-1$ vertices of P-height 1, since every such vertex, *as well as* z_0 *itself*, is a solution of the equation $P(z)=z_0$, and this equation cannot have more than d solutions. Since (as we have already seen) there are no more than d 1-cyclic P-orbits, the total number of vertices of P-height 1 in all the 1-cyclic P-orbits cannot exceed $d(d-1)$, which proves (c) and completes the proof of the theorem.

When $d=2$, the upper bound in Theorem 4 is 2. Thus an immediate consequence is:

THEOREM 5. *If P is a quadratic polynomial defined on the complex plane C, then P has no iterative roots of order $\geqslant 3$.*

Combining Theorems 3 and 5 yields Theorem 1.

6. Epilogue. As pointed out in the introduction, the validity of results like Theorem 1 is a domain-dependent phenomenon. In the proof of Theorem 1 this domain dependence enters through unrestricted appeals to the Fundamental Theorem of Algebra (which implies, incidentally, that Theorem 1 holds in any algebraically closed field of characteristic 0). This domain dependence is further brought out by the following results, which show that the iterative root situation for real quadratic polynomials contrasts sharply with that for complex quadratic polynomials.

First a definition: Let E be a nonempty set. A *1-sided flow on E* is a family $\{f_t | t \geqslant 0\}$ of functions each mapping E into E such that

$$f_s(f_t(x))=f_{s+t}(x) \quad \text{for all } x \text{ in } E \text{ and all real } s,t \geqslant 0. \tag{6.1}$$

A *2-sided flow on E* is a similar family of functions in which the index t ranges over all real numbers.

Clearly, any function which is embeddable in a flow of either type has iterative roots of all orders.

THEOREM 6. *Let R be the real line. Let g be a real quadratic polynomial, so that*

$$g(x)=ax^2+(b+1)x+c,$$

for all real x, where $a \neq 0$, b,c are in R. As in the complex case, set $\Delta(g)=b^2-4ac$. If $\Delta(g)>1$, then g has no iterative roots of any order whatever. If $\Delta(g)=1$, then g can be embedded in a 2-sided flow on R, all of whose members are continuous functions. If $\Delta(g)<1$, then g can be embedded in a 1-sided flow on R, all of whose members are continuous functions; but g cannot be embedded in any 2-sided flow on R.

Lemmas 1 and 9 can be extended to:

LEMMA 12. *Let f be a function and r be an integer $\geqslant 2$. Then any acyclic f-orbit is, as a set, the union of r acyclic f^r-orbits. Correspondingly, any m-cyclic f-orbit is the union of d (m/d)-cyclic f^r-orbits, where d is the greatest common divisor of m and r.*

The notion of an iterative discriminant can be extended from quadratic polynomials to all

polynomials defined on the complex plane as follows:

For any polynomial P of degree $d \geqslant 2$, let $z_1, \ldots, z_d$ be the d fixed points of P, i.e., the d (not necessarily distinct) roots of the equation $P(z) = z$. Set

$$\Delta(P) = a^d \prod_{m<n} (z_m - z_n)^2, \tag{6.2}$$

where $a \neq 0$ is the leading coefficient of P. A routine calculation shows that (6.2) reduces to (3.5) when $d = 2$. We then have the following analog of Lemma 4:

LEMMA 13. *Let P and Q be polynomials on C, each of degree $\geqslant 2$. If P and Q are linearly conjugate, then P and Q have the same degree and $\Delta(P) = \Delta(Q)$.*

Theorem 4 was announced in **[8]** and proved in **[9]**. In some cases the upper bound $d(d-1)$ for the order of an iterative root of a dth degree polynomial in Theorem 4 can be considerably lowered. For example, G. Zimmermann has shown (cf. **[16**, § 3**]**) that $d[d/2]$ is an upper bound for polynomials linearly conjugate to the Čebyšev polynomial of degree d, where $[d/2]$ is the greatest integer $\leqslant d/2$. Also, Theorems 1 and 3 extend to certain nonquadratic Čebyšev polynomials: see **[10]** and **[16**, § 3].

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MISCELLANEA

33. In the beginning, let me as distinctly as possible announce—not the theorem which I hope to demonstrate—for, whatever the mathematicians may assert, there is, in this world at least, *no such thing* as demonstration—but the ruling idea . . . which I shall be continually endeavoring to suggest.

—Edgar Allan Poe, *Eureka*, p. 1 (vol. 16, p. 185 of the Harrison edition of Poe's Works, New York, 1902).

EULER'S VERSION OF THE LAPLACE TRANSFORM

MICHAEL A. B. DEAKIN

1. Historical Introduction. The modern version of the Laplace Transform, now a standard part of most undergraduate mathematics and engineering courses, is, in its detailed working out and its widespread acceptance, a quite recent development. It may most conveniently be dated from the publication of Doetsch's *Theorie und Anwendung der Laplace-Transformation* [3] in 1937. The subject has, however, a much longer history than this. Its early beginnings have been traced back to Euler, by, *inter alia*, Laplace himself [6, p. 88].

It is not the purpose of this paper to trace this history (a detailed account is in preparation for publication elsewhere), but rather to show by specific examples how the earlier versions of the theory can be made to work in practice—made to work, in fact, in cases where the standard modern theory breaks down.

2. Eulerian Theory. Euler in several papers considers particular integrals of the form

$$\int_0^\infty e^{-st}F(t)\,dt,$$

and some of this work anticipates the modern theory of the Laplace Transform. However, this paper relies on a different investigation to be found at Section 1053 in volume 2 of his *Institutiones Calculi Integralis* [4, pp. 242–3].

Here Euler considers the transformation

$$y(u)=\int_b^a e^{K(u)Q(x)}P(x)\,dx. \tag{1}$$

(The notation differs very slightly from Euler's; in what follows, Euler's notation is used with minor adjustments only.)

Euler now considers the expression

$$L(u)\frac{d^2y}{du^2}+M(u)\frac{dy}{du}+N(u)y, \tag{2}$$

and finds this to be equal to

$$\int_b^a e^{KQ}P\left\{N+(MK'+LK'')Q+LK'^2Q^2\right\}dx. \tag{3}$$

He seeks to evaluate this as

$$\int_{x=b}^{x=a} d\left(e^{K(u)Q(x)}R(x)\right), \tag{4}$$

where

$$R(b)e^{K(u)Q(b)}=0. \tag{5}$$

To this end, he seeks an equality between the *indefinite* integrals corresponding to the definite integrals (3), (4). This yields

$$dR+KR\,dQ=P\left\{N+(MK'+LK'')Q+LK'^2Q^2\right\}dx. \tag{6}$$

Euler now assumes that L,M,N are such that

$$\begin{aligned} LK'^2&=A+\alpha K\\ MK'+LK''&=B+\beta K\\ N&=C+\gamma K, \end{aligned} \tag{7}$$

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where A, B, C; α, β, γ are constants. This substitution allows him to replace equation (6) by the two equations:

$$dR = P(C + BQ + AQ^2)\,dx \tag{8}$$

$$R\,dQ = P(\gamma + \beta Q + \alpha Q^2)\,dx. \tag{9}$$

From these it follows that

$$\frac{dR}{R} = \frac{(C + BQ + AQ^2)\,dQ}{\gamma + \beta Q + \alpha Q^2}. \tag{10}$$

Equation (10) can be integrated, so that R is now available to us, and, from equation (9),

$$P\,dx = \frac{R\,dQ}{\gamma + \beta Q + \alpha Q^2}, \tag{11}$$

and

$$y(u) = \int_{x=b}^{x=a} e^{KQ} \frac{R\,dQ}{\gamma + \beta Q + \alpha Q^2}. \tag{12}$$

But now the expression (2) has led to this $y(u)$ on the assumption that expressions (2), (3), (4) were identical. The integral (4), subject to condition (5), may be evaluated as

$$U(u) = R(a)e^{K(u)Q(a)}, \tag{13}$$

so that equation (12) gives a solution to the equation

$$L\frac{d^2y}{du^2} + M\frac{dy}{du} + Ny = U. \tag{14}$$

(This analysis, which appears to a modern reader to be presented backwards, exhibits a methodology which Euler used quite often.)

3. An Example. In order to elucidate Euler's method further, we consider a specific example:

$$\frac{d^2y}{du^2} + \left(\frac{3}{u} + 2u\right)\frac{dy}{du} + 4y = e^{u^2}. \tag{15}$$

This has the general solution

$$y = \tfrac{1}{8}u^{-2}e^{u^2} + c_1 u^{-2} e^{-u^2} + c_2 u^{-2}, \tag{16}$$

where c_1, c_2 are arbitrary constants. We may remark in passing that equation (15) is not amenable to a modern Laplace Transform solution, as e^{u^2} is not exponentially bounded.

We have $L = 1$, $M = 3/u + 2u$, $N = 4$ and $U = e^{u^2}$. The last of these gives us, without loss of generality,

$$K(u) = u^2. \tag{17}$$

Once this equation is given, we must set

$$Q(a) = R(a) = 1. \tag{18}$$

We now have

$$\begin{gathered} L(K')^2 = 4u^2 \\ MK' + LK'' = 8 + 4u^2 \\ N = 4, \end{gathered} \tag{19}$$

so that

$$A = 0, \qquad B = 8, \qquad C = 4 \tag{20}$$

and

$$\alpha=4, \qquad \beta=4, \qquad \gamma=0. \tag{21}$$

It now follows from equation (10) that

$$\frac{dR}{R}=\frac{(4+8Q)\,dQ}{4Q+4Q^2}, \tag{22}$$

giving

$$R=k(Q+Q^2), \tag{23}$$

where k is a constant, whose value may be determined, from equation (18), as $\frac{1}{2}$.

We now have

$$y=\int_{x=b}^{x=a} e^{u^2Q}\frac{k}{4}\,dQ, \tag{24}$$

from equation (12). It now follows by direct integration that

$$y=\tfrac{1}{8}u^{-2}\{e^{u^2Q(a)}-e^{u^2Q(b)}\}. \tag{25}$$

Combining equations (5), (23), we find

$$Q(b)=0 \quad \text{or} \quad -1, \tag{26}$$

and we also know that $Q(a)=1$.

If $Q(b)=0$,

$$y=\tfrac{1}{8}u^{-2}(e^{u^2}-1), \tag{27}$$

while if $Q(b)=-1$,

$$y=\tfrac{1}{8}u^{-2}(e^{u^2}-e^{-u^2}). \tag{28}$$

Equations (27), (28) give two independent particular solutions from which the general solution (16) may be found.

4. Another Example. The same technique may be applied to the equation

$$u\frac{d^2y}{du^2}-u\frac{dy}{du}+y=0, \tag{29}$$

whose general solution is

$$y=c_1u+c_2(e^u-u\,\mathrm{Ei}(u)), \tag{30}$$

where $\mathrm{Ei}(u)$ is the exponential integral, defined [**1**, p. 228] by

$$\mathrm{Ei}(u)=\int_{-\infty}^{u}\frac{e^t}{t}\,dt=-\int_{-u}^{\infty}\frac{e^{-t}}{t}\,dt, \tag{31}$$

and c_1, c_2 are arbitrary constants.

We may note in passing that the modern Laplace Transform method applied to equation (29) does yield the general solution (30), but not easily. If s is the transform variable, we find

$$\mathfrak{L}\{y(u)\}=\frac{c_1}{s^2}-y(0)s^{-2}[s+\ln(s-1)], \tag{32}$$

and the inversion is onerous, although the solution $y=u$ does come out quite readily.

In applying Euler's method to equation (29), we may choose $K(u)$ entirely for our own convenience. For present purposes, take $K(u)=u$. Using equations (7), we find, $A=B=0$, $C=1$; $\alpha=1$, $\beta=-1$, $\gamma=0$. Then

$$\frac{dR}{R}=\frac{dQ}{Q(Q-1)}, \tag{33}$$

and

$$R=k\left(\frac{Q-1}{Q}\right). \tag{34}$$

By equations (5), (34), we have either $Q(b)=1$ or

$$\begin{aligned} Q(b)&=-\infty \quad \text{for } u>0 \\ Q(b)&=\infty \quad \text{for } u<0. \end{aligned} \tag{35}$$

We also require

$$R(a)e^{uQ(a)}=0, \tag{36}$$

so that if $Q(b)=1$, we cannot also take $Q(a)=1$, but must set $Q(a)=-\infty$ for $u>0$, and $Q(a)=\infty$ for $u<0$, while, with the other choice, we find $Q(a)=1$. The two choices give identical results.

We shall suppose first that $u>0$ and set $Q(b)=-\infty$, $Q(a)=1$. Then

$$y=\int_{-\infty}^{1} e^{uQ}k\frac{dQ}{Q^2}; \tag{37}$$

i.e.,

$$y=-k(e^u-u\mathrm{Ei}(u)). \tag{38}$$

If $u<0$, the same form results.

Equation (38) is a solution of equation (29) but not the complete solution.

5. A Different Version. This last example shows the close relationship between Euler's method and a version of the Laplace Transform in vogue last century. This is to be found in such standard texts as Boole's *Differential Equations* [**2**, pp. 461–463], or even as late a work as Ince's *Ordinary Differential Equations* [**5**, pp. 186–189]. We here follow Boole, but use a notation chosen for its correspondence with Euler's.

The equation to be considered is

$$u\phi(D)y+\psi(D)y=0, \tag{39}$$

where $D=d/du$. Set

$$y=\int_b^a e^{ux}P(x)\,dx, \tag{40}$$

where b,a are to be determined. Substitution of equation (40) into equation (39) yields, after an integration by parts,

$$[e^{ux}\phi(x)P]_b^a-\int_b^a e^{ux}\left\{\frac{d}{dx}[\phi(x)P]-\psi(x)P\right\}dx=0. \tag{41}$$

We now require

$$\frac{d}{dx}[\phi(x)P]=\psi(x)P \tag{42}$$

and

$$[e^{ux}\phi(x)P]_b^a=0. \tag{43}$$

Equation (42) is in fact the Laplace Transform of equation (39) if the initial conditions are those of equilibrium. Equation (43) is used to determine b,a. More general analyses replace equation (40) by a contour integral. For details, see (e.g.) Ince's text [**5**, pp. 438–441].

This method is clearly related to Euler's approach. The next section applies it to equation (29).

6. The Second Example Again. Equation (29) is characterized by $\phi(D)=D(D-1)$, $\psi(D)=1$.

We now have, from equation (42),

$$P = kx^{-2}, \tag{44}$$

and, from equation (43),

$$\left[e^{ux}(1-x^{-1})\right]_b^a = 0. \tag{45}$$

Now if $u>0$, choose $b=-\infty$, $a=1$; if $u<0$, choose $b=1$, $a=\infty$. Then, in the former case,

$$y = k\int_{-\infty}^{1} \frac{e^{ux}}{x^2}\,dx$$

$$= ku\int_{-\infty}^{u} \frac{e^{t}}{t^2}\,dt,$$

where we have set $t=ux$. The integral may now be expanded formally to give

$$y = -k(e^u - u\mathrm{Ei}(u)), \tag{46}$$

and a similar form follows in the case $u<0$.

The second solution, $y=c_1 u$, may in fact be found by use of contour integrals as described by Ince [5, pp. 438–441].

7. Relationship to Euler's Method. To see the connection between this approach and Euler's, set, in equation (1), $K(u)=u$, $Q(x)=x$. If equation (29) is to be of second order, then both $\phi(D), \psi(D)$ must be quadratic in D. This yields

$$\phi(x) = \alpha x^2 + \beta x + \gamma \tag{47}$$

$$\psi(x) = Ax^2 + Bx + C. \tag{48}$$

Now, from equations (8), (9)

$$dR = P\psi\,dx, \tag{49}$$

$$R\,dx = P\phi\,dx. \tag{50}$$

From equation (50), $P=R/\phi$, and by logarithmic differentiation

$$\frac{dP}{P} = \frac{dR}{R} - \frac{d\phi}{\phi}. \tag{51}$$

Substitute now from equations (49) and (50) into equation (51) to find

$$\frac{dP}{P} = \left(\frac{\psi - \phi'}{\phi}\right)dx. \tag{52}$$

which is equation (42).

As for the limits, it is clear from equation (50) that Euler's method gives equation (43) in the case $U(u)=0$, under consideration.

8. Acknowledgments. The study of the history and pedagogy of the Laplace Transform, of which this paper forms a part, was first suggested to me, or rather urged upon me, by Mr. D. G. Crawforth, of the University of Exeter. Dr. I. Grattan-Guinness, of Enfield Polytechnic, has also been most helpful in guiding me through a thicket of reference material. Mr. T. Rapke and Dr. S. Bastomsky, of Monash University, helped me in my translation of Euler from the Latin, and Dr. J. Mackenzie, of the CSIRO, corrected an earlier version of Section 6. To all of these helpers, my sincere thanks.

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TWISTED PRISMATIC KLEIN BOTTLES

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1. Introduction. C. J. Matthews, in an article in *Mathematics Teacher* [**1**], and Martin Gardner, in his "Mathematical Games" column [**2**], discuss the properties of twisted prismatic rings. Both start from the well-known Moebius band and consider it to be of a finite cross-section. It is then a special case of a four-sided prism which had been twisted through one half-turn (180°) before the ends were joined together. The discussion is further generalized to include other prismatic bodies, particularly with cross-sections in the shape of regular polygons. If the cross-section is an n-sided polygon, and if the ring is formed without the application of any twist to the prism, then the resulting body obviously has n distinct faces. On the other hand, if the prism was given a twist before the ends were joined together, two separate faces may merge into one another, leading to a body with fewer than n distinct faces. Specifically, if the twist was $360°/n$, a body with a single face will result. A point moving along one of those faces will arrive on an adjacent face after one pass around the loop. In the general case of an n-sided prism which was twisted through t faces, the resulting number of faces is equal to the greatest common divisor of n and t.

Martin Gardner's column [**2**] ends with a question raised by Howard P. Lyons, who wonders what bizarre properties a prismatic ring would have that also possesses an inside bore of prismatic cross-section but which has opposite twists on the inner and outer surfaces. This article will start off with this intriguing question and lead to the exploration of some properties of twisted prismatic Klein bottles.

2. Rings of Prismatic Tubes. If the ring described by H. P. Lyons is opened, one obtains a piece of a prismatic "tubing." If inside prism and outside prism have different twists, i.e., if the tube has not been twisted as a whole, then the cross-section must vary along this piece of tubing from one end to the other. At the two ends the cross-sections must be the same, so that the ends

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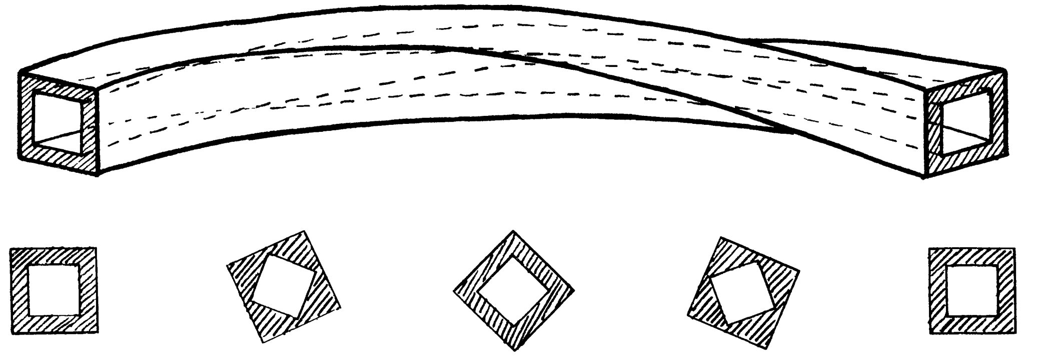
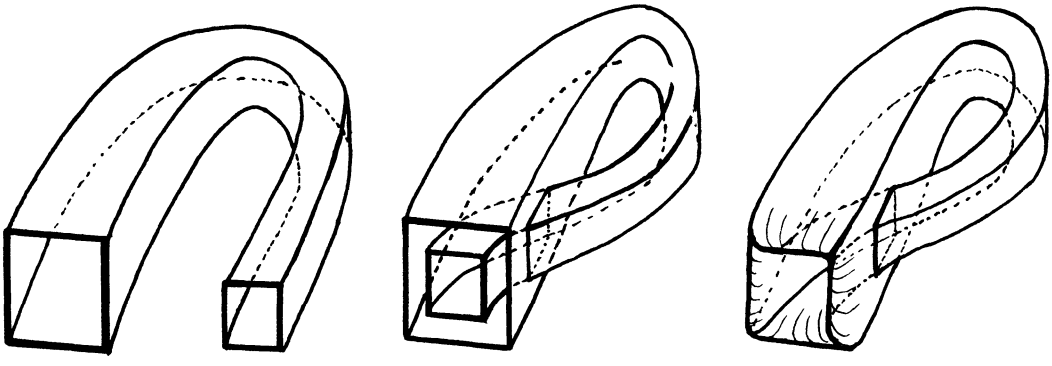
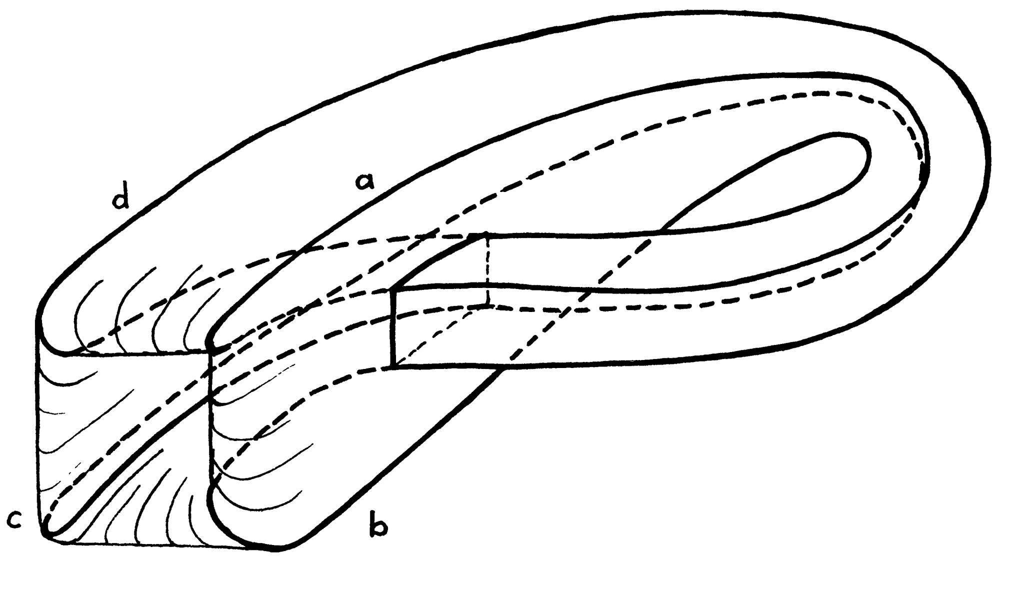


FIG. 1



(a) (b) (c)

FIG. 2



can be joined together without discontinuity. As an example we will consider the special case of a tube in which inside and outside cross-sections are both squares, and where both surfaces are given a twist of 90°, but in opposite directions. Fig. 1 shows some sample cross-sections along the piece of tube forming the described body. Evidently it is assumed that the walls of the tube are thick enough so that the inside square can assume all rotational positions without intersecting the outer surface. The type of analysis presented for rings formed from solid prisms **[1]**, **[2]** can readily be applied to either the inside or the outside of this hollow prismatic ring. Since inside and outside surfaces are completely separate, the analysis can be applied independently, and no new complications arise. However, the issue becomes much more exciting if inside and outside surfaces merge into one another as they do in the single-sided Klein bottle! On such a surface a point moving around the loop will sometimes be on an "inside" surface and sometimes on an "outside," and the twists of the two surfaces may add or subtract and thus make the situation considerably more complicated.

3. A Special Prismatic Klein Bottle. First we look at the construction of a special hollow prismatic body with common inside and outside surfaces. For this purpose we start by narrowing one end of a piece of prismatic tubing, which for simplicity is shown in Fig. 2 (a) with walls of "zero" thickness. The thin end of the tube is then inserted through the side wall near the thick end of the tube and subsequently extracted through the opening at the thick end (Fig. 2 (b)). The thin end is then turned inside out and joined to the surrounding wall of the thick end of the tube, forming what will be referred to as the "mouth" of the bottle (Fig. 2 (c)).

To convince ourselves that this forms a body with a single surface, we trace a path starting somewhere near the mouth on the outermost surface of the "top" face. We continue the path around the loop, or the "handle" of the bottle, toward the narrow end. At some point we have to pass through the side wall of the thick end of the tube, but we keep on going, always staying on the same surface of the original piece of tubing until we reach the mouth of the bottle. At this point, the inside-out roll of the tube will lead us onto the *inside* of the thick part of the tube, and we may end up directly opposite our original starting point.

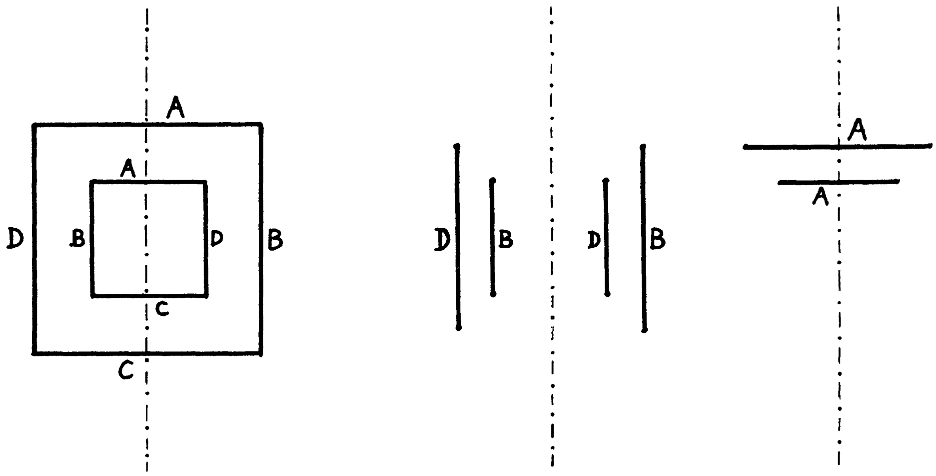
If inside and outside surfaces of the original tube have a regular prismatic cross-section, we can give either surface a twist before we join the ends. Now, as we move around the loop, we not only switch between inside and outside surfaces but also from one face segment to another. If the thickness of the walls is sufficient, so that inside and outside surfaces can have twists of different amounts, then a large and interesting set of possible interconnections exists.

To obtain a better understanding of the issues involved, we will first take a preliminary look at the special case where both surfaces have a square cross-section. If the ends of a quadratic tube are joined as described in Fig. 2, without any twists, one may expect to obtain a body with four distinct faces in the form of Moebius bands. But there is some subtlety involved in the way the ends are joined to form a Klein bottle: Not every side of the prism gets connected to itself! Only top and bottom faces merge with themselves at the mouth of the bottle, but the two side walls merge into one another. Thus, we actually obtain a body formed from two Moebius bands and one double-length, double-sided band.

For the next case, let's take the thin end of the tube and give it a quarter-twist before joining it with the thick end (Fig. 3). Again, our first intuitive guess about the result is wrong, if we assume that this twisted body is constructed from only one single-sided band, by analogy with what happens in solid prismatic rings with a minimum nonzero twist. Actually we obtain four separate faces on two double-length, double-sided bands. This surprising outcome has the following reason: The twist that we experience as we move around the loop is in a different sense when we move on an "inside" surface segment from when we move on the "outside" surface. For example, if we start on the outside of the "top" face, we will end up on the inside of a side wall after one lap around the loop, but the next lap on the inside of this side wall will bring us back to the outside of the "top" wall. If we look at the edges between the faces of this body, we get another surprise: There is a total of only three edges, but one is twice the length $(b+d)$ of the other two (a,c).

4. Skinny Bottles. We now proceed with a study of prismatic Klein bottles in a systematic manner. First we will concentrate on "skinny" bottles with walls of zero thickness. Therefore any twist applied to the outer prism must affect equally the inner prism, and any twist can thus be considered to be applied to the tube as a whole. We will explore the different possibilities for closing tubes with cross-sections of triangles, squares, or higher-order polygons into twisted and untwisted Klein bottles. How many faces and how many edges does each of these bottles have, and are the bodies formed by single-sided or double-sided bands? Many of the intriguing properties can be better understood if we first analyze the necessary boundary conditions that enable us to close a piece of prismatic tubing into a Klein bottle.

Before we join thin and thick ends at the mouth of the bottle (Fig. 2 (b)), there are three necessary requirements to make such a juncture possible. First, the corresponding sides have to line up with one another, which means that the cross-sectional polygons of the large end and of the thin end of the tube must be lined up with one another in a concentric manner. Second, since at this stage of construction of the Klein bottle the piece of tubing has been bent through 180° to form the loop, the sequence of corresponding sides on the two end polygons at the mouth defines figures of opposite orientation. Third, if the piece of tubing is free of any twist, the two cross-sections must be mirror images of one another (except for a scale factor), with the symmetry plane perpendicular to the plane of the loop. This is illustrated schematically in Fig. 4 (a), where corresponding sides have been labeled with the same letters.



(a) (b) (c)

FIG. 4

Thus it becomes clear why a bottle with no twist has exactly n faces, where n is the number of sides in the cross-sectional polygon. Moving once around the loop will transport a point from a particular side of a polygon to the symmetrical side on the other polygon. Symmetrical pairs of faces (Fig. 4 (b)) will thus connect to one another and together form a single double-sided band. Only a side which itself is centered on the symmetry plane (Fig. 4 (c)) merges with itself and thus forms a single-sided Moebius band. Similar rules are valid for the edges between the faces. Corner points that lie on the symmetry plane correspond to short loops, and pairs of symmetrical points correspond to double-length edges.

The behavior discussed above will now be illustrated with a specific example. Fig. 5 (a) shows again the cross-sections of the two ends of the tube near the mouth. Corresponding faces have been labeled with the same letter. According to the analysis in the previous paragraph, we know that faces B and D will form together a double-sided band and faces A and C individually form single-sided Moebius bands. On the other hand, looking at Fig. 5 (b) and using the same technique of analysis, we must conclude that faces A and D as well as faces B and C in pairs form two double-sided bands. Yet both figures involve square prisms and the Klein bottles supposedly are formed with no twist. Intuitively, the bottle represented in Fig. 5 (a) seems less twisted than that in Fig. 5 (b), since at least two faces merge with themselves. Thus we have to take a closer look at the meaning of "twist" in a Klein bottle. It will actually turn out that the two situations shown in Figures 5 (a) and 5 (b) can be generated from one another by introducing a twist, at the time when the Klein bottle is formed.

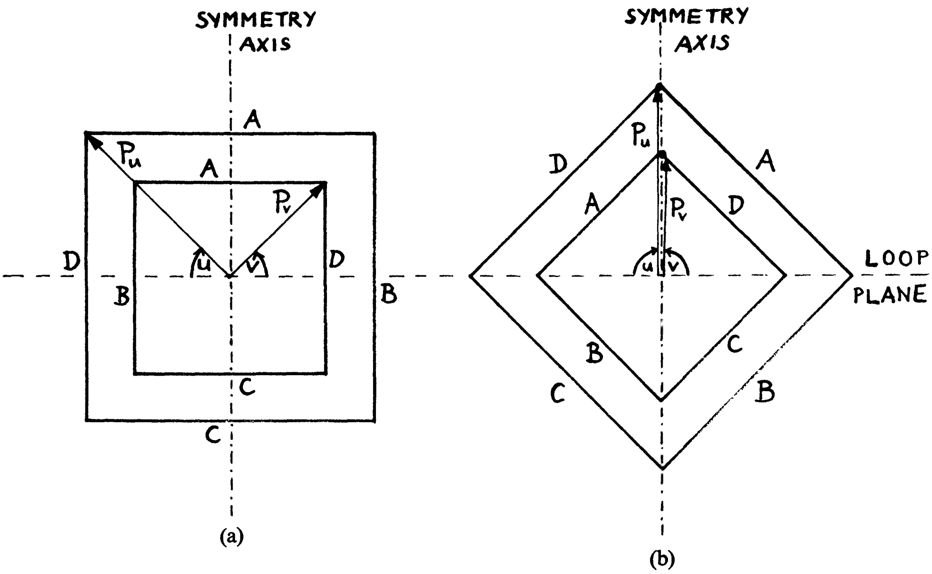


FIG. 5

5. Definition of Twist. For subsequent discussions, a more rigorous definition of "twist" is required. For tubes that can bend freely through three-dimensional space, the definition of twist is not a simple matter. Fortunately for the cases of the rings and Klein bottles considered in this paper, the center line of the bent tube lies in a plane which can provide a suitable frame of reference against which twist can easily be measured. We will call this plane the "loop plane." All cross-sections of the tube lie in planes that are perpendicular to this loop plane. Within any such cross-sectional plane, a vector from the center of the tube to one of the corners of the cross-sectional polygon can be introduced as a reference pointer (P_u, P_v in Fig. 5 (a)). Any change of the angle between this pointer and the loop plane is an indication of some twist in the tube. The total amount of twist can be determined by observing this pointer as the cross-sectional plane is moved along the tube from one end to the other or, in the case of a Klein bottle, by comparing the angles at the large-end and the small-end polygons near the mouth of the bottle. In order for the tube to be free of twist, the two angles (u, v) must be the same, as is the case in Figures 5 (a) and 5 (b).

6. Twisted Skinny Bottles. At first the issues seem to become much more complicated if we are allowed to twist the piece of tubing before the ends are joined. However, twists in a Klein

bottle have some interesting properties which take some effort to understand, but which in the end actually simplify the analysis. In the following we refer to Fig. 2 (b) and to Fig. 6, which show cases of Klein bottles during formation before the ends are joined at the mouth. If in this situation the outer piece of tubing is rotated around its axis, while the plane of the loop is kept the same, we will find that the inner end of the tube must rotate in the opposite direction in order to keep the tube free of twist.

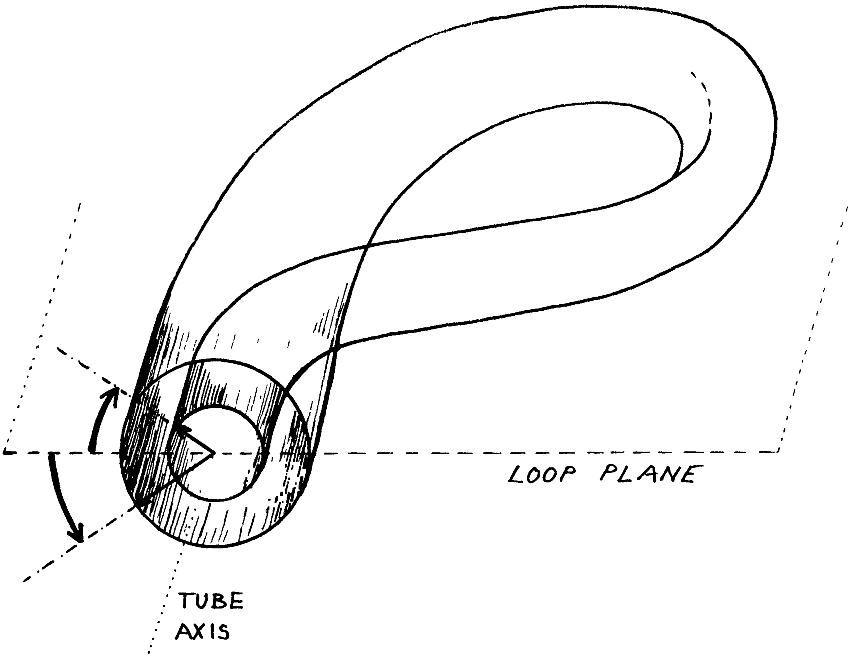


FIG. 6

Alternatively, if we keep the outer polygon at the mouth fixed but rotate the plane of the loop around an axis going perpendicularly through the center of the mouth, then the inner polygon must rotate with respect to the outer polygon at twice the rotational speed of the loop plane to keep the piece of tubing free of twist. This behavior is quite different from that of a torus, where under the same circumstances the two ends facing one another across the juncture plane would keep identical positions. From the behavior above it follows that any twist introduced during the formation of the Klein bottle can subsequently be removed by simply turning the "handle," i.e., the plane of the loop, through one-half the original twist angle. As we rotate the plane of the handle, the intersection of the thin end of the tube with the thick end will shift around the wall of the tube. But since we are dealing here with mathematical bodies, this causes no difficulties.

The intriguing discovery is that, contrary to the situation in prismatic rings, we can produce an *arbitrary* amount of twist in a Klein bottle by simply rotating mouth and loop plane with respect to one another.

7. The Case of the Square Prism. Returning to the example presented in Fig. 5, which is redrawn in Fig. 7, we can convince ourselves that Fig. 7 (b) can indeed be generated from Fig. 7 (a) by introducing a 90° twist before the ends are joined. A bottle of type 7 (a) with a quarter-twist introduced before joining the ends can be represented by Fig. 7 (c) where the inner square appears rotated counterclockwise by 90° with respect to Fig. 7 (a). A closer inspection shows that Fig. 7 (c) has a new symmetry axis, which is tilted to the left by 45°. Fig. 7 (c) thus is

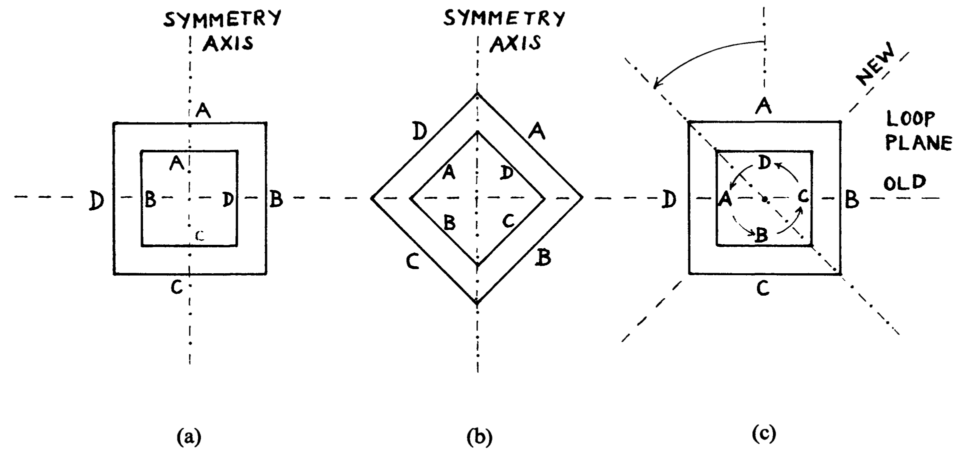


FIG. 7

equivalent to Fig. 7 (b) except for a rotation of 45°. This rotation corresponds to the rotation of the loop plane necessary to remove the introduced twist.

The fact that we can remove the built-in twist by simply rotating the handle still does not mean that the bottles corresponding to Figures 7 (a) and 7 (b) are topologically equivalent. As discussed in Section 4, they have a different number of edges. So there must be at least two generic, twist-free Klein bottles that can be formed from a piece of square tubing.

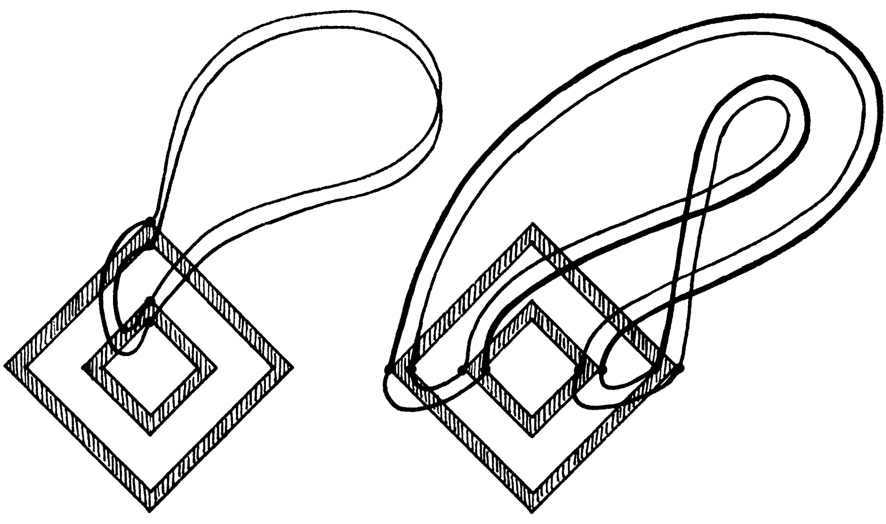
The analysis of all possible cases of twisted or untwisted skinny bottles is now greatly simplified. Any Klein bottle which was given a twist during its formation can be reduced to a Klein bottle without twist by appropriately rotating the handle. Since this operation does not change the interconnection topology or the number of faces or edges, we can restrict our studies to all possible types of twist-free bottles. To be free of twist, the symmetry axis relating the two end-polygons at the mouth has to be perpendicular to the loop plane. Since in a square there are only two types of symmetry axes (one through the corners and one through the center of the faces), it means that there are exactly *two* generic Klein bottles that can be formed from a tube with a square cross-section. In general, polygons with an even number of symmetry axes form two generic bottles, while tubes with a cross-section with an odd number of symmetry axes yield only one type of Klein bottle.

As a consequence, if we look at the specific example of, say, an octagonal tube, all bottles formed by introducing various amounts of twist fall into two distinct classes, and the two untwisted representatives of the two classes can be transformed into one another by cutting the tube and introducing a twist of $360°/8$ before rejoining.

8. Fleshy Bottles. If the walls of the tube are thick enough, so that inner and outer prisms can have independent twists, then the cross-section along the piece of tubing will constantly change and the distinction of single-sided or double-sided bands becomes meaningless. To determine the number of faces in such a body, it is advantageous to consider paths that take two subsequent passes around the loop. Since, with each lap around the loop, the path changes from outer to inner surface and vice versa, it will end up on the same surface on which it started. We add up the total amount of twist, t, experienced during such a double lap, i.e., we subtract the individual twists given to the inner and outer prisms. (In the case of a skinny bottle or a fleshy bottle which was twisted as a whole, the total twist will thus be zero.) With the calculated twist, t, measured in units of $360°/n$, and with the number of sides, n, of the cross-sectional polygons, the number of distinct faces, f, can readily be determined. For $t=0$ it is n, and for all other cases

it is simply the greatest common divisor of t and n. We will illustrate the procedure with the example of a Klein bottle formed from a five-sided prism, where inner and outer surface were both given a twist of 72° but in opposite directions. The number of sides, n, is five; outer twist is $+1$ (units of 360°/5), inner twist is -1 and thus $t=2$. Since the greatest common divisor of 5 and 2 is 1, the surface of this body can be formed by a single strip.

9. Relation Between Skinny and Fleshy Bottles. The analysis described in the last section unambiguously relates the number of edges to the number of sides in the cross-sectional polygon and the total amount of twist on inner and outer surfaces of the tube. In Section 4 we discussed two different Klein bottles formed from a piece of square tubing which have a different number of edges even though both of them can be reduced to bottles with zero twist. This paradox can be resolved if we consider a square tube with fleshy walls. The formula from Section 8 tells us that for twist zero we should obtain four faces and four edges. Depending on the rotational position of the cross-sectional polygons with respect to the loop plane, the four faces will either be located on four sides of two 2-sided fleshy bands or on the two sides of one 2-sided band and on two single-sided fleshy Moebius strips. The situation for the edges is similar. Edges located on the mirror axis of the end-polygons form a double loop in which the inner and outer part are separated by the thickness of the tubing wall (Fig. 8 (a)). Symmetrical pairs of edges off the axis form a pair of double loops separated from one another by the thickness of the wall (Fig. 8 (b)). Thus in bottles formed from a square tube there is always a total of four edges. If the wall thickness is reduced to make the transition to a skinny bottle, edges of the first type become a doubly traversed loop of half the original length, while pairs of edges of the second type merge into one another. This explains the paradoxical results pointed out at the beginning of this section.



(a) (b)

FIG. 8

10. Cutting Faces on Skinny Bottles. For the discussion of what happens as we cut faces along a path parallel to their edges, we will restrict ourselves to the case of skinny bottles. Fleshy twisted bottles, with their constantly changing cross-sections, do not lend themselves for nice cuts the way a Moebius band or skinny prismatic Klein bottles do. In all cases, a cut along the center line of a face will result in double-sided bands of twice the loop length (Figs. 9 (a) and 9 (b)). The only way that a single-sided Moebius band can be generated is to select a piece of a

side of the cross-sectional polygon which is itself centered around the symmetry axis. This is possible, for example, by selecting the center piece (a″) of the single-sided band produced by face A in Fig. 9 (c). Again, because of the symmetry conditions outlined in Fig. 4, it is clear that partial faces will occupy mirror-symmetrical positions on the end-polygons (Fig. 9).

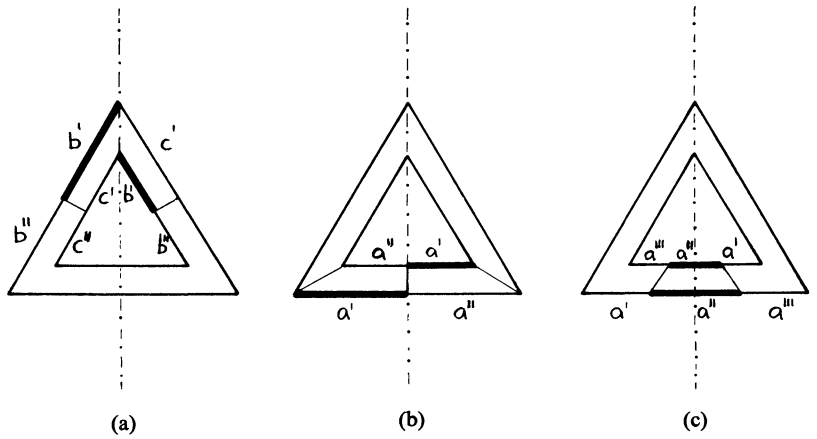


FIG. 9

11. Summary. Klein bottles formed out of prismatic tubing have proved to be a fascinating extension of the Moebius band and of the solid prismatic rings discussed in [**1**], [**2**]. Even though some properties of these bodies appear mind-boggling at first, it turns out that their properties can be understood easily if a double lap around the loop is considered to be the elementary move for analysis, corresponding to a single move around a *solid* prismatic ring. The twists of the internal and external surfaces are properly summed together and determine, in conjunction with the number of tube faces, the number of resulting edges and faces. The results of cutting the faces of these bottles parallel to their edges can readily be understood from the properties of cut Moebius bands and of double-sided twisted bands. The most interesting discovery is the fact that a twist in such a Klein bottle is not permanent but can be changed by arbitrary amounts by simply rotating the handle. This significantly reduces the possible number of topologically different twisted Klein bottles.

12. Acknowledgment. I thank Nelson Max for a critical review of this paper.

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DISPERSION POINTS AND FIXED POINTS

J. COBB AND W. VOXMAN

In this note we examine briefly certain relationships between dispersion points and fixed points. Possible directions for further investigation are also indicated.

A point p in a connected topological space X is said to be a *dispersion point* of X if $X\setminus\{p\}$ is totally disconnected. As might be expected, spaces having dispersion points are in general both rare and unusual, and it can be shown that no space can have more than one dispersion point [2]. Examples of spaces with dispersion points tend to fall into one of two categories: those that are metrizable (and, most commonly, are subsets of the plane) and those that are countable. In [8], Knaster and Kuratowski first introduced the notion of a space with a dispersion point and the example they described in this paper, the "Cantor teepee," remains as perhaps the most famous (and one of the most ingenious) examples of such a space. In 1946 Gustin gave the first example of a countable connected Hausdorff space with a dispersion point [5]; subsequently, a number of such spaces with varying properties have appeared in the literature, [7], [9], [10], [12], and [13]. The construction of the spaces described in [7] and [12] are especially easy to follow.

A space X is said to have the *fixed-point property* if each continuous function $f: X \to X$ has a fixed point, i.e., if $f: X \to X$ is continuous, then there is at least one point $x \in X$ such that $f(x) = x$. Although a number of noncompact spaces have been studied that have the fixed-point property [1], it would appear that the use of spaces with dispersion points has been overlooked in this connection. However, these spaces are particularly interesting in this regard; in fact, we have been able to show that not only do all of the spaces cited in the previous paragraph have the fixed-point property, but, moreover, for each of these spaces, the dispersion point remains fixed under *any* nonconstant continuous function mapping the space into itself. This leads us to make the following conjecture.

CONJECTURE 1: *Suppose that X is a connected space with a dispersion point p and that $f: X \to X$ is a nonconstant continuous function. Then $f(p) = p$.*

As far as we know the conjecture (expressed in this generality) is an open question. In Theorem 1 below we show that the conjecture is valid in the case of the Cantor teepee of Knaster and Kuratowski, and in Theorem 2 we establish a result that can be used to show that certain countable spaces with dispersion points also satisfy the conjecture.

We begin with a brief description of the Knaster-Kuratowski example. Let K be the Cantor set on the interval $[0,1]$ considered as a subset of the plane, and let p be the point $(1/2, 1/2)$. For each $c \in K$, let L_c denote the line segment joining p to c. Let K_E be the set of endpoints of the deleted intervals used in the construction of the Cantor set, and let K_F denote the remaining points of K. Let $S = \{(x,y) \in \cup_{c\in K} L_c \mid c \in K_E \text{ and } y \text{ is rational}\}$, and let $T = \{(x,y) \in \cup_{c \in K} L_c \mid c \in K_F \text{ and } y \text{ is irrational}\}$. Then the Cantor teepee is the set $X = S \cup T$ (with the relative topology inherited from the plane). It can be shown that X is connected and that p is a dispersion point of X (see [2]). In the proof of Theorem 1, we will say that an infinite subset K_1 of the Cantor set K is a *Cantor subset* of K (denoted $K_1 < K$), if $K_1 = K \cap [a,b]$ for some $a, b \in R^1 \setminus K$. The closure of a set A is denoted by cl A. The usual metric for the plane is denoted by d.

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THEOREM 1. *Let X be the Cantor teepee (with dispersion point p) and suppose that $f: X \to X$ is a continuous nonconstant function. Then $f(p) = p$.*

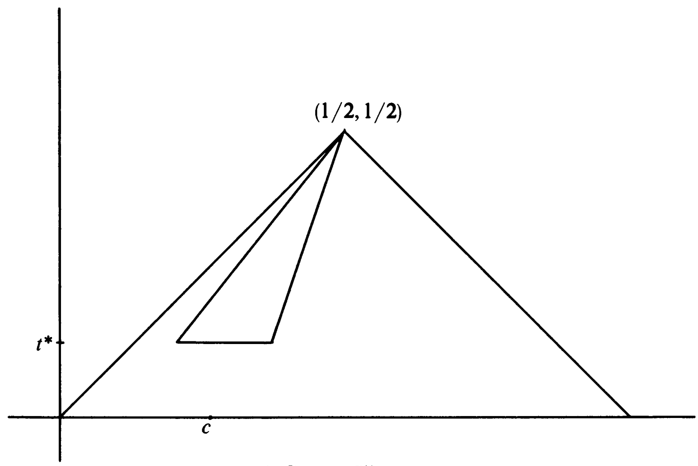
Proof. Suppose that $f(p) = q \neq p$, and let U be a small closed neighborhood of q that does not contain p. Then there is a closed connected neighborhood V of p such that $f(V) \subset U$ (V may be defined by $\{(x,t) \in X \mid t \geqslant \frac{1}{2} - \varepsilon\}$ for a suitably small rational number ε; note that V is homeomorphic to X). Next observe that since $f(V)$ is connected and since $X \setminus \{p\}$ is totally disconnected, it follows that $f(V) = q$.

Since f is assumed to be a nonconstant mapping, there is a point $r = (x,t) \in X$ such that $t \neq 0$ and $f(r) \neq q$. Let W be a small neighborhood of r such that $K \cap \operatorname{cl} W = \varnothing, p \notin f(W)$, and $q \notin f(W)$. Finally let $\hat{K} < K$ be a Cantor subset of K that is contained in the projection of W into K (along the lines L_c), and let $\hat{K}_E = K_E \cap \hat{K}$ and $K_F = \hat{K}_F \cap \hat{K}$.

For each point $w = (c, t^*)$ in the segment $L_c, c \in \hat{K}$, we define a *Cantor sliver* about w to be any set of the form

$$S(K_1, t^*) = \{(x,t) \in X \mid c \in K_1 < \hat{K}, x \in K_1, t \geqslant t^*\}.$$

It is easy to see that each Cantor sliver is connected. (See Fig. 1.)



A Cantor Sliver

FIG. 1

For each $c \in \hat{K}$, let $t_c = \inf \{t \mid (x,t) \in L_c$ and there is a Cantor sliver S about (x,t) such that $f(S) = q\}$. Observe that there are positive numbers α and β such that for each $c \in \hat{K}, 0 < \alpha < t_c < \beta < 1/2$. We now show that if $c \in \hat{K}$ and if w_c denotes the point on L_c with second coordinate t_c, then $w_c \notin X$. To see this, suppose that $w_c \in X$ for some $c \in \hat{K}$. For each positive integer n, let $S_n = S(K_n, r_n)$ be a Cantor sliver where $c \in K_n < \hat{K}, t_c < r_n < t_c + 1/n$, diam $K_n < 1/n$, and $f(S_n) = q$. Also for each n (and K_n), let $\tilde{S}_n = S(K_n, q_n)$ be a Cantor sliver such that $t_c - 1/n < q_n < t_c$. Since $f(\tilde{S}_n)$ is connected and since, by the definition of $t_c, f(\tilde{S}_n) \neq q$, it follows that $p \in f(\tilde{S}_n)$ (all connected subsets of X consisting of more than one point must contain the point p). Thus, for each positive integer n, there are points $w_n \in S_n$ and $\tilde{w}_n \in \tilde{S}_n \setminus S_n$ such that $d(w_n, w_c) < 1/n, d(\tilde{w}_n, w_c) < 2/n, f(w_n) = q$ and $f(\tilde{w}_n) = p$. However, this implies that the sequences $\{w_n\}$ and $\{\tilde{w}_n\}$ both converge to the point w_c, which is impossible since $\lim_{n\to\infty} f(w_n) = q$ and $\lim_{n\to\infty} f(x_n) = p$. We conclude then that the points w_c fail to lie in X.

It now follows from the construction of the Cantor teepee that if $c \in \hat{K}_F$, then t_c is rational, and, hence, $\{t_c \mid c \in \hat{K}_F\}$ ranges over a *countable* number of values $s_1, s_2, \ldots$. For each i, let $K_i = \{c \in \hat{K} \mid t_c = s_i\}$. Since $\hat{K}$ is the disjoint union of the set $\hat{K}_E$ and the sets K_i, we have from the

Baire Category Theorem ([2]) that for some integer i the closure of K_i contains a subset of $\hat{R}$ with nonempty interior, i.e., there is an open interval (a,b) such that $(a,b)\cap\hat{R}\subset\text{cl } K_i$. To conclude the proof, first observe that if $c\in\hat{R}_E\cap(a,b)$, then the point $z=(c,s_i)$ belongs to X; and, furthermore, since $c\in\text{cl } K_i$, there is a sequence of points $\{c_n\}$ in $\hat{R}_F$ such that for each $n, t_{c_n}=s_i$. The same argument that was employed in the previous paragraph may be used to show that there are sequences $\{S_n\}$ and $\{\tilde{S}_n\}$ of Cantor slivers and points $w_n\in S_n$ and $\tilde{w}_n\in\tilde{S}_n\setminus S_n$ such that the sequences $\{w_n\}$ and $\{\tilde{w}_n\}$ converge to z, but such that $\lim_{n\to\infty}f(w_n)=q$ and $\lim_{n\to\infty}f(\tilde{w}_n)=p$. This contradiction completes the proof.

An almost identical proof to that given in Theorem 1 may be used to show that Conjecture 1 is also valid in the case of the example described by Wilder in **[14]**; moreover, relatively straightforward modifications of this proof show that Conjecture 1 also holds for the examples given in **[3]**, **[6]**, and **[11]**.

Proofs that tend in general to be peculiar to the particular space under consideration may be employed to show that the examples of countable spaces with dispersion points found in **[5]**, **[7]**, **[9]**, **[10]**, **[12]**, and **[13]** all satisfy Conjecture 1. In the next theorem we establish a result that is useful in dealing with some of these spaces. We shall say that a pair of points $\{a,b\}$ in a topological space X is *Urysohn-separable* if there are open sets U and V in X such that $a\in U, b\in V$ and $(\text{cl } U)\cap(\text{cl } V)=\varnothing$; if such open sets do not exist then we say that the pair of points $\{a,b\}$ is *Urysohn-inseparable*.

THEOREM 2. *Suppose that X is a Hausdorff space with a dispersion point p. Suppose further that each pair of points $\{p,x\}$ $(x\in X)$ is Urysohn-inseparable, and that each pair of points $\{x,y\}$ $(x\neq p, y\neq p)$ is Urysohn-separable. If $f:X\to X$ is a nonconstant continuous function, then $f(p)=p$.*

Proof. Suppose to the contrary that $f:X\to X$ is a nonconstant continuous function such that $f(p)=q\neq p$. Since $f(X)$ is connected and nondegenerate, and since $\{p\}\cup\{q\}$ is not connected, there is a point $a\in X$ such that $f(a)\neq f(p)$ and $f(a)\neq p$. Since the pair $\{f(p),f(a)\}$ is Urysohn-separable, there are open sets U and V in X such that $f(p)\in U, f(a)\in V$, and $(\text{cl } U)\cap(\text{cl } V)=\varnothing$. This implies that $p\in f^{-1}(U), a\in f^{-1}(V)$, and $(\text{cl } f^{-1}(U))\cap(\text{cl } f^{-1}(V))\subset f^{-1}(\text{cl } U)\cap f^{-1}(\text{cl } V)=\varnothing$. However, this is impossible since the pair $\{p,a\}$ is Urysohn-inseparable; hence, there is no such mapping f.

It is not difficult to apply this theorem to **[5]** and **[9]** to show that Conjecture 1 is valid for these examples.

Question 1: Is Conjecture 1 valid in general, and if not, what conditions (metrizability, countability, planar, etc.) could be imposed on the space X such that the conjecture would hold?

We now briefly consider the other side of the coin. At this point it might seem reasonable to make the following conjecture.

CONJECTURE 2: *If X is a connected topological space that has a unique point x^* with the property that $f(x^*)=x^*$ for every nonconstant continuous function $f:X\to X$, then x^* is a dispersion point of X.*

The following example shows that in general this conjecture is false, although it does leave open the question of whether or not the conjecture would hold if suitable restrictions are placed on X.

Example. Let $Y=R^2\setminus\{(x,y)|x \text{ and } y \text{ are rational}\}$, and let $X=Y\cup\{\infty\}$ be the one-point compactification of Y. It is easy to see that X and Y are connected and that ∞ is not a dispersion point of X. However, if $f:X\to X$ is any nonconstant continuous function, then $f(\infty)=\infty$. To see this, suppose that $f(\infty)=q\neq\infty$, and let U be an arbitrarily small neighborhood of q that lies in R^2. Let V be a neighborhood of ∞ such that $f(V)\subset U$. Note that $\text{cl } V=X$, and, therefore, we have $f(X)=f(\text{cl } V)\subset\text{cl } U$. Since U was an arbitrarily small neighborhood of q, it follows that f is a constant mapping.

Question 2: In the previous example, the space X is not Hausdorff. Do there exist connected Hausdorff (countable, metric, etc.) spaces without dispersion points but having a unique point that is left fixed by all continuous nonconstant functions mapping the space into itself?

Addendum: Utilizing a referee's suggestion, we have been able to give a partially affirmative answer to Question 2. If two copies of the space M_2 (or M_1) described by H. Cook [Continua which admit only the identity mapping onto nondegenerate subcontinua, Fund. Math., 60 (1967) 241–249] are sewn together at a point p, then the resulting metric space X is compact and connected and hence has no dispersion point; the sewing point p is left fixed under all nonconstant mappings of X into itself. Question 2 remains open for spaces more like those having dispersion points, e.g., countable T_2 spaces, noncompact spaces, planar spaces.

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PROGRESS REPORTS

EDITED BY P. R. HALMOS

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It is easy to be too busy to pay attention to what anyone else is doing, but not good. All of us should know, and want to know, what has been discovered since our formal education ended, but new words, and relations between them, are growing too fast to keep up. It is possible for a person to learn of the title of a recent work and of the key words used in it and still not have the faintest idea of what the subject is.

Progress Reports is to be an almost periodic column intended to increase everyone's mathematical information about what others have been up to. Each column will report one step forward in the mathematics of our time. The purpose is to inform, more than to instruct: what is the name of the subject, what are some of the words it uses, what is a typical question, what is the answer, who found it. The emphasis will be on concrete questions and answers (theorems), and not on general contexts and techniques (theories). References will be kept minimal:

usually they will include only one of the earliest papers in which the answer appears and a more recent exposition of the discovery, whenever one is easily available.

Everyone is invited to nominate subjects to be reported on and authors to prepare the reports. The ground rules are that the principal theorem should be old enough to have been published in the usual sense of that word (and not just circulated by word of mouth or in preprints); it should be of interest to more than just a few specialists; and it should be new enough to have an effect on the mathematical life of the present and near future. In practice most reports will probably be on progress achieved somewhere between 5 and 15 years ago.

THE DUAL OF H^1

SHELDON AXLER

Let T denote the unit circle in the complex plane $\mathbb{C}$ and let L^p denote the usual Lebesgue space: $L^p = L^p(T, d\theta)$. Thus for $p \in [1, \infty)$, L^p is the space of measurable functions $f: T \to \mathbb{C}$ such that

$$\|f\|_p = \left(\int_0^{2\pi} |f(e^{i\theta})|^p d\theta \right)^{1/p} < \infty.$$

The space L^∞ consists of the bounded measurable functions from T to $\mathbb{C}$. (Functions which agree except on a set of measure zero are identified with each other.) The Hardy space H^p is the closed subspace of L^p defined by

$$H^p = \left\{ f \in L^p : \int_0^{2\pi} f(e^{i\theta}) e^{in\theta} d\theta = 0 \text{ for } n = 1, 2, \dots \right\}.$$

The Hardy spaces arise naturally as the spaces of boundary values of certain classes of analytic functions. This connection with analytic functions has been exploited to develop a rich theory of H^p spaces.

A linear map $\phi: X \to \mathbb{C}$ on a Banach space X with norm $\|\cdot\|$ is called bounded (or continuous) if

$$\|\phi\| = \sup\{|\phi(x)| : x \in X, \|x\| \leq 1\} < \infty.$$

The space X^* of all bounded linear maps on a Banach space X is called the dual of X. Because of the Hahn-Banach Theorem and other standard tools of functional analysis, it is often tremendously useful in studying a Banach space to have a concrete representation of its dual.

For $p \in [1, \infty)$, let p' denote the element of $(1, \infty]$ which satisfies $1/p + 1/p' = 1$. Since the beginning of the theory of Banach spaces it has been known that if $p \in [1, \infty)$ then the dual of L^p can be identified with $L^{p'}$. A deeper result (known for over half a century) states that if $p \in (1, \infty)$ then the dual of H^p is isomorphic to $H^{p'}$. More precisely, for $g \in H^{p'}$, define a linear function $\phi_g: H^p \to \mathbb{C}$ by

$$\phi_g(f) = \int_0^{2\pi} f(e^{i\theta}) g(e^{-i\theta}) d\theta.$$

Then the map $g \mapsto \phi_g$ is a one-to-one, linear mapping of $H^{p'}$ onto $(H^p)^*$. Furthermore, there is a constant $K_p > 0$ such that

$$\frac{1}{K_p} \|\phi_g\| \leq \|g\|_{p'} \leq K_p \|\phi_g\|$$

for all $g \in H^{p'}$.

What happens when $p = 1$? For $g \in H^\infty$, it is still possible to define the linear map $\phi_g: H^1 \to \mathbb{C}$ as above. The mapping $g \mapsto \phi_g$ is still a one-to-one linear mapping of H^∞ into $(H^1)^*$, but it turns out that this mapping is not onto all of $(H^1)^*$. So somehow H^∞ is not big enough to be naturally identified with $(H^1)^*$.

To attempt to identify $(H^1)^*$, consider the inclusion map $j: H^2 \to H^1$ defined by $j(f)=f$. The map j is continuous, and so if $\phi \in (H^1)^*$, then the composition $\phi \circ j: H^2 \to \mathbb{C}$ is an element of $(H^2)^*$. An identification of $(H^2)^*$ with H^2 was described above. Thus we can conclude that for each $\phi \in (H^1)^*$, there exists a function $g_\phi \in H^2$ corresponding to $\phi \circ j \in (H^2)^*$. For $\phi \in (H^1)^*$, g_ϕ is the element of H^2 which satisfies the equation

$$\phi(f) = \int_0^{2\pi} f(e^{i\theta}) g_\phi(e^{-i\theta}) d\theta \quad \text{for all } f \in H^2.$$

It is not hard to see that the map $\phi \mapsto g_\phi$ is a one-to-one, linear map of $(H^1)^*$ into H^2 (but not onto H^2). To represent $(H^1)^*$ concretely, it is necessary to find the range of this mapping. In other words, we want to find an intrinsic characterization of a function in H^2 which tells us whether or not this function is of the form g_ϕ for some $\phi \in (H^1)^*$.

This problem was solved by Charles Fefferman and Elias Stein. To state their solution, it is necessary to introduce the notion of the mean oscillation of a function. For I a subinterval of the circle T, let $|I|$ denote the measure (arc length) of I. For $g \in L^2$, let $\mathrm{av}_I g$ denote the average of g over I; so $\mathrm{av}_I g = \int_I g/|I|$. Thus $\int_I |g - \mathrm{av}_I g|/|I|$ is the average amount over I by which g differs from its average over I. The mean oscillation of g, denoted $\|g\|_{\mathrm{MO}}$, is defined to be the supremum of this quantity as I ranges over all subintervals of the circle T;

$$\|g\|_{\mathrm{MO}} = \sup\left\{ \int_I |g - \mathrm{av}_I g|/|I| : I \text{ is a subinterval of } T \right\}.$$

The result of Fefferman and Stein is that the dual of H^1 can be identified with $\{g \in H^2 : \|g\|_{\mathrm{MO}} < \infty\}$. More precisely, the map $\phi \mapsto g_\phi$ defined above is a one-to-one, linear map of $(H^1)^*$ onto $\{g \in H^2 : \|g\|_{\mathrm{MO}} < \infty\}$. Furthermore, there is a constant $K>0$ such that

$$\frac{1}{K}\|\phi\| \leq \|g_\phi\|_{\mathrm{MO}} + \|g_\phi\|_2 \leq K\|\phi\|$$

for every $\phi \in (H^1)^*$.

There have been many useful applications of this characterization of $(H^1)^*$. The space BMO (functions of bounded mean oscillation) defined by $\mathrm{BMO} = \{g \in L^2 : \|g\|_{\mathrm{MO}} < \infty\}$ has turned out to be important in several contexts. The methods associated with BMO tend to be real variable techniques, as opposed to the complex variable approach (analytic functions) which dominates much of the theory of Hardy spaces. Finally, the BMO techniques work in Hardy spaces of several variables, which is the setting in which Fefferman and Stein proved their results.

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MISCELLANEA

34. Topology in more than three dimensions exists. I do not say that it is an easy subject—I have devoted too much effort to it not to appreciate how difficult it is; but it is a possible field for investigation, and it is not based exclusively on calculation; we do not know how to cultivate it fruitfully without constantly appealing to intuition. Consequently there really are intuitive ideas about spaces of more than three dimensions; and if they require more intense effort than ordinary geometric intuition, this is presumably a matter of habit, as well as the influence of the rapidly increasing complexity of spaces as the number of dimensions grows.

—H. Poincaré, Pourquoi l'espace a trois dimensions, in *Dernières Pensées*, Paris, 1913.

MATHEMATICAL NOTES

EDITED BY DEBORAH TEPPER HAIMO AND FRANKLIN TEPPER HAIMO

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PRINCIPAL VERTICES, EXPOSED POINTS, AND EARS

G. H. MEISTERS

In [1] Guggenheimer states that *a Jordan polygon has two principal vertices that are exposed points of its convex hull*, and he refers to Meisters's paper [3]. Such a statement cannot be found in Meisters's paper, and in fact it is false. The polygon illustrated in the figure below provides a counterexample.

By a *Jordan polygon* $P = V_1 \dots V_N$ is meant a simple closed polygonal plane curve with *N sides* $V_1V_2, V_2V_3, \dots, V_{N-1}V_N, V_NV_1$ joining the *N vertices* $V_1, \dots, V_N$. In [3] any consecutive vertices V_{i-1}, V_i, and V_{i+1} of a Jordan polygon P are said to form an *ear* (regarded as the region enclosed by the triangle $V_{i-1}V_iV_{i+1}$) *at the vertex* V_i if the open chord joining V_{i-1} and V_{i+1} lies entirely inside the polygon P. Two such ears are called *nonoverlapping* if the interiors of their triangular regions are disjoint. The following Two-Ears Theorem was proved in [3].

TWO-EARS THEOREM. *Except for triangles, every Jordan polygon has at least two nonoverlapping ears.*

Guggenheimer's false statement was perhaps an attempt to express this Two-Ears Theorem in terms of the concept of "principal vertex." A vertex V_i of the polygon $P = V_1 \dots V_N$ is called *principal* if no vertex of P is in the interior of the triangle $V_{i-1}V_iV_{i+1}$ or on the open chord $(V_{i-1}V_{i+1})$. See [1]. But it is doubtful that the concept of "ear vertex" (i.e., a vertex at which there is an ear) can be expressed in terms of the concept of "principal vertex" without in some way referring to the *interior* of the polygon, because the definition of the former depends on the Jordan Curve Theorem for polygons while that of the latter depends only on the Jordan Curve Theorem for triangles. For example, a principal vertex is an ear vertex if and only if the *interior angle* at this vertex is less than a straight angle. Every ear vertex is a principal vertex, but there need not be an ear at every principal vertex. In fact, every Jordan polygon has at least three principal vertices but need have no more than two ears.

An "exposed point" of a set X is defined by Klee in [2] and can also be found in many books on convex sets. (An *exposed point* of a set X in a topological linear space is a point p in X such

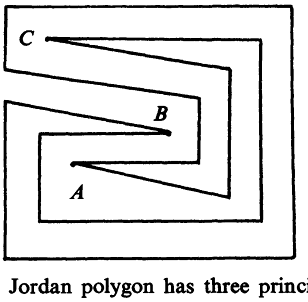

FIG. 1. This Jordan polygon has three principal vertices (A, B, and C) and has ears at two of these vertices (A and B), but it has no principal vertex on the boundary of its convex hull.

that X is supported at p by a closed hyperplane which intersects X only at p.) The important thing here is that an exposed point is a special kind of boundary point. But the Jordan polygon in Figure 1 above has no principal vertex on the boundary of its convex hull, so that Guggenheimer's statement (italicized in the first sentence of this article) is false.

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THE CONNECTION BETWEEN limsup AND UNIFORM INTEGRABILITY

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Let us begin by entertaining the following question. What happens to Fatou's lemma when liminf is replaced by limsup? More specifically, given a probability space (Ω,Σ,μ), find a nice, sufficient condition in order that a sequence $\{f_n\}$ of nonnegative integrable functions satisfy the inequality

$$\limsup\int_\Omega f_n\,d\mu\leqslant\int_\Omega f\,d\mu,\quad \text{where } f=\limsup f_n. \tag{*}$$

An answer is suggested by the following lemma.

LEMMA. *Let $\{f_n\}$ be a sequence of nonnegative measurable functions and let $\delta>0$. Then there exists a positive integer N and $A\in\Sigma$ satisfying* (i) *$x\in A$ and $n\geqslant N$ imply $f_n(x)\leqslant f(x)+\delta$, where $f=\limsup f_n$; and* (ii) *$\mu(A)>1-\delta$.*

Proof. Let $g_n=(V_{i\geqslant n}f_i)-f=V_{i\geqslant n}(f_i-f)$; let $F_n=\{x;f_n(x)>f(x)+\delta\}$, and let $G_n=\cup_{i\geqslant n}F_i$. Then $G_n=\{x;g_n(x)>\delta\}$. Since $\lim_n g_n=0$, $\lim_n\mu(G_n)=0$; so we choose N with $\mu(G_N)<\delta$ and let $A=\Omega-G_N$.

Using the lemma, it is easy to see that uniform integrability of the sequence $\{f_n\}$ (i.e., for each $\varepsilon>0$ there exists $\delta>0$ such that if $\mu(E)<\delta$, then $\int_E f_n\,d\mu<\varepsilon, n\geqslant 1$) is a nice, sufficient condition:

COROLLARY. *Let $\{f_n\}$ be a uniformly integrable sequence of nonnegative integrable functions; then* (*) *is satisfied.*

Toward understanding the connection between limsup and uniform integrability, we give an example of a sequence which satisfies (*) but is not uniformly integrable.

(1) Let (Ω,Σ,μ) be half of Lebesgue measure on $[0,2]$. Let

$$f_n=\sum_{k=1}^{2^{(n-1)}}2I(a_{n,k},b_{n,k}]+2^nI(1+2^{-n},1+2^{-(n-1)}],$$

where $a_{n,k}=(2k-1)/2^n$, $b_{n,k}=(2k)/2^n$, and IE denotes the indicator function of a set E. Thus, $\int_0^1 f_n=\int_1^2 f_n=\frac{1}{2}$, $\int_0^1 f=1$, and $\int_1^2 f=0$. Indeed, if $\{f_{n_k}\}$ is a subsequence of $\{f_n\}$ then $\limsup_k f_{n_k}=2$ a.e. on $[0,1]$; and $f\equiv 0$ on $(1,2]$.

However, if $\{f_n\}$ is uniformly integrable, then the stronger condition

$$\limsup \int_E f_n \, d\mu \leqslant \int_E f \, d\mu, \quad E \in \Sigma, \tag{**}$$

is satisfied. (Looking back at (1), $\int_1^2 f_n > \int_1^2 f$, so (**) is not satisfied in that example.)

Next, we give an example of a sequence which satisfies (**) but is not uniformly integrable.

(2) Let (Ω, Σ, μ) be Lebesgue measure on $(0,1]$. Let $f_j = 2^n I([i-1]/2^n,\ i/2^n]$, where $j = 2^n - 2 + i$, $1 \leqslant i \leqslant 2^n$, and $n \geqslant 1$. Then $\{f_n\}$ is not uniformly integrable; but $f \equiv \infty$, so (**) is satisfied.

Example (2) tells us that an assumption about f is needed; so we suppose that f is an integrable function and state the following theorem.

THEOREM. *Let (Ω, Σ, μ) be a probability space. Let $\{f_n\}$ be a sequence of nonnegative integrable functions. Let $f = \limsup f_n$ and suppose that $\int_\Omega f \, d\mu < \infty$. Then uniformly integrability is equivalent to* (**).

The proof is completed by supposing that $\{f_n\}$ is not uniformly integrable. Then there exists $\varepsilon > 0$, sequences $\{n_k\}$ and $\{E_k\}$, and $\delta > 0$ such that (i) if $\mu(E) < \delta$ then $\int_E f \, d\mu < \varepsilon$, (ii) $\mu(E_k) < \delta/2^k$, and (iii) $\int_{E_k} f_{n_k} \, d\mu > \varepsilon$. Let $E = \cup E_k$ and (**) is not satisfied.

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ON FINITE CLASSES OF CONVEX SETS THAT PERMIT SPACE COVERINGS

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Let $K_1, K_2, \ldots, K_m$ denote closed convex subsets (not necessarily distinct) of d-dimensional euclidean space R^d. We shall say that the finite sequence $(K_i) = (K_1, K_2, \ldots, K_m)$ permits a covering of R^d if there are rigid motions $\sigma_1, \sigma_2, \ldots, \sigma_m$ of R^d so that $R^d \subset \sigma_1 K_1 \cup \sigma_2 K_2 \cup \cdots \cup \sigma_m K_m$. This note is concerned with the problem of finding necessary and sufficient conditions that a given finite sequence (K_i) permits a covering of R^d. It will be shown that the problem can be reduced to a covering problem for convex sets on the $(d-1)$-dimensional unit sphere. If $d = 2$, this leads to a very satisfactory solution of the original covering problem. These results are corollaries of a theorem that establishes a relationship between covering properties of unbounded convex sets and certain inscribed convex cones.

Covering problems of this kind were first posed by Chakerian [1]. Related results and problems are discussed as a recent Research Problem in this MONTHLY [2].

If p is a point of R^d, and u is a unit vector in R^d, we write (p,u) to denote the ray $\{p + \lambda u : \lambda \geqslant 0\}$. More generally, if U is a set of unit vectors, we denote by (p, U) the cone $\{p + \lambda u : \lambda \geqslant 0, u \in U\}$. If $U = \varnothing$, then $(p, U) = \{p\}$.

Our results depend on the following known lemma (cf. Grünbaum [3, p. 23]). We include a very short proof here for the sake of completeness.

LEMMA. *If K is a closed convex subset of R^d and $(p,u) \in K$, $q \in K$, then $(q,u) \in K$.*

Proof. If $x \in (q,u)$, then $x = q + \lambda u$ with some $\lambda \geqslant 0$. For any μ with $0 < \mu \leqslant 1$ we have $x + \mu(p-q) = (1-\mu)q + \mu(p + (\lambda/\mu)u)$ where $q \in K$ and $p + (\lambda/\mu)u \in K$. By the convexity of K, it follows that $x + \mu(p-q) \in K$. Since K is closed, and x is a boundary point of the subset $\{x + \mu(p-q) : 0 < \mu \leqslant 1\}$ of K, we obtain the desired result $x \in K$.

If p is a given point of a closed convex set K, one can associate with p an inscribed maximal cone, i.e., the cone (p, U) consisting of all rays (p,u) with $(p,u) \subset K$. The Lemma shows immediately that any two such cones that are associated in this way with different points of K

are translates of each other. Moreover, from the fact that $\operatorname{cl}\operatorname{conv}(p,U)$ is again a cone in K with apex p, it follows that $\operatorname{cl}\operatorname{conv}(p,U)\subset(p,U)$, and this implies that (p,U) is closed and convex. Following the terminology of Grünbaum (loc. cit.) we call such an inscribed maximal cone a *characteristic cone* of K.

If the characteristic cones of closed convex subsets $K_1,K_2,\dots,K_m$ of R^d have equal apices and form a covering of R^d, then it is obvious that the sets $K_1,K_2,\dots,K_m$ themselves also form a covering of R^d. Our main result is a kind of converse of this statement. Instead of coverings of R^d we consider the slightly more general situation of coverings of a given cone in R^d (which may be R^d itself).

THEOREM. *Let $C=(p,U)$ be a cone in R^d, and let $K_1,K_2,\dots,K_m$ be closed convex subsets of R^d with characteristic cones $(p_1,U_1),(p_2,U_2),\dots,(p_m,U_m)$. Then*

$$C\subset K_1\cup K_2\cup\cdots\cup K_m \tag{1}$$

implies

$$C\subset(p,U_1)\cup(p,U_2)\cup\cdots\cup(p,U_m).$$

Proof. We may obviously assume that $U\neq\emptyset$. Let (p,u) be an arbitrary but fixed ray in C. All we have to show is that, under the assumption (1), (p,u) is in some (p,U_i). Since $(p,u)\subset C\subset K_1\cup K_2\cup\cdots\cup K_m$, one of the intervals $(p,u)\cap K_i$, say, $(p,u)\cap K_1$, must be unbounded. Therefore there is a point q in K_1 so that $(q,u)\subset K_1$. Because of $p_1\in K_1$ it follows from the Lemma that $(p_1,u)\subset K_1$, and the definition of the characteristic cone shows that $(p_1,u)\subset(p_1,U_1)$. This relation obviously implies that $(p,u)\subset(p,U_1)$. This concludes the proof of the Theorem.

Let us now denote by S^{d-1} the $(d-1)$-dimensional unit sphere in R^d, centered at the point $o=(0,0,\dots,0)$. If K is a closed convex subset of R^d with characteristic cone (p,U), we can associate with K the set $S(K)=(o,U)\cap S^{d-1}$ on S^{d-1}. We shall refer to $S(K)$ as the *characteristic cap* of K. If the characteristic cone of K is d-dimensional and $d\geqslant 2$, then $S(K)$ is spherically convex (in the wider sense that includes hemispheres and the total sphere).

The following corollary is an immediate consequence of our theorems.

COROLLARY 1. *Let (K_i) be a finite sequence of closed convex subsets of R^d. Then (K_i) permits a covering of R^d if and only if the sequence $(S(K_i))$ of the corresponding characteristic caps permits a covering of S^{d-1}.*

The statement that $(S(K_i))$ permits a covering of S^{d-1} means of course that $S^{d-1}\subset\bigcup\rho_iS(K_i)$ for suitable rotations ρ_i of R^d about o.

If $d=2$, the situation is particularly simple. In this case the characteristic caps are arcs of the unit circle. If the lengths of these arcs are called the *characteristic angles*, then Corollary 1 can obviously be reformulated in the following way.

COROLLARY 2. *Let (K_i) be a finite sequence of closed convex subsets of R^2, and let (α_i) be the sequence of the corresponding characteristic angles. Then, (K_i) permits a covering of R^2 if and only if $\sum\alpha_i\geqslant 2\pi$.*

For example, the characteristic angle of any parabola (i.e., of the closed convex domain bounded by a parabola) is zero. Consequently, there exists no finite sequence of parabolas that would permit a covering of R^2. A finite sequence of hyperbolas of the form $\{(x,y):x^2/a_i^2-y^2/b_i^2\geqslant 1, x\geqslant 0\}$ permits a covering of R^2 if and only if $\sum\tan^{-1}(b_i/a_i)\geqslant 2\pi$.

The following result is another immediate consequence of the theorem above, or of Corollary 1.

COROLLARY 3. *Let (K_i) be a finite sequence of closed convex subsets of R^d, and let $(K_i)'$ be the*

subsequence that is obtained from (K_i) *by removing all those sets that contain no d-dimensional convex cones. Then* (K_i) *permits a covering of* R^d *if and only if* $(K_i)'$ *permits a covering of* R^d.

For example, if $d \geqslant 2$ and (K_i) contains any bounded sets, or cylinders with a bounded base, then these sets can be removed from (K_i) without altering the covering property of (K_i).

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AN ELEMENTARY PROOF OF THE POLAR DECOMPOSITION THEOREM

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Inspired by a paper of G. Grioli [1], we have noticed that by patching together some well-known results we can get a new proof of the theorem of the title.

In this paper we consider only the real case; for the complex case, the proof goes along the same lines. In comparison with standard proofs to be found in treatises on linear algebra (cf., e.g., [2, Section 83]), our proof has two distinguishing features: first, in the spirit favored by J. von Neumann in [3], it is fully "vector-geometrical" even in those parts which are usually established by analytic arguments; second, it is elementary, in that it does not make use of such tools as the spectral theorem and the square root theorem.

Let Lin denote the space of all linear transformations of a finite-dimensional (real) inner product space $\mathcal{V}$ into itself. Furthermore, let Sym be the subspace of the symmetric elements of Lin, and let Orth be the set of orthogonal elements of Lin. Let us denote the inner product of $a, b \in \mathcal{V}$ by $a \cdot b$. An element A of Sym is said to be positive semi-definite if $Av \cdot v \geqslant 0$ for all $v \in \mathcal{V}$; we denote by Pos the collection of all such elements. We endow Lin with an inner product space structure in the usual way: $A \cdot B = \operatorname{tr}(AB^T)$, where tr stands for the trace functional and B^T is the transpose of B. In particular, $\operatorname{tr} A = A \cdot I$, where I is the identity transformation of $\mathcal{V}$; if $A \in \text{Pos}$, then $A \cdot I \geqslant 0$.

We begin by stating a lemma whose proof we postpone.

LEMMA. *Assume that* $A \cdot H \leqslant A \cdot I$ *for all* $H \in \text{Orth}$; *then,* $A \in \text{Pos}$. *Conversely, if* $A \in \text{Pos}$, *then* $A \cdot H \leqslant A \cdot I$ *for all* $H \in \text{Orth}$. *Moreover, if equality holds for* $H = H_0 \in \text{Orth}$, *then* $H_0 A = A$, *i.e.,* H_0 *and* I *agree on the range of* A.

REMARK. The direct and converse statements of this lemma are less general forms, respectively, of Lemma 7 and Theorem I of [3]. Apparently, J. von Neumann did not pursue a characterization of the set of maximizers such as the last statement.

We now indulge in a simple computation. We state a problem:

Given $F \in \text{Lin}$, find maximum $F \cdot H$ for $H \in \text{Orth}$.

As Orth is a compact set, the problem makes sense and has a nonempty solution set $\mathbb{S}$. If $R \in \mathbb{S}$, we have

$$F \cdot H \leqslant F \cdot R \qquad \forall H \in \text{Orth},$$

or, equivalently,

$$FR^T \cdot HR^T \leqslant FR^T \cdot I \qquad \forall H \in \text{Orth},$$

where we have used the identity $A \cdot B = AR \cdot BR$ for $R \in$ Orth. In view of the lemma, we can conclude that $V = FR^T \in$ Pos, and hence

$$F = VR, \qquad \text{where} \qquad V \in \text{Pos} \quad \text{and} \quad R \in \text{Orth}.$$

This is the existence part of the following theorem.

POLAR DECOMPOSITION THEOREM. *If $F \in$ Lin, then there are a (uniquely determined) element $V \in$ Pos and an element $R \in$ Orth such that $F = VR$. Moreover, if $VR = V\bar{R}$, then $\bar{R}R^TV = V$.*

Proof. (*Uniqueness.*) Suppose

$$F = VR = \bar{V}\bar{R}, \quad \text{where} \quad V, \bar{V} \in \text{Pos} \quad \text{and} \quad R, \bar{R} \in \text{Orth}. \tag{1}$$

Then we can conclude that $V \cdot HR^T = \bar{V} \cdot H\bar{R}^T$ for all $H \in$ Orth. Choose $H = H_0$ such that $V \cdot HR^T$ attains its maximum at H_0 for $H \in$ Orth. Thus, by the lemma, we have

$$V = H_0 R^T V \quad \text{and} \quad \bar{V} = H_0 \bar{R}^T \bar{V}.$$

But (1) is equivalent to $F^T = R^T V = \bar{R}^T \bar{V}$. Hence

$$V = H_0 R^T V = H_0 F^T = H_0 \bar{R}^T \bar{V} = \bar{V}.$$

The last part of the theorem follows from $H_0 R^T V = H_0 \bar{R}^T V$. As a bonus we have obtained a characterization of the solution set $\mathbb{S}$ of our problem: if $F = VR$ is a polar decomposition of F, then $\bar{R} \in \mathbb{S}$ iff $\bar{R}R^TV = V$.

In proving the lemma, the following facts will be helpful. Given $a, b \in \mathcal{V}$, the tensor product $a \otimes b$ is the linear map defined by the formula: $(a \otimes b)v = (b \cdot v)a$ for any $v \in \mathcal{V}$. Also, $\operatorname{tr}(a \otimes b) = a \cdot b$, $(a \otimes b)^T = b \otimes a$, and $(a \otimes b)(c \otimes d) = (b \cdot c) a \otimes d$.

Proof of the lemma. (Direct statement.) Assume $A \cdot H \leqslant A \cdot I$ for all $H \in$ Orth. We first prove that $A \in$ Sym. Let $e, f \in \mathcal{V}$ be unit vectors and orthogonal to each other. A simple computation shows that, for any real number α,

$$H_\alpha = I - (1 - \cos\alpha)(e \otimes e + f \otimes f) + \sin\alpha(e \otimes f - f \otimes e)$$

is an element of Orth. From the hypothesis we have

$$0 \leqslant A \cdot (I - H_\alpha) = A \cdot ((1 - \cos\alpha)(e \otimes e + f \otimes f) - \sin\alpha(e \otimes f - f \otimes e))$$

and hence

$$0 \leqslant A \cdot (e \otimes e + f \otimes f - \cot(\alpha/2)(e \otimes f - f \otimes e)) \quad \text{for } \cos\alpha \neq 0. \tag{2}$$

Thus we conclude from (2) that $A \cdot (e \otimes f - f \otimes e) = 0$, or, equivalently, $e \cdot Af = f \cdot Ae$, which implies the symmetry of A.

To show that $A \in$ Pos, note that $H = I - 2e \otimes e$ is in Orth for each unit vector e. Then $A \cdot H = A \cdot I - 2Ae \cdot e$, and, from the hypothesis, $Ae \cdot e \geqslant 0$.

(*Converse statement.*) Assume now that $A \in$ Pos. If $H \in$ Orth, then the symmetric part of $(H - I)$ is $-\frac{1}{2}(H - I)(H - I)^T$, and we have

$$A \cdot (H - I) = -\tfrac{1}{2} A \cdot (H - I)(H - I)^T = -\tfrac{1}{2}(H - I)^T A (H - I) \cdot I. \tag{3}$$

Moreover, as $A \in$ Pos, $(H - I)^T A (H - I) \in$ Pos, and (3) gives $A \cdot (H - I) \leqslant 0$.

(*Characterization of maximizers.*) Suppose that $H_0 \in$ Orth satisfies $A \cdot H_0 = A \cdot I$. Then $A \cdot H_0^T = A \cdot I$ (since $(H_0 - H_0^T)$ is skew-symmetric). From (3) we get that $(H_0^T - I)^T A (H_0^T - I)$, which is in Pos, has vanishing trace. Therefore,

$$(H_0^T - I)v \cdot A(H_0^T - I)v = 0. \tag{4}$$

For given $v \in \mathcal{V}$, set $w=\alpha(H_0^T-I)v+A(H_0^T-I)v$, where α is any real number. Then, using (4),

$$0 \leqslant w \cdot Aw = 2\alpha A(H_0^T-I)v \cdot A(H_0^T-I)v + A(H_0^T-I)v \cdot AA(H_0^T-I)v$$

for all α's, from which it follows that $A(H_0^T-I)v=0$ for all $v \in \mathcal{V}$, or, equivalently, that $A(H_0^T-I)=0$ and hence $(H_0-I)A=0$. This completes the proof.

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BOUNDS ON THE NEGATIVE EXPONENTIAL FUNCTION

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1. Introduction. It is known that the negative exponential function is bounded between consecutive partial sums of its Maclaurin series. (See, e.g., [**3**, p. 104].) That is, for every positive integer n

$$\sum_{r=0}^{2n+1} \frac{(-x)^r}{r!} \leqslant e^{-x} \leqslant \sum_{r=0}^{2n} \frac{(-x)^r}{r!}, \qquad x \geqslant 0. \tag{1.1}$$

We extend this result, showing that

$$\sum_{r=0}^{2n} \frac{(-x)^r}{r!} - \frac{x^{\nu+2n}}{\Gamma(\nu+2n+1)} \leqslant e^{-x} \leqslant \sum_{r=0}^{2n+1} \frac{(-x)^r}{r!} + \frac{x^{\nu+2n+1}}{\Gamma(\nu+2n+2)} \tag{1.2}$$

for all ν such that $0 \leqslant \nu \leqslant 1$. We also generalize the inequalities (1.1) and (1.2) to the confluent hypergeometric and hypergeometric functions. Note that (1.2) reduces to (1.1) for $\nu=0,1$.

2. Preliminary lemma. *A function f on $(0,\infty)$ is said to be completely monotone if it possesses derivatives $f^{(n)}$ of all orders and if for all $x>0$*

$$(-1)^n f^{(n)}(x) \geqslant 0.$$

Typical examples of completely monotone functions are e^{-x} and $(1+x)^{-a}$, where a is a positive number. Other examples are the confluent hypergeometric and the hypergeometric functions, given by

$$\Phi(a,b;-x)=\sum_{r=0}^{\infty} \frac{(a)_r}{(b)_r} \cdot \frac{(-x)^r}{r!}, \qquad b \geqslant a > 0 \tag{2.1}$$

$$F(a,b;c;-x)=\sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{(c)_r} \cdot \frac{(-x)^r}{r!}, \qquad c \geqslant \min(a,b) > 0 \tag{2.2}$$

where $(a)_0=1$ and $(a)_r=a(a+1)\cdots(a+r-1)$. That the functions (2.1) and (2.2) are completely monotone is clearly seen from their integral representation formulas (see, e.g., Abramowitz and Stegun [**1**, 13.2.1, 15.3.1]). Note that $\Phi(a,b;-x)=e^{-x}$ for $a=b$, and $F(a,b;c;-x)=(1+x)^{-a}$ for $b=c$. Let

$$N(x)=\Phi(a,b;-x)-\sum_{r=0}^{n}\frac{(a)_r}{(b)_r}\cdot\frac{(-x)^r}{r!}.$$

COROLLARY 3.1. *If $b\geqslant a\geqslant 1$ and n is even (odd), then for all $x\geqslant 0$ and $0\leqslant\nu\leqslant 1$*

$$N(x)+(-1)^n\frac{(a)_{\nu+n}}{(b)_{\nu+n}}\cdot\frac{(-x)^{\nu+n}}{\Gamma(\nu+n+1)}\geqslant(\leqslant)\,0. \tag{3.4}$$

Note that (3.4) reduces to (1.2) for $a=b$.

Inequalities similar to those given by (3.1) and (3.2) have been derived by Askey [2] for the Jacobi polynomials and Bessel functions.

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RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

THE FIXED POINT PROPERTY FOR NON-EXPANSIVE MAPPINGS, II

SIMEON REICH

The purpose of this paper is to update our previous note with the same title [14]. We feel such a sequel is necessary because many interesting results have been found since [14] was written, although the problem presented there is still open:

Does every weakly compact convex subset C of a Banach space E have the fixed point property for non-expansive mappings?

(Recall that a mapping is said to be non-expansive if it is Lipschitzian with constant 1.)

Let X_r be the space l^2 renormed by

$$|x|_r=\max\{\|x\|_2,r\|x\|_\infty\},$$

where $\|\cdot\|_2$ denotes the l^2 norm and $\|\cdot\|_\infty$ denotes the l^∞ norm. This space has normal structure only if $r<\sqrt{2}$. Nevertheless, Karlovitz [9] was able to show that the answer to our problem was affirmative for $X_{\sqrt{2}}$. (This result can also be obtained by combining Theorem 3 in his earlier paper [8] with the work of Ishikawa [6].) In fact, Karlovitz's proof works for all $r<2$. Much simpler proofs have since been found by Baillon [1] and Baillon and Schöneberg [2]. Baillon's

argument, which works for other spaces as well, has been extended by Bynum [4]. The case $r=2$ is more complicated, but it is known [2] that our problem has an affirmative answer in this case too. All the proofs use the following interesting lemma [8], [5]:

Let C be a weakly compact convex subset of a Banach space E, and let $T: C \to C$ be non-expansive. Suppose that C is minimal in the sense that it contains no proper closed convex subsets which are invariant under T. If $\{x_n\} \subset C$ and $x_n - Tx_n \to 0$, then for each x in C,

$$\lim_{n\to\infty} \|x_n - x\| = \text{diameter}(C).$$

This lemma was also used by Baillon [1] in his proof that the answer to our problem is affirmative if E is uniformly smooth (equivalently, if its dual E^* is uniformly convex). Thus Theorem 1.7 and Proposition 1.10 in [15] are valid for all such spaces. See [16] for another application of this result.

In the setting of the problem, there is always a *separable* closed convex subset of C which is invariant under T. Hence we may assume that E is separable, so that in some sense the problem is about $C[0,1]$. Odell and Sternfeld [12] considered the simpler space c_0 and showed that the closed convex hull of a weakly convergent sequence in c_0 has the fixed point property for non-expansive mappings. Their proof is quite intricate.

Still another direction was pursued by Lifschitz in [10]. He showed that to each complete metric space X there corresponds a number $k(X) \geqslant 1$, such that if $T: X \to X$ has bounded iterates and all of its powers have Lipschitz constant $q<k$ then T has a fixed point. For Hilbert space, $\sqrt{2} < k \leqslant \pi/2$ [1].

We remark in passing that Ray [13] has shown that if a closed convex subset of a Hilbert space is *unbounded* then it does *not* possess the fixed point property for non-expansive mappings.

What about weak* compact convex subsets of E^*? Some of them have the fixed point property for non-expansive mappings: For example, those with normal structure, subsets of l^1 [7], and balls in L^∞ [18], [17]. But not all of them do—the positive part of the unit ball in l^1 renormed by $|x| = \max\{\|x^+\|_1, \|x^-\|_1\}$ provides us with a counterexample [11].

This seems to indicate that our problem may also have a negative answer. On the other hand, it is reasonable to expect that the answer is affirmative if E is reflexive and strictly convex.

Finally, we wish to point out that an affirmative answer to our problem would imply that every weakly compact convex subset of a Banach space has the *common* fixed point property for non-expansive mappings [3]. It would also yield interesting properties of accretive operators. It would imply, for example, that an m-accretive operator with a bounded domain in a reflexive space is surjective, and that [15, Theorem 1.4] is true for all reflexive spaces.

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CLASSROOM NOTES

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ON THE REFLECTIVE PROPERTY OF ELLIPSES

JAMES H. FOSTER AND JEAN J. PEDERSEN

Ellipses have the interesting reflective property that, if light rays emanating from one focus reflect off the ellipse, they will converge on the other focus. The proof of this, using calculus, appears as a worked example or an exercise in most first-year calculus textbooks. A simple, noncalculus proof and an argument using force/energy concepts are given here.

For the noncalculus approach we use the following result attributed to Heron of Alexandria and illustrated in Figure 1. Its proof is based on the observation that any possible path AQB is equal in length to the reflected path AQB' and the shortest distance between the points A and B' is the straight line AB'.

Heron's Result. Given a straight line L and two points A and B on the same side of L, the point P on L minimizes the sum of the distances $AP+PB$ if and only if the angles between AP and L and between BP and L are equal.

An ellipse may be characterized by the

DEFINITION. An ellipse is the locus of points whose sum of distances from two fixed points, called the foci, is constant.

Since light travels so that the angle of incidence equals the angle of reflection, the reflective property of the ellipse follows from the

THEOREM. *If P is any point on an ellipse with foci F_1 and F_2, then the focal radii F_1P and F_2P make equal angles with the tangent line L to the ellipse at point P.* (See Fig. 2.)

Proof. By Heron's result it suffices to show that P minimizes the sum of the distances F_1Q+QF_2 over all points Q on L. So for any point Q, distinct from P, on L let R be the point of intersection of F_1Q and the ellipse. Then, using in sequence the definition of an ellipse, the triangle inequality, and a substitution, it follows that

$$F_1P+PF_2=F_1R+RF_2<F_1R+RQ+QF_2=F_1Q+QF_2. \blacksquare$$

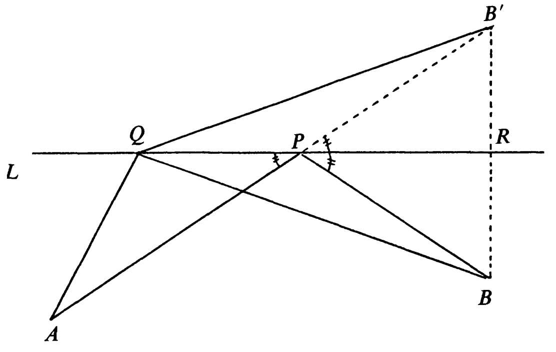

FIG. 1

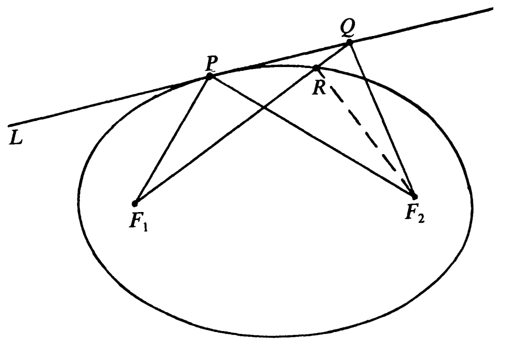

FIG. 2

The following physical interpretation can also be used to demonstrate this reflective property of ellipses. It is a version by G. Pólya of a well-known type of argument that uses the concepts of vector force and potential energy from mechanics. Suppose at each focus there is a central, attractive force with constant magnitude f. For example, if the ellipse is on a horizontal plane, each force could be produced by the pull of a weightless string which passes over a frictionless pulley located at the respective focus and supports an object of weight f (see Fig. 3.) Since the potential energy of a force with constant magnitude is force times distance, the combined potential energy of these two forces at any point on the plane is P.E. $=fd_1+fd_2$, where d_1 and d_2 are the distances from the point to the foci and the potential energy of each force is taken to be zero at its focus. Hence, the curve on which the potential energy is equal to a constant E_0 is the ellipse $d_1+d_2=E_0/f$. Let P be a point on this equipotential ellipse, $\vec{f_1}$ and $\vec{f_2}$ be the vector forces acting toward the foci F_1 and F_2 at point P, and α and β be the respective angles between these forces and the line tangent to this equipotential ellipse at P (see Fig. 4). Since these forces have the same magnitude, their resultant force $\vec{f_R}$ bisects the angle between them. This, together with the fact that the force at any point is perpendicular to the equipotential curve (see [**4**, p. 121]), implies that the angles α and β are equal.

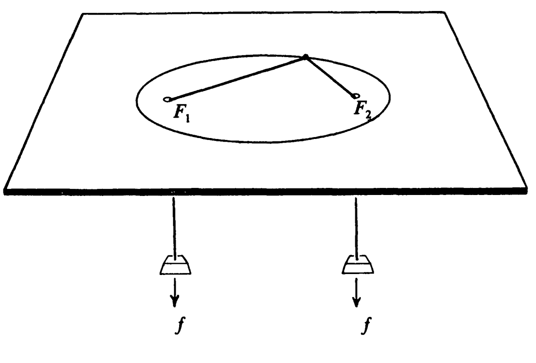

FIG. 3

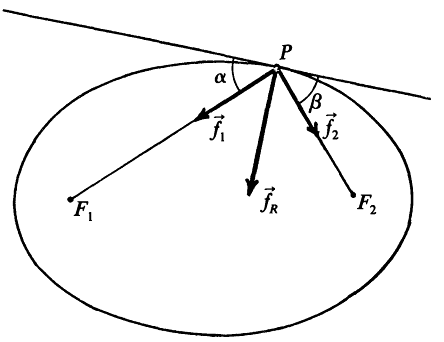

FIG. 4

REMARKS. (1) See [**5**] for a similar mathematical proof (pp. 142–145) and a different physical interpretation (pp. 146–147) of this reflective property. A nice vector calculus proof may be found in [**1**].

(2) This reflective property has been used to study aircraft noise in wind-tunnel tests (see [**3**]). A microphone placed at one focus of an ellipsoidal mirror picks up noises originating at the opposite focus only and effectively rejects noises originating at other locations (see Fig. 5). The noise "scene" can be scanned by moving the mirror-microphone assembly. Thus, specific noise

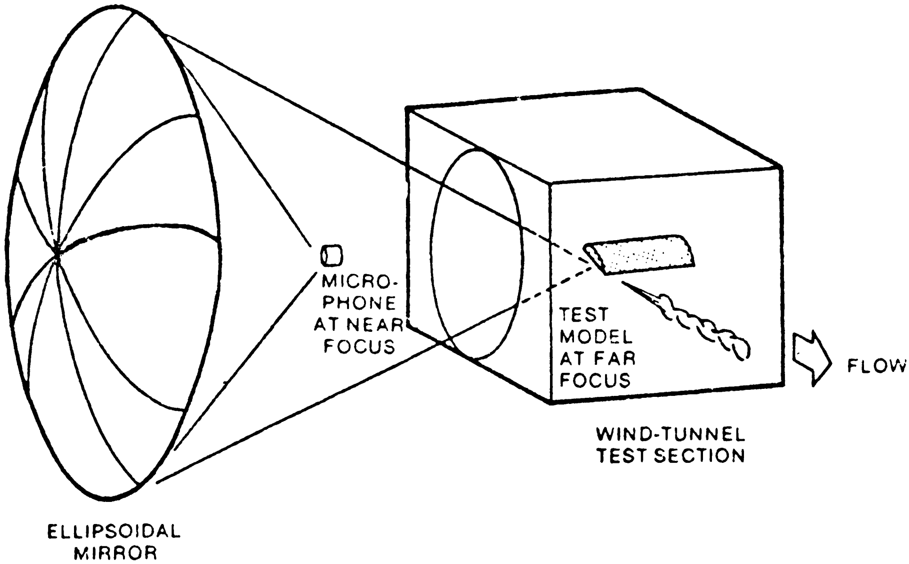

FIG. 5

sources can be located and studied without interference from other sources.

(3) This reflective property has also been used to send relatively large amounts of light through very small openings, as in pinhole lamps used in some theaters (see [2]). This is illustrated in Figure 6, where the reflective surface is an ellipsoid whose foci are at the light source and pinhole.

(4) Other reflective properties of the ellipse for light rays not passing through the foci are discussed in [7, p. 239].

(5) The corresponding reflective properties of the parabola and hyperbola (as stated, for example, in [6, pp. 406 and 415]) can be demonstrated by a modification of the force/energy argument above.

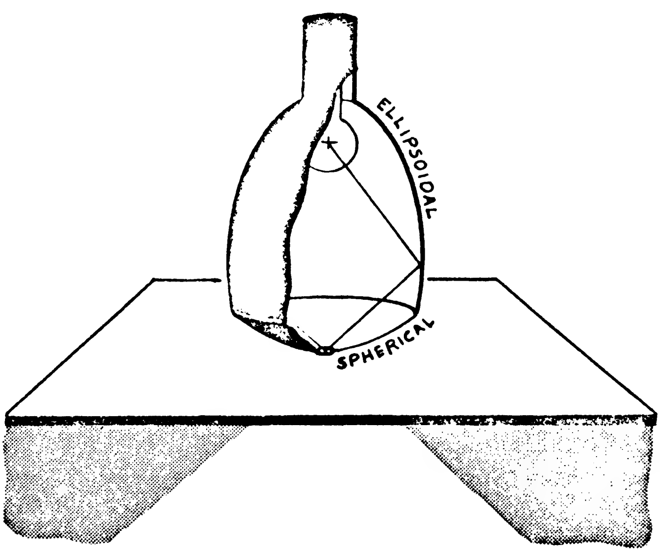

FIG. 6

Acknowledgment. The authors thank Professor Pólya for his interest in and comments about this problem.

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FOUR OF A KIND IN PINOCHLE

STANLEY P. GUDDER

1. Introduction. In finite mathematics and elementary probability courses one frequently computes probabilities for certain hands in poker and bridge. The game of pinochle is seldom considered. The probabilities in pinochle are considerably harder to compute and contain subtleties that are not present in most other card games. We shall confine ourselves to finding the probability of four of a kind in pinochle. To compute this probability most efficiently, we shall need careful counting techniques, the inclusion-exclusion principle, Bayes's rule, and inequality estimates. Thus, in a single example, we can illustrate many of the basic principles of probability theory.

The pinochle deck is formed by combining two regular 52 card decks and then discarding all cards below the nine. Hence, there are 48 cards in all, with eight cards of each rank from nine through ace. There are four players and each has a twelve-card hand. A hand is said to contain four aces if it has an ace from each of the four different suits. Four kings, queens, jacks are defined similarly. A hand is said to contain four of a kind if it has four aces or four kings or four queens or four jacks. If a hand contains four aces and four kings, for example, it contains four of a kind.

2. Four of a Kind in a Hand. Let A be the event that a single pinochle hand contains four aces. We would like to find the probability $P(A)$. Since the binomial symbol $\binom{n}{r}=n!/(n-r)!r!$ gives the number of combinations of n objects taken r at a time, the number of pinochle hands is $N_H=\binom{48}{12}$. If N_A is the number of hands containing four aces, then $P(A)=N_A/N_H$. Because there are two choices for each of the four aces and the other eight cards are combinations from the remaining 44 cards, one might suspect that $N_A=2^4\binom{44}{8}$. Unfortunately, this reasoning counts some of the hands twice. To see this, let AS_1, AS_2 denote the two aces of spades; AH_1, AH_2 the two aces of hearts; etc. Then for example we have counted

$$AS_1, AH_1, AD_1, AC_1, AS_2, X_1,\ldots,X_7$$

and

$$AS_2, AH_1, AD_1, AC_1, AS_1, X_1,\ldots,X_7$$

(where $X_1,\ldots,X_7$ represent seven of the remaining cards) as two different hands, when in fact they are the same. The correct expression is

$$N_A=2^4\binom{40}{8}+4\cdot 2^3\binom{40}{7}+6\cdot 2^2\binom{40}{6}+4\cdot 2\binom{40}{5}+\binom{40}{4}. \tag{1}$$

The nth term in equation (1) is the number of ways of getting four aces with precisely n side

aces, $n=0, 1, 2, 3, 4$. We thus obtain

$$P(A)=N_A/N_H=105293/3811606\approx 1/36.2.$$

What is the probability $P(F)$ of four of a kind in a single pinochle hand? If $P(K)$, $P(Q)$, $P(J)$ give the probability of four kings, queens, and jacks, respectively, applying the inclusion-exclusion principle gives

$$\begin{aligned}P(F)=&P(A\cup K\cup Q\cup J)=P(A)+P(K)+P(Q)+P(J)\\&-P(A\cap K)-P(A\cap Q)-P(A\cap J)-P(K\cap Q)-P(K\cap J)-P(Q\cap J)\\&+P(A\cap K\cap Q)+P(A\cap K\cap J)+P(A\cap Q\cap J)+P(K\cap Q\cap J)\\&-P(A\cap K\cap Q\cap J)\\=&4P(A)-6P(A\cap K)+4P(A\cap K\cap Q).\end{aligned}\tag{2}$$

Using reasoning similar to that in (1) we obtain

$$N_{A\cap K}=2^8\binom{32}{4}+8\cdot 2^7\binom{32}{3}+2^6\binom{8}{2}\binom{32}{2}+2^5\binom{8}{3}\binom{32}{1}+2^4\binom{8}{4}.\tag{3}$$

Now $P(A\cap K)=N_{A\cap K}/N_H$. Since $A\cap K\cap Q$ allows no repeated cards, $P(A\cap K\cap Q)=2^{12}/N_H$. We can evaluate the expressions in (2) to obtain

$$P(F)=0.1091857\approx 1/9.2.$$

3. Four of a Kind in a Deal. What is the probability of four of a kind in a pinochle deal? That is, what is the probability $P(D_4)$ that at least one of the four hands in a single pinochle deal contains four of a kind? Let B, C, D, and E be the events that the first, second, third, and fourth hands, respectively, contain four of a kind. Applying the inclusion-exclusion principle gives

$$P(D_4)=4P(F)-6P(B\cap C)+4P(B\cap C\cap D)-P(B\cap C\cap D\cap E).\tag{4}$$

The last three terms on the right-hand side of equation (4) are extremely tedious to compute. For this reason, we shall compute only $P(B\cap C)$ exactly (we shall approximate the last two terms later). This will enable us to solve the following problem exactly. What is the probability $P(F_2)$ that four of a kind shows in at least one of the first two hands of a pinochle deal? This probability is

$$\begin{aligned}P(F_2)=&P(B\cup C)=P(B)+P(C)-P(B\cap C)\\=&2P(F)-P(B\cap C).\end{aligned}\tag{5}$$

The probability $P(B\cap C)$ is most easily obtained by using the formula $P(B\cap C)=P(B)P(C|B)$ where $P(C|B)$ is the conditional probability of C given B. We can find $P(C|B)$ by applying the inclusion-exclusion principle

$$\begin{aligned}P(C|B)=&P(A\cup K\cup Q\cup J|B)\\=&4P(A|B)-6P(A\cap K|B)+4P(A\cap K\cap Q|B).\end{aligned}\tag{6}$$

Of course, $P(A|B)$ is the probability of four aces in the second hand given four of a kind in the first hand, etc. We now evaluate the three terms on the right-hand side of equation (6). Using Bayes's rule we have $P(A|B)=P(B|A)P(A)/P(B)$. We have already found $P(A)$ and $P(B)=P(F)$, and the factor $P(B|A)$ is again computed using the inclusion-exclusion principle.

$$\begin{aligned}P(B|A)=&P(A\cup K\cup Q\cup J|A)=P(A|A)+3P(K|A)-3P(A\cap K|A)\\&-3P(K\cap Q|A)+3P(A\cap K\cap Q|A)+P(K\cap Q\cap J|A)\end{aligned}\tag{7}$$

The terms on the right-hand side of equation (7) are evaluated using reasoning similar to that in equations (1) and (3). For example, $P(A|A)$, that is, the probability of four aces in the first hand, given four aces in the second hand, is $\binom{44}{12}^{-1}\binom{40}{8}$,

$$P(K|A)=\binom{44}{12}^{-1}\left[2^4\binom{36}{8}+4\cdot 2^3\binom{36}{7}+6\cdot 2^2\binom{36}{6}+4\cdot 2\binom{36}{5}+\binom{36}{4}\right],$$

and

$$P(A\cap K|A)=\binom{44}{12}^{-1}\left[2^4\binom{32}{4}+4\cdot 2^3\binom{32}{3}+6\cdot 2^2\binom{32}{2}+4\cdot 2\binom{32}{1}+1\right].$$

Similarly,

$$P(A\cap K|B)=P(B|A\cap K)P(A\cap K)/P(B)$$

where

$$\begin{aligned}P(B|A\cap K)=&\,2P(A|A\cap K)+2P(Q|A\cap K)\\&-P(A\cap K|A\cap K)-4P(A\cap Q|A\cap K)-P(Q\cap J|A\cap K)\\&+2P(A\cap K\cap Q|A\cap K)+2P(A\cap Q\cap J|A\cap K).\end{aligned}$$

Finally,

$$P(A\cap K\cap Q|B)=P(B|A\cap K\cap Q)P(A\cap K\cap Q)/P(B)$$

where

$$\begin{aligned}P(B|A\cap K\cap Q)=&\,3P(A|A\cap K\cap Q)+P(J|A\cap K\cap Q)\\&-3P(A\cap K|A\cap K\cap Q)-3P(A\cap J|A\cap K\cap Q)\\&+P(A\cap K\cap Q|A\cap K\cap Q)+3P(A\cap K\cap J|A\cap K\cap Q).\end{aligned}$$

Evaluating all these expressions gives

$$P(B\cap C)=0.0126830$$

and

$$P(F_2)=0.2056884\approx 1/4.9.$$

Let us now return to the original problem, equation (4). Although the author has carried out the tedious calculations ($P(D_4)=0.36596$), it is more instructive to show how to estimate $P(D_4)$ in terms of the probabilities we have computed above. Since

$$P(B\cap C\cap D\cap E)<P(B\cap C\cap D)<P(B\cap C),$$

we have

$$\begin{aligned}P(D_4)&>4P(F)-6P(B\cap C)+3P(B\cap C\cap D)>4P(F)-6P(B\cap C)\\P(D_4)&<4P(F)-6P(B\cap C)+4P(B\cap C\cap D)<4P(F)-2P(B\cap C).\end{aligned}$$

These are closely related to the Bonferroni inequalities. Applying (5), we have

$$2P(F_2)-4P(B\cap C)<P(D_4)<2P(F_2).$$

Substituting the values of these quantities gives $0.3606<P(D_4)<0.4114$.

Since the last two terms of (4) would be considerably smaller than the others, the approximation

$$P(D_4)\cong 2P(F_2)-4P(B\cap C)=0.3606\cong\frac{1}{2.8}$$

is good enough for practical purposes.

Similar methods can be used to find the probabilities of occurrence of A, K, Q, J, 10 in a given hand or in a deal. These numbers are also of interest at the pinochle table.

Acknowledgment. The author would like to thank Daniel Shine, who originally introduced him to this problem and checked some of the above calculations.

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SIMILARITY OF MATRICES

WILLIAM WATKINS

It is well known that if A and B are $n\times n$ real matrices and if they are similar over the complex numbers then they are similar over the real numbers. Although this theorem is understandable to a beginning linear algebra student, its proof is usually part of the more advanced theory of similarity invariants [2, p. 144], [1, p. 203]. The purpose of this note is to give an elementary proof that requires only some simple facts about determinants and polynomials. First, if $\det X\neq 0$, then X is nonsingular. Second, a nonzero polynomial has only finitely many roots.

THEOREM. *Let A and B be real $n\times n$ matrices. If A is similar to B over the complexes, then A is similar to B over the reals.*

Proof. Suppose that $A=S^{-1}BS$, for some nonsingular complex matrix S. There are real matrices P and Q such that $S=P+iQ$. Then $(P+iQ)A=B(P+iQ)$; and since A, B, P, and Q are all real, we have $PA=BP$ and $QA=BQ$. If either P or Q is nonsingular, then we are finished. Even if both P and Q are singular, we are finished if there is a real number r such that $P+rQ$ is nonsingular. For then $(P+rQ)A=B(P+rQ)$.

We know that the polynomial $p(x)=\det(P+xQ)$ is not identically zero since $p(i)=\det S\neq 0$. It follows that in any infinite set there is an element r such that $p(r)\neq 0$. In particular there is a real number r such that $p(r)\neq 0$. But then $P+rQ$ is a real nonsingular matrix and we are finished.

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2. S. Perlis, Theory of Matrices, Addison-Wesley, Reading, Mass., 1952.

DEPARTMENT OF MATHEMATICS, CALIFORNIA STATE UNIVERSITY, NORTHRIDGE, CA 91330.

MATHEMATICAL EDUCATION

EDITED BY W. E. MASTROCOLA

Material for this department should be sent to W. E. Mastrocola, Department of Mathematics, Colgate University, Hamilton, NY 13346.

THE MATHEMATICS CONTEST AT STOCKTON STATE COLLEGE

CHARLES W. HERLANDS

Stockton State College is the newest of the eight New Jersey State Colleges. Practically all its students are from New Jersey, and a substantial majority of newly admitted students are recent high school graduates.

In 1976 the Faculty of the Mathematics Program at Stockton State College developed a mathematics contest for New Jersey high school students. The first contest took place in February 1977, with more than 300 competitors from over 50 high schools; participation has increased each year, to more than 350 students in 1979.

The contest consists of a $2\frac{1}{2}$-hour examination containing 25 multiple-choice problems that can be solved without using college-level mathematics and yet are nontrivial, plus 3 or 4 "tie-breaker" problems. Twelve-question "sample tests," prepared before the preliminary announcements are mailed, are sent on request.

Students work the examination individually. The top five individual efforts receive cash prizes and trophies. The sum of the top five scores from each school constitutes that school's "team" score; the top three schools and their fifteen "team" members also receive trophies.

The contest takes place in the morning; the results are announced at an afternoon Awards Ceremony. While the examinations are being marked, students may eat lunch, tour campus facilities, and attend demonstrations of computer graphics, analog computing, and a physics "magic show." In 1978 a panel discussion on careers in mathematics was also featured.

The Mathematics Contest has stimulated interest in mathematics at the high schools and has encouraged and rewarded excellence in the high school mathematics programs. Some schools begin practicing early for the contest; some schools (the same ones?) use the contest as preparation, or as a competitive selection tool, for the MAA Exam in March.

The contest also affords our faculty an opportunity to get to know some of the high school mathematics teachers, to hear their concerns, and to answer their questions. The divisional faculty and the high school teachers meet informally while the students are taking the written examination in the morning.

The Mathematics Program has greatly increased its contact with nearby professional organizations and commercial enterprises which encourage or employ mathematicians. Several actuarial societies, the Institute of Electrical and Electronics Engineers, and the Atlantic City Electric Company, for example, have helped to sponsor the contest and have contributed generously toward the prizes.

The students obviously enjoy the challenging questions on the examination. Students turn in only their answer sheets; they may keep the examination itself, and answers (but not complete solutions) are available as soon as all answer sheets have been collected. Many students and teachers spend much of the early afternoon discussing and arguing over examination questions.

Reference

1. D. L. Sherry and J. R. Weaver, The mathematics olympics at the University of West Florida, this MONTHLY, 86 (1979) 125–126.

DIVISION OF NATURAL SCIENCES AND MATHEMATICS, STOCKTON STATE COLLEGE, POMONA, NJ 08240.

PROBLEMS AND SOLUTIONS

EDITED BY VLADIMIR DROBOT

*Send all **proposed** problems, in duplicate if possible, to Professor Vladimir Drobot, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053. Please include solutions, relevant references, etc.*

An asterisk (*) indicates that neither the proposer nor the editors supplied a solution.

***Solutions** should be sent to the addresses given at the head of each problem set.*

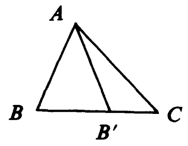
A publishable solution must, above all, be correct. Given correctness, elegance and conciseness are preferred. The answer to the problem should appear right at the beginning. If your method yields a more general result, so much the better. If you discover that a MONTHLY *problem has already been solved in the literature, you should of course tell the editors; include a copy of the solution if you can. Before taking the trouble to write up a very lengthy solution, you may ask the appropriate editor whether any solutions have already been received. Enclose a self-addressed card or envelope.*

PROBLEMS DEDICATED TO EMORY P. STARKE

Solutions of these problems dedicated to E. P. Starke should be mailed to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131, by August 31, 1980. To facilitate consideration, type with double spacing in the format used below. If acknowledgment is desired, include a self-addressed card, label, or envelope.

S 29. *Proposed by Clark Kimberling, University of Evansville.*

Suppose $T=ABC$ is a triangle having sides $AB<AC<BC$ and a point B' on segment BC satisfying $AB'=AB$. Call T *admissible* if the shortest side of triangle $T'=AB'C$ does not touch the shortest side of T, i.e., the shortest side of T' is $B'C$.



(a). Characterize all T for which the sequence $T_1=T, T_2=T_1', T_3=T_2', \ldots$ consists exclusively of admissible triangles.

(b). For such T, let s_n be the length of the shortest side of T_n and determine $\lim_{n\to\infty}(s_n/s_{n+1})$.

*(c). For such T, let P be the limit point of the nested triangles T_n and determine the angle APB.

SOLUTIONS OF PROBLEMS DEDICATED TO EMORY P. STARKE

A Cyclic Power Inequality

S 6 [1979, 222]. *Proposed by M. S. Klamkin and A. Meir, University of Alberta.*

Let $x_i>0$ for $i=1,2,\ldots,n$ with $n\geqslant 2$. Prove that

$$(x_1)^{x_2}+(x_2)^{x_3}+\cdots+(x_{n-1})^{x_n}+(x_n)^{x_1}\geqslant 1.$$

Solution by David Hammer, University of California, Davis. For $n=2$, the solution is in E 1342 [1959, 513]. Hence we assume that $n>2$. We also may assume that $0<x_i<1$ for all i. Let S be the given sum. Since S is invariant under cyclic permutation of the x_i, we may assume that x_3 is minimal among the x_i and hence that $x_3\leqslant x_1$. Then

$$S\geqslant (x_1)^{x_2}+(x_2)^{x_3}\geqslant (x_3)^{x_2}+(x_2)^{x_3}\geqslant 1$$

as desired.

The sharpness of the inequality is shown by the example with $n=4$ and

$$a=r^{-r^r},\quad b=r^{-r},\quad c=r^{-1},\quad d=1.$$

Then $a^b+b^c+c^d+d^a=3r^{-1}+1$ and one can let $r\to\infty$.

A slight generalization is the following: Let $x_i>0$ for $i=1,2,\ldots,n$ and let s be a permutation of $1,2,\ldots,n$, which is the product of m disjoint cycles of length at least 2 and has k fixed points. Then

$$\sum_{i=1}^{n}(x_i)^{x_{s(i)}}\geqslant m+k\left(\frac{1}{e}\right)^{1/e}.$$

This follows immediately from S 6 and the well-known fact that the minimum of x^x occurs when $x=1/e$.

Editorial note. F. S. Cater gave the generalization that

$$(x_1)^{x_2}+(x_2)^{x_3}+\cdots+(x_{n-1})^{x_n}+(x_n)^{x_1}>1+(n-2)\min(x_1^{x_2},x_2^{x_3},\ldots,x_{n-1}^{x_n},x_n^{x_1}).$$

Also solved by Robert Breusch, D. Carlson, CWRU Problem Solving Team, F. S. Cater, Don Deal, Bob Dickinson (Canada), Thomas H. Foregger, Gustaf Gripenberg (Finland), Steven Janke & Stephen J. Schiffman, Robert Kowalski, L. Kuipers (Switzerland), Man Kam Kwong, J. P. Lambert, O. P. Lossers (Netherlands), Aries Matsoukas (Greece) & Achilles Venetoulias, L. E. Mattics, Tim McMillan, R. G. Nath, David Richman, Adam Riese, Michael Skalsky, Freek Wiedijk & Jan van de Craats (Netherlands), one anonymous solver and the proposers.

ELEMENTARY PROBLEMS

Solutions of these Elementary Problems should be mailed in duplicate to Dr. J. L. Brenner, 10 Phillips Road, Palo Alto, CA 94303 (U.S.A.), by August 31, 1980. Please place the solver's name and mailing address on each (double-spaced) sheet. Include a self-addressed card or label (for acknowledgment).

E 2826. *Proposed by Heinz W. Engl and Lewis Lum, University of Delaware.*

Prove or disprove: Let X be an arcwise connected compact metric space with more than one point. Suppose no two points are connected by more than one arc. Let $\{D_n\}_{n=1}^{\infty}$ be a decreasing $(D_{n+1}\subseteq D_n)$ sequence of dense subsets with arcwise connected complements. If $\bigcap_{n=1}^{\infty}\operatorname{int}(D_n)\neq\varnothing$ then $\bigcap_{n=1}^{\infty}D_n$ is not a single point.

E 2827. *Proposed by Gérard Letac, Université Paul Sabatier, Toulouse, France.*

The n vertices of a tree are labeled with the integers $1,2,\ldots,n$. Let $d(i,j)$ denote the number of edges between i and j. Compute the determinant of the $n\times n$ matrix with (i,j) element $x^{d(i,j)}$.

E 2828. *Proposed by Jerrold W. Grossman and Hai-Ping Ko, Oakland University, Rochester, Mich.*

Prove that $\sum C_i^n(i+1)^{i-1}(j+1)^{j-1}=2(n+2)^{n-1}$, the sum being extended over all nonnegative integers i,j such that $i+j=n$.

E 2829. *Proposed by Mark Meyerson, U.S. Naval Academy, Annapolis.*

Prove or disprove: In a connected metric space of finite diameter, in which every point has a neighborhood homeomorphic to E^n for some fixed n, every pair of points can be connected by a path (a continuous map of an interval) of finite length. (The length of a path is defined as the limit of the sum of the distances between the images of the endpoints of the subintervals in a subdivision of the interval as the norm of the subdivision approaches zero.)

SOLUTIONS OF ELEMENTARY PROBLEMS

An Iterated Function

E 984 [1951, 564; 1952, 252, 408; 1977, 739]. *Proposed by Joseph Rosenbaum, Hartford Conn.*

(a) Find $f(x)$ when $f[f(x)]=x^2-2$.

(b) More generally, find $f_1(x)$ when $f_n(x)=x^2-2$, where $f_n(x)$ is defined by the relation $f_{r+1}(x)=f_1[f_r(x)]$.

The following references are pertinent to this problem.

1. The article of Rice, Schweizer, and Sklar that appears in this issue (pp. 252–263).
2. A technical report by Rufus Isaacs, On fractional iteration, Johns Hopkins University (Department of Mathematical Sciences) Tech. Report 320. The report offers some new directions for research.

Triangle Centroid

E 2715 [1978, 384; 1979, 705]. *Proposed by Jack Garfunkel, Flushing, N.Y.*

Let G be the centroid of the triangle $A_1A_2A_3$ and let

$$\theta_i = \angle(\overrightarrow{A_iA_{i+1}}, \overrightarrow{A_iG}), \qquad (i=1,2,3).$$

Prove or disprove that $\Sigma \sin\theta_i \leqslant 3/2$.

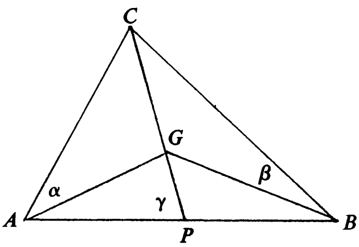
II. *Solution by C. S. Gardner, University of Texas at Austin.* In triangle ABC, let h_a, h_b, h_c be the altitudes from A, B, C, respectively; let m_a, m_b, m_c be the medians. Suppose $AC \leqslant BA \leqslant CB$. Then $h_b \geqslant h_c \geqslant h_a$, $m_b \geqslant m_c \geqslant m_a$. Thus $(h_b - h_c)/m_c \leqslant (h_b - h_c)/m_a$, so that $h_b/m_c + h_c/m_a \leqslant h_b/m_a + h_c/m_c$. It is known that the proposed inequality is equivalent to the relation

$$h_a/m_b + h_b/m_c + h_c/m_a \leqslant 3. \qquad (*)$$

Clearly (*) will follow if the stronger inequality

$$h_a/m_b + h_b/m_a + h_c/m_c \leqslant 3 \qquad (**)$$

is established.



See the figure. Let G be the centroid of the triangle; let P be the midpoint of side AB. The angles α, β, γ are: $\alpha = GAC$, $\beta = GBC$, $\gamma = CPA$. Note that $\gamma \leqslant 90°$. The inequality (**) is the same as

$$2\sin\alpha + 2\sin\beta + \sin\gamma \leqslant 3, \qquad (***)$$

which will now be proved. Let B' be the point (on PB) at which a circle through G and C is tangent to PB. Then the angle β' subtended by segment GC at B' exceeds the angle subtended by GC at any other point of the ray PB. (See Heinrich Dorrie, *100 Great Problems of Elementary Mathematics*, translated by David Antin, Dover, 1965.)

Change the scale so that $PC = 3$, $PG = 1$. Then (since $PB'^2 = PG \cdot PC$) $PB' = \sqrt{3}$. Moreover, $\beta \leqslant \beta' \leqslant 90°$, so that $\sin\beta \leqslant \sin\beta' = \sin\gamma/(2 + \sqrt{3}\cdot\cos\gamma)$. Similarly, $\alpha \leqslant \alpha'$, where α' is the greatest angle that GC subtends at any point on the ray PA. Note $\sin\alpha' = \sin\gamma/(2 - \sqrt{3}\cdot\cos\gamma)$. There are two cases.

Case 1. Suppose $\gamma \geqslant 30°$. Then $\alpha' \leqslant 90°$, $\sin\alpha \leqslant \sin\alpha'$, and thus $2\sin\alpha + 2\sin\beta + \sin\gamma - 3 \leqslant 2\sin\alpha' + 2\sin\beta' + \sin\gamma - 3$. Since $(4 - 3\cos^2\gamma)\ (2\sin\alpha' + 2\sin\beta' + \sin\gamma - 3) \leqslant 3(\sin\gamma - 1)^3 \leqslant 0$, (***) is proved in this case.

Case 2. Suppose $\gamma \leqslant 30°$. Then $2\sin\alpha + 2\sin\beta + \sin\gamma - 3 \leqslant 2 + 2\sin\beta' + \sin\gamma - 3 \leqslant 2 + 2(\frac{1}{7}) + (\frac{1}{2}) - 3 < 0$.

Sequence of Polynomials

E 2737 [1978, 764]. *Proposed by Robert Ross Wilson, California State University, Long Beach.*

Define a sequence of polynomials by $P_0 = 1$, $P_1 = x + 1$, and $P_{n+1} = P_n + xP_{n-1}(n \geqslant 1)$. Show that all roots of each P_n are real.

I. *Solution by A. J. Douglas and G. T. Vickers, University of Sheffield, England.* Let $Q_0=f$, $Q_1=g+f^2$ be any two functions with $g\neq 0$. Define Q_{n+1} recursively by

$$Q_{n+1}=fQ_n+gQ_{n-1}. \qquad (*)$$

We show below that for any n, $Q_n=0$ if and only if

$$f^2+4g\cos^2 k\pi/(n+2)=0, \quad 1\leqslant k\leqslant\left[\tfrac{1}{2}(n+1)\right]. \qquad (1)$$

Setting $f=1$, $g=x$, we obtain the sequence of polynomials P_n. It follows that $P_n=0$ if and only if $1+4x\cos^2 k\pi/(n+2)=0$, i.e., $x=-\frac{1}{4}\sec^2 k\pi/(n+2)$. Hence all the roots of each P_n are real.

Let α,β be the characteristic roots of the second-order linear difference equation (*), obtained by solving the auxiliary equation $\lambda^2-f\lambda-g=0$. Then Q_n is given by

$$Q_n=\begin{cases} A\alpha^n+B\beta^n, & \text{if } \alpha\neq\beta,\\ (C+Dn)\alpha^n, & \text{if } \alpha=\beta,\end{cases} \qquad (2)$$

where A,B,C,D are constants.

Let $s=(f^2+4g)^{1/2}$. Using $Q_0=f$, $Q_1=f^2+g$, the following is valid for $n=0,1$, and hence for all n:

$$Q_n=\begin{cases} s^{-1}2^{-n-2}\left((f+s)^{n+2}-(f-s)^{n+2}\right), & \text{if } s\neq 0,\\ (n+2)\left(\tfrac{1}{2}f\right)^{n+1}, & \text{if } s=0.\end{cases}$$

Now let $Q_n=0$. Since $g\neq 0$, we have $(z+1)^{n+2}=(z-1)^{n+2}$, $z=f/s$. Thus $z+1=(z-1)\omega$, where ω is an $(n+2)$nd root of unity, and $z=f/s=(\omega+1)/(\omega-1)=i\cot\pi k/(n+2)$, $1\leqslant k\leqslant n+1$. This implies (1). Conversely, if (1) is satisfied, then $Q_n=0$.

II. *Solution by F. B. Strauss, University of Texas at El Paso.* For $x=u+iv$ define r_n and s_n to be the real and imaginary parts, respectively, of $P_n(x)$. Then $r_{n+1}+is_{n+1}=r_n+is_n+(u+iv)(r_{n-1}+is_{n-1})$ and hence $r_{n+1}=r_n+ur_{n-1}-vs_{n-1}$ and $s_{n+1}=s_n+vr_{n-1}+us_{n-1}$. Suppose there is a number $a+bi$ with $b\neq 0$ such that the set of positive integers n for which $P_n(a+bi)=0$ is nonempty. Suppose m is the least positive integer so that P_{m+1} has $a+bi$ as a root. Since the coefficients of each polynomial are real, $a-bi$ is also a root of P_{m+1}. It follows that $r_{m+1}=0=r_m+ar_{m-1}-bs_{m-1}=r_m+ar_{m-1}+bs_{m-1}$ and that $s_{m+1}=0=s_m+br_{m-1}+as_{m-1}=s_m-br_{m-1}+as_{m-1}$. Hence $-bs_{m-1}=bs_{m-1}$ and $br_{m-1}=-br_{m-1}$. Thus, since $b\neq 0$, $s_{m-1}=0=r_{m-1}$. But then $a+bi$ is a root of P_{m-1}, which contradicts our choice of m. Hence all the roots of each P_n are real.

Editor's Remark. Many solvers proved the assertion by showing that the zeros of P_n and P_{n+1} interlace. It may be noted that the same property is enjoyed by several sets of classical polynomials—the polynomials of Chebyshev, Hermite, Legendre, Jacobi, Gegenbauer.

Also solved by Michael H. Albert (Canada), John Avila, C. W. Barnes, Irl Bivens, D. M. Bloom, Theodore S. Bolis, Duane M. Broline, Case Western Reserve University Problem Solving Team, Robert T. Cunningham, Boris Datskovsky, Mary Dowlen, W. O. Egerland & J. L. Graham, C. T. Giel, Graceland College Mathematics Research Group, Gustaf Gripenberg (Finland), JoAnne Growney, Doug Hensley, Eli L. Isaacson, Thomas Jager, A. A. Jagers (Netherlands), W. Lam, R. Sherman Lehman, O. P. Lossers (Netherlands), W. Fred Martens, L. E. Mattics, Wendell H. Mills, J. Navratil (Czechoslovakia), J. W. Nienhuys (Netherlands), S. Ricci, Otto G. Ruehr, Santa Clara Problem Solving Ring, Harold Shulman, Joseph Silverman, Michael Skalsky, J. M. Stark, L. van Hamme (Belgium), Paul A. Vojta, Robert Weinstock, David Zeitlin, and the proposer.

A Determinant with Reciprocal Factorials

E 2747 [1978, 824]. *Proposed by H. L. Krall, Pennsylvania State University, and E. Grosswald, Temple University.*

Compute the determinant of the matrix $A=(a_{ij})$ where $0\leqslant i,j\leqslant n-1$ and $a_{ij}=1/(i+j+1)!$.

Solution by Eli L. Isaacson, New York University. Let D denote the determinant. Multiplying

the ith row by $(n+i)!$ $(0\leqslant i\leqslant n-1)$ and dividing the jth column by $(n-j-1)!$ $(0\leqslant j\leqslant n-1)$ yields

$$\left[\prod_{k=0}^{n-1}\frac{(n+k)!}{(n-k-1)!}\right]D=\det\begin{vmatrix}\binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n}\\ \binom{n+1}{2} & \binom{n+1}{3} & \cdots & \binom{n+1}{n+1}\\ \vdots & \vdots & & \vdots\\ \binom{2n-1}{n} & \binom{2n-1}{n+1} & \cdots & \binom{2n-1}{2n-1}\end{vmatrix}.$$

Now subtract row i from row $i+1$ for $i=n-2,n-3,\ldots,0$; then for $i=n-2,n-3,\ldots,1$; then for $i=n-2,n-3,\ldots,2$; etc., to obtain

$$\left[\prod_{k=0}^{n-1}\frac{(n+k)!}{(n-k-1)!}\right]D=\det\begin{vmatrix}\binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n-1} & \binom{n}{n}\\ \binom{n}{2} & \binom{n}{3} & \cdots & \binom{n}{n} & 0\\ \vdots & \vdots & & \vdots & \vdots\\ \binom{n}{n-1} & \binom{n}{n} & \cdots & 0 & 0\\ \binom{n}{n} & 0 & \cdots & 0 & 0\end{vmatrix}$$

$$=(-1)^{n(n-1)/2}.$$

Thus,

$$D=(-1)^{n(n-1)/2}\frac{0!\quad 1!\quad 2!\ \cdots\ (n-1)!}{n!(n+1)!(n+2)!\cdots(2n-1)!}.$$

Also solved by Anders Bager (Denmark), Theodore S. Bolis, Aage Bondesen (Denmark), Paul Chauveheid (Belgium), Jose Luis de Miguel (Spain), W. O. Egerland, L. Kuipers (Switzerland), Yasuhiko Ikeda, Hermann Schmidt (Germany), George N. Trytten, Ellis Von Eschen, and the proposers.

Editor's note. Hermann Schmidt identified the problem as a special case of a determinant calculated by Stern, *J. Reine Angew. Math.*, 66 (1866) 288, quoted also (without proof) in Pascal, *Determinanten*, Leipzig, 1900, p. 135. John McFall (Canada) referred to *Muir's History of the Theory of Determinants* (Macmillan, 1920), pp. 448-454, where the problem is generalized (but the generalization is not solved). Aage Bondesen (Denmark) considered the $(q+1)\times(q+1)$ matrix with (i,j) element $1/(p+i+j)!$. Denoting the determinant of this matrix by $D_{p,q}$, he proved the relation $D_{p,q}=(-1)^q q!D_{p+2,q-1}/(p+q)!$. From this relation, he computed $D_{1,n-1}$, the determinant of the problem.

Intersections and Unions of Subsets

E 2764 [1979,223]. *Proposed by Ioan Tomescu, University of Bucharest, Rumania.*

Let X be a finite set. Prove that $\Sigma|A_1\cup A_2\cup\cdots\cup A_k|=(2^k-1)\Sigma|A_1\cap A_2\cap\cdots\cap A_k|$ where sums are over all choices of $A_1,\ldots,A_k\subseteq X$. (Here $|S|$ is the number of elements in S).

Solution by David Carlson, Colorado Department of Agriculture. For any subset S of X, let $S(1)=S$ and $S(0)=X-S$. Given any collection $\mathcal{Q}=\{A_1,\ldots,A_k\}$ of k subsets of X, a simple argument shows that

$$A_1\cup A_2\cup\cdots\cup A_k=\cup(A_1^{v_1}\cap A_2^{v_2}\cap\cdots\cap A_k^{v_k}),$$

where the disjoint union on the right side is over all choices of the vector $\mathbf{v}=(v_1,\ldots,v_k)$ except the zero vector. Since each v_i is 0 or 1, there are 2^k-1 such vectors.

Let $\mathbb{S}=\mathbb{S}(k)$ denote the family of all collections of k subsets of X. Each fixed vector $\mathbf{v}$ maps $\mathbb{S}$ onto itself by $\{A_1,\ldots,A\}\to\{A_1^{v_1},\ldots,A_k^{v_k}\}$. Hence

$$\begin{aligned}\sum_{\mathcal{C}\in\mathcal{S}}|A_1\cup\cdots\cup A_k| &= \sum_{\mathcal{C}\in\mathcal{S}}\sum_{\mathbf{v}\neq\mathbf{0}}|A_1^{v_1}\cap\cdots\cap A_k^{v_k}|\\ &= \sum_{\mathbf{v}\neq\mathbf{0}}\sum_{\mathcal{C}\in\mathcal{S}}|A_1^{v_1}\cap\cdots\cap A_k^{v_k}|\\ &= \sum_{\mathbf{v}\neq\mathbf{0}}\sum_{\mathcal{C}\in\mathcal{S}}|A_1\cap\cdots\cap A_k|\\ &=(2^k-1)\sum_{\mathcal{C}\in\mathcal{S}}|A_1\cap\cdots\cap A_k|.\end{aligned}$$

Also solved by Robert A. Melter, A. D. Sands (Scotland), Hann Tzang Wang (Republic of China), and the proposer.

Editorial note. A number of writers made the additional assumption that the sums in question were taken over ordered k-tuples of subsets of X and proceeded to evaluate both sides of the proposed equality.

Change of Variable Formula for Definite Integrals

E 2765 [1979, 223]. *Proposed by Naoki Kimura, University of Arkansas, Fayetteville.*

Establish the two following equations:

$$\int_{-1/2}^{3/2} f(3x^2-2x^3)\,dx=2\int_0^1 f(3x^2-2x^3)\,dx,$$
$$\int_{-1/2}^{3/2} xf(3x^2-2x^3)\,dx=2\int_0^1 xf(3x^2-2x^3)\,dx,$$

for all functions f continuous on $-1/2\leqslant x\leqslant 3/2$.

Is there a quadratic polynomial $g(x)$ such that

$$\int_{-1/2}^{3/2} f(3x^2-2x^3)\,dx=\int_0^1 g(x)f(3x^2-2x^3)\,dx$$

for every continuous function f?

Solution by Otto G. Ruehr, Michigan Technological University. With the change of integration variable, $3-2x=4\cos^2\theta$, the first equation becomes

$$\int_0^{\pi/6}(\sin 2\theta)f(\cos^2 3\theta)\,d\theta+\int_{\pi/3}^{\pi/2}(\sin 2\theta)f(\cos^2 3\theta)\,d\theta=\int_{\pi/6}^{\pi/3}(\sin 2\theta)f(\cos^2 3\theta)\,d\theta$$

and the second is similar with $\sin 2\theta$ replaced by $\sin\theta\ \cos 3\theta$. In each case the changes $\theta\to\pi/3-\theta$ in the first integral and $\theta\to 2\pi/3-\theta$ in the second reduce the left side to the right. To answer the question, set $f(z)=1,z,z^2$ and find that $g\equiv 2$ (as in the first equation) is the only quadratic polynomial with the required property.

An interesting alternative approach would be to use the Weierstrass Approximation Theorem to reduce the problem to that of polynomial f and, in turn, by linearity to that of establishing the two given equations for $f(z)=z^n$. The next natural step, introducing binomial expansions and integrating term by term, leads, however, to several combinatorial identities such as

$$\sum_{j=0}^{2n}(-4)^j\binom{3n+1}{n+j+1}=\sum_{j=0}^{n}2^j\binom{3n+1}{2n+j+1}$$

and

$$\sum_{j=0}^{2n}(-3)^j\binom{3n-j}{n}=\sum_{j=0}^{n}4^j\binom{3n-j}{2n},$$

which are apparently new and no easier to prove than the original equations. Their validity, of course, follows from the trigonometric proof.

Also solved by Robert Breusch, A. B. Farnell, Thomas Jager, L. Kuipers (Switzerland), O. P. Lossers (Netherlands), L. E. Mattics, N. Miku (Netherlands), Gerald Rogers, Buck Ware, and the proposer.

These solvers separated the interval $[-1/2,3/2]$ into three subintervals $[-1/2,0]$, $[0,1]$, and $[1,3/2]$ and noted that the relation $3x^2-2x^3=u$ has a unique solution on each interval.

The proposer noted that the relation

$$\int_{-1/2}^{3/2} f(3x^2-2x^3)x^2\,dx=\int_0^1 f(3x^2-2x^3)g(x)\,dx$$

holds for every continuous function f, provided $g(x)=-x^2+3x$.

Distance Between Lines in $\mathbb{R}^3$.

E 2769 [1979, 307]. *Proposed by Harry D. Ruderman, Hunter College, Manhattan.*

Let λ and λ' be (not necessarily coplanar) lines in space. On each of these lines, set up a real number coordinate system, with possibly different units of length. Let XX' be the line segment joining a point X on λ to the point X' on λ' with the same coordinate. Describe how to obtain X such that XX' has minimal length for all such segments.

Solution by Charles Vanden Eynden, Illinois State University at Normal, and Milton Eisner, J. Sargeant Community College (independently). Let λ,λ' have the respective parametric equations $X=P+tV, X'=P'+tV'$, where P,V,P',V' are fixed points of $\mathbb{R}^3$. The distance $\|P-P'+t(V-V')\|$ between X and X' is minimal when $t=-(P-P')\cdot(V-V')/\|V-V'\|^2$. (If $V=V'$ the distance XX' is constant.)

Also solved by F. S. Cater, J. Dou (Spain), Mark D. Meyerson, L. A. Ringenberg, D. L. Shell, and the proposer.

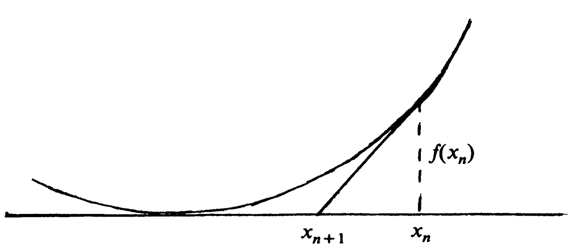
ADVANCED PROBLEMS

Solutions of these Advanced Problems should be mailed in duplicate to Professor Roger C. Lyndon, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109 (USA) by August 31, 1980. The solver's full post-office address should be on each sheet.

6293. *Proposed by W. M. Kahan, University of California, Berkeley.*

Assume that $f(x)$ never takes negative values but that it does vanish somewhere inside an interval throughout which the graph of $f(x)$ is convex. To find where $f(x)$ vanishes one might use Newton's iteration

$$x_{n+1}=x_n-f(x_n)/f'(x_n)\quad \text{for } n=0,1,2,3,\ldots.$$



This iteration will converge to the desired zero of $f(x)$ from any starting iterate x_0 in the interval, though convergence may be arbitrarily slow; so much is already well known. The problem is to prove that, no matter how slowly the iterates $\{x_n\}$ converge, $\Sigma_1^\infty(2^n f(x_n))^2$ is finite.

6294. *Proposed by Murray S. Klamkin, University of Alberta.*

If $n=n_1+n_2+\cdots+n_r$ where $n_r\geqslant 0$, prove that

$$\frac{n^n}{\Pi n_i^{n_i}}\geqslant\frac{\Gamma(1+n)}{\Pi\Gamma(1+n_i)}\geqslant\frac{(n+1)^{n+1}}{\Pi(n_i+1)^{n_i+1}}.$$

6295. *Proposed by N. Bromberg, Rutgers University.*

Let $\{X_j\}_{j=0}^{\infty}$ be a sequence of independent, identically distributed Poisson random variables with mean $\lambda<1$. Let $A_k=\{X_0+\cdots+X_k\leqslant k\}$. Show that $P\{\cap_{k=0}^{\infty}A_k\}=1-\lambda$.

SOLUTIONS OF ADVANCED PROBLEMS

Linear Systems for Coloring Maps

6215 [1978, 390]. *Proposed by Ki Hang Kim and Fred Roush, Alabama State University.*

Heawood's system for the four-color theorem for a map with n faces amounts to a linear system of rank $n-2$ in a $2n-4$ dimensional vector space over GF(3). Prove that for a random rank n system in a $2n$ dimensional vector space over GF(3), the probability that there is at least one solution vector with no zero component tends to 1 as $n\to\infty$.

Solution by R. W. K. Odoni, University of Exeter, England. Let w be the number of subspaces W of dimension n in GF(3^{2n}). For vectors $x=(x_1\cdots x_{2n})$ we put $\delta(x)=1$ if $x_i\neq 0$ for any i, and $\delta(x)=0$ otherwise. We write $F(W)=\sum_{x\in W}\delta(x)$. We calculate the mean $\mu=w^{-1}\sum_W F(W)$ and the variance $\sigma^2=w^{-1}\sum_W(F(W)-\mu)^2$. First we find that

$$w\mu=\sum_W F(W)=\sum_{x\neq 0}\delta(x)\left\{\sum_{\substack{W\\ x\in W}}1\right\}=4^nw/(1+3^n),$$

whence $\mu=4^n/(1+3^n)$. Further we have

$$w(\mu^2+\sigma^2)=\sum_W F^2(W)=\sum_x\sum_y\delta(x)\delta(y)\left\{\sum_{\substack{W\\ x,y\in W}}1\right\}.$$

Straightforward calculations now show that

$$\mu^2+\sigma^2=2\mu+(3^n-3)(2^{4n}-2^{1+2n})/(1+3^n)(3^{2n}-3).$$

Recalling that $\mu=4^n/(1+3^n)$, we find that $\sigma^2=2\mu+O(16/27)^n$.

Now let $\varepsilon(W)=1$ if $F(W)>0$ and $\varepsilon(W)=0$ otherwise. Then, by the Cauchy-Schwarz inequality, we have

$$\left\{\sum_W F(W)\right\}^2=\left\{\sum_W F(W)\varepsilon(W)\right\}^2\leqslant\sum_W F^2(W)\cdot\sum_W\varepsilon(W),$$

that is,

$$w^{-1}\sum_W\varepsilon(W)\geqslant\frac{\mu^2}{\mu^2+\sigma^2}$$

and

$$\left|1-\frac{\mu^2}{\mu^2+\sigma^2}\right|\leqslant O(3/4)^n,$$

which tells us that the probability that $\varepsilon(W)=0$ is only $O(3/4)^n$.

Also solved by the proposers.

Chains and Antichains

6220* [1978, 500]. *Proposed by Mohammad Ismail, Auburn University, Alabama*

A collection K of sets is called a chain (resp. antichain) if for any $A,B\in K$, either $A\subseteq B$ or $B\subseteq A$ (resp. for any $A,B\in K$, $A\not\subseteq B$ and $B\not\subseteq A$). Let ω_1 be the first uncountable ordinal. Does there exist a family $\mathcal{F}=\{K_\alpha:\alpha<\omega_1\}$ of collections of subsets of a set X satisfying the following conditions:

(1) Each K_α is an infinite countable antichain.

(2) If $\alpha<\beta<\omega_1$, then every member of K_β is contained in some member of K_α and no member of K_α is contained in any member of K_β.

(3) If $\mathcal{F}^*=\cup_{\alpha<\omega_1}K_\alpha$, then every chain, and every antichain, in $\mathcal{F}^*$ is countable.

Solution by Fred Galvin, University of Colorado. More generally, consider any ordinal Ω such that $|\Omega|\leqslant 2^{\aleph_0}$. Let R be the real line, and let $f:\Omega\to R$ be a 1-to-1 mapping. For $\alpha<\Omega$ let $K_\alpha=\{X_{\alpha,n}:n<\omega\}$, where

$$X_{\alpha,n}=\{\langle\xi,i\rangle\in\Omega\times\omega:\xi\geqslant\alpha,f(\xi)\geqslant f(\alpha),i=n\text{ or else }\xi>\alpha,i<n\}.$$

Note that $X_{\beta,m}\subseteq X_{\alpha,n}$ if and only if $\langle\beta,m\rangle\in X_{\alpha,n}$. With this, one can easily verify (1)–(3). For (3), note that, if there is an uncountable chain or antichain in $\{X_{\alpha,n}:\alpha<\Omega,n<\omega\}$, then there is a chain or antichain of the form $K=\{X_{\alpha,n}:\alpha\in A\}$ for some fixed $n<\omega$ and some uncountable set $A\subseteq\Omega$. Now, if $\alpha<\beta$, then $X_{\alpha,n}$ and $X_{\beta,n}$ are comparable if and only if $f(\alpha)\leqslant f(\beta)$. Since f is 1-to-1, this means that f is either strictly increasing on A (if K is a chain) or else strictly decreasing on A (if K is an antichain). But this is impossible, since no uncountable subset of R is well-ordered or anti-well-ordered by the natural ordering of R.

Galvin goes on to generalize the problem further. He replaces ω_1 by an arbitrarily linearly ordered set S and proves among other things that, under the continuum hypothesis, the problem has a positive solution if and only if the cardinality of S is less than or equal to $2^{\aleph_0}$.

Also solved by Arnold W. Miller.

Groups and Cardinal Numbers

6221 [1978, 500]. *Proposed by F. David Hammer, University of California, Davis.*

Recently, Shelah found a group of cardinality $\aleph_1$ with no proper subgroups of that cardinality. Prove that such cannot happen with abelian groups. In fact every uncountable abelian group has a proper subgroup of the same cardinality.

Editorial note. William Scott, University of Utah, points out that in his paper, Groups and Cardinal Numbers, Amer. J. Math, 74 (1952) 187–197, there appears (paraphrased) the following Theorem 9. If G is an uncountable abelian group of order A, then G has 2^A subgroups of order A. He remarks that this theorem, together with results of I. Szélpál, Die unendlichen Abelschen Gruppen mit lauter endlichen echten Untergruppen, Pub. Math. Debrecen, 1 (1949) 63–64, shows that the only infinite abelian groups without proper subgroups of the same order are the p^∞-groups.

We refer to results of Ol'shanskiĭ and Rips (see solution to 6052* [1980, 68]) for countably infinite groups, all of whose proper subroups are finite cyclic.

Also solved by Anders Bager (Denmark), Michael Barr, F. S. Cater, Jeffrey Mitchell Cohen, P. K. Garlick, Robert Gilmer, William H. Gustafson, Barbara Osofsky, D. A. Overdijk (Netherlands), Kenneth A. Ross, William Staton, and the proposer. Cater proves the theorem of Scott stated above. Gilmer remarks that it follows that any uncountable group with no proper subgroup of the same cardinality must be equal to its own derived group. Ross points out that the solution to the problem follows immediately from 16.13.c in E. Hewitt and K. A. Ross, Abstract Harmonic Analysis I, Springer-Verlag, 1963.

Mapping the 3-Sphere onto the 2-Sphere

6225 [1978, 600]. *Proposed by Edmund H. Anderson, Louisiana State University, Baton Rouge.*

Construct a homotopically trivial mapping from the three-sphere onto the two-sphere such that the pre-images of points are simple closed curves.

I. Solution by L. R. King, Davidson College. Let D^2 denote the closed unit disk in the complex plane and put $S^1=\partial D^2$. Define $f_i: S^1\times D^2 \to D^2$, $i=1,2$, by $f_1(z_1,z_2)=|z_2|z_1$ and $f_2(z_1,z_2)=z_2$. Note in both cases that the (continuous) mappings are onto such that pre-images of points are circles. The mapping defined by $H(z_1,z_2,t)=((1-t)|z_2|+t)z_1$ defines a homotopy rel $\partial(S^1\times D^2)$ of f_1 to a mapping onto ∂D^2. Now the three-sphere S^3 is the union of solid tori $T_1, T_2 (T_i \cong S^1\times D^2, i=1,2)$ sewn together along $\partial T_1=\partial T_2$ via homeomorphism $h:\partial T_2\to\partial T_1$ given by $h(z_1,z_2)=(z_2,z_1): S^3=T_1\cup_h T_2$. Also, the two-sphere S^2 is the union of closed unit disks D_1, D_2 sewn together along $\partial D_1=\partial D_2$ via the identity: $S^2=D_1\cup_{\mathrm{id}} D_2$. The mappings $f_i: T_i\to D_i, i=1,2$, agree on $T_1\cap T_2$ and define a mapping $f: S^3\to S^2$ with the desired properties.

II. Solution by Jerome Minkus, Berkeley, California. Let $B^3=\{(x,y,z)\in R^3 | x^2+y^2+z^2\leqslant 1\}$ be the unit 3 ball in Euclidean 3-space R^3. One of the standard ways to define the 3-sphere S^3 is by identifying points in the northern hemisphere of $\partial B^3=\{(x,y,z)|x^2+y^2+z^2=1\}$ with points in the southern hemisphere of ∂B^3 by reflection through the equatorial disk $B^2=\{(x,y,0)|x^2+y^2\leqslant 1\}\subset B^3$. In other words, identify $(x,y,\sqrt{1-x^2-y^2})$ and $(x,y,-\sqrt{1-x^2-y^2})\in\partial B^3$ for each $(x,y)\in B^2$. (See for example, Seifert-Threlfall, Lehrbuch der Topologie, p. 53). Let $S^1\subset S^3$ be the image of the equator $\partial B^2\subset B^3$ under these identifications. Let $\tilde{\pi}: B^3\to B^2$ be the projection $\tilde{\pi}(x,y\,z)=(x,y)$. Then $\tilde{\pi}$ is compatible with the above identifications on ∂B^3 and therefore induces a map $\pi: S^3\to B^2$. If we identify all the points on ∂B^2 to a single point p_0, then we get a 2-sphere S^2. Let $\phi: B^2\to S^2$ denote this identification map and let $\mu=\phi\circ\pi: S^3\to S^2$. Then μ is homotopically trivial and $\mu^{-1}(p_0)=S^1$ (=the "equatorial circle" in S^3). Since $\pi^{-1}(x,y)$ is a simple closed curve in S^3 for each point (x,y) in the interior of B^2, it follows that $\mu^{-1}(p)$ is also a simple closed curve in S^3 for each $p\in S^2-\{p_0\}$.

Also solved by N. Miku (Netherlands), William Myers, Daniel S. Silver, Dorothea K. Stillinger, Jussi Väisälä (Finland), and the proposer.

Legendre Polynomial Integral Inequality

6227* [1978, 600]. *Proposed by D. M. Milosević, Pranjani, Yugoslavia.*

Prove the following inequality in which $P_n(x)$ is a Legendre polynomial:

$$\int_{-1}^{+1}\frac{1-P_n(x)}{(1-x)^{5/4}}\,dx<2^{5/4}\left(\sum_{k=1}^{n}\frac{n}{k}\right)^{1/2}.$$

Solution by David Borwein, University of Western Ontario. It is known that, for $y>1$,

$$\begin{aligned}\int_{-1}^{1}\frac{1-P_n(x)}{y-x}\,dx&=\log\frac{y+1}{y-1}-\int_{-1}^{1}\frac{P_n(x)}{y-x}\,dx\\&=(1-P_n(y))\log\frac{y+1}{y-1}+2\sum_{k=1}^{n}\frac{1}{k}P_{k-1}(y)P_{n-k}(y).\end{aligned}$$

Letting $y\to 1+$, we get

$$\int_{-1}^{1}\frac{1-P_n(x)}{1-x}\,dx=2\sum_{k=1}^{n}\frac{1}{k}.$$

It is also known that $P_n'(x)=P_{n-2}'(x)+(2n-1)P_{n-1}(x)$ and that

$$(2n-1)\int_{-1}^{1}\frac{P_{n-1}(x)}{(1-x)^{1/2}}dx=2\sqrt{2}\ .$$

It follows that

$$\int_{-1}^{1}\frac{1-P_n(x)}{(1-x)^{3/2}}dx-\int_{-1}^{1}\frac{1-P_{n-2}(x)}{(1-x)^{3/2}}dx=2\int_{-1}^{1}\frac{P_n'(x)-P_{n-2}'(x)}{(1-x)^{1/2}}dx=4\sqrt{2}\ ,$$

and hence that

$$\int_{-1}^{1}\frac{1-P_n(x)}{(1-x)^{3/2}}dx=2\sqrt{2n}\ .$$

Consequently, by the Cauchy-Schwarz inequality, we have

$$\int_{-1}^{1}\frac{1-P_n(x)}{(1-x)^{5/4}}dx<\left(\int_{-1}^{1}\frac{1-P_n(x)}{(1-x)^{3/2}}dx\right)^{1/2}\left(\int_{-1}^{1}\frac{1-P_n(x)}{1-x}dx\right)^{1/2}$$
$$=2^{5/4}\sqrt{n}\left(\sum_{k=1}^{n}\frac{1}{k}\right)^{1/2}.$$

Also solved by Paul F. Byrd, O. P. Lossers (Netherlands), C. C. Rousseau, Otto G. Ruehr, Robert E. Schafer, and B. J. Venkatachala & C. R. Pranesachar (India), who prove the following stronger result: for $0<s<\frac{1}{2}$,

$$\int_{-1}^{+1}\frac{1-P_n(x)}{(1-x)^{1+s}}dx<2^{1+s}\sum_{j=1}^{n}j^{2s-1},$$

with equality only when $s=0$ and when $s=\frac{1}{2}$.

Sums of Five Distinct Cubes

6232* [1978, 686]. *Proposed by Allan Wm. Johnson, Jr., Defense Communications Agency, Washington, D.C.*

Prove or disprove: Given any integer $G>13$, there exist distinct integers $x_i>0$ such that $G^3=\sum_{i=1}^{5}x_i^3$.

Discussion by the proposer. I can neither prove nor disprove the conjecture embodied in this proposed problem, but I offer numerical evidence which seems to favor it. I investigated this conjecture by computer, using a program whose function was to represent a given cube as the sum of five distinct cubes. The computer showed that, for each G in the interval $14\leqslant G\leqslant 500$, the Diophantine equation $G^3=\sum_{i=1}^{5}x_i^3$ has at least one solution in distinct $x_i>0$. To gauge the scarcity or abundance of solutions as G increases in value, the computer was programmed to enumerate all distinct $x_i>0$ solutions of $G^3=\sum_{i=1}^{5}x_i^3$ for each G in the interval $14\leqslant G\leqslant 113$. This enumeration suggests that, as G increases, solutions for a given G tend to be numerous rather than sparse.

This conjecture has an obvious similarity to Waring's Problem except that Waring's Problem does not require distinct cubes. Small [**1**, p. 15] states that when Jacobi decomposed each of the first 12000 integers into a sum of as few cubes as possible, he nearly allowed himself to speculate that most integers can be represented as the sum of five not necessarily distinct cubes. Hardy and Wright [**2**, p. 335] state that numerical tabulations suggest that most integers can be represented as the sum of either four or five not necessarily distinct positive cubes. In view of these comments on Waring's Problem, it is perhaps not surprising that most cubes seem to be representable as the sum of five distinct cubes. In fact, it is possible that most cubes can be represented as the sum of four distinct cubes: the computer showed that, for each G in the interval $122\leqslant G\leqslant 500$, the equation $G^3=\sum_{i=1}^{4}x_i^3$ has a solution in distinct $x_i>0$.

Postscript: The following conjecture is a generalization suggested by numerical evidence:

> Given any integer $k>3$, there exists an integer $h>0$ such that for any integer $G \geqslant h$ we can find distinct integers $x_i>0$ for which $G^3=\sum_{i=1}^{k} x_i^3$.

For each $6 \leqslant k \leqslant 10$, a computer search verified this conjecture within the range $h(k) \leqslant G \leqslant 500$, where, for $k=6,\ldots,10$, $h(k)=13,14,16,18,19$.

This conjecture was also tested for $k=3$ and $1 \leqslant G \leqslant 500$, but the results of this test do not suggest a candidate for h.

References

1. C. Small, Waring's Problem, Mathematics Magazine, 50 (1977) 12–16.
2. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford University Press, London, 4th ed. (1975 printing).

Editor's Note: Numerical evidence for the conjecture was also supplied by Nick Franceschine III.

REVIEWS

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How to Use (and Misuse) Statistics. By Gregory A. Kimble. Prentice-Hall, Englewood Cliffs, New Jersey, 1978. xi+290 pp., \$14.95, \$11.95 (P). (Telegraphic Review, February 1979.)

Classic is a strong adjective, but this book is at least a candidate for classic status. Superficially it is an elementary text, written by a psychologist, for the intelligent but mathematically illiterate person. In actuality anyone with an interest in statistics, even a professional statistician, will find material of interest and hours of pleasure in this book.

Kimble has taken a radical approach to elementary statistics, and his book differs from a standard text in content, philosophy, and style. The only common ground is that the standard introductory topics are covered, but even here the order is drastically revised: hypothesis testing, descriptive statistics, experimental design, elementary probability, correlation, regression, and analysis of variance. The author states his guiding precepts in the preface: "What I want you to come away with is an appreciation of a style of thought and a respectable level of statistical literacy. I see no necessity, with these as my objectives, to dwell on formulas and computations. For those of you who find security in arithmetic, a final section of the book presents some of the technical tools. But what I want you to take away from your reading does not require mastery of that section."

Kimble succeeds beyond any rational expectation. In his hands understanding really is independent of computational expertise. He succeeds because he is not teaching the reader a subject but, rather, making a personal statement as to the relevance and meaning of statistics. This is an area, and a way of thought, which has great importance to the author, both professionally and personally, and in this book he shares with the reader what, how, and why.

Kimble uses "clearly" important examples, such as pupillary dilation, IQ tests, and the swine flu vaccination program, to make his points, but he also uses purely personal examples to add another dimension. An interest in oriental rugs provides a demonstration of parameter estimation, where the value of interest is the "reserve" price on items at Sotheby's auction house in New York. A quotation from his mother, upon considering a questionable activity, illustrates negative correlation. "The more I think of it, the less I think of it." The great penis experiment, drawn from his nights at the Psychological Round Table, shows problems with experimental studies.

Above all, the book is varied, interesting, and lively. In an area where most elementary texts read like cookbooks, this book reads like a novel. The author is sometimes flippant, sometimes irreverent, and sometimes a bit bawdy; but always to good purpose. Kimble has more than enough to say—his paragraphs would rate pages from less imaginative authors—and he says it with style and vigor.

Since Kimble is a psychologist, some of his mathematics is nonstandard. The result can be irritating, such as his use of positive and negative for increasing and decreasing functions; or even wrong, as when he claims all of probability deals with finite sample spaces. These problems are minor since the overall impressions are accurate, and some of his approaches are worthy of imitation. For instance, regression, introduced after correlation, is treated in terms of normalized variables. The regression equation then becomes $Z_y = rZ_x$, where r is the product correlation coefficient. This has the dual advantages of being the simplest form of a straight line and, since $-1 < r < 1$, of making "regression to the mean" obvious. Another example of a different, but highly effective, treatment is Kimble's discussion of the Uses and Misuses of Correlation in Chapter 9. Focusing on reliability and validity of tests and the consequences of less than perfect correlations, this chapter alone is worth the price of the book.

Overall, Kimble has convincingly demonstrated that statistical literacy can be taught independently of statistical techniques. The question remains: Is this desirable? Students of Kimble's book will have an understanding of statistics at least comparable, and probably superior, to that gained by students of any standard text. Even with the appendix, however, Kimble's students will not have the computational facility of other students. It is not clear that this is important. For most students the emphasis on techniques may be a holdover of the Puritan ethic that pain is good (as Kimble half-facetiously suggests). On the other hand, without the ability to do the calculations the student is dependent on authority. The results of a study may be understandable, but the techniques, design, and even the basic relevance of the study must be taken on faith; and my main reservation about Kimble's book is that it does promote faith. Making computations an integral part of an elementary course removes some of the mystery and some of the reverence for the expert, and this also seems an important part of a first course.

How you use Kimble's book will depend on your feelings about the importance of calculations, but no matter your feelings this is a book worth having. It is a reasonable choice for an elementary course either alone, supplemented by a standard text, or as a supplement. (There will be problems meshing this text with any other because of the nonstandard order of topics, and there may be an ego problem because any series of lectures I've heard (or given) suffers in comparison with this book.) It is also a good source of ideas and examples for use in any applied statistics course. I would require any undergraduate taking mathematical statistics to read this book for balance. Most of all, however, this book is fun to read, and that is a rare quality in a mathematics text.

Larry E. Knop, Hamilton College

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
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1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, P, *Mathematics into Type, Revised Edition.* Ellen Swanson. AMS, 1979, x + 90 pp, $8 (P). [ISBN: 0-8218-0053-1] A slightly revised version of the 1971 original, largely involving changes related to the shift from monotype to computerized phototypesetting. An essential reference for meticulous authors of mathematics papers. LAS

GENERAL, P, *Lecture Notes in Mathematics-753: Applications of Sheaves.* Ed: M.P. Fourman, C.J. Mulvey, D.S. Scott. Springer-Verlag, 1979, xiv + 779 pp, $37.60 (P). [ISBN: 0-387-09564-0] Proceedings of the research symposium on applications of sheaf theory to logic, algebra, and analysis, Durham, England, July 9-21, 1977. Includes papers in physics and differential geometry. JAS

GENERAL, P, L, *Engineering Formulas, Third Edition.* Kurt Gieck. McGraw, 1979, 433 pp, $11.95. [ISBN: 0-07-023216-4] Third American edition. Pocket-sized format. Blank page faces each printed page for note-making. Over 1500 formulas, 300 diagrams, numerous tables. Elementary mathematics from arithmetic through Fourier Series, followed by statics, dynamics, hydraulics, heat, strength, electrical engineering, optics, chemistry, machine parts. S.I. units. Brief logarithm and trigonometry tables. Every working engineer should have a copy within reach. JK

GENERAL, P, *Transactions of the Moscow Mathematical Society, 1979, Issue 2.* AMS, 1980, v + 298 pp, $48 (P). Translation of Volume 36, 1978. LAS

BASIC, T(13: 1), *Intermediate Algebra.* Daniel L. Auvil. A-W, 1979, xi + 494 pp, $13.95. [ISBN: 0-201-00135-7] Standard treatment of second year high school algebra for college students. Attractive format and readable explanations make this a good choice for insecure or poorly prepared students. Difficulty and level of rigor increases throughout book, although never to a very high level. Many applications problems from a variety of fields. MW

BASIC, T, *Intermediate Algebra, Fifth Edition.* William Wooton, Irving Drooyan. Wadsworth, 1980, xii + 481 pp. [ISBN: 0-534-00704-X] Six chapters review beginning algebra whereas five chapters are organized around the function concept. Changes include the omission of formal proofs from the presentation of real numbers and a rewriting of the section on logarithms to reflect the wide use of calculators. Attractive two-color format. (*Third Edition,* TR, May 1972.) JNC

BASIC, T(13: 1), *Algebra, Second Edition.* Robert T. Stephens. HR&W, 1979, xiv + 543 pp, $13.95 (P). [ISBN: 0-03-046371-8] Essentially ninth grade algebra for college students. Presents topics in a chatty, non-threatening manner with many examples. Some exercises in each section have answers in margin, all others in back of book. Good text for basic skills work or a learning center. MW

PRECALCULUS, T(13: 1), *Algebra and Trigonometry with Analytic Geometry.* Arthur B. Simon. Freeman, 1979, xii + 533 pp, $15.95. [ISBN: 0-7167-1016-1] Preparation for calculus, with strong emphasis on function concept. Stresses evaluation and application of functions and curve sketchings. Excellent treatment of these areas, but weak development of concepts of logarithmic and trigonometric functions. Superficial review of fundamentals assumes fairly strong high school preparation. Many exercises, differentiated by degree of difficulty. MW

PRECALCULUS, T(13: 1), *Analytic Geometry, Fifth Edition.* Gordon Fuller. A-W, 1979, xii + 334 pp, $13.95. [ISBN: 0-201-02414-4] Traditional text emphasizes basic concepts needed for calculus. New topics include transcendental functions and intersection of lines, as well as additional exercises. (*Fourth Edition,* TR, August-September 1973.) MW

PRECALCULUS, T(13: 1), *College Algebra with Applications.* Sabah Al-hadad, C.H. Scott. Winthrop, 1979, xii + 481 pp, $14.95. [ISBN: 0-87626-140-3] Adequate but unexceptional text. Uses set notation throughout. Begins each section with a list of objectives and includes many examples. More applications problems than average. Detailed explanations of "word problem analysis" techniques help students learn solution skills. MW

EDUCATION, P*, L*, *New Trends in Mathematics Teaching, Volume IV.* ICMI. UNESCO, 1979, 280 pp, (P). [ISBN: 92-3-101546-X] Thirteen comprehensive, well-documented reports on the world-wide status of mathematics education. Each chapter was presented at the Third International Congress of Mathematical Education in Karlsruhe (August 1976) as a keynote paper for one of the Congress's thirteen sections, and subsequently revised for this volume. Six chapters slice across the curriculum by grade level; the remaining seven slice vertically by topic (e.g., goals and objectives; curriculum development; algorithms and computers). LAS

HISTORY, S, L, *Albert Einstein, The Human Side: New Glimpses from His Archives.* Ed: Helen Dukas, Banesh Hoffmann. Princeton U Pr, 1979, viii + 167 pp, $8.95. Quotations (in English original, or in English translation) from unpublished informal letters and other documents that Einstein wrote "without thought of publication." The editors provide just enough background commentary to display Einstein's impish wit and penetrating insight to best advantage. Concludes with German originals, and with a short Einstein chronology. LAS

HISTORY, P, L. *Aepinus's Essay on the Theory of Electricity and Magnetism.* R.W. Home. Trans: P. J. Connor. Princeton U Pr, 1979, xiv + 514 pp, $37.50. [ISBN: 0-691-08222-7] First English translation of this 1759 milestone of mathematical physics. His rigorous mathematical investigation was a significant departure from the qualitative and nonmathematical treatments of his predecessors. The lengthy introductory monograph by Home provides much new material about Aepinus's career, the background and impact of his research, as well as the general scientific climate of eighteenth century Germany and Russia. GHM

HISTORY, P, L. *Norbert Wiener: Collected Works with Commentaries, Volume II.* Ed: P. Masani. MIT Pr, 1979, xiii + 969 pp, $50. [ISBN: 0-262-23092-5] Papers on harmonic analysis, Tauberian theorems and complex anlaysis, supplemented with expert commentary placing each in the context of present day research. (*Volume I* (TR, May 1979) includes Wiener's early papers; two more volumes will complete the work.) LAS

HISTORY, L. *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert.* Felix Klein. Chelsea, 1967, xiii + 385 + x + 208 pp, $17.50. Reprint in one volume of the two volume work originally published in 1926 and 1927. LAS

FOUNDATIONS, T?(15-17), P. *A Formal Background to Mathematics: Logic, Sets and Numbers.* R.E. Edwards. Springer-Verlag, 1979. [ISBN: 0-387-90431-X] *Ia,* xxxiv + 467 pp; *Ib,* 465 pp, $29.80 set (P). Intended to provide readers familiar with conventional informal mathematics (particularly teachers and college students) with a very detailed account of the formal logical and set theoretical background to mathematics. Considerable attention, in fact the major emphasis, is given to comparing formal and informal methods, with unusually lengthy discussions of conventional mathematical practices and their interpretation in the strictest formal terms. GHM

FOUNDATIONS, T(18), P. *Lecture Notes in Mathematics-759: Degrees of Unsolvability: Structure and Theory.* Richard L. Epstein. Springer-Verlag, 1979, xiv + 240 pp, $16 (P). [ISBN: 0-387-09710-4] Basic definitions and properties, exercises, notes and comments, and conjectures; assumes some background in logic and recursive functions. Some new results and proofs scattered throughout. LCL

FOUNDATIONS, T(16-17), L. *Introduction to Mathematical Logic, Second Edition.* Elliott Mendelson D. Van Nostrand, 1979, viii + 328 pp, $15.95. [ISBN: 0-442-25307-9] This revised edition of a widely used, shall we say "classical," text differs little from the original (1964). Two new sections treat elementary equivalence and non-standard analysis, and many new exercises have been added with selected answers in the back. To this reviewer's mind both editions suffer greatly from crowded typography and excessive formalism (especially in defining the semantics). Almost completely lacks clear informal discussions of the intuitive meanings of the definitions and theorems presented. The basic material is "all there," very accurately presented to be sure, but the book is not to be recommended as a first introduction to mathematical logic except for the very sophisticated student. GHM

FOUNDATIONS, P. *O Teorema de Gödel e a Hipôtese do Contínuo.* Manuel Lourenco. Fundacao Calouste Gulbenkian, xciv + 900 pp, (P). Portugese translations of major works (in their entirety) of Gödel, Cohen, Rosser, Turing, Feferman and Dummett on the theme of incompleteness of mathematical systems.GHM

COMBINATORICS, P. *Packing and Covering in Combinatorics.* Ed: A. Schrijver. Math. Centre Tracts, No. 106. Math Centrum, 1979, 313 pp, Dfl. 38 (P). [ISBN: 90-6196-180-7] This book is based on lectures given during a Study Week on Packing and Covering, June 5-9, 1978, organized by the Mathematical Centre in Amsterdam. Fourteen articles by eleven different authors. Most of the articles include extensive lists of references. CEC

NUMBER THEORY, P. *C.P. Ramanujam--A Tribute.* Ed: K.G. Ramanathan. Springer-Verlag, 1978, vii + 361 pp, $16.50. [ISBN: 0-387-08770-2] This collection of C.P. Ramanujam's papers is published as a tribute to his memory. The volume also includes eleven papers which have been contributed by colleagues. The papers in this collection are in number theory or algebraic geometry. There are also biographical sketches and photographs. CEC

LINEAR ALGEBRA, T(13-14: 2). *Elementary Linear Algebra.* Stanley I. Grossman. Wadsworth, 1980, x + 387 pp, $18.95. [ISBN: 0-534-00746-5] Many options given for covering such topics as isomorphisms, isometries, and Cayley-Hamilton Theorem, "if time permits." Includes a chapter on numerical methods. Does not get to vector spaces until Chapter 5. LLK

ALGEBRA, P. *Lecture Notes in Mathematics-715: Group Rings and Their Augmentation Ideals.* Inder Bir S. Passi. Springer-Verlag, 1979, vi + 137 pp, $9 (P). [ISBN: 0-387-09254-4] A report on dimension subgroups and the nilpotence of the augmentation ideal. JAS

ALGEBRA, S(18), P. *Lecture Notes in Mathematics-764: Representations of Finite Chevalley Groups, A Survey.* Bhama Srinivasan. Springer-Verlag, 1979, xi + 177 pp, $11.80 (P). [ISBN: 0-387-09716-3] An attempt to survey the more recent main developments for ordinary representations of finite Chevalley groups. The book includes results of Harish-Chandra, Lusztig, Deligne, Springer, and Kazhdan as well as a chapter on the l-adic cohomology. Bibliography, index. JS

ALGEBRA, T(16-18: 1, 2), S, P, L. *Fundamentals of the Theory of Groups, Second Edition.* M.I. Kargapolov, Ju. I. Marzljakov. Trans: Robert G. Burns. Grad. Texts in Math., V. 62. Springer-Verlag, 1979, xvii + 203 pp, $17.50. [ISBN: 0-387-90396-8] A compact introduction which avoids excessive detail and generalization; it proceeds quickly to the fundamentals necessary for reading current specialized literature. This *Second Edition* includes material on polycyclic groups, but is otherwise much the same as the *First Edition* (which appeared in 1971). LCL

ALGEBRA, T(18: 1), S, P. *Notes on the Witt Classification of Hermitian Innerproduct Spaces over a Ring of Algebraic Integers.* P.E. Conner. U of Texas Pr, 1979, xii + 145 pp, $15.95. [ISBN: 0-292-

75516-3] The book is aimed toward preparation for applications "to topological problems in knot concordance, cobordism classification of diffeomorphisms and actions of finite groups" and written at a level accessible to the reader with a background of (at least) a year's course in algebraic number theory. Brief bibliography, no index. JS

ALGEBRA, S(18), P. *Introduction to Harmonic Analysis on Reductive P-adic Groups.* Allan J. Silberger. Princeton U Pr, 1979, iv + 371 pp, $11 (P). [ISBN: 0-691-08246-4] An extended and extensive introduction to harmonic analysis on reductive p-adic groups based on work by Harish-Chandra from 1971-1973. Includes Jacquet and Bruhat theory, Maass-Selberg relations, Schwartz space, Eisenstein integral, commuting algebra theorem, rank one groups. Bibligraphy, index. JS

ALGEBRA, T(16: 1), S, L. *Introduction to Topological Semigroups.* Anthony Connors Shershin. U Pr of Florida, 1979, xii + 151 pp, $15.75. [ISBN: 0-8130-0664-3] Written at a level to attract undergraduates, the book develops the basic ideas of topological semigroups in an elementary and careful treatment with the request that it "be judged on its clarity rather than its completeness." Topics include ideals, homomorphisms, quotients, constructions. Exercises (solutions included), bibliography, index. JS

FINITE MATHEMATICS, T(13: 1). *Finite Mathematics and its Applications.* Larry J. Goldstein, David I. Schneider. P-H, 1980, xvii + 448 pp, $15.95. [ISBN: 0-13-317263-5] Good organization of topics: linear mathematics, probability and statistics, plus applications (Markov processes, theory of games, etc.). Applications are realistic and not cutesy. LLK

FINITE MATHEMATICS, T(13: 2). *Mathematics Applied to Business and the Social Sciences.* Robert F. Brown, Brenda W. Brown. Wadsworth, 1980, xi + 628 pp, $18.95. [ISBN: 0-534-00754-6] A unique format: introduction of a problem, discussion of appropriate mathematical topics, then solution of the posed problem. Topics are those of finite mathematics with two chapters of calculus. LLK

CALCULUS, T(13-14: 4), L. *Analysis.* Einar Hille. Krieger, 1979. *V. I,* xiv + 626 pp, $27.50 [ISBN: 0-88275-910-8]; *V. II,* xii + 672 pp, $25. [ISBN: 0-88275-895-0] Corrected reprinting of the original edition published by Blaisdell in 1964 and 1966. A modern, theoretical *Cours d'Analyse,* encompassing elementary and advanced calculus. Contains many excellent problems. LAS

DIFFERENTIAL EQUATIONS, T(17-18), S, P. *Differential and Integral Equations: Boundary Value Problems and Adjoints.* Stefan Schwabik, Milan Tvrdý, Otto Vejvoda. Reidel, 1979, 248 pp, $39.50. [ISBN: 90-277-0802-9] A rigorous, austere, terse treatment of "the basic mathematical theory with solutions of bounded variation as well as general boundary value problems." Classical linear theory and some non-linear perturbation theory. Bibliography, index. JS

DIFFERENTIAL EQUATIONS, P. *Lectures on Pseudo-Differential Operators: Regularity Theorems and Applications to Non-elliptic Problems.* Alexander Nagel, E.M. Stein. Princeton U Pr, 1979, 159 pp, $6.75 (P). [ISBN: 0-691-08247-2] An exposition of some new classes of pseudo-differentiable operators relevant to functions of several complex variables. LCL

DIFFERENTIAL EQUATIONS, P. *Rings of Differential Operators.* J.-E. Björk. Math. Lib., V. 21. North-Holland, 1979, xvii + 374 pp, $61. [ISBN: 0-444-85292-1] A systematic presentation of the theory of rings of differential operators with polynomial or analytic coefficients. The methods and approach to the major results rely heavily on homological algebra, algebraic geometry, and sheaf theory. Note price! LCL

NUMERICAL ANALYSIS, P. *Numerical Analysis of Singular Perturbation Problems.* Ed: P.W. Hemker, J.J.H. Miller. Acad Pr, 1979, xi + 499 pp, $40.50. [ISBN: 0-12-340250-6] Proceedings of the conference held at the University of Nijmegen, The Netherlands, May 30-June 2, 1978. Contains all 14 invited lectures and a number of shorter contributed papers. JAS

NUMERICAL ANALYSIS, P. *Multivariate Approximation Theory.* Ed: Walter Schempp, Karl Zeller. Int. Ser. Num. Math., V. 51. Birkhäuser, 1979, 455 pp, $38 (P). [ISBN: 3-7643-1102-9] Proceedings of an international symposium held in Oberwolfach, February 4-10, 1979. Treats splines, interpolation, cubature formulas and error estimates in several variables. LAS

NUMERICAL ANALYSIS, P. *Solution Methods for Integral Equations, Theory and Applications.* Ed: Michael A. Golberg. Math. Concepts and Methods in Sci. and Eng., V. 18. Plenum Pr, 1979, ix + 350 pp, $35. [ISBN: 0-306-40254-8] Thirteen contributors attempt to bridge the gap between the work of mathematicians and engineers in the numerical solution of integral equations. Attempts to classify types of problems that arise, then to indicate the methods that work. Good bibliographies and suggestions for future investigation. AWR

OPTIMIZATION, P. *Dynamic Programming and Its Applications.* Ed: Martin L. Puterman. Acad Pr, 1978, xv + 410 pp, $19.50. [ISBN: 0-12-568150-X] 20 papers on dynamic programming and Markov decision processes from an international conference held at the University of British Columbia in 1977. Includes surveys, several recent theoretical results and diverse applications. One bibliography, on computational advances, is extensive. RWN

ANALYSIS, T(15-16: 1, 2), S, P, L. *Essential Mathematics for Applied Fields.* Richard M. Meyer. Springer-Verlag, 1979, xvi + 555 pp, $16.80 (P). [ISBN: 0-387-90450-6] Broad self-contained gathering, requiring only calculus through differential equations "for use by workers in a variety of applied fields." Includes basic real analysis, Riemann-Stieltjes integration, finite calculus, basic complex analysis, applied linear algebra. Suggested for use as a remedial reference, for independent (or guided) study, as a source for review, or entry into more advanced and/or related literature. JK

ANALYSIS, P. *Representations of Locally Compact Groups with Applications.* Ed: T.H. Koornwinder. Math Centrum, 1979. *Part I,* iv + 220 pp; *Part II,* 273 pp, (P). [ISBN: 90-6196-181-5] A collection of

thirteen papers by Dutch mathematicians (five in *Part I*, eight in *Part II*). The applications are primarily to quantum theory. *Part II* includes an index of terminology and a list of addresses of authors. JAS

ANALYSIS, P. *Lecture Notes in Mathematics-726: Schwartz Spaces, Nuclear Spaces and Tensor Products.* Yau-Chuen Wong. Springer-Verlag, 1979, viii + 418 pp, $19.50 (P). [ISBN: 0-387-09513-6] A unified treatment of Schwartz spaces, nuclear spaces, and λ-nuclear spaces using the topology of uniform convergence on order-bounded sets. JAS

ANALYSIS, P. *Lecture Notes in Mathematics-707: Discontinuous Čebyšev Systems.* Roland Zielke. Springer-Verlag, 1979, vi + 111 pp, $9 (P). [ISBN: 0-387-09125-4] Following Karlin's and Studden's monograph on continuous Čebyšev systems (1966), research papers have explored the discontinuous case. Here is perhaps the first book collecting and developing these results. AWR

DIFFERENTIAL GEOMETRY, S(18), P. *Lecture Notes in Mathematics-708: Equations de Pfaff algébriques.* J.P. Jouanolou. Springer-Verlag, 1979, v + 255 pp, $15.70 (P). [ISBN: 0-387-09239-0] The five chapters deal first with Jacobi forms on a projective space, then specialize to algebraic equations and systems of Pfaff, concluding with results on the density of forms with algebraic solutions, and applications to some classical results in differential equations. Bibliography, index. JS

GEOMETRY, T(13: 1), S. *Geometry: An Exercise in Reasoning.* Ken Seydel. Saunders, 1980, xiii + 416 pp. [ISBN: 0-7216-8070-4] Attractive. Tailored for the community college student in a one semester course. Compiled from class-tested notes. Two column proofs. Informally presented; unexciting but honest. Author failed in his resolve "to resist the temptation to include many beautiful and interesting theorems." Have a look. JK

GEOMETRY, S**(11-16), P, L**. *Spherical Models.* Magnus J. Wenninger. Cambridge U Pr, 1979, xii + 147 pp, $19.95; $7.95 (P). [ISBN: 0-521-22279-6; 0-521-29432-0] Worthy companion volume to author's superb *Polyhedron Models*. Regular, semiregular and star-faced polyhedra projected onto a circumscribing sphere. Secrets of model construction are revealed in clear diagrams and stunning photographs. Directed toward use by students and teachers on high school level, but will appeal to artists, designers, engineers and builders of geodesic domes, whose relationship to polyhedra is explicitly revealed. JK

TOPOLOGY, P. *Algebraic Cobordism and K-theory.* Victor P. Snaith. Memoirs No. 221. AMS, 1979, vii + 152 pp, $7.60 (P). [ISBN: 0-8218-2221-7] A new construction of cobordism as a localization of the stable homotopy ring of a classifying space. This construction includes the classical cobordism theories and generalizes to define the algebraic cobordism of any ring. JAS

TOPOLOGY, P. *Lecture Notes in Mathematics-763: Algebraic Topology, Aarhus 1978.* Ed: J.L. Dupont, I.H. Madsen. Springer-Verlag, 1979, vi + 695 pp, $33.60 (P). [ISBN: 0-387-09721-X] Proceedings of the Symposium held at Aarhus, Denmark, August 7-12, 1978 in connection with the 50th anniversary of Aarhus University. JAS

PROBABILITY, S. *The Theory of Blackjack: The Compleat Card Counter's Guide to the Casino Game of Twenty-One.* Peter A. Griffin. GBC Pr, 1979, 180 pp, $8.95. Collection of mathematical results (with proofs in appendices) for those already familiar with basic strategies. RSK

PROBABILITY, P. *Probabilităţi şi Procese Stocastice, V. II.* George Ciucu, Constantin Tudor. Editura Academiei (Romania), 1979, 318 pp, Lei. 25. This second volume of a series on probability treats stochastic processes. (*V. I*, TR, November 1979). JAS

STATISTICS, P. *Tratat de Statistică Matematică, Voluml III: Analiză Secvenţială.* Gheorghe Mihoc, Virgil Craiu. Editura Academiei (Romania), 1979, 396 pp. This scholarly treatise, the third of a series, treats sequential analysis. Earlier volumes: *Voluml I*, TR, November 1976; *Voluml II*, TR, April 1978. JAS

STATISTICS, P. *Analysis of Economic Time Series: A Synthesis.* Marc Nerlove, David M. Grether, José L. Carvalho. Acad Pr, 1979, xvi + 468 pp, $29.50. [ISBN: 0-12-515750-9] Integrated presentation of distributed-lag models, spectral analysis, and unobserved components models. Includes many illustrative examples and a good set of references. RSK

COMPUTER PROGRAMMING, T*(13: 1), S, L. *Programming Fortran 77: A Structured Approach.* J.N.P. Hume, R.C. Holt. Reston Pub, 1979, 340 pp, $10.95 (P). [ISBN: 0-8359-5671-7] An introduction to Fortran 77 which uses SF/k. (SF/k is a nested sequence of eight levels of Fortran 77.) The structured approach is emphasized. Includes a lot of general programming information. Lots of good exercises. Well written. CEC

COMPUTER PROGRAMMING, T*(13: 1). *Fundamentals of Structured Programming Using Fortran with SF/k and WATFIV-S.* R.C. Holt, J.N.P. Hume. Reston Pub, 1977, xiii + 349 pp, $10.50 (P). [ISBN: 0-87909-302-1] The same book as the above except that it includes a six-page appendix on the WATFIV-S compiler. CEC

COMPUTER PROGRAMMING, S(13-18). *Computer Graphics Primer.* Mitchell Waite. Howard W. Sams, 1979, 184 pp, $12.95 (P). [ISBN: 0-672-21650-7] An introduction to the vocabulary, equipment manufacturers, and programming (mostly in enhanced Basic) relevant to graphics in the personal computer market. Big on pong games, short on hidden line algorithms. JAS

COMPUTER PROGRAMMING, T(13: 1), S, P, L. *Structured PL/I Programming: An Introduction.* John J. Xenakis. Duxbury Pr, 1979, xv + 413 pp, $12.95 (P). [ISBN: 0-87872-190-8] A well-written introduction to PL/I. Includes a good introduction to general programming practices and emphasises structured programming. Lots of good exercises. CEC

SYSTEMS THEORY, P. *Lecture Notes in Control and Information Sciences-17: Qualitative Aspects of Large Scale Systems: Developing Design Rules Using APL.* O.I. Franksen, P. Falster, F.J. Evans. Springer-Verlag, 1979, xii + 119 pp, $9 (P). [ISBN: 0-387-09609-4] Application of tensors and graphs to the study of design concepts that represent the properties of controllability and observability of linear, time-invariant systems. LCL

APPLICATIONS (ECONOMICS), P. *Expected Utility Hypotheses and the Allais Paradox.* Ed: Maurice Allais, Ole Hagen. Theory and Decision Lib., V. 21. Reidel, 1979, vii + 714 pp, $86.85. [ISBN: 90-277-0960-2] Presentation (in English for the first time) and analysis of Maurice Allais's 1952 critique of von Neumann and Morgenstern's theory of utility. The "Allais paradox" arises when, in special circumstances, reasoning with classical utility theory yields conflicting conclusions. LAS

APPLICATIONS (ECONOMICS), P. *Studies in the Economics of Search.* Ed: S.A. Lippman, J.J. McCall. North-Holland, 1979, viii + 225 pp, $41.50. [ISBN: 0-444-85222-0] This Volume 123 of Contributions to Economic Analysis contains 9 papers with significant quantitative contents in a new field of study.JAS

APPLICATIONS (ENGINEERING), T(17-18: 1, 2), P. *Stochastic Models, Estimation, and Control, Volume 1.* Peter S. Maybeck. Math. in Sci. and Eng., V. 141. Acad Pr, 1979, xix + 423 pp, $29.50. [ISBN: 0-12-480701-1] Introduction to fundamental concepts with attention to examples, practical aspects, and algorithms. The mathematics is relatively sophisticated. LCL

APPLICATIONS (ENGINEERING), P. *Computer-Aided Design of Digital Electronic Circuits and Systems.* Ed: Gerald Musgrave. North-Holland, 1979, viii + 325 pp, $44. [ISBN: 0-444-85374-X] Proceedings of a 1978 symposium held in Brussels, organized by the Commission of the European Communities. Provides a (two-year-old) state-of-the-art technical survey of computer-aided design. LAS

APPLICATIONS (FLUID MECHANICS), P. *Annual Review of Fluid Mechanics, Volume 12.* Ed: Milton Van Dyke, J.V. Wehausen, John L. Lumley. Annual Reviews, 1980, 490 pp, $17. [ISBN: 0-8243-0712-7] Sixteen collected papers on a wide range of theoretical and applied topics: e.g., "Solitary waves" by John Miles, and "Fluid mechanics of the duodenum" by Enzo Macagno and James Christensen. JAS

APPLICATIONS (INFORMATION THEORY), T(14-16: 1), S, L. *Elementary Information Theory.* D.S. Jones. Clarendon Pr, 1979, 182 pp, $24.50; $9.95 (P). [ISBN: 0-19-859636-7; 0-19-859637-5] A tidy and largely self-contained introduction (including basic concepts of probability) to coding theory, channel capacity, error correction, and continuous signals. Provides solid mathematical background for more sophisticated units in coding, cryptography, linguistics, and optimal communications literature. LCL

APPLICATIONS (MECHANICS), P. *Trends in Applications of Pure Mathematics to Mechanics, V. II.* Ed: Henryk Zorski. Pitman, 1979, viii + 341 pp, $58.50. [ISBN: 0-273-08421-6] 20 papers presented in September, 1977 at a symposium in Kozubnik, Poland, the second in a series of cross-disciplinary symposia. *Volume I* from the 1975 symposia, appeared in 1976. The price is nearly 20¢ per page! LAS

APPLICATIONS (PHYSICS), P. *Fundamentals of Maxwell's Kinetic Theory of a Simple Monatomic Gas, Treated as a Branch of Rational Mechanics.* C. Truesdell, R.G. Muncaster. Pure and Appl. Math., V. 83. Acad Pr, 1980, xxviii + 593 pp, $54. [ISBN: 0-12-701350-4] A mathematical development, as a deductive theory, of one of the most mathematized areas of physics. The author aims to give an essentially complete picture of the theorems and conjectures of this theory as currently developed. This book presents a detailed mathematical theory with some physical and philosophical interpretations and discussion. JAS

APPLICATIONS (PHYSICS), T(16-18: 1, 2). *A Course in Mathematical Physics 2: Classical Field Theory.* Walter Thirring. Trans: Evans M. Harrell. Springer-Verlag, 1979, viii + 249 pp, $22. [ISBN: 0-387-81532-5] An introduction to classical field theory using the vocabulary of manifolds, forms, and connections. The differential geometry is presented very sketchily since the reader is presumed to have access to *Volume 1* (TR, December 1979) or to one of a number of suggested references on the subject. The aim is to present some theoretical physics in a mathematically efficient (and pleasing) way. The book contains some exercises, a small index, and a useful but intentionally incomplete bibliography. It is a text, not a treatise. However, it is physically and mathematically quite sophisticated. JAS

APPLICATIONS (PHYSICS), P. *Relativistic Quantum Fields.* C. Nash. Acad Pr, 1978, viii + 223 pp, $31.50. [ISBN: 0-12-514350-8] This is really a physics book which presents a number of mathematical techniques that have proved useful in quantum electrodynamics, e.g., the dimensional and renormalization group methods. JAS

APPLICATIONS (PHYSICS), P. *Lecture Notes in Physics-93: Stochastic Behavior in Classical and Quantum Hamiltonian Systems.* Ed: G. Casati, J. Ford. Springer-Verlag, 1979, vi + 375 pp, $17.80 (P). [ISBN: 0-387-09120-3] Proceedings of the Volta Memorial Conference which was held at Como, Italy in the summer of 1977. Although much of the content is physics, the aim of the conference was to bring together researchers from a wide variety of disciplines with a common concern. JAS

APPLICATIONS (PHYSICS), T(16-17: 1), S, L. *A Short Course in General Relativity.* J. Foster, J.D. Nightingale. Longman, 1979, xv + 192 pp, $13.50 (P). [ISBN: 0-582-44194-3] The first half of this short and to-the-point text presents a foundation in differential geometry. The notation is relatively classical, as is the geometry. The presentation is clear but unembroidered with intuition. The problem sets are short and give essential practice with the basic notation. The rest of the book presents "field equations and curvature," "physics in the vicinity of a massive object," "gravitational radiation," and "elements of cosmology." JAS

APPLICATIONS (PHYSICS), P. *Complex Manifold Techniques in Theoretical Physics.* Ed: D.E. Lerner, P.D. Sommers. Research Notes in Math., No. 32. Pitman, 1979, 242 pp, $21.95 (P). [ISBN: 0-273-08437-2] Papers from a July 1978 workshop in Lawrence, Kansas: "Over the past few years, a remarkable relationship between complex analysis and mathematical physics has emerged. Through the use of Penrose's twistor theory, the solutions to some fundamental differential equations of mathematical physics can be converted into complex analytic objects on projective space. The corresponding classification problems of complex geometry can, in principle, be solved completely, with the result that one now has an explicit description of the set of all soutions to the (self-dual) Yang-Mills, Einstein, and zero rest-mass equations." LAS

APPLICATIONS (PHYSICS), P. *Lecture Notes in Physics-94: Group Theoretical Methods in Physics.* Ed: W. Beiglböck, A. Böhm, E. Takasugi. Springer-Verlag, 1979, xiii + 540 pp, $23 (P). [ISBN: 0-387-09238-2] The proceedings of the Seventh International Colloquium and Integrative Conference on Group Theory and Mathematical Physics which was held in Austin, Texas, September 11-16, 1978. These proceedings are divided into some eleven subsections by topic and begin with tributes to Eugene P. Wigner and Valentine Bargman on the occasion of their being presented with the Wigner Medal. JAS

APPLICATIONS (PHYSICS), P. *Lecture Notes in Physics-84: Stochastic Processes in Nonequilibrium Systems.* Ed: L. Garrido, P. Seglar, P.J. Shepherd. Springer-Verlag, 1978, xi + 352 pp, $17.80 (P). [ISBN: 0-387-08942-X] Lectures given at the Fifth Sitges International School of Statistical Mechanics in June 1978. Eight major papers and 20 shorter notes on a variety of topics related to stochastic processes in nonequilibrium systems. TAV

APPLICATIONS (PHYSICS), T(16-18), S, P. *Frequency and Time.* P. Kartaschoff. Acad Pr, 1978, xv + 260 pp, $26.50. [ISBN: 0-12-400150-5] This readable survey is the first of a series devoted to the recent advances in measurement of frequency and time. Interestingly written, it includes such topics as frequency stability, quartz and atomic clocks, time scales and coordination, counters, phase-time measurement, and radio-signal comparision methods. MU

APPLICATIONS (PHYSICS), S(15-18), P, L?. *Theoretical Kinematics.* O. Bottema, B. Roth. Appl. Math. and Mech., V. 24. North-Holland, 1979, xiv + 558 pp, $87.75. [ISBN: 0-444-85124-0] The treatment is mainly analytical, although geometric interpretations are given occasionally. The text employs the mathematics of algebraic geometry, vector and matrix algebra and elementary calculus. No exercises are provided. MU

APPLICATIONS (PHYSICS), T(16-18: 1, 2), S, P. *Elongational Flows: Aspects of the Behaviour of Model Elasticoviscous Fluids.* C.J.S. Petrie. Fearon-Pitman, 1979, 254 pp, $17.50 (P). [ISBN: 0-273-08406-2] Concerned primarily with theoretical rheology and non-Newtonian fluid mechanics. MU

APPLICATIONS (PHYSICS), P. *Functional Integration and Quantum Physics.* Barry Simon. Pure and Appl. Math., V. 86. Acad Pr, 1979, ix + 296 pp, $29.50. [ISBN: 0-12-644250-9] An exposition of powerful Wiener integral techniques in quantum physics. For mathematical physicists and probabilists. TRS

APPLICATIONS (PHYSICS), P, L. *Works on the Foundations of Statistical Physics.* Nikolai Sergeevich Krylov. Trans: A.B. Migdal, Ya. G. Sinai, Yu. L. Zeeman. Princeton U Pr, 1979, xxviii + 283 pp, $19.50; $7.50 (P). [ISBN: 0-691-08230-8; 0-691-08227-8] Translation of 1950 Russian edition of an unfinished monograph on the foundations of statistical physics, including a short related paper and the doctoral dissertation of the author, who died at the age of 29 in 1947. Evidence of its worth as an inspiration for further work is given in an afterword citing progress made since its publication. A valuable historical imprint. JK

APPLICATIONS (PHYSICS), P. *Groups in Physics: Collective Model of the Nucleus, Canonical Transformations in Quantum Mechanics.* Marcos Moshinsky. Pr U Montreal, 1979, 99 pp, (P). [ISBN: 2-7606-0458-6] Highly technical survey, for experts only. LCL

APPLICATIONS (PHYSICS), S, P, L*. *Surprises in Theoretical Physics.* Rudolf Peierls. Princeton U Pr, 1979, viii + 167 pp, $15; $3.95 (P). Lectures presenting examples, mostly from quantum physics, in which plausible expectation is not borne out by careful analysis. LAS

APPLICATIONS (SOCIAL SCIENCE), T(14-16: 1), S*, P, L*. *Games as Models of Social Phenomena.* Henry Hamburger. Freeman, 1979, xi + 264 pp, $16.95; $7.95 (P). [ISBN: 0-7167-1011-0; 0-7167-1010-2] An introduction to game theory and its applications in political science, economics, and psychology. A good introduction to model building in the social sciences. FLW

APPLICATIONS (SOCIAL SCIENCE), P, L**. *Measurement Theory with Applications to Decisionmaking, Utility, and the Social Sciences.* Fred S. Roberts. Ency. Math. and its Appl., V. 7. A-W, 1979, xxii + 420 pp, $24.50. [ISBN: 0-201-13506-X] An introductory survey of an important interdisciplinary research area--the theory of measurement--which has no disciplinary home. Significant, albeit uncommon, mathematical theories--relations and orders, representation of measures, product structures and conjoint measurement--support diverse applications to such areas as psychophysics, policy science, decision theory. A rich source of accessible mathematical models. LAS

Reviewers Whose Initials Appear Above

Judith N. Cederberg, St. Olaf; Clifton E. Corzatt, St. Olaf; Lorraine L. Keller, St. Olaf; Richard S. Kleber, St. Olaf; Joseph Konhauser, Macalester; Loren C. Larson, St. Olaf; George H. Mills, St. Olaf; R.W. Nau, Carleton; A. Wayne Roberts, Macalester; Thomas R. Savage, St. Olaf; John Schue, Macalester; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; Milton Ulmer, Carleton; T.A. Vessey, St. Olaf; Martha Wallace, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY PAUL A. HAEDER, University of Nebraska at Omaha

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036

SUGGESTION BOX

Members of the MAA are encouraged to send in suggestions, questions, etc., about the operations of the Association. Communications will be referred to the appropriate officer of the Association for answering; from time to time, those of general interest may also be answered in one or both of the official journals. Communications should be addressed to: Suggestion Box, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036

PERSONAL ITEMS

Dr. Paul F. Cohen, Central State University, Wilberforce, Ohio, is a Visiting Assistant Professor at the University of Santa Clara, Santa Clara, California.

Mourad E. H. Ismail of McMaster University, Barbara Keyfitz of Princeton University, and Hal L. Smith of the University of Utah have been appointed Assistant Professors at Arizona State University.

Assistant Professor John McCleary, formerly of Bates College, Lewiston, Maine, is now a member of the faculty at Vassar.

Dr. Charles D. Reinauer, formerly of Houston, has been appointed Instructor at San Jacinto College, Pasadena, Texas.

Dr. Linda Lesniak-Foster, Louisiana State University, has been appointed Assistant Professor at Western Michigan University, Kalamazoo.

Dr. D. A. Quarles, Jr., I.B.M. Research Division, has been appointed Visiting Professor for 1979-80 at the Stevens Institute of Technology, Hoboken, New Jersey.

Dr. Stephen C. King, California State University, Los Angeles, has been appointed Assistant Professor at the University of South Carolina, Aiken.

Assistant Professor Andrew T. Kitchen, St. John Fisher College, Rochester, New York, has been promoted to Associate Professor.

Assistant Professors Michael McAsey and Valerian Nita, Western Michigan University, have accepted appointments at Bradley University, Peoria, Illinois and the Ford Motor Company, respectively.

Dr. Jerry Tate is now the Coordinator of Mathematics at the new South Campus of San Jacinto College, Pasadena, Texas.

Professor David C. Haines, has replaced Professor Stephen Hoffman as Chairman of the Mathematics Department at Bates College, Lewiston, Maine.

Associate Professor Philip Leonard, Arizona State University, Tempe, has been promoted to Professor.

Professor G. L. Alexanderson, University of Santa Clara, has been named the Michael and Elizabeth Valeriote Professor.

Associate Professors Erik Schreiner and Arthur White, Western Michigan University, have been promoted to Professors.

Assistant Professor Dean W. Hoover, Alfred University, Alfred, New York, has been promoted to Associate Professor.

Professor Kurt Bing, Rensalaer Polytechnic Institute, Troy, New York, was given the title Professor Emeritus in June, 1979.

Professor Gary Chartrand, Western Michigan University, was given the Distinguished Professor Award ($1,500 Honorarium), based on outstanding achievement and wide recognition in the academic community.

Professor Carl C. Steyer, University of Arkansas, died on January 28, 1980, at the age of 61. He was a member of the Association for fourteen years.

Professor John T. Moore, University of Western Ontario, died in February, 1980, at the age of 64. He was a member of the Association for twenty seven years.

Chairman Bobby J. Jimerson, Department of Mathematics, East Texas Baptist College, died in February, 1980, at the age of 47. Bobby was a member of the Association for two years.

Dr. F. A. Ficken, Pelham, New York, died in December, 1978. He was a member of the Association for thirty seven years.

Dr. William H. Reynolds, Cortland, New York, died in February, 1980. He was a member of the Association for ten years.

Dr. Louise J. Rosenbaum, Middletown, Connecticut, died on January 16, 1980. She was a member of the Association for forty eight years.

Ms. Alexandra I. Forsythe, Stanford, California, died on February 1, 1980. She was a member of the Association for six years.

CHANGE OF DATE FOR MAINE CONFERENCE

A short course on Application of Mathematics in the Managerial Sciences to be held in Orono, Maine, was reported in the January, 1980, *Mathematical Monthly* as scheduled for June 16-20, 1980. The date has been changed to June 23-27, 1980.

MATHEMATICAL LIBRARIES OF DEVELOPING COUNTRIES

The developing countries (D.C.) have serious problems to get the mathematical documentation they need and to start the mathematical libraries of their national universities. So, in agreement with the I.M.U. Executive Committee, the I.M.U. Development and Exchanges Commission has decided to launch an international programme of help to the mathematical libraries of developing countries. The International Centre of Pure and Applied Mathematics (I.C.P.A.M.) recently created in Nice (France) has decided to give its help to the realization of this important project.

The programme consists of collecting throughout the world textbooks, journals and varied mathematical publications and sending them to the mathematical documentation centres of the developing countries, the centres with a regional or sub-regional location coming first. The I.M.U. Executive Committee and the I.M.U. Development and Exchanges Commission do call upon all the National Mathematical Committees, all the National Mathematical societies, all the Mathematical Institutes and Departments and upon all the mathematicians all over the world to take part in this most important programme.

If you are interested, please send the complete *list* of publications you offer to the following address, where the operation will be duly coordinated. (Send only *lists*, not books.) PROGRAMME "MATHEMATICAL DOCUMENTATION for D.C.", c/o International Centre for Pure and Applied Mathematics (I.C.P.A.M.), 1, Avenue Edith Cavell, 06000 NICE/FRANCE Call. (93) 53.18.43.

MATHEMATICS AND STATISTICS CONFERENCE

The Eighth Annual Mathematics and Statistics Conference at Miami University, Oxford, Ohio, will be held September 26-27, 1980. The theme for this year's conference will be "Statistics". Featured speakers will include Robert V. Hogg of the University of Iowa and William H. Lawton of the Eastman Kodak Company. There will be sessions of contributed papers and a poster session which should be suitable for a diverse audience of statisticians, mathematicians, and students. Abstracts should be sent by June 1, 1980 to Professor John Skillings, Department of Mathematics and Statistics, Miami University, Oxford, Ohio 45056. Information regarding preregistration and housing may also be obtained from Professor Skillings.

COLLECTIONS OF *MONTHLYS* AVAILABLE

Occasionally this editorial department receives information about available collections (or partial collections) of *Mathematical Monthlys*. If space permits, an effort will be made to disseminate such information to our readers. As an example, W. L. Williams, 213 King Street, Columbia, South Carolina, wishes to sell a collection of all *Monthlys* from over fifty years.

OHIO SECTION SHORT COURSE

Professor Harry Pollard of Purdue University will present a series of lectures on the history of mathematics. The lectures will be of interest to college and university teachers. This course will be held June 10-13 at Kenyon College, Gambier, Ohio. For more information contact: Phillip Schmidt, Department of Mathematical Sciences, The University of Akron, Akron, Ohio 44325.

MEETING HONORING PROFESSOR HILLE

A meeting in honor of Professor Einar Hille's 85th birthday was held on January 8 and 9, 1980, at the Surf and Sand Hotel, Laguna Beach, California. Professor Frank B. Cannonito, University of California Irvine, was Chairman of the Organizing Committee. This timely meeting preceded Professor Hille's death by only a month.

Professor Hille was a Member of the National Academy of Science and a distinguished former President of the American Mathematical Society. He was a member of the original faculty of the Department of Mathematics, University of California Irvine, in 1965. He had the title of Professor Emeritus from Yale University and, in the past few years, was Resident Mathematician at the University of California San Diego.

The four distinguished invited speakers were: Professor Hille, J. Dieudonne from France, I. Segal from M.I.T., and A. Gleason from Harvard. In addition to the invited one-hour addresses, there were contributed talks.

SYMPOSIUM ON PROBABILITY THEORY

The TENTH ANNUAL SOUTHERN CALIFORNIA SYMPOSIUM ON PROBABILITY THEORY was held at the University of California Irvine, on Monday, December 17, 1979. The Coordinator for the Symposium was Professor Donald A. Darling, UCI, and the Department of Mathematics, UCI, was the sponsor. The Mathematics Department at UCSD (San Diego) was responsible for the program this year. Speakers were: Professor Abel Klein, UCI; Professor Richard Vitale, Claremont Graduate School; Professor Peter Walters, University of Warwick, England (visiting USC); Professor Herbert Heyer, University of Tübingen (visiting UCSD).

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

OCTOBER MEETING OF THE OHIO SECTION

The Ohio Section of MAA held its Fall meeting at the College of Wooster, Wooster, Ohio, October 26-27, 1979. One hundred and thirty people registered for the meeting. Section Chairman D.O. Koehler presided: H.W. Vayo was the Program Chairman.

Invited addresses included: *Catastrophe Theory*, by Y-C Lu, Ohio State University; and *Mathematical Modeling and Existence Theorems*, by Dorothy L. Bernstein, Brown University.

The following contributed papers were also presented:

Applications of Complex Analysis to Operator Theory, J.J. Buoni, Youngstown State University and B.L. Wadhwa, Cleveland State University
Mathematics—'In State Nascendi', K. Cummins, Kent State University
An Analysis of Variance for Exponential Random Variables, C. Davis, University of Toledo
Operators Defined by Multiplication by Analytic Functions, J.A. Deddens, University of Cincinnati
On the Monotonicity of a Class of Exponential Sequences, T.P. Dence, Bowling Green State University, Firelands Campus
Calculus and the Hinterlands, D.O. Koehler, Miami University
An Example of Complex Variables in Operator Theory, R. Lange, Youngstown State University
Computer Controlled Milling of Models of Surfaces, C.A. Long, Bowling Green State University
Nahuat[1] Mathematics, S.E. Payne, Miami University
Life Insurance - A Computer Project for Students, L.J. Schneider, John Carroll University
Nth Order Tensor Operators, G.L. Szoke, University of Akron
Analytic Functions and Range Inclusion of the Operator $X \longrightarrow (AX\text{-}XB)$, R.E. Weber, Pennsylvania State University, Sharon Campus.

Meeting highlights included discussion sessions and special presentations. *A Panel Discussion: Modeling Courses in the Curriculum* was led by D.L. Bernstein, Brown University; D. Hull, Ohio State University (moderator); and Z. Karian, Denison University. *A 'Swap' Session: Freshman Placement* was moderated by D.J. Horwath, John Carroll University. *Special Sessions: Operator Theory* were moderated by J.J. Buoni, Youngstown State University and B.L. Wadhwa, Cleveland State University. Also, a special 'Heroes of Mathematics' lecture was presented: *The Mystery of Ancient Britain's Mathematics*, by L. Peck, Miami University.

The officers for the academic year 1979-80 are: *Executive Committee:* D.O. Koehler (Miami University), Section Chairman; D.L. Deever (Otterbein College), Section Chairman-Elect; M.D. Wetzel (Denison University), Section Past-Chairman; G. Mavrigian (Youngstown State University), Secretary-Treasurer; S.W. Hahn ((Wittenberg University), Sectional Governor; and H.W. Vayo (University of Toledo), Program Committee Chairman.

GUS MAVRIGIAN, *Secretary-Treasurer*

NOVEMBER MEETING OF THE NORTHEASTERN SECTION

The twenty-fifth annual meeting of the Northeastern Section of the MAA was held at the University of Hartford, Hartford, Connecticut, on November 17, 1979. Among the 141 people who attended the meeting was Anne F. O'Neil, the newly elected Sectional Governor. The following talks were given at the morning session:

The Poincare Model as a Barbilian Geometry, Howard Eves, University of Maine
The Christie Lecture: "Is the Universe Simply Connected", John Milnor, Institute of Advanced Study.

At the business meeting the following By-Law Amendments were presented and passed: A representative from the two-year colleges, elected to a two year term at the annual meeting in even-numbered years, is to be added to the Executive Committee; the term of the Vice-Chairman is to be shortened to one year with election to this office occurring in even-numbered years. The Chairman will be elected in odd-numbered years to serve a two year term of office.

The following officers were elected: Chairman: Roger L. Cooke, University of Vermont; Secretary-Treasurer: George W. Best, Phillips Academy; Two-Year College Representative: Nancy Meyers, Bunker Hill Community College. The business meeting closed with the announcement that the section will hold a summer meeting in June 1980.

The afternoon program included a lecture and a panel discussion: *Hand Calculators and Microprocessors*, panelists: Louise Gould, Ethel Walker School; William Hudnall, The University of Hartford; John Morris, Pratt and Whitney Aircraft, Inc. *Recreational Mathematics in the Classroom*, Richard S. Dolliver, Greater Hartford Community College

EASTERN PENNSYLVANIA AND DELAWARE SECTION

The annual section meeting was held at Drexel University on November 17, 1979. Those elected with terms: Chair - Howard Anton (1980), Vice Chair - Bing Wong (1980), Secretary-Treasurer - Willard Baxter (1982), At Large Executive Committee - Peter Jessup (1982), Pat Overdeer (1982), and Bruce Scranton (1981). Invited talks with title were: *Problems and results in Unimodal Sequences*, Curtis Greene; *Rings with involution - an overview*, Willard Baxter; and *Mathematical precocity - identifying and developing the potential*, W.C. George. There was a panel discussion: *The machine in the garden - the relationship of computer sciences and the undergraduate mathematics major*, with panelists: John Kellett, John Koch, and Walter Brown.

WILLARD E. BAXTER, *Secretary-Treasurer*

MATHEMATICAL ASSOCIATION OF AMERICA
Officers and Committees
February 1, 1980

General Offices: Dolciani Mathematical Center
1529 Eighteenth Street, N.W., Washington, D. C. 20036
Telephone 202-387-5200

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Executive Director Emeritus: Harry M. Gehman
Editorial Director: Raoul Hailpern
Secretary Emeritus: Henry L. Alder

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President-Elect, Richard D. Anderson, Louisiana State University (1980-81)
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Secretary, David P. Roselle, Virginia Tech (1980-84)
Treasurer, Leonard Gillman, University of Texas (1978-82)

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G. Baley Price, University of Kansas (1980-83)

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Gloria F. Gilmer, Atlanta University (1979-81)
Katherine P. Layton, Beverly Hills High School (1978-80)
Timothy J. Robertson, University of Iowa (1978-80)
Susan J. Devlin, Bell Telephone Laboratories (1980-82)
Richard K. Guy, University of Calgary (1980-82)

Editor of MATHEMATICS MAGAZINE: J. A. Seebach, Jr., St. Olaf College (1980)

Editor of TWO-YEAR COLLEGE MATHEMATICS JOURNAL: Donald J. Albers, Menlo College (1979-83)

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Florida, Charles W. McArthur, Florida State University
Illinois, Jon M. Laible, Eastern Illinois University
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Iowa, James L. Cornette, Iowa State University
Louisiana-Mississippi, Larry S. Haw, Nicholls State University
Maryland-D. C.-Virginia, Theodore J. Benac, U. S. Naval Academy
Michigan, Yousef Alavi, Western Michigan University
North Central, Warren Loud, University of Minnesota
Eastern Pennsylvania-Delaware, Jerry P. King, Lehigh University
Southern California, Alice M. King, California State Polytechnic Institute
Texas, James N. Younglove, University of Houston

Sectional Governors (July 1, 1978-June 30, 1981):

Allegheny Mountain, Melvin R. Woodard, Indiana University of Pennsylvania
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Nebraska, Gary H. Meisters, University of Nebraska
Northern California, Kenneth R. Rebman, California State University at Hayward

Oklahoma-Arkansas, Paul E. Long, University of Arkansas
Rocky Mountain, A. Duane Porter, University of Wyoming
Wisconsin, Gary B. Klatt, University of Wisconsin-Whitewater

Sectional Governors (July 1, 1979-June 30, 1982):

Kansas, John J. Hutchinson, Wichita State University
Missouri, Troy L. Hicks, University of Missouri, Rolla
New Jersey, Michael I. Aissen, Rutgers University
Northeastern, Anne F. O'Neill, Wheaton College
Ohio, Samuel W. Hahn, Wittenberg University
Pacific Northwest, Ivan Niven, University of Oregon
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Southwestern, Gerald S. Rogers, New Mexico State University
Southeastern, John D. Neff, Georgia Institute of Technology

COMMITTEES OF THE ASSOCIATION

Terms of members expire, except where otherwise noted, at the Annual Meeting in January following the last year of service listed below. For temporary committees, no terms are listed since they are automatically discharged at the expiration of the President's term of office, which is the Annual Meeting in January, 1981.

EXECUTIVE COMMITTEE

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Investment Committee: Leonard Gillman, Chairman (1978-82), ex-officio, Henry L. Alder (1978-80), Edward Cameron (1979-81), David P. Roselle (1980-84), ex-officio, Howard E. Zink (1980-82).

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Editor of the New Mathematical Library: Anneli Lax.

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CALENDAR OF FUTURE MEETINGS

Sixtieth Summer Meeting, University of Michigan, Ann Arbor, August 18–20, 1980.

Sixty-fourth Annual Meeting, San Francisco, California, January 9–11, 1981.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, West Virginia Wesleyan College, Buckhannon, April 25–26, 1980.

EASTERN PENNSYLVANIA AND DELAWARE, Cedar Crest College, Allentown, Pennsylvania, April 26, 1980.

FLORIDA, early March. Deadline for paper titles two weeks before meeting.

ILLINOIS, John A. Logan College, Carterville, April 25–26, 1980.

INDIANA, Valparaiso University, Valparaiso, April 26, 1980.

INTERMOUNTAIN, Utah State University, Logan, late April or early May 1980.

IOWA, Simpson College, Indianola, April 18–19, 1980.

KANSAS, Kansas State University, Manhattan, April 12, 1980.

KENTUCKY, Western Kentucky University, Bowling Green, April 11–12, 1980.

LOUISIANA–MISSISSIPPI, Friday-Saturday before February 20. Deadline for papers three months before meeting.

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, University of Richmond, Richmond, Virginia, April 12, 1980.

METROPOLITAN NEW YORK, Mercy College, Dobbs Ferry, May 4, 1980.

MICHIGAN, Hope College, Holland, May 2–3, 1980.

MISSOURI, Westminster College, Fulton, April 25–26, 1980.

NEBRASKA, Doane College, Crete, April 18–19, 1980.

NEW JERSEY, Union College, Cranford, October 25, 1980.

NORTH CENTRAL, Gustavus Adolphus College, St. Peter, Minnesota, April 25–26, 1980.

NORTHEASTERN, Saturday after Thanksgiving, and third week in June in odd-numbered years.

NORTHERN CALIFORNIA, first or second Saturday in February.

OHIO, Wittenberg University, Springfield, April 25–26, 1980.

OKLAHOMA–ARKANSAS, (approx.) Friday and Saturday of first weekend in April. Deadline for papers three weeks before meeting.

PACIFIC NORTHWEST, Central Washington University, Ellensburg, Washington, June 20–21, 1980.

ROCKY MOUNTAIN, last weekend in April or first in May. Deadline for papers eight weeks before meeting.

SEAWAY, Herkimer County Community College, Herkimer, New York, May 2–3, 1980.

SOUTHEASTERN, Appalachian State University, Boone, North Carolina, April 11–12, 1980.

SOUTHERN CALIFORNIA, first or second Saturday in March.

SOUTHWESTERN, Northern Arizona University, Flagstaff, spring 1980.

TEXAS, East Texas State University, Commerce, April 11–12, 1980.

WISCONSIN, Friday and Saturday between mid-April and first week in May. Deadline for papers six weeks before meeting.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Toronto, January 3–8, 1981.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES

AMERICAN MATHEMATICAL SOCIETY, University of Michigan, Ann Arbor, August 19–22, 1980.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Amherst, Massachusetts, June 23–26, 1980.

ASSOCIATION FOR COMPUTING MACHINERY, Nashville, Tennessee, October 27–29, 1980.

ASSOCIATION FOR SYMBOLIC LOGIC

ASSOCIATION FOR WOMEN IN MATHEMATICS, University of Michigan, Ann Arbor, August 18–22, 1980.

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHÉMATIQUES

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS, University of Michigan, Ann Arbor, August 18–21, 1980.

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Seattle, Washington, April 16–19, 1980.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Shoreham Hotel, Washington, D.C. May 5–7, 1980.

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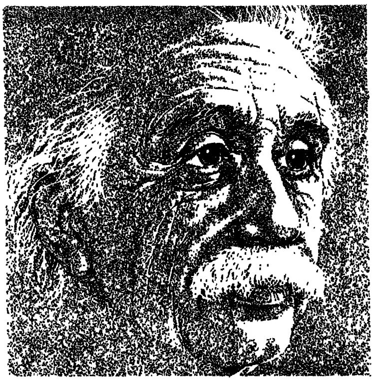

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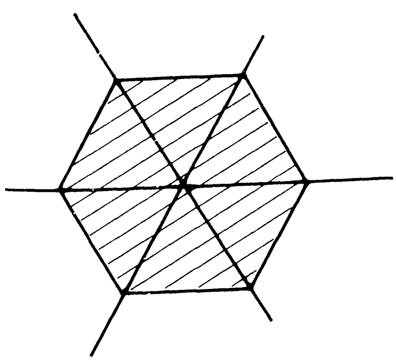

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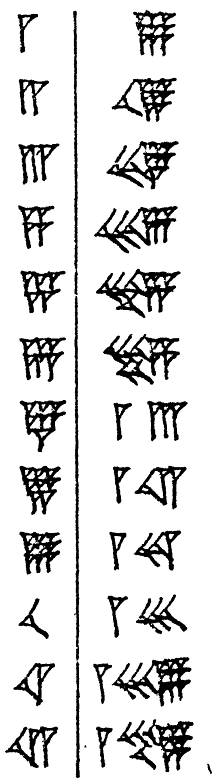
THE AMERICAN MATHEMATICAL MONTHLY

Volume 87, Number 5



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AN OFFICIAL JOURNAL OF
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VOLUME 87 **NUMBER 5**

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Research papers per se are unsuitable; see statement of policy (Vol. 85, p. 1).

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SHERLOCK HOLMES IN BABYLON

R. CREIGHTON BUCK

Let me begin by clarifying the title "Sherlock Holmes in Babylon." Lest some members of the Baker Street Irregulars be misled, my topic is the archaeology of mathematics, and my objective is to retrace a small portion of the research of two scholars: Otto Neugebauer, who is a recipient of the Distinguished Service Award, given to him by the Mathematical Association of America in 1979, and his colleague and long-time collaborator, Abraham Sachs. It is also a chance for me to repay both of them a personal debt. I went to Brown University in 1947, and as a new Assistant Professor I was welcomed as a regular visitor to the Seminar in the History of Mathematics and Astronomy. There, with a handful of others, I was privileged to watch experts engaged in the intellectual challenge of reconstructing pieces of a culture from random fragments of the past. (See [4], [5].)

This experience left its mark upon me. While I do not regard myself as a historian in any sense, I have always remained a "friend of the history of mathematics"; and it is in this role that I come to you today.

Let me begin with a sample of the raw materials. Figure 1 is a copy of a cuneiform tablet, measuring perhaps 3 inches by 5. The markings can be made by pressing the end of a cut reed into wet clay. Dating such a tablet is seldom easy. The appearance of this tablet suggests that it may have been made in Akkad in the city of Nippur in the year -1700, about 3,700 years ago.

Confronted with an artifact from an ancient culture, one asks several questions: (i) What is this and what are its properties? (ii) What was its original purpose? (iii) What does this tell me about the culture that produced it? In the History of Science, one expects neither theorems nor rigorous proofs. The subject is replete with conjectures and even speculations; and in place of proof, one often finds mere confirmation: "I believe P implies Q; and because I also believe Q, I therefore also believe P."

In Figure 1, we draw a vertical line to separate the first two columns. In the first column, we recognize what seem to be counting symbols for the numbers from 1 through 9. Paired with these, in the second column we see 9, then 1 and 8, then 2 and 7, and then 3 and 6. This suggests that what we have is a "table of 9's," a multiplication table for the factor 9. Checking further, we see 5 and 4 across from the counting symbol for 6, which confirms the conjecture. However, in the next line we see 7 and then across from it what seems to be a 1 and a 3.

We modify our conjecture; instead of an ordinary decimal system, we are dealing with a hybrid. There is a decimal substratum, using one type of wedge for units and another for tens, but the system is base 60 in the large. The 1 and 3 in fact represent $60+3=63$. We then immediately conjecture that the same wedge symbol will be used for 10, for 60, for $(60)^2$, $(60)^3$, and so on, while the digits will be given in a decimal form.

Thus from a single tablet we might have conjectured a complete sexagesimal numeral system. We would then seek confirmation of this by examining other tablets, hoping to see the same patterns there. Indeed, this was done in the last century, and among the thousands of

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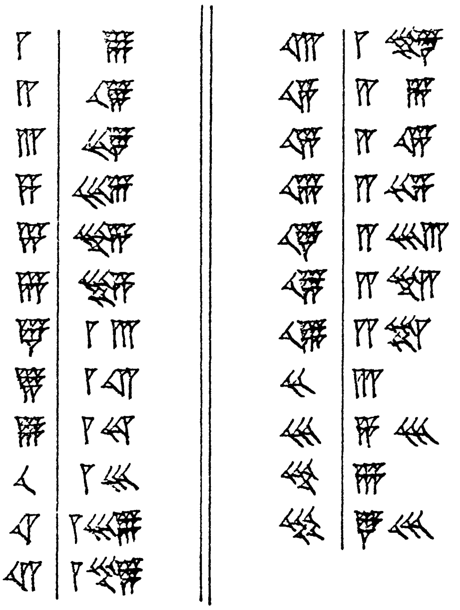


FIG. 1

Babylonian tablets many were found that bear multiplication tables of the same general type as that given in Figure 1, generated by various multiplication factors. There are a great many duplicates.

We find the Babylonian numeral system cumbersome to write. In this paper, base 60 numerals will be written by putting the digits (0 through 59) in ordinary Arabic base ten, and separating consecutive digits by the symbol "/". The "units place" will be on the right as usual. Thus,

$$7/13/28 \quad \text{represents} \quad 28+13(60)+7(60)^2=26{,}008$$

Addition is easy:

$$\begin{array}{r} 14/28/31 \\ 3/35/45 \\ \hline 18/4/16 \end{array}$$

If the tablets that bear multiplication tables are catalogued, something strange is seen. Many tables of 9's, 12's, etc., are found; but there are also multiplication tables for unlikely factors, while many tables we would have expected never appear. In Figure 2, we list those that occur frequently.

We are left with three puzzles: (i) Why are some tables missing? (For example, 7, 11, 13, 14, etc.?) (ii) Why are there tables with factors such as 3/45, 7/12, 7/30, and 44/26/40? (iii) Why are there so many tablets with exactly the same multiplication tables on them? Some clues are found; for example, there are tablets that contain two versions of the same multiplication table, one done neatly and one less neatly and perhaps with an error or two. I am sure that a familiar picture comes immediately to your mind: a cluster of students, all engaged in copying a model table provided by the teacher who will shortly be grading their efforts. Are we not correct to infer that in Nippur there was probably an extensive school for scribes who were in training to become bureaucrats or priests?

To help answer the first two questions, let us examine another tablet, which for convenience I have transcribed into the slash notation. (See Fig. 3.) This again fits the pattern of two matched

Factors Used for
Multiplication Tables

2	18	1/15=75	7/12=432
3	20	1/20=80	7/30=450
4	24	1/30=90	8/20=500
5	25	1/40=100	12/30=750
6	30	2/15=135	16/40=1000
8	36	2/24=144	22/30=1350
9	40	2/30=150	44/26/40=160,000
10	45	3/20=200	
12	48	3/45=225	and a scattering of others
15	50	4/30=270	
16		6/40=400	

FIG. 2

2	30	16	3/45	45	1/20
3	20	18	3/20	48	1/15
4	15	20	3	50	1/12
5	12	24	2/30	54	1/6/40
6	10	25	2/24	1/4	56/15
8	7/30	27	2/13/20	1/12	50
9	6/40	30	2	1/15	48
10	6	32	1/52/30	1/20	45
12	5	36	1/40	1/21	44/26/40
15	4	40	1/30		

FIG. 3

columns, and we look for an explanation. We note at once that in the first few rows the product of the adjacent column numbers is always 60. There seem to be some exceptions, however. With the pair 9 and 6/40, this product is

$$(9)\times(6/40)=(9)\times(400)=3600$$

and again

$$(16)\times(3/45)=(16)\times(225)=3600$$

while still further down, we see

$$(27)\times(2/13/20)=(27)\times(8000)=216{,}000.$$

The solution becomes obvious if we write these products in Babylonian form; since 60 is 1/0, 3600 is 1/0/0, and 216,000 is 1/0/0/0. For confirmation, look at the last entry in the table:

$$(1/21)\times(44/26/40)=(81)\times(160{,}000)=12{,}960{,}000$$
$$=1/0/0/0/0.$$

If we now follow the Babylonian practice of omitting terminal zeros, we see that Figure 3 is merely a table of reciprocals, written in "sexagesimal floating point." If A is an integer in the first column, the integer paired with it in the second column, A^R, is one chosen so that their product would be written as "1," meaning *any* suitable power of 60. The integers that appear in the table will always be factorable into powers of 2, 3, and 5, since these have terminating reciprocals in base 60. The term "floating-point arithmetic" is today a computer concept but is also understandable to anyone who has used a slide rule or worked with logarithms; the concept

would also have been familiar to medieval astronomers who multiplied large numbers by the device called "posthaphaeresis."

Now that Figure 3 is understood, we can answer the two puzzles left hanging on the previous page. Observe that the integers used to generate multiplication tables, as seen in Figure 2, mostly come from the standard reciprocal table. (There are also tablets that contain nonstandard reciprocals, reciprocals of such numbers as 7, 11, etc., of necessity given in terminating approximate form.) In floating point, $B \div A = B \times A^R$. Thus the combination of a set of multiplication tables and a reciprocal table makes it easy to carry out floating-point division, provided that the divisor is one of the "nice" numbers in base 60, of the form $2^\alpha 3^\beta 5^\gamma$. For example, let us divide 417 by 24; in base 60, this will be $6/57 \div 24 = 17/22/30$.

Method: $6/57 \div 24 = (6/57) \times (24)^R = (6/57) \times (2/30)$:

$$\begin{aligned} 6/57 \times 2 &= 12 + 1/54 = 13/54 \\ 6/57 \times 30 &= 3 + 28/30 = \underline{3/28/30} \\ \text{answer} \quad &= 17/22/30 \end{aligned}$$

The last steps in this calculation are easier if one recalls that $30 = 2^R$, so that multiplication by 30 is the same as halving. (Of course the scribe must be sure to keep track of the actual magnitudes and place values.)

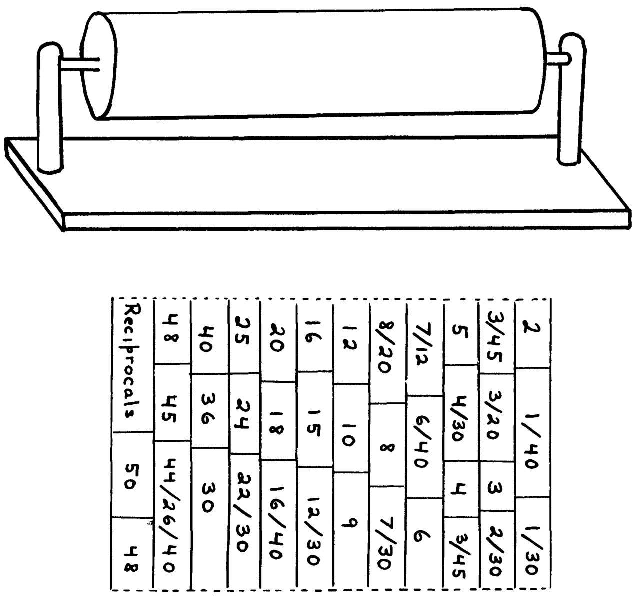


FIG. 4

That common calculations were made in this fashion becomes even more plausible in the light of one remarkable discovery. This is an inscribed cylinder, carrying on its curved face a copy of the standard reciprocal table and each of the standard multiplication tables. (In Figure 4, we show this restored, with each multiplication table indicated by its generator.) With the help of this cylinder, perhaps mounted on a stand, a scribe could easily keep track of taxes and calculate wages; perhaps we have here the Babylonian version of a slide rule or desk calculator!

With this brief introduction to the arithmetic of the Babylonians, we turn to another tablet whose mathematical nature had been overlooked until the work of Neugebauer and Sachs. It is

in the George A. Plimpton Collection, Rare Book and Manuscript Library, at Columbia University, and usually called Plimpton 322. (See Fig. 5, which is reproduced here by permission of the Library.) The left side of this tablet has some erosion; traces of modern glue on the left edge suggest that a portion that had originally been attached there has since been lost or stolen. Since it was bought in a marketplace, one may only conjecture about its true origin and date,

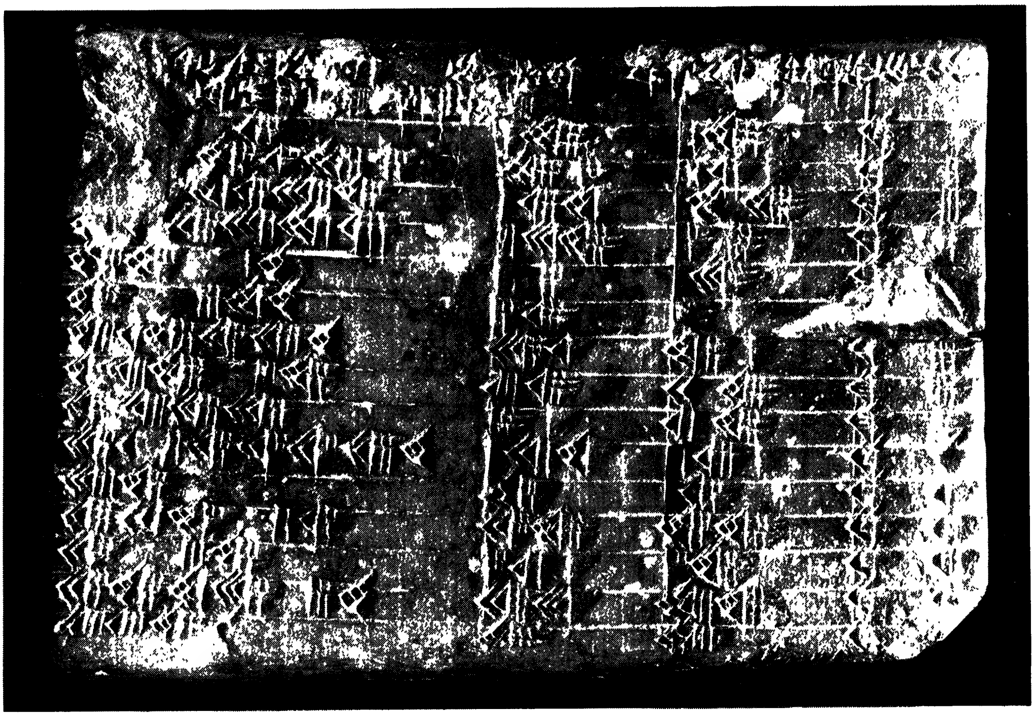


FIG. 5. Plimpton 322

Plimpton 322

Column *A*	Column *B*	Column *C*
15	1/59	2/49
58/14/50/6/15	56/7	3/12/1
1/15/33/45	1/16/41	1/50/49
5 29/32/52/16	3/31/49	5/9/1
48/54/ 1/40	1/5	1/37
47/ 6/41/40	5/19	8/1
43/11/56/28/26/40	38/11	59/1
41/33/59/ 3/45	13/19	20/49
38/33/36/36	9/1	12/49
35/10/2/28/27/24/26/40	1/22/41	2/16/1
33/45	45	1/15
29/21/54/ 2/15	27/9	48/49
27/ 3/45	7/12/1	4/49
25/48/51/35/6/40	29/31	53/49
23/13/46/40	56	53

FIG. 6

although the style suggests about -1600 for the latter. As with most such tablets, this had been assumed to be a commercial account or inventory report. We will attempt to show why one can be led to believe otherwise.

First, let us transcribe it into the slash notation, as seen in Figure 6. We have reproduced the three main columns, which we have labeled A, B, and C. We note that there are gaps in column A, due to the erosion. However, it seems apparent that the numbers there are steadily decreasing. We note that some of the numerals there are short and some long, apparently at random. In contrast with this, all the numerals in columns B and C are rather short, and we do not see any evidence of general monotonicity.

B	C
119	169
3367	11521
4601	6649
12709	18541
65	97
319	481
2291	3541
799	1249
541	769
4961	8161
45	75
1679	2929
25921	289
1771	3229
56	53

FIG. 7

$C+B$	$C-B$
288	50
14888	8154
11250	2048
31250	5832
162	32
800	162
5832	1250
2048	450
1310	228
13132	3200
120	30
4608	1250
26210	−25632
5000	1458
109	−3

FIG. 8

Since it is easier for us to work with Arabic numerals, let us translate columns B and C into these numerals and look for patterns. (See Fig. 7.) We see at once that B is smaller than C, with only two exceptions. Also, playing with these numbers, we find that column B contains exactly one prime, namely, 541, while column C contains eight numbers that are prime.

In the first 20,000 integers, there are about 2,300 primes, which is about 10 percent; among 15 integers, selected at random from this interval, we might, then, expect to see one or two primes, but certainly not eight! This at once tells us that the tablet is mathematical and not merely

B	C	(a, b)
119	169	12, 5
3367	11521	?
4601	6649	75, 32
12709	18541	125, 54
65	97	9, 4
319	481	20, 9
2291	3541	54, 25
799	1249	32, 15
541	769	?
4961	8161	81, 40
45	75	?
1679	2929	48, 25
25921	289	?
1771	3229	50, 27
56	53	?

FIG. 9

Corrected Version

B	C	(a, b)
119	169	12, 5
3367	4825	64, 27
4601	6649	75, 32
12709	18541	125, 54
65	97	9, 4
319	481	20, 9
2291	3541	54, 25
799	1249	32, 15
481	769	25, 12
4961	8161	81, 40
45	75	$1, \frac{1}{2} = 30$
1679	2929	48, 25
161	289	15, 8
1771	3229	50, 27
56	106	9, 5

FIG. 10

arithmetical. (Imagine your feelings if you were to find a Babylonian tablet with a list of the orders of the first few sporadic simple groups.)

Encouraged, one attempts to find further visible patterns, for example, by combining the entries in columns B and C in various ways. One of the earliest tries is immediately successful. In Figure 8, we show the results of calculating $C+B$ and $C-B$. If you are sensitive to arithmetic you will note that, in almost every case, the numbers are each twice a perfect square.

If $C+B=2a^2$ and $C-B=2b^2$, then $B=a^2-b^2$ and $C=a^2+b^2$. Thus the entries in these columns could have been generated from integer pairs (a,b). In passing, we note that b, being $(a-b)(a+b)$, is not apt to be prime; on the other hand, when a and b are relatively prime, every prime of the form $4N+1$ can be expressed as a^2+b^2.

In Figure 9, we have recopied columns B and C, together with the appropriate pairs (a,b) in the cases where this representation is possible. As a further confirmation that we are on the right track, we note that in every such pair the numbers a and b are both "nice," that is, factorable in terms of 2, 3, and 5. In five cases, the pattern breaks down and no pair exists. It will be a further confirmation if we can explain these discrepancies as errors made by the scribe who produced the tablet. We make a simple hypothesis and assume that B and C were each computed independently from the pair (a,b) and that a few errors were made but each affected only one number in each row. Thus in each vacant place we will assume that either B or C is correct and the other wrong, and attempt to restore the correct entry. Since we do not know the correct pair (a,b) we must find it; because of the evidence in the rest of the table, we insist that an acceptable pair must be composed of "nice" sexagesimals.

We start with line 9; here, $B=541$, which happens to be the only prime in Column B. We therefore assume B is wrong and C is correct, and thus write $C=769=a^2+b^2$. This has a single solution, the pair (25,12). (We also note that both happen to be nice sexagesimals.) If this is correct, then B should have been $(25)^2-(12)^2=481$, instead of 541 as given. Is there an obvious explanation for this mistake? Yes, for in slash notation, $541=9/1$ and $481=8/1$. The anomaly in line 9 seems to be merely a copy error.

Turn now to line 13; here, B is far larger than C, which is contrary to the pattern. Assume that B is in error and C is correct, and again try $C=289=a^2+b^2$. There is a "nice" unique solution, (15,8), and using these, we are led to conjecture that the correct value of B is $(15)^2-(8)^2=161$. Again, we ask if there is an obvious explanation for arriving at the incorrect value given, 25921. A partial answer is immediate: $(161)^2=25921$; so that for some reason the scribe recorded the *square* of the correct value for B.

Continuing, consider line 15. Since $B=56$ and $C=53$, we have $B>C$, which does not match the general pattern. However, it is not clear whether B is too large or C too small. Trying the first, we assume C is correct and solve $53=a^2+b^2$, obtaining the unique answer (7,2). We reject this, since 7 is not a nice sexagesimal. Now assume that B is correct, and write $56=a^2-b^2=(a+b)(a-b)$. This has two solutions, (15,13) and (9,5). We reject the first and use the second, obtaining $9^2+5^2=106$ as the correct value of C. Seeking an explanation, we note that the value given by the scribe, 53, is exactly *half* of the correct value.

Turning now to line 2 of Figure 9, we have $B=3367$ and $C=11521$, either of which might be correct. Assume that $C=a^2+b^2$ and find two solutions (100,39) and (89,60). While 100 and 60 are nice, 39 and 89 are not, so we reject both pairs and assume that B is correct. Writing $3367=(a-b)(a+b)$ and factoring 3367 in all ways, we find four pairs: (1684, 1683), (244,237), (136, 123), (64,27), of which we can accept only the last. This yields $(64)^2+(27)^2=4825$ as the correct C. Comparing this with the number 11521 that appeared on the tablet, we see no immediate naive explanation for the error. For example, since $4825=1/20/25$ and $11521=3/12/1$, it does not seem to be a copy error. Without an explanation, we may have a little less confidence in this reconstruction of the entries in line 2.

The last misfit in the table is line 11, where we have $B=45$ and $C=75$. This is unusual also because this is the only case where B and C have a common factor. The sums-and-differences-of-squares pattern failed because neither $C+B=120$ nor $C-B=30$ is twice a square. However,

everything becomes clearer if we go back to base 60 notation and remember that we use floating point; for $120=2/0$, which is twice $1/0$ and which we can also write as 1, clearly a perfect square. In the same way, 30 is twice 15, which is also 4^R and which is the square of 2^R. The pattern is preserved and no corrections need be made in the entries: with $a=1=1/0$ and $b=\frac{1}{2}=2^R=30=0/30$, we have $a^2=1/0$ and $b^2=0/15$, and

$$C=a^2+b^2=1/0+0/15=1/15=75$$
$$B=a^2-b^2=1/0-0/15=0/45=45.$$

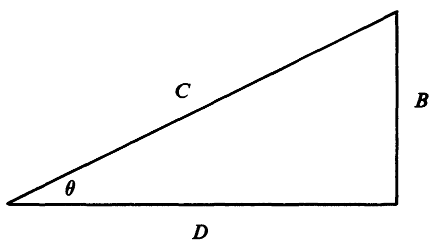
(Another aspect of the line 11 entries will appear later.)

With this, we have completed the work of editing the original tablet. In Figure 10, we give a corrected table for columns B and C, together with the appropriate pairs (a,b) from which they can be calculated.

It is now the time to raise the second canonical question: What was the purpose behind this tablet? Speculation in this direction is less restricted, since the road is not as well marked. We can begin by asking if numbers of the form a^2-b^2 and a^2+b^2 have any special properties. In doing so, we run the risk of looking at ancient Babylonia from the twentieth century, rather than trying to adopt an autochthonous viewpoint. Nevertheless, one relation is extremely suggestive, involving both algebra and geometry. For any numbers (integers) a and b,

$$(a^2-b^2)^2+(2ab)^2=(a^2+b^2)^2 \qquad (*)$$

In addition, if we introduce $D=2ab$, then B, C, and D can form a right-angled triangle with $B^2+D^2=C^2$. And finally, these formulas generate *all* Pythagorean triplets (triangles) from the integer parameters (a,b). (See Fig. 11.)



$B=a^2-b^2, \quad D=2ab, \quad C=a^2+b^2$

FIG. 11

There is no independent information showing that these facts were known to the Babylonians at the time we conjecture that this tablet was inscribed, although, as will appear later, their algebra had already mastered the solution of quadratic equations. If the tablet indeed is connected with this observation, then the unknown column A numbers ought to be connected in some way with the same triangle. The next step is, then, to proceed as before and try many different combinations of B, C, and D, in hopes that one of these will approximate the entries in column A. Slopes and ratios are an obvious starting point, so one calculates $C\div B, C\div D, B\div D$, etc. After discarding many failures, one arrives at the combination $(B\div D)^2$. In Figure 12, we give the values of this expression, calculated from the corrected values of B and using the hypothetical values of (a,b) to find D. (We remark that it was very helpful to have a programmable pocket calculator that could be trained to work in sexagesimal arithmetic!)

If we now return to Figure 6 and compare the numerals given there in column A with those that appear in Figure 12, we see that there is almost total agreement. For example, in line 10 we have exact duplication of an eight-digit sexagesimal! On probabilistic grounds alone, this is an

overwhelming confirmation. Of course, at the top of the tablet where there were gaps due to erosion, Figures 6 and 12 are not the same, but it is evident that the calculated data in Figure 12 can be regarded as filling in the gaps. There are two minor disagreements in the two tables. In line 13, the tablet does not show an internal "0" that is present in Figure 12. This could have been the custom of the scribe in dealing with such an event. In line 8, the scribe has written a digit "59" where there should have been a consecutive pair of digits, "45/14". Since $59=45+14$, it is not difficult to invent several different ways in which an error of this sort could have been made.

Calculated Values of $(B+D)^2$

line		
	1	59/0/15
	2	56/56/58/14/50/6/15
	3	55/7/41/15/33/45
	4	53/10/29/32/52/16
	5	48/54/1/40
	6	47/6/41/40
	7	43/11/56/28/26/40
	8	41/33/45/14/3/45
	9	38/33/36/36
	10	35/10/2/28/27/24/26/40
	11	33/45
	12	29/21/54/2/15
	13	27/0/3/45
	14	25/48/51/35/6/40
	15	23/13/46/40

FIG. 12

It should be remarked that Neugebauer and Sachs did not use $(B+D)^2$ as a source for column A but rather $(C+D)^2$. Because of the relationship between B and C, and formula (*), one sees that $(C+D)^2=(B+D)^2+1$. Thus, the only effect of the change would be to introduce an initial "1/" before all the sexagesimals that appear in Figure 12, and the reason for their choice was that they believed that this was true for column A on the Plimpton tablet. Others who have examined the tablet do not agree. (I have not seen the tablet, and I do not believe it matters which alternative is used.)

We now know the relationship of columns A, B, and C. Referring to Figure 11, C is the hypotenuse, B the vertical side, and A is the square of the slope of the triangle; thus, in modern notation $A=\tan^2\theta$. It is interesting to observe that the anomalous case of line 11, with $B=45$ and $C=75$, turns out to be the familiar 3,4,5 triangle; in the Babylonian case, this would seem to have been the $\frac{3}{4}, 1, \frac{5}{4}$ triangle, since $45=3\times 4^R$ and $75=1/15=5\times 4^R$. Of course the triangle, the side D, and the parameters (a,b) are all constructs of ours and not immediately visible in the original tablet. All that we can assert without controversy is that $A=B^2+(C^2-B^2)$.

Let us reexamine some of our reasoning. In lines 2, 9, 13, and 15, the scribe recorded correct values for A but incorrect values for C, B, B, and C, respectively. This suggests strongly that A was not calculated directly from the values of B and C, but that A, B, and C were all calculated independently from data that do not appear on the tablet; our hypothetical pair (a,b) gains life. (Of course there is the possibility that the tablet before us is merely a copy from another master tablet.) In either case, it seems odd that column A should be error free while columns B and C, involving simpler numbers, should have four errors.

Other questions can be raised. If, as argued by Neugebauer, the purpose of the tablet was to record a collection of integral-sided Pythagorean triangles (triplets), why do we not see the values of D, or at least the useful parameters (a,b)? And why would one want the values in column A which are squares of the slope? And why should the entries be arranged in an order that makes the numbers A decrease monotonically?

Variants of this explanation have been proposed. If one computes the values of the angle θ for each line of the tablet, they are seen to decrease steadily from about 45° to about 30°, in steps of about 1°. Is this an accident? Could this tablet be a primitive trigonometric table, intended for engineering or astronomic use? But again, why is $\tan^2\theta$ useful [3],[6]?

Additional confirmation of such a hypothesis could be given by an outline of a computation procedure leading to the tablet, which makes all of the errors plausible and also shows why they would have occurred preferentially in columns B and C. (See [1],[4],[7].)

Building upon an earlier suggestion of Bruins, an intriguing explanation has been recently proposed by Voils. In Nippur, a large number of "school texts" have been found, many containing arithmetic exercises. Among these, a standard puzzle problem is quite common. The student is given the difference (or sum) of an unknown number and its reciprocal and asked to find the number. If x is the number (called "igi") and x^R is its reciprocal (called "igibi"), then the student is to solve the equation $x - x^R = d$. Thus, the "igi and igibi" problems are quadratic equations of a standard variety.

The school texts teach a specific solution algorithm: "Find half of d, square it, add 1, take the square root, and then add and subtract half of d." This is easily seen to be nothing more than a version of the quadratic formula, tailored to the "igi and igibi" problems. Voils connects this class of problems, and the algorithm above, with the Plimpton tablet as follows.

First, assume with Bruins that the tablet was computed not from the pair (a,b) but from a single parameter, the number $x = a \div b$. Since a and b are both "nice," the number x and its reciprocal x^R can each be calculated easily. Indeed, $x = a \times b^R$ and $x^R = b \times a^R$, and a^R and b^R, each appear in a standard reciprocal table. Next observe that

$$B = a^2 - b^2 = (ab)(x - x^R)$$

$$C = a^2 + b^2 = (ab)(x + x^R)$$

$$A = \left(\frac{B}{D}\right)^2 = \left\{\frac{1}{2}(x - x^R)\right\}^2.$$

This shows that the entries A, B, C in the Plimpton tablet could have been easily calculated from a special reciprocal table that listed the paired values x and x^R. Indeed, the numbers B and C can be obtained from $x \pm x^R$ merely by multiplying these by integers chosen to simplify the result and shorten the digit representation. (See [1],[2],[7].)

Voils adds to this suggestion of Bruins the observation that the numbers A are exactly the results obtained at the end of the second step in the solution algorithm, $(d/2)^2$, applied to an igi-igibi problem whose solution is x and x^R. Furthermore, the numbers B and C can be used to produce other problems of the same type but having the same intermediate results in the solution algorithm. Thus Voils proposes that the Plimpton tablet has nothing to do with Pythagorean triplets or trigonometry but, instead, is a pedagogical tool intended to help a mathematics teacher of the period make up a large number of igi-igibi quadratic equation exercises having known solutions and intermediate solution steps that are easily checked [7].

It is possible to point to another weak confirmation of this last approach. Suppose that we want a graduated table of numbers x and their reciprocals x^R. We start with the class of *all* pairs (a,b) of relatively prime integers such that $b < a < 100$ and each integer a and b is "nice," factorable into powers of 2, 3, and 5. It is then easy to find the terminating Babylonian representation for both $x = a \div b$ and for $x^R = b \div a$. Make a table of these, arranged with x decreasing. Impose one further restriction:

$$\sqrt{3} < x < 1 + \sqrt{2}\ .$$

(This corresponds to the limitation $30° < \theta < 45°$, where θ is the base angle in the triangle in Figure 11.)

Then, the resulting list of pairs will coincide with that given in Figure 10, the corrected

Plimpton table, except for three minor points. The pair (16,9) does not appear, the pair (125,54) does appear, and instead of the pair (2,1) we have the pair $(1,\frac{1}{2})$; in passing, we recall that the last pair yields the standard 3,4,5 Pythagorean triangle.

Unlike Doyle's stories, this has no final resolution. Any of these reconstructions, if correct, throws light upon the degree of sophistication of the Babylonian mathematician and breathes life into what was otherwise dull arithmetic. For other vistas into the past, especially those that show us the beginnings of computational astronomy, I refer the reader to the bibliography. I can do no better than to close with an analogy used by Neugebauer:

> In the "Cloisters" of the Metropolitan Museum in New York there hangs a magnificent tapestry which tells the tale of the Unicorn. At the end we see the miraculous animal captured, gracefully resigned to his fate, standing in an enclosure surrounded by a neat little fence. This picture may serve as a simile for what we have attempted here. We have artfully erected from small bits of evidence the fence inside which we hope to have enclosed what may appear as a possible living creature. Reality, however, may be vastly different from the product of our imagination; perhaps it is vain to hope for anything more than a picture which is pleasing to the constructive mind, when we try to restore the past.
>
> —*The Exact Sciences in Antiquity* (p. 177)

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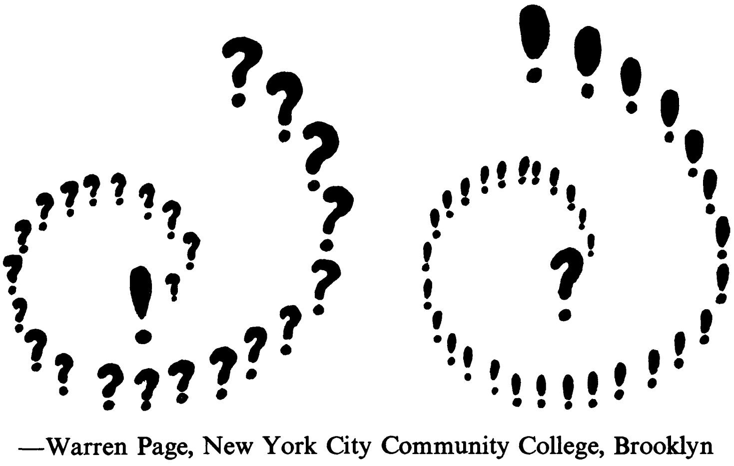
35.

OPEN AND CLOSED APPROACHES TO UNDERSTANDING



—Warren Page, New York City Community College, Brooklyn

RAMANUJAN'S EXTENSIONS OF THE GAMMA AND BETA FUNCTIONS

RICHARD ASKEY

1. Introduction. Hardy closed his obituary notice of Ramanujan [**14**] by considering how Ramanujan's work would ultimately be judged. While it is always risky to make such judgments immediately after a death, Hardy ventured the following: "It has not the simplicity and the inevitableness of the very greatest work; it would be greater if it were less strange. One gift it has which no one can deny, profound and invincible originality." Hardy reaffirmed this judgment sixteen years later [**15**, pp. 6–7].

The second statement has held up well. Ramanujan's work still seems very original. As a good indication of this originality we can consider the "lost notebook" that Andrews discovered [**2**]. This work contains hundreds of results Ramanujan found; primarily, and possibly completely, in the last year of his life. Very few of these results have been rediscovered in the more than half a century between when Ramanujan discovered them and when his pages were rediscovered. However, the first statement is not quite so clear. Some of the results of Ramanujan that seemed strange in the 1920's and 1930's seem much less strange now. This will be illustrated by two identities.

The first is

$$\int_0^\infty \frac{(1+at)(1+atq)\cdots}{(1+t)(1+tq)(1+tq^2)\cdots} t^{x-1}\,dt = \frac{\pi}{\sin \pi x} \prod_{k=1}^\infty \frac{(1-q^{k-x})(1-aq^{k-1})}{(1-q^k)(1-aq^{k-x-1})} \tag{1.1}$$

where $0<q<1$, $x>0$, and $0<a<q^x$. The right-hand side must be interpreted using a limit when x is an integer.

The second is a sum which seems even more complicated than (1.1). It will not be stated here because its complexity might dissuade the reader from continuing. Both of these identities are extensions of the beta function given as an integral on $[0,\infty)$, and the sum contains a few of Jacobi's elliptic function identities. The gamma function was discovered 250 years ago [**10**], so this is an appropriate time to reconsider it, the beta function, and the extensions found by Ramanujan. The connection with elliptic functions will not be given here. See [**5**] if you are interested in it.

Ramanujan stated (1.1) in [**21**] and started the section containing it with: "Another curious formula is the following." Hardy gave the first proof of (1.1) in [**13**]. He closed this paper with the evaluation of "another curious integral," which is another important integral. Hardy gave a nice treatment of Ramanujan's method of evaluating integrals of this type in his book on Ramanujan [**15**]. The one thing missing in this treatment is any indication that these integrals related to anything that had ever been considered before. The reader should look carefully at (1.1) and try to discover what Hardy missed.

The gamma function is the most natural extension of the factorial

$$n! = 1\cdot 2\cdots n.$$

Euler's original definition was

$$\Gamma(x+1) = \prod_{k=1}^\infty \frac{k}{(k+x)}\left(\frac{k+1}{k}\right)^x. \tag{1.2}$$

The author received his Ph.D. from Princeton in 1961 under the direction of S. Bochner. After teaching at Washington University and the University of Chicago, he joined the faculty of the University of Wisconsin (Madison) where he is now Professor of Mathematics. He was a Guggenheim Fellow in 1969–70. His main mathematical interests are in special functions and their applications, especially (more recently) those involving q-series—he says that he has a bad case of the q-disease. He has recently written a volume on orthogonal polynomials and special functions (SIAM, 1975) and contributed a section on special functions to the *Encyclopedia of Mathematics and Its Applications*.—*Editors*

Later he found the integral representation

$$\Gamma(x)=\int_0^{\infty} t^{x-1}e^{-t}\,dt, \qquad \operatorname{Re} x>0. \tag{1.3}$$

The q-factorial is defined by

$$\begin{aligned} n!_q &= 1(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}) \\ &=(1-q)(1-q^2)\cdots(1-q^n)(1-q)^{-n}. \end{aligned} \tag{1.4}$$

When $0<q<1$ this can be rewritten as

$$(1-q)^{-n}\prod_{k=1}^{\infty}\frac{(1-q^k)}{(1-q^{k+n})}.$$

In analogy with $\Gamma(x)$, F. H. Jackson **[19]** defined $\Gamma_q(x)$ by

$$\Gamma_q(x)=\frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad 0<q<1, \tag{1.5}$$

where

$$(a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k). \tag{1.6}$$

In the future we will always take $0<q<1$. Most of the results extend to other q's, but sometimes with a bit of a change. The functional equation for $\Gamma(x)$,

$$\Gamma(x+1)=x\Gamma(x), \tag{1.7}$$

becomes

$$\Gamma_q(x+1)=\frac{1-q^x}{1-q}\Gamma_q(x) \tag{1.8}$$

for the q-gamma function. A few useful facts about the q-gamma function follow:

THEOREM A (q-analogue of the Bohr-Mollerup theorem). *The q-gamma function defined by* (1.5) *satisfies*

$$\log\Gamma_q(x) \textit{ is convex for } x>0. \tag{1.9}$$

It is the only solution of (1.8) *with this property with* $\Gamma_q(1)=1$.

THEOREM B. *The following inequalities hold*:

$$\Gamma_r(x)\leqslant\Gamma_q(x)\leqslant\Gamma(x),\ 0<x\leqslant 1 \quad \textit{or} \quad x\geqslant 2,\ 0<r<q<1, \tag{1.10}$$

$$\Gamma(x)\leqslant\Gamma_q(x)\leqslant\Gamma_r(x), \quad 1\leqslant x\leqslant 2, \quad 0<r<q<1. \tag{1.11}$$

Also

$$\lim_{q\to 1^-}\Gamma_q(x)=\Gamma(x).$$

See **[5]** for these results. In this paper two discrete approximations to the beta function were given. But no integrals were given that involve $\Gamma_q(x)$ as the value of an integral. This problem will be considered in Sections 2 and 5.

Jackson's definition (1.5) is analogous to Euler's definition (1.2). Neither one is particularly useful, for these infinite products do not occur very often in this form. However, Euler's integral (1.3) occurs regularly and it is the real reason for studying the gamma function.

There are a number of reasons for considering q-extensions of ordinary functions. One is the study of partitions of integers. For example, Euler proved the identity

$$(q;q)_\infty=\sum_{-\infty}^{\infty}(-1)^k q^{k(3k+1)/2}. \tag{1.12}$$

This has the following interpretation.

A partition of an integer n is a set of integers $\lambda_1,\dots,\lambda_j$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_j > 0$, $\lambda_1 + \cdots + \lambda_j = n$. The numbers λ_j are called the parts of the partition. If $\lambda_i \neq \lambda_{i+1}$, $i = 1,2,\dots,j-1$, the partition has distinct parts. Let $P_d(n,O)$ be the number of partitions of n into distinct parts with an odd number of parts and $P_d(n,E)$ the number with an even number of distinct parts. As an example, the partitions of 7 into distinct parts are 7, 6+1, 5+2, 4+3, 4+2+1; so $P_d(7,O)=2$, $P_d(7,E)=3$. Euler's identity (1.12) says that

$$\begin{aligned} P_d(n,O) &= P_d(n,E), && n \neq k(3k+1)/2, \\ P_d(n,O) &= P_d(n,E) + (-1)^{k+1}, && n = k(3k \pm 1)/2, \quad k = 0,1,\dots. \end{aligned} \tag{1.13}$$

Observe that $7 = 2(3\cdot 2+1)/2$ so $P_d(7,O) = P_d(7,E) + (-1)^3$. Euler found a very complicated proof of (1.12). Gauss, in some unpublished work **[11]**, and Jacobi **[20]** found a more general identity which is easier to prove.

$$\sum_{-\infty}^{\infty} (-1)^n q^{n^2} x^n = (q^2;q^2)_\infty (qx;q^2)_\infty (qx^{-1};q^2)_\infty. \tag{1.14}$$

If q^2 is replaced by q^3 and x is specialized to be $q^{\frac{1}{2}}$, then (1.14) becomes Euler's identity (1.12). For obvious reasons formula (1.14) is called the triple product identity. The series in (1.14) is an example of a theta function. Jacobi used this function and some identities he found from elliptic functions to determine the number of ways n can be written as the sum of 2, 4, 6, or 8 squares.

If $-x = e^{i\theta}$ and $q = e^{-t}$ then (1.14) can be rewritten as

$$1 + 2\sum_{n=1}^{\infty} e^{-n^2 t}\cos n\theta = \prod_{n=0}^{\infty} (1 - e^{-(n+1)t})(1 - 2e^{-(2n+1)t}\cos\theta + e^{-(4n+2)t}), \qquad t > 0. \tag{1.15}$$

The function

$$1 + 2\sum_{n=1}^{\infty} r^n \cos n\theta = (1 - r^2)/(1 - 2r\cos\theta + r^2), \qquad |r| < 1 \tag{1.16}$$

is well known and plays a central role when studying harmonic functions in the disc $x^2 + y^2 < 1$, $x = r\cos\theta$, $y = r\sin\theta$. The series in (1.15) is not as well known, but it plays the same role when studying solutions to the heat equation on a finite rod **[31]**. The positivity of both of these series is crucial and is far from obvious from the series. However, it is obvious from the functions on the right-hand side. It is not surprising that the Gauss-Jacobi identity (1.14) can be used to demonstrate the positivity of the series in (1.15). What better way is there to tell when a function changes sign than to locate all of its zeros? Usually this cannot be done, but that is exactly the essence of (1.15).

Ramanujan's second identity, the sum which was alluded to above, is an even better identity than (1.14). It contains (1.14) as a special case, and also contains the binomial theorem as a special case. Clearly, some important mathematics is contained in these identities; they are more than "curious."

2. Ramanujan's Integral Extension of the Beta Function. The beta function is defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt. \tag{2.1}$$

Euler evaluated it as

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{2.2}$$

It is possible to change variables in an integral and so transform (2.1) to many other integrals. One of the most fruitful arises on setting $s = t/(1-t)$, or $t = s/(s+1)$. The integral is then

$$\int_0^\infty \frac{t^{x-1}\,dt}{(1+t)^{x+y}} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{2.3}$$

This is the integral Ramanujan extended. Consider the case $x+y=j$. Then a natural extension of (2.3) is

$$\int_0^\infty \frac{t^{x-1}dt}{(1+t)(1+tq)\cdots(1+tq^{j-1})}, \qquad j=1,2,\ldots, \qquad 0<x<j. \tag{2.4}$$

For if $q=1$, (2.4) reduces to the integral in (2.3). As in Section 1 the finite product can be written as a quotient of two infinite products, so the integral is then

$$\int_0^\infty t^{x-1}\frac{(-tq^j;q)_\infty dt}{(-t;q)_\infty}. \tag{2.5}$$

This makes sense for real j satisfying the conditions given in (2.4). Ramanujan evaluated this integral. Before reading the derivation below, the reader should try to evaluate this integral.

Define $f(a)$ by

$$f(a)=\int_0^\infty t^{x-1}\frac{(-at;q)_\infty}{(-t;q)_\infty}dt, \qquad 0<q<1.$$

This integral converges when $x>0$ and $|a|<q^x$. Assume for the moment that $x\neq 1,2,\ldots$. Removing the first factor of $(-at;q)_\infty$ gives

$$\begin{aligned} f(a)&=\int_0^\infty t^{x-1}\frac{(-aqt;q)_\infty}{(-t;q)_\infty}[1+a(t+1)-a]dt \\ &=(1-a)f(aq)+a\int_0^\infty t^x\frac{(-aqt;q)_\infty}{(-qt;q)_\infty}\frac{dt}{t} \\ &=(1-a)f(aq)+aq^{-x}f(a). \end{aligned}$$

Thus

$$f(a)=\frac{(1-a)f(aq)}{(1-aq^{-x})}=\frac{(a;q)_\infty}{(aq^{-x};q)_\infty}f(0). \tag{2.6}$$

Unfortunately $f(0)$ is hard to evaluate if it is all you have. An integral equivalent to $f(0)$ with $x=\frac{1}{2}$ was contained in Ramanujan's first letter to Hardy. Hardy had the following to say about the integrals in this letter: "I thought that, as an expert in definite integrals, I could probably prove (1.5) and (1.6), and did so, though with a good deal more trouble than I had expected."

Fortunately we shall not have to work as hard as Hardy, since we still have some freedom left. Formula (2.6) gives $f(a)$ in terms of $f(0)$, but it also gives $f(0)$ as a function of $f(a)$. All we need to do is choose an appropriate value of a for which $f(a)$ can be evaluated easily. The value $a=q$ is one such. For

$$f(q)=\int_0^\infty \frac{t^{x-1}dt}{1+t}=\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin \pi x}. \tag{2.7}$$

Use (2.6) to find $f(0)$ in terms of $f(q)$ and substitute in (2.6) to obtain

$$f(a)=\frac{(a;q)_\infty(q^{1-x};q)_\infty}{(q;q)_\infty(aq^{-x};q)_\infty}\frac{\pi}{\sin \pi x}. \tag{2.8}$$

When $x\to k$, an appropriate limit needs to be taken in (2.8). The result is

$$\int_0^\infty t^{k-1}\frac{(-at;q)_\infty dt}{(-t;q)_\infty}=\frac{(q;q)_{k-1}\log q^{-1}}{(a^{-1};q)_k a^k(-1)^k}=\frac{-(1-q)(1-q^2)\cdots(1-q^{k-1})(\log q)}{(1-a)(q-a)\cdots(q^{k-1}-a)} \tag{2.9}$$

where

$$(a;q)_k=(1-a)(1-aq)\cdots(1-aq^{k-1})=\frac{(a;q)_\infty}{(aq^k;q)_\infty}.$$

When $a=q^{x+y}$, it is possible to rewrite (2.8) in terms of the q-gamma function and the ordinary gamma function

$$\int_0^\infty t^{x-1}\frac{(-tq^{x+y};q)_\infty\,dt}{(-t;q)_\infty}=\frac{\Gamma_q(y)\Gamma(x)\Gamma(1-x)}{\Gamma_q(x+y)\Gamma_q(1-x)}. \tag{2.10}$$

When $q\to 1$, this reduces to (2.3).

3. Ramanujan's Sum Extension of the Beta Function. One of the fundamental series expansions is the binomial theorem. It can be written as

$$(1-x)^{-a}=\sum_{k=0}^\infty \frac{(a)_k}{k!}x^k \tag{3.1}$$

where

$$(a)_k=a(a+1)\cdots(a+k-1)=\frac{\Gamma(k+a)}{\Gamma(a)}. \tag{3.2}$$

This has been extended to the q-binomial theorem

$$\sum_{k=0}^\infty \frac{(a;q)_k}{(q;q)_k}x^k=\frac{(ax;q)_\infty}{(x;q)_\infty}. \tag{3.3}$$

Simple proofs of (3.3) can be found in [**1**, Theorem 2.1] and [**8**, p. 66]. There is another version of these proofs, all of which use functional equations, that is very easy to remember and, after the principles have been learned, it is even easy to discover. To find it, first consider the problem of trying to sum the series (3.1). Set

$$f_a(x)=\sum_{k=0}^\infty \frac{(a)_k}{k!}x^k. \tag{3.4}$$

Then

$$f_a'(x)=\sum_{k=1}^\infty \frac{(a)_k}{(k-1)!}x^{k-1}=af_{a+1}(x). \tag{3.5}$$

To eliminate $f_{a+1}(x)$ consider

$$\begin{aligned} f_a(x)-f_{a+1}(x)&=\sum_{k=1}^\infty \frac{[(a)_k-(a+1)_k]}{k!}x^k \\ &=\sum_{k=1}^\infty \frac{(a+1)_{k-1}[a-(a+k)]x^k}{k!}=-xf_{a+1}(x) \end{aligned} \tag{3.6}$$

so

$$f_a'(x)=\frac{a}{(1-x)}f_a(x).$$

This gives

$$f_a(x)=(1-x)^{-a}.$$

A natural analogue of $(a)_n$ is $(q^a;q)_n=(1-q^a)\cdots(1-q^{a+n-1})$. For

$$\lim_{q\to 1}\frac{(q^a;q)_n}{(q^b;q)_n}=\frac{(a)_n}{(b)_n}.$$

To simplify printing, and for other reasons, define

$$h_a(x)=\sum_{k=0}^\infty \frac{(a;q)_k}{(q;q)_k}x^k. \tag{3.7}$$

The use of $(a;q)_k$ rather than $(q^a;q)_k$ allows one to take $a<0$ and to have it independent of q. The function of the derivative operator in (3.5) is to remove the last factor in the denominator. The last factor in the denominator in (3.7) is $(1-q^k)$ so

$$h_a(x)-h_a(qx)=\sum_{k=1}^{\infty}\frac{(a;q)_k}{(q;q)_{k-1}}x^k=(1-a)xh_{aq}(x). \tag{3.8}$$

To replace $h_{aq}(x)$ observe that

$$\begin{aligned} h_a(x)-h_{aq}(x) &= \sum_{k=1}^{\infty}\frac{(aq;q)_{k-1}}{(q;q)_k}\left[(1-a)-(1-aq^k)\right]x^k \\ &= -axh_{aq}(x). \end{aligned} \tag{3.9}$$

Thus

$$h_a(x)-h_a(qx)=\frac{(x-ax)}{(1-ax)}h_a(x)$$

or

$$h_a(x)=\frac{(1-ax)}{(1-x)}h_a(qx). \tag{3.10}$$

Iteration gives

$$h_a(x)=\frac{(ax;q)_k}{(x;q)_k}h_a(q^kx)=\frac{(ax;q)_\infty}{(x;q)_\infty}h_a(0)=\frac{(ax;q)_\infty}{(x;q)_\infty}. \tag{3.11}$$

This sum has been discovered many times. Gauss **[11]**, Cauchy **[9]**, and Heine **[17]** all rediscovered it independently. It was given earlier by Schweins **[30]** and Rothe **[29]**. Gauss **[12]** and Cauchy **[9]** both observed that taking $a=q^{-2n}$ in (3.3) and symmetrizing the sum so it extends from $-n$ to n leads in the limit to

$$\sum_{-\infty}^{\infty}(-1)^n q^{\binom{n}{2}}x^n=(x;q)_\infty(qx^{-1};q)_\infty(q;q)_\infty. \tag{3.12}$$

If q is replaced by q^2 and x by qx, formula (3.12) becomes the Gauss-Jacobi triple product (1.14).

Ramanujan found a better sum, one that contains both the q-binomial theorem and the triple product for the theta function as special cases. He considered

$$\sum_{-\infty}^{\infty}\frac{(a;q)_k}{(b;q)_k}x^k. \tag{3.13}$$

Here

$$(a;q)_k=\frac{(a;q)_\infty}{(aq^k;q)_\infty} \tag{3.14}$$

so

$$(a;q)_{-k}=\frac{1}{(aq^{-k};q)_k}=\frac{(-1)^k q^{\binom{k+1}{2}}}{a^k(qa^{-1};q)_k}.$$

The series (3.13) is composed of two series, the one when $k\geqslant 0$, which converges when $|x|<1$; and the one when $k<0$, which converges when $|b/(ax)|<1$. Thus the whole series converges when

$$|ba^{-1}|<|x|<1.$$

Observe that

$$\frac{1}{(q;q)_n}=\frac{(q^{n+1};q)_\infty}{(q;q)_\infty}$$

vanishes when $n=-1,-2,\ldots$. Thus (3.13) reduces to the series in (3.3) when $b=q$. There are a number of evaluations of (3.13) including one **[18]** that shows it can be directly obtained from the q-binomial theorem and the analyticity of (3.13) in b for $|b|$ sufficiently small. A very natural proof can be given by exactly the same argument used in Section 2. To see this set

$$\begin{aligned}\sum_{-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}x^n&=\frac{(a;q)_\infty}{(b;q)_\infty}\sum_{-\infty}^{\infty}\frac{(bq^n;q)_\infty}{(aq^n;q)_\infty}x^n\\&=\frac{(a;q)_\infty}{(b;q)_\infty}f(b).\end{aligned}$$

Then

$$\begin{aligned}f(b)&=\sum_{-\infty}^{\infty}\frac{(bq^{n+1};q)_\infty}{(aq^n;q)_\infty}x^n\left[1-b(q^n-a^{-1})-ba^{-1}\right]\\&=\left(1-\frac{b}{a}\right)f(bq)+\frac{b}{ax}\sum_{-\infty}^{\infty}\frac{(bq^{n+1};q)_\infty}{(aq^{n+1};q)_\infty}x^{n+1}\\&=\left(1-\frac{b}{a}\right)f(bq)+\frac{b}{ax}f(b)\end{aligned}$$

so

$$f(b)=\frac{\left(1-\frac{b}{a}\right)}{\left(1-\frac{b}{ax}\right)}f(bq).$$

Iterating this relation gives

$$f(b)=\frac{\left(\frac{b}{a};q\right)_\infty}{\left(\frac{b}{ax};q\right)_\infty}f(0)$$

since $f(b)$ is continuous in b for $|b|$ small enough. This is the analogue of (2.6). The special case that is easy to evaluate is $b=q$. So

$$\begin{aligned}f(b)&=\frac{\left(\frac{b}{a};q\right)_\infty\left(\frac{q}{ax};q\right)_\infty}{\left(\frac{b}{ax};q\right)_\infty\left(\frac{q}{a};q\right)_\infty}f(q)\\&=\frac{\left(\frac{b}{a};q\right)_\infty\left(\frac{q}{ax};q\right)_\infty(q;q)_\infty(ax;q)_\infty}{\left(\frac{b}{ax};q\right)_\infty\left(\frac{q}{a};q\right)_\infty(a;q)_\infty(x;q)_\infty}.\end{aligned}$$

Thus we have Ramanujan's sum

$$\sum_{-\infty}^{\infty}\frac{(a;q)_nx^n}{(b;q)_n}=\frac{(ax;q)_\infty\left(\frac{q}{ax};q\right)_\infty(q;q)_\infty\left(\frac{b}{a};q\right)_\infty}{(x;q)_\infty\left(\frac{b}{ax};q\right)_\infty(b;q)_\infty\left(\frac{q}{a};q\right)_\infty}.\tag{3.15}$$

Hardy's comment on (3.15) was that it is "a remarkable formula with many parameters" [**15**, p. 222]. Hardy does not give a proof of (3.15) but says that it can be deduced from a formula that is equivalent to the q-binomial theorem. Ramanujan does not give a proof of (3.15), but he and Hardy were both very familiar with the type of proof given above. It is likely this is the proof Hardy had in mind. It was essentially given in [**4**], but the current arrangement is easier to remember.

The formula for the sum of (3.15) looks formidable, but it can be understood and remembered. The special case $b=q$ is so important that it should be learned. However, even *it* can be partially explained. The series (3.3) converges for $|x|<1$, and $x=1$ is a pole of order one. So it is natural to expect the factor $(x;q)_\infty$ in the denominator. From a theorem of Abel: If $\lim_{n\to\infty}a_n=a$, then $\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}a_nx^n=a$, it is clear that

$$\lim_{x\to 1}(1-x)\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}x^k=\frac{(a;q)_\infty}{(q;q)_\infty}.$$

So the factor $(ax;q)_\infty$ is reasonable.

The series (3.13) converges for $|ba^{-1}|<|x|<1$ and has a pole of order one at $x=ba^{-1}$ so the factor $(b/ax;q)_\infty$ in the denominator can be remembered. When $b=q$ the sum has to reduce to the q-binomial theorem so the factor $(q/ax;q)_\infty$ in the numerator can be recalled. The factors $(ax;q)_\infty/(x;q)_\infty$ occur because (3.13) reduces to the series in (3.3) when $b=q$. The remaining factors are independent of x and can be computed by the theorem of Abel mentioned above. Heuristic arguments like this can be of great aid in trying to remember formulas, but they are almost never mentioned outside of the classroom and many people never become aware of them.

The triple product (3.12) is contained in (3.15). Set $b=0$, $a=c^{-1}$, replace x by cx, and then set $c=0$. Formulas (1.12), (3.12) and (3.15) occur in increasing order of generality and decreasing order of difficulty when trying to prove them. The more freedom there is in a formula the easier it is to prove. Euler's formula (1.12) is too tight to be easy to prove. There is nothing free to use for differentiating or differencing. The theta series (3.12) is better in this regard, and it is easy to find the factors that contain x by difference equations. However, the factor that is independent of x is harder to find, since there are no values of x for which the series can be evaluated trivially. Ramanujan's sum is easy to evaluate as we have just shown. This is so because of the freedom to work with the free parameters that it contains.

4. Connections with the q-gamma Function and an Indeterminate Moment Problem. The integral representation for the gamma function arises from (2.3) by setting $t=sy^{-1}$ and letting $y\to\infty$.

$$\begin{aligned}\lim_{y\to\infty}\int_0^\infty s^{x-1}(1+sy^{-1})^{-x-y}\,ds&=\lim_{y\to\infty}\frac{\Gamma(x)\Gamma(y)y^x}{\Gamma(x+y)}=\Gamma(x)\\&=\int_0^\infty s^{x-1}e^{-s}\,ds.\end{aligned}\tag{4.1}$$

Since

$$\frac{\Gamma_q(y)}{\Gamma_q(x+y)}=\frac{(q^{x+y};q)_\infty}{(q^y;q)_\infty}(1-q)^x$$

we have

$$\int_0^\infty\frac{t^{x-1}\,dt}{(-t;q)_\infty}=\frac{\Gamma(x)\Gamma(1-x)(1-q)^x}{\Gamma_q(1-x)}.\tag{4.2}$$

Rescaling, this is

$$\int_0^\infty\frac{t^{x-1}\,dt}{(-(1-q)t;q)_\infty}=\frac{\Gamma(x)\Gamma(1-x)}{\Gamma_q(1-x)}.\tag{4.3}$$

Since

$$\frac{1}{(x;q)_\infty}=\sum_{n=0}^{\infty}\frac{x^n}{(q;q)_n},\qquad (a=0\text{ in }(3.11)),$$

$$\frac{1}{(-(1-q)t;q)_\infty}=\sum_{n=0}^{\infty}\frac{(1-q)^n}{(q;q)_n}(-1)^n t^n$$

and this converges to $\exp(-t)$ as $q\to 1^-$. Thus (4.3) is an extension of Euler's integral for the gamma function.

It is unclear at present if it is better to write this integral representation as (4.2) or (4.3). One can easily be reduced to the other, so there is no significant difference. Usually one adopts the simpler formula, which is probably (4.2), but to facilitate comparison with the case $q=1$ we will use (4.3). This integral exists for all x with $\operatorname{Re} x>0$, so moments of all positive orders exist. A routine calculation gives

$$\frac{\displaystyle\int_0^\infty \frac{t^{\alpha+n-1}\,dt}{(-(1-q)t;q)_\infty}}{\displaystyle\int_0^\infty \frac{t^{\alpha-1}\,dt}{(-(1-q)t;q)_\infty}}=\frac{(q^{1-n-\alpha};q)_n(-1)^n}{(1-q)^n}=\frac{(q^\alpha;q)_n q^{-n\alpha-\binom{n}{2}}}{(1-q)^n},\qquad n=0,1,\ldots. \tag{4.4}$$

Ramanujan's sum (3.15) can be written as

$$\sum_{-\infty}^{\infty}\frac{(bq^k;q)_\infty}{(aq^k;q)_\infty}x^k=\frac{(ax;q)_\infty\left(\frac{q}{ax};q\right)_\infty(q;q)_\infty\left(\frac{b}{a};q\right)_\infty}{(x;q)_\infty\left(\frac{b}{ax};q\right)_\infty(a;q)_\infty\left(\frac{q}{a};q\right)_\infty}. \tag{4.5}$$

Set $b=0$, $a=-(1-q)$, $x=q^{\alpha+n}$. Then a calculation gives

$$\frac{\displaystyle\sum_{-\infty}^{\infty}\frac{q^{(\alpha+n)k}}{(-(1-q)q^k;q)_\infty}}{\displaystyle\sum_{-\infty}^{\infty}\frac{q^{\alpha k}}{(-(1-q)q^k;q)_\infty}}=\frac{(q^\alpha;q)_n q^{-n\alpha-\binom{n}{2}}}{(1-q)^n}. \tag{4.6}$$

Observe that the right-hand sides of (4.4) and (4.6) are the same.

The sum in (4.6) can be considered as an integral on $(0,\infty)$ with respect to a measure that is constant on the intervals (q^{k+1},q^k), $k=0,\pm 1,\pm 2,\ldots$, and has a jump of size

$$j_k=\frac{At^\alpha}{(-(1-q)t;q)_\infty}$$

at $t=q^k$. Here A is determined to normalize the measure to have total mass equal to one. So

$$A^{-1}=\sum_{-\infty}^{\infty}\frac{q^{j\alpha}}{(-(1-q)q^j;q)_\infty}.$$

Since the measures in (4.4) and (4.6) are both nonnegative with support contained in $[0,\infty)$, the sequence

$$m_n=\frac{(q^\alpha;q)_n}{(1-q)^n}q^{-n\alpha-\binom{n}{2}}$$

is an indeterminate Stieltjes moment sequence. Many indeterminate moment sequences are known but few have measures that have been given explicitly. Even fewer of them have explicit measures that have been found and are as different as those in (4.4) and (4.6)—i.e., one absolutely continuous and the other discrete, and yet at the same time both measures are very natural.

5. Another Extension of the Beta Function. L. J. Rogers was one of those unfortunate mathematicians whose work was not appreciated when it was done. One example is Hölder's inequality, which he published a year before Hölder. See [**16**, p. 25]. Another is a pair of identities.

$$\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}, \tag{5.1}$$

$$\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}=\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}. \tag{5.2}$$

Rogers published these in 1894 [**25**], and they were ignored for over twenty years. Ramanujan rediscovered them empirically about 1910 but could not prove them. Neither could Hardy nor others that Hardy consulted. One of these was MacMahon, who was fascinated by these identities because of their combinatorial implications. If the left-hand side of (5.1) is expanded in a power series, the coefficient of q^n is the number of ways of writing n as the sum of integers congruent to 1 or 4 (mod 5). For example, $10=9+1=6+4=6+1^4=4+4+1+1=4+1^6=1^{10}$, where 1^4 means four ones. There are six ways of writing 10 in this form. The right-hand side can be interpreted in a similar fashion. The details are a bit more complicated, but if n^2 is written as $n^2=1+3+\cdots+(2n-1)$ and each one of the factors of q^{n^2} is associated with a factor of $(q;q)_n$ (observe there are n factors in both cases) the coefficient of q^n in the power series is the number of ways of writing n as the sum of integers that differ by at least two. For example, $10=9+1=8+2=7+3=6+4=6+3+1$. Again there are six ways of writing 10. The first Rogers-Ramanujan identity (5.1) says that these numbers are equal for each n. Another interpretation of the right-hand side, which is due to Andrews [**3**], is that the coefficient of q^n in the power series expansion is the number of ways of writing n as the sum of integers so that each integer is at least as large as the number of integers used. For example, $10=8+2=7+3=6+4=5+5=4+3+3$. With these interesting combinatorial interpretations it is natural that many people became interested in trying to understand and prove (5.1) and (5.2). Ramanujan finally figured out a method of proving them [**28**], but only after he read a paper of Rogers [**25**] that contains a more general result than (5.1) and (5.2). The problem here is exactly the same problem that arose with Euler's sum (1.12). There is no freedom in the series. It is natural to consider

$$\sum_{n=0}^{\infty}\frac{q^{n^2}x^n}{(q;q)_n},$$

but one still does not know what to do without some help in knowing what to look for. Ramanujan found this help in Rogers's paper [**25**], and he and Rogers worked out direct proofs of (5.1) and (5.2). These were published jointly [**28**], and Rogers also gave another relatively direct proof [**27**]. However, buried in a paper of Rogers [**26**], the sequel to [**25**] was the real key to these identities. Rogers discovered a very interesting set of polynomials and found many of their important identities. The one thing he did not do was to show that his polynomials are orthogonal. They are, and their weight function is a very nice extension of the symmetric beta distribution.

The ultraspherical polynomials $C_n^\lambda(x)$ satisfy the three-term recurrence relation

$$2x(n+\lambda)C_n^\lambda(x)=(n+1)C_{n+1}^\lambda(x)+(n+2\lambda-1)C_{n-1}^\lambda(x), \qquad n=0,1,\ldots,\quad C_0^\lambda(x)=1,\quad C_{-1}^\lambda(x)=0. \tag{5.1}$$

Their orthogonality relation is

$$\int_{-1}^{1}C_n^\lambda(x)C_m^\lambda(x)(1-x^2)^{\lambda-\frac{1}{2}}dx=\begin{cases}0, & \text{if } m\neq n\\ \dfrac{2\pi\lambda}{(n+\lambda)}\dfrac{(2\lambda)_n}{n!}\dfrac{\Gamma(2\lambda)}{\Gamma(\lambda)\Gamma(\lambda+1)}, & \text{if } m=n,\quad \lambda>-\frac{1}{2}.\end{cases} \tag{5.2}$$

The polynomials introduced by Rogers have the recurrence relation

$$2x(1-q^{n+\lambda})C_n^\lambda(x|q)=(1-q^{n+1})C_{n+1}^\lambda(x|q)+(1-q^{n+2\lambda-1})C_{n-1}^\lambda(x|q), \tag{5.3}$$

$C_0^\lambda(x|q)=1$, $C_{-1}^\lambda(x|q)=0$, $n=0,1,\ldots$. When $0<q<1$ and $\lambda>0$ their orthogonality relation is

$$\int_{-1}^1 C_n^\lambda(x|q)C_m^\lambda(x|q)\prod_{k=0}^\infty \frac{(1-2(2x^2-1)q^k+q^{2k})}{(1-2(2x^2-1)q^{k+\lambda}+q^{2k+2\lambda})}\frac{dx}{\sqrt{1-x^2}}$$

$$=\begin{cases} 0, & \text{if } m\neq n,\\ 2\pi\dfrac{(1-q^\lambda)}{(1-q^{n+\lambda})}\dfrac{(q^{2\lambda};q)_n}{(q;q)_n}\dfrac{\Gamma_q(2\lambda)}{\Gamma_q(\lambda)\Gamma_q(\lambda+1)}, & \text{if } m=n\end{cases} \tag{5.4}$$

This orthogonality relation is given in [6] and used to obtain the main identity Rogers used in deriving the Rogers-Ramanujan identities. This identity is

$$C_n^\lambda(x|q)=\sum_{k=0}^{\lfloor n/2\rfloor} q^{\lambda k}\frac{(q^{\mu-\lambda};q)_k(q^\mu;q)_{n-k}}{(q;q)_k(q^{\lambda+1};q)_{n-k}}\frac{(1-q^{n-2k+\lambda})}{(1-q^\lambda)}C_{n-2k}^\mu(x|q). \tag{5.5}$$

In his earlier paper [25] Rogers had this identity when $\lambda\to\infty$ and $\mu=1,q$, and also the other way around. To find the general result he computed the coefficients in (5.5) for low values of n and these coefficients factored. After taking a few values of n there was only one possible form for the coefficients, so he assumed it and proved it by induction. In [6] a direct proof is given, but it uses a fairly complicated identity to obtain the simple form of the coefficients and so will not be given here. Also the orthogonality proofs that are known use a formula slightly more complicated than those given in section 3. However, it is possible to evaluate the integral (5.4) when $m=n=0$ using Ramanujan's sum. For

$$\begin{aligned} I(\lambda)&=\int_{-1}^1\prod_{k=0}^\infty\frac{(1-2(2x^2-1)q^k+q^{2k})}{(1-2(2x^2-1)q^{k+\lambda}+q^{2k+2\lambda})}(1-x^2)^{-\frac12}dx\\ &=\int_0^\pi\prod_{k=0}^\infty\frac{(1-2q^k\cos 2\theta+q^{2k})}{(1-2q^{k+\lambda}\cos 2\theta+q^{2k+2\lambda})}d\theta\\ &=\int_0^\pi\frac{(e^{2i\theta};q)_\infty(e^{-2i\theta};q)_\infty}{(q^\lambda e^{2i\theta};q)_\infty(q^\lambda e^{-2i\theta};q)_\infty}d\theta\\ &=\int_0^\pi\frac{(e^{2i\theta};q)_\infty(qe^{-2i\theta};q)_\infty}{(q^\lambda e^{2i\theta};q)_\infty(q^\lambda e^{-2i\theta};q)_\infty}(1-e^{-2i\theta})\,d\theta.\end{aligned}$$

In Ramanujan's sum (3.15) take $x=q^\lambda e^{2i\theta}$, $a=q^{-\lambda}$, $b=q^\lambda$. Then

$$I(\lambda)=\frac{(q^\lambda;q)_\infty(q^{\lambda+1};q)_\infty}{(q;q)_\infty(q^{2\lambda};q)_\infty}\sum_{-\infty}^\infty\frac{(q^{-\lambda};q)_n}{(q^\lambda;q)_n}q^{\lambda n}\int_0^\pi e^{2in\theta}(1-e^{-2i\theta})\,d\theta.$$

The only integrals that do not vanish occur when $n=0$ or $n=1$. Then

$$\begin{aligned} I(\lambda)&=\frac{\pi\Gamma_q(2\lambda)}{\Gamma_q(\lambda)\Gamma_q(\lambda+1)}\left[1-\frac{(1-q^{-\lambda})q^\lambda}{(1-q^\lambda)}\right]\\ &=\frac{2\pi\Gamma_q(2\lambda)}{\Gamma_q(\lambda)\Gamma_q(\lambda+1)}.\end{aligned}$$

To get some idea of this integral, consider the case $\lambda=2$. There is cancellation in the infinite products and

$$I(2)=\int_{-1}^{1}\frac{(4-4x^2)((1+q)^2-4qx^2)}{(1-x^2)^{\frac{1}{2}}}dx$$

$$=4^2\int_{-1}^{1}\left(\left(\frac{1+q}{2}\right)^2-qx^2\right)(1-x^2)^{\frac{1}{2}}dx=2\pi\frac{(1-q^3)}{(1-q)}.$$

Using (3.15) and a special case of a more complicated identity it is possible to prove the following:

$$\int_{-1}^{1}\prod_{k=0}^{\infty}\frac{(1-2xq^k+q^{2k})(1+2xq^k+q^{2k})}{(1-2xq^{k+\alpha}+q^{2k+2\alpha})(1+2xq^{k+\beta}+q^{2k+2\beta})}(1-x^2)^{-\frac{1}{2}}dx \tag{5.6}$$

$$=\frac{2\pi\Gamma_q(\alpha+\frac{1}{2})\Gamma_q(\beta+\frac{1}{2})(1+q)}{\Gamma_q(\alpha+\beta+1)(\Gamma_{q^2}(\frac{1}{2}))^2}\,\frac{(-q;q)_\infty(-q;q)_\infty(-q;q)_\infty}{(-q^{\alpha+\frac{1}{2}};q)_\infty(-q^{\beta+\frac{1}{2}};q)_\infty(-q^{\alpha+\beta};q)_\infty}.$$

When $q\to 1$ this converges to

$$\int_{-1}^{1}(2-2x)^{\alpha}(2+2x)^{\beta}(1-x^2)^{-\frac{1}{2}}dx=\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})\pi}{\Gamma(\alpha+\beta+1)[\Gamma(\frac{1}{2})]^2}2^{2\alpha+2\beta}$$

and since $\Gamma(\frac{1}{2})=\pi^{\frac{1}{2}}$ this is the correct value of the integral.

There is an even more general integral, with two more parameters, which can be evaluated. However this integral seems to be more complicated, so it will not be given here. See [7].

6. Reflections. Now that Ramanujan's notebooks have been published **[23]**, **[24]** we can see how Ramanujan saw these results. The first notebook has two parts. The right-hand pages are relatively systematic. They are divided into chapters which are relatively homogeneous. The left-hand pages were working pages. These have even numbers in the published version. They contained results that were discovered later, or work that did not yet fit into a framework. Results on basic hypergeometric series and integrals are contained on pages 124, 126, 128, 130, 132, 134, 136, 146, 160, 164, 182, 186, 239, 241, 243, in the first notebook. A basic hypergeometric series is a series Σa_n with a_{n+1}/a_n a rational function of q^n for a fixed q. The series (3.15) is on page 136. The integral (2.8) is given on page 182. The general theorem Ramanujan mentioned in **[21]** is also stated on this page. Both the integral (2.8) and the series (3.15) are also given in the second notebook. By this time Ramanujan understood these results better and most of Chapter 16 concerns basic hypergeometric series and integrals. Formula (2.8) is formula 14 on page 195. Formula (3.15) is 17 on page 196. It is interesting that the integral (2.8) is the only integral given in Chapter 16. So Ramanujan must have been aware of some connection between (2.8) and (3.15) by the time he wrote his second notebook. However, it is unlikely that he was aware of the very close connection between the special cases given in Section 4 of this paper.

With the advent of inexpensive copying machines it is possible to examine documents much more easily than was the case not too many years ago. Brian Birch sent me a copy of the three scholarship reports Ramanujan submitted to the Board of Studies in Mathematics. Hardy used these reports as source material for Chapter 11, "Definite Integrals", of **[15]**. In the first report Ramanujan says: "For instance, the integral treated in Ex. (v) *note* Art. 5 in the paper, Mr. G. H. Hardy, M.A., F.R.S. of Trinity College, Cambridge, considers to be 'new and interesting.'' Example (v) is the case $a=0$ of (2.8) and the *note* explains what happens when x is a positive integer. So Hardy had some idea that this was an interesting result. However, I think it is much more interesting than Hardy seems to have thought it was. Who is right will depend to a large extent on how often this integral arises. Up to this point Hardy has been right, for this integral does not seem to have arisen very often. However, now that we understand a bit more about it, I think it will start to arise. Hardy would most likely be pleased if history proves him

wrong and Ramanujan's work becomes more natural as we learn more. In any case, Ramanujan's sum (3.15) has arisen often enough so that it must be considered an important result. As we saw, it was missed by Gauss and Jacobi in an area in which they both worked extensively. Ramanujan was a remarkable mathematician in many ways.

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THE EQUATIONS FOR LARGE VIBRATIONS OF STRINGS

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1. Introduction. Many elementary books on partial differential equations ostensibly show that the wave equation in one spatial dimension describes the small transverse vibrations of an elastic string. Of these books I know of but one, namely **[21]**, whose development of the wave equation does not invoke such unjustifed simplifications as the assumption that the motion of each particle of the string is confined to a plane perpendicular to the line joining the ends of the string. I fear that slipshod derivations of the equations of mathematical physics, like those relying on such ad hoc assumptions, succeed only in convincing novice scientists and mathematicians that applications are inherently dirty and incompatible with analysis. Indeed, good students of science can only be found among those who find such presentations incomprehensible.

This article has several objectives: (i) To show that the equations governing the large motion of a string of any material can be cleanly, simply, and honestly derived from fundamental principles. (ii) To show that the weak form of these equations can be obtained by a simple, yet rigorous, procedure that, unlike the standard methods, is not based upon tacit hypotheses that are invalid in the very instances when the weak form is most useful. (The weak form of the equations, which is formally equivalent to the physicists' Principle of Virtual Work, plays a central role in the modern theory of differential equations.) (iii) To examine some as yet unstudied aspects of the relationship between the nonlinear problem for elastic strings and its linearization about a straight equilibrium state. (iv) To discuss a number of analytical questions closely connected with the use of the weak formulation.

We begin our study in Section 2 by presenting a naive yet honest derivation of the classical equations of motion under the smoothness assumption that all derivatives appearing in the governing equations are continuous. In this setting we discuss the nature of elastic and viscoelastic strings and sketch the standard derivation of the weak equations. In Section 3 we carefully reconsider this derivation without these smoothness assumptions, which may well be unwarranted on both physical and mathematical grounds. Here we show the equivalence of the integral formulation of the governing equations as an impulse-momentum law to the weak formulation when the variables of the problem are merely required to be such that all the integrals appearing in the analysis are well defined in the sense of Lebesgue. Although some

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basic results of real variable theory are needed to make this development precise, the underlying construction, which is new, is completely elementary. Though quite general, this setting cannot directly handle concentrated and impulsive forces. We content ourselves with commenting but briefly on such distributional forces because their comprehensive treatment entails some deep, open mathematical questions, which would seem to demand a far more sophisticated approach than that used here. In Section 4, we first study the elementary, but nonetheless illuminating, question of existence and uniqueness of a straight equilibrium state. We reduce this problem (which is paradigmatic for quasilinear elliptic boundary value problems) to that of finding the zeros of a real-valued function. We extend this analysis to treat a concentrated load. We next describe a systematic but formal perturbation scheme that approximates the exact equations by a sequence of nonhomogeneous linear wave equations. The first approximation consists of three classical, uncoupled, wave equations for the transverse and longitudinal motion about the straight equilibrium state. We then discuss some strategies for the rigorous determination of the relation between the linear and nonlinear problems. In Section 5 we discuss the historical background of this problem, the role of discretized models, pertinent research, and some open problems.

2. The Classical Equations. Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be an orthonormal basis for the Euclidean 3-space $\mathbb{E}^3$. A *configuration* of a string, which we think of as a region of space that a string could occupy, is defined to be a curve (not necessarily simple) in $\mathbb{E}^3$. We define the *reference configuration* of the string to be the unit segment $\{x\mathbf{i}\colon x\in[0,1]\}$. We may think of the string in its unstretched state as occupying this configuration. We identify a *material particle* of the string by its coordinate x in this configuration. Let $\mathbf{r}(x,t)$ denote the position of particle x at time t. We take the domain of $\mathbf{r}$ to be $[0,1]\times[0,\infty)$. (The function $\mathbf{r}(\cdot,t)$ thus defines the *configuration at time t*.) Then $\mathbf{r}_x(x,t)\equiv(\partial\mathbf{r}/\partial x)(x,t)$ is tangent to the curve $\mathbf{r}(\cdot,t)$ at the point $\mathbf{r}(x,t)$. In this section we assume that all functions of (x,t) and of x that are exhibited here are continuous on the interiors of their domains. (Thus, by this convention, $\mathbf{r}_x$ is continuous on $(0,1)\times(0,\infty)$.)

We assume that the ends $x=0$ and $x=1$ of the string are held fixed at the points $\mathbf{0}$ and $L\mathbf{i}$. In the optimistic spirit that led us to assume that $\mathbf{r}$ is continuous on $(0,1)\times(0,\infty)$, we may further suppose that $\mathbf{r}(\cdot,t)$ is continuous on $[0,1]$. In this case these boundary conditions are defined by the following pointwise limits:

$$\lim_{x\to 0}\mathbf{r}(x,t)=\mathbf{0},\qquad \lim_{x\to 1}\mathbf{r}(x,t)=L\mathbf{i}\qquad (\text{for } t>0). \tag{2.1a}$$

Conditions (2.1a) are conventionally denoted by

$$\mathbf{r}(0,t)=\mathbf{0},\qquad \mathbf{r}(1,t)=L\mathbf{i}. \tag{2.1b}$$

We assume that at time $t=0$, the string is released from configuration $\mathbf{u}$ with velocity field $\mathbf{v}$. If $\mathbf{r}_t(x,\cdot)$ is assumed to be continuous on $[0,\infty)$, then these initial conditions have the pointwise interpretations

$$\lim_{t\to 0}\mathbf{r}(x,t)=\mathbf{u}(x),\qquad \lim_{t\to 0}\mathbf{r}_t(x,t)=\mathbf{v}(x), \tag{2.2a}$$

which are conventionally written as

$$\mathbf{r}(x,0)=\mathbf{u}(x),\qquad \mathbf{r}_t(x,0)=\mathbf{v}(x)\qquad \text{for } x\in(0,1). \tag{2.2b}$$

(In Section 3 we examine interpretations of (2.1b) and (2.2b) that are less restrictive than those of (2.1a) and (2.2a).) For $\mathbf{r}$ to have some chance of being smooth, we should require $\mathbf{u}$ to meet the *compatibility conditions*

$$\mathbf{u}(0)=\mathbf{0},\qquad \mathbf{u}(1)=L\mathbf{i}. \tag{2.3}$$

We complete our study of the geometry of the deformed string by noting that the *elongation* at (x,t), which is the local ratio at x of the length in the configuration at time t to that of the

reference configuration, is $|\mathbf{r}_x(x,t)|$. We assume that

$$|\mathbf{r}_x(x,t)|>0 \tag{2.4}$$

to prevent this length ratio from being reduced to zero.

For $x\in(0,1)$ let $\mathbf{n}^+(x,t)$ be the contact force exerted on the material segment $[0,x)$ by the material segment $[x,1]$ at time t and let $-\mathbf{n}^-(x,t)$ be the contact force exerted on $(x,1]$ by $[0,x]$ at time t. (By definition of a contact force, $\mathbf{n}^+(x,t)$ is also the contact force exerted at time t on $[a,x)$ by $[x,b]$ for any a and b satisfying $0\leqslant a<x<b\leqslant 1$; i.e., $\mathbf{n}^+$ depends solely on (x,t).) Let $\mathbf{f}(x,t)$ be the force per unit reference length at (x,t) exerted by any other agent. $\mathbf{f}$ could depend on $\mathbf{r}$ in quite complicated ways; e.g., $\mathbf{f}$ could be given by $\mathbf{f}(x,t)=\mathbf{g}(\mathbf{r}(x,t),\ \mathbf{r}_t(x,t),x,t)$, where $\mathbf{g}$ is prescribed. (The dependence of $\mathbf{g}$ on $\mathbf{r}_t$ would account for air resistance, while its dependence on $\mathbf{r}$, which is of less physical importance, could account for the spring-like resistance of an ambient elastic medium or for variable gravitational attraction.) Let $\rho(x)$ be the mass density per unit length at x in the reference configuration.

The requirement that the resultant force on any material segment $(a,b)\subset(0,1)$ equal the time derivative of the linear momentum of that segment yields the *equations of motion* for the string:

$$\begin{aligned}\mathbf{n}^+(b,t)-\mathbf{n}^-(a,t)+\int_a^b\mathbf{f}(x,t)\,dx&=\frac{d}{dt}\int_a^b\rho(x)\mathbf{r}_t(x,t)\,dx\\&=\int_a^b\rho(x)\mathbf{r}_{tt}(x,t)\,dx\quad\text{for all }(a,b)\subset(0,1).\end{aligned}\tag{2.5}$$

Now the continuity of $\mathbf{n}^+$ implies that $\mathbf{n}^+(a,t)=\lim_{b\to a}\mathbf{n}^+(b,t)$. Since $\mathbf{f}$ and $\mathbf{r}_{tt}$ have been assumed to be continuous, we may let $b\to a$ in (2.5) to obtain

$$\mathbf{n}^+(a,t)=\mathbf{n}^-(a,t)\quad\text{for all }a\in(0,1). \tag{2.6}$$

We may accordingly drop the superscripts from $\mathbf{n}$. If we now differentiate (2.5) with respect to b and then replace b by x, we obtain the *classical equations of motion of a string*:

$$\mathbf{n}_x(x,t)+\mathbf{f}(x,t)=\rho(x)\mathbf{r}_{tt}(x,t),\quad\text{for }x\in(0,1),\ \ t>0. \tag{2.7}$$

We describe the material properties of the string by specifying how the force $\mathbf{n}$ is related to the motion $\mathbf{r}$. Such a relation is necessary if we are to obtain a formally determinate system containing (2.7). The defining property of a string, which distinguishes it from a rod (which resists bending), is that $\mathbf{n}(x,t)$ must be tangent to the curve $\mathbf{r}(\cdot,t)$ at x. (The motivation for this condition comes from the classical equations expressing the equality of the resultant torque on any segment to the time derivative of angular momentum for that segment. These equations are $\mathbf{m}_x+\mathbf{r}_x\times\mathbf{n}+\mathbf{g}=\mathbf{w}_t$ where $\mathbf{m}(x,t)$ is the couple exerted on $[0,x)$ by $[x,1]$ at t, $\mathbf{g}(x,t)$ is the applied couple per unit length, and $\mathbf{w}$ is the angular momentum. Suppose $\mathbf{g}=\mathbf{0}$. If we assume that our one-dimensional body has zero thickness, then we can take $\mathbf{m}=\mathbf{0}$ and $\mathbf{w}=\mathbf{0}$. In this case, the angular momentum equation reduces to $\mathbf{r}_x\times\mathbf{n}=\mathbf{0}$, which is our assumption for strings.) We accordingly take

$$\mathbf{n}(x,t)=n(x,t)\mathbf{r}_x(x,t)/|\mathbf{r}_x(x,t)|. \tag{2.8}$$

$n(x,t)$ is the *tension* at (x,t). Let

$$\mathbf{r}^t(x,s)=\mathbf{r}(x,t-s)\quad\text{for }s\geqslant 0. \tag{2.9}$$

$\mathbf{r}^t(x,\cdot)$ is called the *history* of $\mathbf{r}(x,\cdot)$ up to time t. We account for a very large class of materials for strings by assuming that there is a functional N such that

$$n(x,t)=N(|\mathbf{r}_x^t(x,\cdot)|,x), \tag{2.10}$$

i.e., that the tension at (x,t) is determined by the past history of the elongation $|\mathbf{r}_x|$ at x. We do not allow N to depend upon $\mathbf{r}^t(x,\cdot)$ or upon $\mathbf{r}_x^t(x,\cdot)$ because such a dependence would imply

that the material properties of the string could be altered by causing it to undergo a rigid motion. We do not allow N to depend upon t because such a dependence would imply that material properties would be influenced by the choice of a clock. N could depend upon $|\mathbf{r}_x^t(\cdot,\cdot)|$, but such generality has not proved particularly fruitful in mechanics. In the special case for which there is a function N_0 such that

$$n(x,t)=N_0(|\mathbf{r}_x(x,t)|,x), \tag{2.11}$$

the string is called *elastic*. That an increase in tension accompany an increase in elongation is ensured by requiring that $N_0(\cdot,x)$ be increasing. If there is a function N_1 such that

$$n(x,t)=N_1(|\mathbf{r}_x(x,t)|,|\mathbf{r}_x(x,t)|_t,x), \tag{2.12}$$

the string is called *viscoelastic* (of a *differential type*). That an increase in tension accompany an increase in the rate of elongation is ensured by demanding that $\beta\mapsto N_1(\alpha,\beta,x)$ be increasing. If we substitute (2.11) or (2.12) into (2.8) and then substitute this into (2.7) we obtain a quasilinear system of partial differential equations. If we use the more general constitutive assumption (2.10), then we get what may be called a quasilinear partial functional-differential equation. In this case the initial conditions (2.2) would have to be supplemented by giving $\mathbf{r}^0(\cdot,\cdot)$.

Let $\boldsymbol{\eta}$ have compact support in $(0,1)\times(0,\infty)$. We can formally obtain the *weak form* of (2.7) or the *Principle of Virtual Work* by dotting (2.7) with $\boldsymbol{\eta}$ and integrating the resulting expression by parts over $(0,1)\times(0,\infty)$. We obtain

$$\int_0^\infty\int_0^1[\mathbf{n}\cdot\boldsymbol{\eta}_x-\mathbf{f}\cdot\boldsymbol{\eta}+\rho\mathbf{r}_t\cdot\boldsymbol{\eta}_t]\,dx\,dt=0 \tag{2.13}$$

for all such $\boldsymbol{\eta}$. By using the arbitrariness of $\boldsymbol{\eta}$ and the smoothness of our other variables, we could easily reverse our steps and recover (2.7) and (2.5) from (2.13).

3. The Weak Form of the Equations. It has long been known that the solutions of the equations for purely longitudinal motion of elastic strings can exhibit shocks, i.e., discontinuities in $\mathbf{r}_x$ and $\mathbf{r}_t$ (cf. [**14**], [**16**]). The same is also true for strings governed by certain forms of (2.10), in which N depends on the past history of $\mathbf{r}_x$ (cf. [**9**], e.g.). The shock structure of spatial motions of elastic strings has recently been analyzed by [**13**]. Thus the smoothness assumptions made in Section 2 are completely unwarranted for elastic strings at least. It is clear that the integral formulations (2.5) and (2.13) would make sense under far weaker smoothness assumptions than used in Section 2; it is not clear, however, that (2.5) and (2.13) are equivalent. In this section, we study the formulation of the problem under these weaker smoothness assumptions and we give a simple direct proof of the equivalence of precisely formulated generalizations of (2.5) and (2.13). This proof replaces the demonstration of formal equivalence given in Section 2, which is universally propounded by mathematicians and physicists alike despite the pivotal role played in it by the classical equation (2.7), an equation that may be devoid of mathematical meaning. In consonance with the goals of this section and in contrast with the methods of Section 2, we must state regularity restrictions on our variables with great care.

If we formally integrate (2.5) with respect to t over $[0,\tau]$ and take account of (2.2), then we obtain the *(Linear) Impulse-Momentum Law*:

$$\int_0^\tau[\mathbf{n}^+(b,t)-\mathbf{n}^-(a,t)]\,dt+\int_0^\tau\int_a^b\mathbf{f}(x,t)\,dx\,dt = \int_a^b\rho(x)[\mathbf{r}_t(x,\tau)-\mathbf{v}(x)]\,dx. \tag{3.1}$$

The left side of (3.1) is the *(linear) impulse of the force system* $\{\mathbf{n}^\pm,\mathbf{f}\}$ and the right side of (3.1) is the *change in linear momentum* for the segment (a,b) over time interval $[0,\tau)$. We regard (3.1) as the natural generalization of the equations of motion (2.5). We now state virtually the weakest possible restrictions on the functions entering (3.1) for the integrals of (3.1) to make sense as Lebesgue integrals and for our boundary and initial conditions to have consistent generalizations. We then study these generalizations by appealing to results from real analysis. We resume the main thread of our development in the paragraph containing (3.5). The reader unfamiliar

with real analysis may wish to skim over or even skip the intervening material, which appears in small type.

We assume that there are numbers ρ^- and ρ^+ such that

$$0<\rho^- \leq \rho(x) \leq \rho^+ < \infty \quad \text{for all } x\in[0,1]. \tag{3.2}$$

We assume that $\mathbf{r}_x$ and $\mathbf{r}_t$ are locally integrable on $[0,1]\times[0,\infty)$, that $\mathbf{r}$ satisfies the boundary conditions (2.1) in the sense of trace (cf. [1], [17]), i.e., that

$$\lim_{x\to 0}\int_{t_1}^{t_2}\mathbf{r}(x,t)\,dt=\mathbf{0}, \qquad \lim_{x\to 1}\int_{t_1}^{t_2}[\mathbf{r}(x,t)-L\mathbf{i}]\,dt=\mathbf{0} \qquad \text{for each } (t_1,t_2)\in[0,\infty), \tag{3.3}$$

that $\mathbf{u}$ is integrable on $[0,1]$, that the first initial condition of (2.2) is assumed in the sense of trace:

$$\lim_{t\to 0}\int_a^b[\mathbf{r}(x,t)-\mathbf{u}(x)]\,dx=\mathbf{0} \quad \text{for all } (a,b)\subset(0,1), \tag{3.4}$$

and that $\mathbf{v}$ is integrable on $[0,1]$. (Conditions (3.2) and (3.3) are consistent with the local integrability of $\mathbf{r}_x$ and $\mathbf{r}_t$; cf. [1], [17].) We do not prescribe a corresponding generalization of the second initial condition of (2.2) because we shall show that it is inherent in (3.1) as the presence there of $\mathbf{v}$ might suggest.

We assume that $\mathbf{n}^+$, $\mathbf{n}^-$, and $\mathbf{f}$ are locally integrable on $[0,1]\times[0,\infty)$. After we determine in what sense $\mathbf{n}^+$ equals $\mathbf{n}^-$ we shall see that this requirement imposes growth restrictions on the functional N of (2.10), which we do not explore.

We must show that the first and third integrals of (3.1) make sense. This will enable us to obtain a weaker version of the intepretation of $\mathbf{n}^\pm$ than that prevailing in Section 2. Since $\mathbf{n}^+$ is locally integrable on $[0,1]\times[0,\infty)$, Fubini's Theorem implies that for each $\tau\in(0,\infty)$, there is a set $A^+(\tau)\in[0,1]$ with Lebesgue measure $|A^+(\tau)|=1$ such that $\mathbf{n}^+(x,\cdot)$ is integrable over $[0,\tau]$ for $x\in A^+(\tau)$. Moreover, the Lebesgue Differentiation Theorem implies that there is a subset $A_0^+(\tau)$ of $A^+(\tau)$ with $|A_0^+(\tau)|=1$ such that for $x\in A_0^+(\tau)$, $\int_0^\tau\mathbf{n}^+(x,t)\,dt$ has the "right" value in the sense that it is the limit of its averages over intervals centered at x. The corresponding statements obtained by replacing the superscript "+" by "−" are likewise true. Let $A(\tau)=A_0^+(\tau)\cap A_0^-(\tau)$ ($|A(\tau)|=1$). Let B be the set of t's for which $\rho(\cdot)\mathbf{r}_t(\cdot,t)$ is integrable over $[0,1]$ and for which $\int_0^1\rho(x)\mathbf{r}_t(x,t)\,dx$ has the "right" value. (By Fubini's Theorem and Lebesgue's Differentiation Theorem we have that $|B\cap[0,T]|=T$ for all $T>0$.) Thus each term in (3.1) is well defined for each $\tau\in B$ and for each a and b in $A(\tau)$ with $a<b$.

We now derive some important consequences from (3.1). Since Fubini's Theorem allows us to interchange the order of integration of the double integral in (3.1), we find that $b\mapsto\int_0^\tau\mathbf{n}^+(b,t)\,dt$ is absolutely continuous on $A(\tau)$. Consequently,

$$\int_0^\tau\mathbf{n}^+(a,t)\,dt=\lim\int_0^\tau\mathbf{n}^+(b,t)\,dt \quad \text{as } b\to a \text{ through } A(\tau).$$

Then (3.1) implies that

$$\int_0^\tau\mathbf{n}^+(a,t)\,dt=\int_0^\tau\mathbf{n}^-(a,t)\,dt \qquad \text{for each } \tau\in B \text{ and for each } a\in A \tag{3.5}$$

Thus the superscripts "+" and "−" are superfluous in (3.1) and will accordingly be dropped. Next, we observe that the properties of the Lebesgue integral imply that if $a,b\in A(\tau)$, then $a,b\in A(\sigma)$ for each $\sigma\in[0,\tau]$. Let us fix $\tau>0$ and replace τ in (3.1) by σ. Let $a,b\in A(\tau)$. By our preceding remarks (3.1) makes sense for each $\sigma\in B$. Letting $\sigma\to 0$ through B we obtain

$$\lim_{B\ni\sigma\to 0}\int_a^b\rho(x)[\mathbf{r}_t(x,\sigma)-\mathbf{v}(x)]\,dx=\mathbf{0} \quad \text{for all } a,b \text{ in } A(\tau). \tag{3.6}$$

This generalization of the second initial condition of (2.2) is thus implicit in (3.1). It is somewhat weaker than the analogous (3.4), which may be interpreted as saying that $\mathbf{r}(\cdot,t)$ converges weakly to $\mathbf{u}$ in $L_1(0,1)$. (Cf. [22] for a discussion of modes of convergence weaker than weak convergence.)

Without indulging in the artificial exploitation of (2.7), we now show how (3.1) is equivalent to a precisely formulated version of (2.13). Let ϕ be a piecewise linear function with support in

(a,c) and let ψ be a piecewise linear function with support in $[0,\tau)$. Let $\mathbf{e}$ be a fixed unit vector. Then (3.1) implies that

$$\int_0^\tau\int_a^c \phi_b(b)\psi_t(t)\left\{\int_0^t \mathbf{e}\cdot[\mathbf{n}(b,s)-\mathbf{n}(a,s)]\,ds+\int_0^t\int_a^b \mathbf{e}\cdot\mathbf{f}(x,s)\,dx\,ds\right\}db\,dt$$
$$=\int_0^\tau\int_a^c \phi_b(b)\psi_t(t)\int_a^b \rho(x)\mathbf{e}\cdot[\mathbf{r}_t(x,t)-\mathbf{v}(x)]\,dx\,db\,dt. \tag{3.7}$$

Since ψ is absolutely continuous we can integrate the left side of (3.7) by parts with respect to t over $[0,\tau]$. Since $\psi(\tau)=0$ and $\phi(a)=0$, this integration yields the result that the triple integral on the left side of (3.7) equals

$$-\int_0^\tau\int_a^c \phi_b(b)\psi(t)\mathbf{e}\cdot\mathbf{n}(b,t)\,db\,dt. \tag{3.8a}$$

Similarly, the quadruple integral on the left side of (3.7) reduces to

$$\int_0^\tau\int_a^c \phi(b)\psi(t)\mathbf{e}\cdot\mathbf{f}(b,t)\,db\,dt \tag{3.8b}$$

and the right side of (3.7) equals

$$-\int_0^\tau\int_a^c \phi(b)\psi_t(t)\rho(b)\mathbf{e}\cdot[\mathbf{r}_t(x,t)-\mathbf{v}(x)]\,db\,dt. \tag{3.9}$$

Let us set

$$\boldsymbol{\eta}(x,t)=\phi(x)\psi(t)\mathbf{e}. \tag{3.10}$$

Since $\boldsymbol{\eta}$ has support in $(a,c)\times[0,\tau)$, we can use (3.8) and (3.9) to write (3.7) as

$$\int_0^\infty\int_0^1[\mathbf{n}(x,t)\cdot\boldsymbol{\eta}_x(x,t)-\mathbf{f}(x,t)\cdot\boldsymbol{\eta}(x,t)]\,dx\,dt$$
$$=\int_0^\infty\int_0^1 \rho(x)[\mathbf{r}_t(x,t)-\mathbf{v}(x)]\cdot\boldsymbol{\eta}_t(x,t)\,dx\,dt \tag{3.11}$$

for all $\boldsymbol{\eta}$'s of the form (3.10) and more generally for all $\boldsymbol{\eta}$'s in the space V that is the completion of finite linear combinations of functions of the form (3.10) in the norm $\|\boldsymbol{\eta}\|=\operatorname{ess\,sup}(|\boldsymbol{\eta}_x|+|\boldsymbol{\eta}_t|)$. (Some properties of this space V are discussed in [3].) Note that this class of $\boldsymbol{\eta}$'s is larger than that used in (2.13) because these $\boldsymbol{\eta}$'s need not have support in $(0,1)\times(0,\infty)$. Consequently, the form of (3.11) is more general than that of (2.13). Equation (3.11) is the *Principle of Virtual Work* or the *Weak Form* of (2.7). The Weak Form of the Initial-Boundary Value Problem for Elastic Strings is obtained by inserting (2.8) and (2.11) into (3.11). Conversely, we can recover (3.1) (without the superscripts "+" and "−") from (3.11) by taking $\boldsymbol{\eta}$ to have the form (3.10) with

$$\phi(x)=\begin{cases}0 & \text{for } 0\leqslant x\leqslant a,\\ \dfrac{x-a}{\varepsilon} & \text{for } a\leqslant x\leqslant a+\varepsilon,\\ 1 & \text{for } a+\varepsilon\leqslant x\leqslant b-\varepsilon,\\ \dfrac{b-x}{\varepsilon} & \text{for } b-\varepsilon\leqslant x\leqslant b,\\ 0 & \text{for } b\leqslant x,\end{cases} \tag{3.12a}$$

$$\psi(t)=\begin{cases}1 & \text{for } 0\leqslant t\leqslant\tau,\\ 1+\dfrac{\tau-t}{\varepsilon} & \text{for } \tau\leqslant t\leqslant\tau+\varepsilon,\\ 0 & \text{for } \tau+\varepsilon\leqslant t,\end{cases} \tag{3.12b}$$

and then letting $\varepsilon \to 0$. In this process, we must evaluate

$$\lim_{\varepsilon\to 0}\int_0^{\tau+\varepsilon}\frac{1}{\varepsilon}\int_a^{a+\varepsilon}\mathbf{n}(x,t)\cdot\mathbf{e}\psi(t)\,dx\,dt, \tag{3.13}$$

which Fubini's Theorem and (3.12b) allow us to rewrite as

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_a^{a+\varepsilon}\int_0^{\tau}\mathbf{n}(x,t)\cdot\mathbf{e}\,dt\,dx+\lim_{\varepsilon\to 0}\varepsilon\left[\frac{1}{\varepsilon^2}\int_a^{a+\varepsilon}\int_\tau^{\tau+\varepsilon}\mathbf{n}(x,t)\cdot\mathbf{e}\psi(t)\,dt\,dx\right]. \tag{3.14}$$

Now the Lebesgue Differentiation Theorem implies that the first limit in (3.14) is

$$\int_0^{\tau}\mathbf{n}(a,t)\cdot\mathbf{e}\,dt \tag{3.15}$$

for almost all a in $(0,1)$ and that the least upper bound of the absolute value of the bracketed term in the second limit of (3.14) is finite for almost all $(a,\tau)\in(0,1)\times(0,\infty)$. Thus (3.13) equals (3.14). The other terms are treated similarly. The arbitrariness of $\mathbf{e}$ allows it to be canceled in the final expression.

If $\mathbf{f}$ is not locally integrable, then the development culminating in (3.5) and (3.6) is not valid. $\mathbf{f}$ would not be locally integrable if there were concentrated or impulsive forces applied to the string. Such forces could be described by measures in the context of (3.1) and by distributions in the context of (3.11). It is not evident what class of force measures, when introduced into (3.1), will both yield an effective generalization of (3.5) and support a proof of equivalence of suitably generalized versions of (3.1) and (3.11). A mathematically natural scale of generalizations of (3.11) can be obtained by the following standard procedure: We write (3.11) as

$$\langle\boldsymbol{\sigma},\boldsymbol{\eta}\rangle=0 \tag{3.16}$$

where $\boldsymbol{\sigma}$ accounts for contributions from $\mathbf{n}$, $\mathbf{f}$, and $\rho(\mathbf{r}_t-\mathbf{v})$. If we restrict $\boldsymbol{\eta}$ to belong to a class of functions E whose members are smoother than those of V, then we could correspondingly extend $\langle\cdot,\boldsymbol{\eta}\rangle$ for $\boldsymbol{\eta}\in E$ to a class of $\boldsymbol{\sigma}$'s larger than those studied above. (For example, if E were to consist of infinitely differentiable functions $\boldsymbol{\eta}$ with support in $(0,1)\times(0,\infty)$, then the class of σ's are *distributions*. This choice of E would, however, strip (3.11) of its possession of a generalization of (3.6).) This process of extending the bilinear form $\langle\cdot,\cdot\rangle$, however elegant, still avoids confronting the fundamental physical question of generalizing (3.5).

Even if a physically satisfactory extension of (3.1) were available to account for data as measures, the exceptionally challenging problem of actually analyzing such an initial-boundary value problem remains. Some, but not all, of the difficulties to be faced in such an analysis can be observed in the recent study [4] of a boundary value problem for a single semi-linear elliptic equation. In the beginning of the next section, we actually work out the solution of a degenerately simple static problem for a concentrated force on a string. One question of physical importance that arises in such analyses is to determine how a sequence of solutions corresponding to a sequence of locally integrable data converges as the sequence of data converges in some suitable topology to a measure (such as the Dirac measure). The study of concentrated and impulsive loads in the linear equations for strings, and in fact for linear differential equations in general, is of central importance because the theory yields by superposition solutions for much larger classes of data. This is not the case for nonlinear problems. One manifestation of this in the context of elastic strings, say, is that not all the derivatives on $\mathbf{r}$ can be shifted from $\boldsymbol{\sigma}$ to the test function $\boldsymbol{\eta}$ in (3.16) by integration by parts. Thus, in nonlinear problems concentrated and impulsive forces must be studied only for their intrinsic physical or mathematical interest.

4. The Linearized Equations. Suppose that (2.11) holds and that $\alpha\mapsto N_0(\alpha,x)$ is strictly increasing from $-\infty$ to ∞ as α increases from 0 to ∞. Let $p:[0,1]\to\mathbb{R}$ be integrable. Let $\mathbf{u}(x)=\sigma(x)\mathbf{i}$ where σ satisfies the equations

$$[N_0(\sigma_x(x),x)]_x+p(x)=0,\ \sigma(0)=0,\ \sigma(1)=L. \tag{4.1a,b,c}$$

If $\mathbf{v}=\mathbf{0}$ and if $\mathbf{f}(x,t)=p(x)\mathbf{i}$, then to each solution σ of (4.1) there corresponds a static solution of an appropriately generalized version of the initial boundary value problem (2.1), (2.2), (2.7),

(2.8), (2.11) given by $\mathbf{r}(x,t)=\sigma(x)\mathbf{i}$. In other words, this $\mathbf{r}$ satisfies (3.1) or (3.11) subject to the side conditions (3.3) and (3.4), as is easily verified. (This solution $\mathbf{r}$ can be shown to be unique among sufficiently regular solutions of the initial-boundary value problem.)

The boundary value problem (4.1) has a unique solution σ as the following elementary argument shows. Equation (4.1a) is equivalent to

$$N_0(\sigma_x(x),x)+P(x)=\nu\ (\text{const.}),\ P(x)\equiv\int_0^x p(y)\,dy, \tag{4.2}$$

which in turn is equivalent to

$$\sigma_x(x)=R(\nu-P(x),x). \tag{4.3}$$

Here $R(\cdot,x)$ is the inverse of $N_0(\cdot,x)$; R exists by virtue of the assumptions imposed on N_0. Moreover, $\beta\mapsto R(\beta,x)$ is strictly increasing from 0 to ∞ as β increases from $-\infty$ to ∞. If N_0 is continuously differentiable, then so is R by the implicit function theorem. The integration of (4.3) shows that

$$\sigma(x)=\int_0^x R(\nu-P(y),y)dy \tag{4.4}$$

satisfies (4.1a, b) and would satisfy (4.1c) if ν could be chosen so that

$$\int_0^1 R(\nu-P(y),y)dy=L. \tag{4.5}$$

But $\nu\mapsto\int_0^1 R(\nu-P(y),y)dy$, just like $R(\cdot,x)$, is strictly increasing from 0 to ∞ as ν increases from $-\infty$ to ∞, so that (4.5) has a unique solution for ν in terms of L. This means that (4.1) has a unique solution for σ in terms of L and P, which is obtained by substituting the solution ν of (4.5) into (4.4). Since P is absolutely continuous, equation (4.9) shows that σ is continuously differentiable, if R is continuous. If p were continuous and R were continuously differentiable, then σ would be twice continuously differentiable. Note that if the string is uniform (so that N_0 is independent of x) and if $P=0$, then $\sigma=xL$. Since $N_0(\alpha,x)\to-\infty$ as $\alpha\to0$, equation (4.3) ensures that $\sigma_x(x)>0$ for all x in $[0,1]$ (see (2.4)), so that σ is invertible. If $N_0(1,x)=0$, which reflects the eminently reasonable assumption that the reference state is free of tension, and if $P=0$, then the solution ν of (4.5) has the same sign as $L-1$. (This elementary analysis of the existence and uniqueness of σ is a primitive prototype of the application of methods of monotone operator theory to quasilinear elliptic equations.)

Let us note that (4.2) is equivalent to (2.11) and (3.1) without the superscripts "+" and "−". Equation (4.2) makes sense even when P is not absolutely continuous (i.e., when P is not the indefinite integral of an integrable function). The analysis goes through without modification as long as P is a real-valued function, e.g., if P were the Heaviside function H_c, $c\in(0,1)$. ($H_c(x)=0$ for $x<c$ and $=1$ for $x>c$.) In this case p would be the Dirac delta concentrated at c. The simplicity of this highly degenerate problem is misleading. See the discussion at the end of the last section.

Now let us consider the problem in which $\mathbf{u},\mathbf{v}$, and $\mathbf{f}$ have the form

$$\mathbf{u}(x)=\sigma(x)\mathbf{i}+\varepsilon\mathbf{u}_1(x),\mathbf{v}(x)=\varepsilon\mathbf{v}_1(x),\mathbf{f}(x,t)=p(x)\mathbf{i}+\varepsilon\mathbf{f}_1(x,t). \tag{4.6}$$

Here ε represents a small real parameter. Suppose that $N_0(\cdot,x)$ is $(p+1)$ times continuously differentiable. We seek formal solutions of the initial-boundary value problem whose dependence on the parameter ε is specified by

$$\mathbf{r}(x,t,\varepsilon)=\sigma(x)\mathbf{i}+\sum_{k=1}^{p}\frac{\varepsilon^k}{k!}\mathbf{r}_k(x,t)+o(\varepsilon^{p+1}). \tag{4.7}$$

Since

$$\mathbf{r}_k(x,t)=\frac{\partial^k\mathbf{r}(x,t,\varepsilon)}{\partial\varepsilon^k}\Big|_{\varepsilon=0},$$

we can find the problem formally satisfied by $\mathbf{r}_k$ by substituting (4.7) into the equations of the nonlinear problem, differentiating the resulting equations k times with respect to ε, and then setting $\varepsilon=0$. We find that the equation for $\mathbf{r}_k$ involves $\mathbf{r}_1,\ldots,\mathbf{r}_{k-1}$; thus the system of equations for $\mathbf{r}_1,\ldots,\mathbf{r}_p$ can be solved recursively. In particular, the equation for $\mathbf{r}_1$ reduces to

$$\left\{N_0'(\sigma_x(x),x)(\mathbf{r}_{1x}\cdot\mathbf{i})\mathbf{i}+N_0(\sigma_x(x),x)[\sigma_x(x)]^{-1}[(\mathbf{r}_{1x}\cdot\mathbf{j})\mathbf{j}+(\mathbf{r}_{1x}\cdot\mathbf{k})\mathbf{k}]\right\}_x -\rho(x)\mathbf{r}_{1tt}=-\mathbf{f}_1(x,t), \tag{4.8}$$

where N_0' is the partial derivative of N_0 with respect to its first argument. $\mathbf{r}_1$ must satisfy the boundary conditions

$$\mathbf{r}_1(0,t)=\mathbf{0},\qquad \mathbf{r}_1(1,t)=\mathbf{0} \tag{4.9}$$

and initial conditions

$$\mathbf{r}_1(x,0)=\mathbf{u}_1(x),\qquad \mathbf{r}_{1t}(x,0)=\mathbf{v}_1(x). \tag{4.10}$$

If $\mathbf{f}_1\cdot\mathbf{i}=0$, then (4.8) implies that $\dot{\mathbf{r}}_1\cdot\mathbf{i}$ satisfies the scalar wave equation

$$[N_0'(\sigma_x(x),x)w_x]_x=\rho(x)w_{tt}\quad \text{for } x\in(0,1),t>0 \tag{4.11}$$

where σ_x is given by (4.3). If we set

$$s=\sigma(x),\ \hat{\mathbf{r}}_1(s,t)=\mathbf{r}_1(\sigma^{-1}(s),t),\hat{\rho}(s)=\frac{\rho(\sigma^{-1}(s))}{\sigma_x(\sigma^{-1}(s))}, \tag{4.12}$$

if $\mathbf{f}_1\cdot\mathbf{j}=0$ and $\mathbf{f}_1\cdot\mathbf{k}=0$, and if we use (4.2), then we obtain from (4.8) that $\hat{\mathbf{r}}_1\cdot\mathbf{j}$ and $\hat{\mathbf{r}}_1\cdot\mathbf{k}$ each satisfy the scalar wave equation

$$\left\{\left[\nu-P(\sigma^{-1}(s))\right]w_s\right\}_s=\hat{\rho}(s)w_{tt}\quad \text{for } s\in(0,L),t>0. \tag{4.13}$$

Note that equations (4.11) and (4.13) are uncoupled.

Equation (4.11) describes the small longitudinal motion of the string (or of a rod) about its straight, stretched equilibrium state. Note that both the nonuniformity of the string and the presence of P cause the coefficients of (4.11) to depend upon x. (Equation (4.11) is frequently cited as a source of Sturm-Liouville problems for ordinary differential equations; these are obtained from (4.11) by separation of variables.) Since N_0' is positive by hypothesis, equation (4.11) is hyperbolic.

Equation (4.13) describes the small transverse vibrations of the string. $\nu-P(\sigma^{-1}(s))$ is the tension of the string at $\sigma^{-1}(s)$ in the configuration $\sigma\mathbf{i}$ and $\hat{\rho}$ is the mass per unit length in this configuration. If $P=0$, this tension, which is the coefficient of w_{ss} in (4.13), is constant whether or not the string is uniform. This tension need not be positive. Where it is, the equation is hyperbolic; where it is negative, the equation is elliptic. In the latter case, our initial value problem is not well posed. That this is not surprising is apparent from the erratic behavior of a rubber band under compression. This same absence of hyperbolicity (where the tension is negative) can occur in the full nonlinear equations for elastic strings and represents a source of serious technical difficulty. This difficulty can be removed by endowing the string with resistance to bending and twisting (i.e., by replacing the string theory with a rod theory) at the cost of enlarging the number of equations. These enlarged systems have a number of attractive analytic features, which compensate to some extent for their complexity (cf. [2]).

The left side of (4.8) is exactly the Gâteaux differential at $\sigma\mathbf{i}$ in the direction $\mathbf{r}_1$ of the operator $\mathbf{r}\mapsto[N_0(|\mathbf{r}_x|,\cdot)\mathbf{r}_x/|\mathbf{r}_x|]_x-\rho(\cdot)\mathbf{r}_{tt}$ from $C^2([0,1]\times[0,T])$ to $C^0([0,1]\times[0,T])$. For the reasons mentioned at the beginning of Section 3, the domain $C^2([0,1]\times[0,T])$ of this nonlinear operator is singularly inappropriate. Thus, we find ourselves in the paradoxical situation of being unable to reconcile (4.8), which is easy to analyze, which describes the physics of small vibrations with good accuracy, and which is obtained from (2.7), (2.8), (2.11) by a natural but unfortunately purely formal process, with the nonlinear system (2.7), (2.8), (2.11), which is derived in a geometrically exact way from fundamental physical principles and from a general assumption

on the material behavior of the string. I know of no work that gives a mathematically precise resolution of this incompatibility, an incompatibility due to the characteristic irregularity of solutions of quasilinear hyperbolic systems.

By examining some related problems, however, it is possible to gain insight into the nature of this difficulty and thereby to be in a position to suggest ways to clarify this issue. Let us first study the quasilinear system (2.7), (2.8), (2.11). If the partial derivative of N_1 with respect to its second argument is positive, then this system, which describes the motion of a viscoelastic string with internal friction, has a parabolic character. This system can be linearized to produce a system like (4.8). If the nonlinear problem is posed in a suitable space of Hölder continuous functions, then the work of [5] shows that for small ε, the (unique) solution of (2.1), (2.2), (2.7), (2.8), (2.12) under assumption (4.6) is approximated by (4.7) in the norm of the function space, where $\mathbf{r}_1, \mathbf{r}_2, \ldots$ satisfy linear problems analogous to (4.8). (This proof is based upon an implicit function theorem. For a global analysis of purely longitudinal motion of such viscoelastic problems, see [7].) Many weaker frictional mechanisms, described by (2.10), do not destroy the hyperbolic character of the governing equations (where the tension is positive). The same is true of an elastic string subject to air resistance; this is governed by (2.7), (2.8), (2.11) with $\mathbf{f}$ having the form $-\mathbf{A}\mathbf{r}_t$, where $\mathbf{A}$ is a positive-definite matrix possibly depending on $\mathbf{r}$. The work of [9], [18] suggests that for such problems there is a threshold distinguishing "small" from "large" initial conditions; solutions with large initial conditions exhibit shocks, those with small initial conditions do not. It therefore seems possible to relate an amplitude of the frictional effect to the amplitude ε of the initial data so that the equation for $\mathbf{r}_1$ is the vectorial wave equation (4.8), so that the nonlinear problem nevertheless has sufficient frictional dissipation to establish the requisite threshold for initial data, and so that an implicit function theorem can be used in a suitable space to justify (4.7) in a rigorous way. An alternative approach to justifying (4.7) might be fashioned on the observation that the equations for an elastic string should have classical solutions on a time interval aproaching infinity as the initial data approach zero.

We have only exhibited the equations for $\mathbf{r}_1$ of (4.7). A physically illuminating determination of $\mathbf{r}_2$ and $\mathbf{r}_3$ for a string made of a special elastic material was carried out in [6]. A general account of such perturbation methods, which describes efficient methods for handling the approximating systems (by means of alternative theorems) and which contains an extensive bibliography, is given in [12].

5. Conclusion. The first steps toward correctly formulated equations for the vibrating string were made by Taylor in 1713. In 1743 d'Alembert derived the first explicit partial differential equation for the small motion of a heavy string. The correct equations for the large vibrations of a string in a plane, equivalent to the planar version of (2.7), were derived by Euler in 1744 by taking the limit of the equations for a discrete model. The correct linear equations for the small planar transverse motion of a string, which is just the wave equation, was obtained and brilliantly analyzed by d'Alembert in 1746. In 1750, Euler stated "Newton's equations of motion" and used them to derive the equations of motion for a string in a manner related to the one we used in Section 2. The spatial equations of motion for strings were obtained by Lagrange in 1761. A critical historical appraisal of these pioneering researches superficially outlined here, with full bibliographic references, is given in [20]. The status of the traditional ad hoc assumption that every particle on an elastic string moves in a plane perpendicular to $\mathbf{i}$ and through its position in $\sigma\mathbf{i}$ was clarified by J. B. Keller's [11] study of the exact equations. He showed that there is but one material for which the particles execute such a motion with the string having a sinusoidal shape.

The proof that $\mathbf{n}^+ = \mathbf{n}^-$ culminating in (2.6) was obtained by Euler in 1771 and followed earlier work of Pardies in 1673 and Jas. Bernoulli in 1691-1704. The theory of stress, which generalizes (2.6) to three dimensions, was obtained by Cauchy in 1823-1827. A modern generalization of Cauchy's result in a setting that does not rely on his smoothness assumptions is

given by [10]. Our proof in Section 3 that $\mathbf{n}^+ = \mathbf{n}^-$ (roughly speaking, as measures) generalizes the proof of Section 2 just as the proof of [10] generalizes that of Cauchy.

The Principle of Virtual Work in the form commonly used today was laid down by Lagrange in 1788. The proof in Section 3 that a precisely formulated version of this principle is equivalent to precisely formulated Impulse-Momentum Law is roughly modeled on the development of [3] for three-dimensional problems of continuous mechanics. The difficulties associated with the nature of boundaries of three-dimensional bodies immerse [3] in far deeper questions of analysis and measure theory than those confronted in Section 3. Our entire development can be easily and naturally extended to describe the behavior of rods, which resist bending and which can suffer shear, torsion, and other modes of deformation in addition to the stretching and bending that a string undergoes.

In his derivation of 1744 of the equations of motion of the vibrating string, Euler followed the earlier example of Huygens in 1673 and Joh. Bernoulli in 1727 of regarding the equations as the limit of those for a finite collection of beads joined by massless springs as the number of beads approach infinity while their total mass remains fixed. (Lagrange used a similar approach in 1759 and in part of his work of 1761. Euler's work subsequent to 1760 did not rely on this artifice.) The motion of the system of beads is described by a finite system of ordinary differential equations. It is natural to ask: In what sense does the solution of this or of related systems of ordinary differential equations approximate the solution of the partial differential equations? This question has not been answered for the quasilinear equations of the string. Many of the difficulties for elastic strings are similar to those described in Section 4. Accessible information for viscoelastic strings would come from the modern exploitation of the weak equations (3.11), (2.10) by the Faedo-Galerkin method (cf. [15]). An analysis along these lines for a quasilinear engineering model of an elastic string was performed in [8]. This work proves the convergence of solutions of a system of ordinary differential equations to the classical solution of the partial differential equation on the time interval before the advent of shocks.

The equations for an elastic string are formally equivalent to the Euler-Lagrange equations for the extremization of the Lagrangian functional and the weak version of these equations is formally equivalent to the vanishing of the first variation (Gâteaux differential) of this functional. The difficulties associated with the irregularity of solutions of the governing equations have so far prevented the exploitation of variational methods for treating these equations. The internal friction of viscoelastic strings means that their motion is not conservative and that their equations do not have a natural variational characterization.

Nonlinear wave equations arising in modern physics have been subjected to an intensive, fruitful, but by no means exhaustive analysis (cf. [15], [19]). This analysis is possible because these equations are semilinear and do not have solutions with shocks. These semilinear equations, which are obtained by adding a nonlinear perturbation to the linear wave equation, have the form $u_{tt} - u_{xx} + f(u, u_x) = 0$; they should be contrasted with the quasilinear system (2.7), (2.11), in which the coefficients of the highest x-derivatives depend upon the derivative of the unknown function. Thus, the nonlinear wave equations arising from the conceptually simple field of classical continuum mechanics are harder to analyze than those arising from conceptually difficult fields of modern physics.

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MISCELLANEA

36. What we above all things want is, I believe, a varied production of modernized didactic text-books. I have congratulated the Society on the work of recent years, largely inspired by itself, in the production of ambitious treatises calculated to exhibit to the inner circle of accomplished mathematicians a fuller knowledge of recent mathematical advances, calculated to induce those who are already real researchers to research nearer the present confines of known mathematical truth, to give larger views to those who are to lead on coming mathematicians. The next thing is for those whose views are enlarged to do their duty as leaders by endeavouring to secure that the elementary teaching of mathematics be as captivating as ever, but so conveyed that thought be encouraged, that attention to logical soundness in fundamentals be enforced as essential in real mathematics, and by providing lucid and suggestive introductory works on higher subjects, suited to be at once studied by those who have acquired the gift of accurate thought and the possession of elementary knowledge.

— E. B. Elliott, Retiring Presidential Address to the London Mathematical Society, 1898 (Proc. London Math. Soc., (1) 30 (1899) 17).

SUBSERIES OF THE POWER SERIES FOR e^x

LEE RUBEL AND KENNETH STOLARSKY

One of the mysteries of life is that e^x is bounded for negative x, and even tends to 0 as $x \to -\infty$, at least from the point of view of its power series

$$g(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

For the question of just boundedness, it is simpler to consider

$$f(x) = e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}.$$

The question arises whether there is any proper subseries

$$f_1(x) = \sum_{n \in N_1} \frac{x^n}{n!},$$

where N_1 is a subset of $\mathbb{N} = \{1,2,3,\dots\}$ with $N_1 \neq \varnothing$ and $N_1 \neq \mathbb{N}$, so that $f_1(x)$ is also bounded for all $x<0$. We prove here that there are exactly *four* such subseries f_1. They are simple and are directly exhibited. (For none of them does $f_1(x)$ have a limit as $x \to -\infty$.) They yield exactly two nontrivial decompositions of $e^x - 1$ as a sum $f_1 + f_2$ of subseries that are each bounded for negative x. There is no such decomposition into *three* or more parts.

Our proof uses a powerful theorem of Szegö on noncontinuable power series. It would be interesting to find a more direct and elementary proof. The other main ingredients of our proof are some considerations about cyclotomic polynomials and about Laplace transforms.

Finally, we turn our methods to a brief consideration of subseries of the series $k(e^x - 1)$, which is a formal notation for the series that for each $n \in \mathbb{N}$ has a block of k terms of the form $x^n/n!$.

THEOREM. *The only proper subseries of $e^x - 1$ that are bounded for $x<0$ are*

(a) $\displaystyle\sum_{n \equiv 0,1 \bmod 4} \frac{x^n}{n!}$, (b) $\displaystyle\sum_{n \equiv 2,3 \bmod 4} \frac{x^n}{n!}$,

(c) $\displaystyle\sum_{n \equiv 0,3 \bmod 4} \frac{x^n}{n!}$, *and* (d) $\displaystyle\sum_{n \equiv 1,2 \bmod 4} \frac{x^n}{n!}$.

The corresponding functions are

(a′) $\frac{1}{2}(e^x + \cos x + \sin x) - 1$ (b′) $\frac{1}{2}(e^x - \cos x - \sin x)$

(c′) $\frac{1}{2}(e^x + \cos x - \sin x) - 1$ (d′) $\frac{1}{2}(e^x - \cos x + \sin x)$.

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COROLLARY 1. *There is no nontrivial decomposition of $e^x - 1$ as a sum of three or more subseries, each of which is bounded for $x<0$.*

COROLLARY 2. *There is no proper subseries of $e^x - 1$ that approaches a limit as $x \to -\infty$.*

Before we begin the proof, we state two classical results. The first can be found in **[BO**, Theorem 5.3.1, p. 73], and the second in **[DI**, Theorem II, p. 324].

THEOREM A. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}$$

be an entire function of exponential type (i.e. $|f(z)| \leqslant Ae^{B|z|}$ for some constants A and B). If

$$F(z) = \int_0^{\infty} e^{-tz} f(t)\,dt$$

is the complex Laplace transform of f, then $F(z)$ is analytic for $|z|>B$ (including the point at ∞) and is represented there by

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}}.$$

THEOREM B (Szegö). *If among the coefficients A_n of a Taylor series $f(z) = \sum_{n=0}^{\infty} A_n z^n$ there are only a finite number of different numbers, then either $f(z) = P(z)/(1-z^q)$, where $P(z)$ is a polynomial and q a positive integer, or else $f(z)$ has no analytic continuation over any arc of the unit circle.*

REMARK 1. It is easy to see that when $f(z) = P(z)/(1-z^q)$, then the coefficients A_n are eventually periodic of period q.

The proof of our theorem is based on several lemmas.

LEMMA 1. *Let q be a positive integer and let $r = 0, 1, \ldots, q-1$. Further let*

$$S_{r,q}(x) = \sum_{n \equiv r \bmod q} \frac{x^n}{n!}.$$

If ω is any primitive qth root of unity, then

$$S_{r,q}(\lambda) = \frac{1}{q}\left\{ e^x + \frac{e^{\omega x}}{\omega^r} + \frac{e^{\omega^2 x}}{\omega^{2r}} + \cdots + \frac{e^{\omega^{q-1} x}}{\omega^{(q-1)r}} \right\}. \tag{1}$$

Proof. Compare the terms in the Taylor series of the two sides of equation (1) and use the fact that $1 + \omega^k + \omega^{2k} + \cdots + \omega^{(q-1)k}$ is 0 when $k \not\equiv 0 \bmod q$, and is q when $k \equiv 0 \bmod q$. No special property of the exponential function is needed for this proof.

DEFINITION. An exponential polynomial is an entire function of the form

$$E(z) = \sum_{j=1}^{n} P_j(z) e^{\lambda_j z}, \tag{2}$$

where the λ_j are distinct complex numbers (one of them possibly 0) and the $P_j(z)$ are polynomials.

LEMMA 2. *If $E(z)$ is an exponential polynomial as in (2) and if $E(x)$ is bounded for $x<0$, then $P_j(z) \equiv 0$ for each λ_j with $\operatorname{Re}\lambda_j < 0$ and $P_j(z)$ is a constant for each λ_j with $\operatorname{Re}\lambda_j = 0$.*

Proof. In our case, the λ_j are either 0 or roots of unity, so we will sketch the simple proof only for that case. Throw away the λ_j with $\operatorname{Re}\lambda_j > 0$, since $e^{\lambda_j x}$ approaches zero exponentially fast for

these λ_j, as $x \to -\infty$. Now look at the $\lambda_j = \alpha_j + i\beta_j$ with $-\alpha_j$ maximal. Either this set is a singleton, or consists of a conjugate pair. In the latter case it contributes

$$T(z) = e^{\alpha_j z}\left[P_1(z)e^{i\beta_j z} + P_2(z)e^{-i\beta_j z}\right].$$

Since $T(z)$ must be bounded as $z \to -\infty$ through real values, we see that the lead terms (say they have degree n) of P_1, P_2 must be identical, and hence

$$T(z) = c_1 z^n e^{\alpha_j z}\cos\beta_j z + O(z^{n-1}).$$

By again examining $T(z)$ we find that $c_1 = 0$, so $P_1(z)$ and $P_2(z)$ vanish identically. The singleton case is even easier. A similar argument handles the terms with $\alpha_j = 0$.

LEMMA 3. *Let $P(z) = \sum_{j=0}^{q-1}\delta_j z^j$ be a polynomial of degree $\leqslant q-1$, all of whose coefficients δ_j are either* 0 *or* 1. *Suppose $P(\xi) = 0$ for every qth root of unity ξ such that* $\operatorname{Re}\xi < 0$. *Then either all the δ_j are* 0 *or all the δ_j are* 1 *or else* $q \equiv 0 \bmod 4$ *and the δ_j are periodic* mod 4 *with one of the following four patterns*

(a) 1, 1, 0, 0 (b) 0, 0, 1, 1
(c) 1, 0, 0, 1 (d) 0, 1, 1, 0.

In other words, the function δ_j is the characteristic function of a union of two congruence classes mod 4. These are exactly the correspondingly labeled unions of two arithmetic progressions that appear in (a), (b), (c), and (d) in the statement of our theorem.

This lemma is the heart of our proof, and uses the next lemma in *its* proof.

LEMMA 4. *For every positive integer d other than* 1, 4, *and* 6, *there is a primitive dth root of unity ξ with* $\operatorname{Re}\xi < 0$.

Proof. It is clear that 1, 4, and 6 are actually exceptions. Now given any other d, we require an integer $m \geqslant 1$ such that $(m, d) = 1$ and

$$\frac{\pi}{2} < 2\pi\frac{m}{d} < \frac{3\pi}{2};$$

that is,

$$d < 4m < 3d. \tag{3}$$

We choose the integer m according to $d \bmod 4$.

(i)	$d = 4k+1$	$m = 2k$	$k \geqslant 0$
(ii)	$d = 4k+2$	$m = 2k-1$	$k \geqslant 0$
(iii)	$d = 4k+3$	$m = 2k+1$	$k \geqslant 0$
(iv)	$d = 4k$	$m = 2k+1$	$k \geqslant 1$.

The inequality (3) is valid except in (i) and (ii) for $k = 0$, and in (ii) and (iv) for $k = 1$. These correspond to $d = 1$, 2, 4, and 6. The case $d = 2$ is trivial, so only 1, 4, and 6 are exceptional, as promised.

Proof of Lemma 3. If $d \neq 1$, 4, and 6, and $d \mid q$, it is clear by Lemma 4 that $P(z)$ has a root in common with the dth cyclotomic polynomial $\Phi_d(z)$. The Φ_d are irreducible over the rational field and are relatively prime in pairs, so that

$$\prod_{d \mid q}{}^{*}\Phi_d(z) \mid P(z), \tag{4}$$

where * indicates $d \neq 1$, 4, and 6. Note that $\Phi_1(z) = z - 1$, $\Phi_4(z) = z^2 + 1$, $\Phi_6(z) = z^2 - z + 1$, and

$$x^q - 1 = \prod_{d \mid q}\Phi_d(z).$$

There are four cases; we will treat them in order.

I. $6 \nmid q, 4 \nmid q$. III. $6 \nmid q, 4 | q$.
II. $6 | q, 4 \nmid q$. IV. $6 | q, 4 | q$.

Case I. In this case $(x^q - 1)/(x-1)$ divides $P(z)$ and hence equals $P(z)$ if $P(z) \neq 0$.

Case II. Let $q = 6k$. From (4) we see that there is a polynomial $Q(z)$ such that

$$P(z) = \frac{Q(z)(z^{6k}-1)}{(z-1)(z^2-z+1)}. \tag{5}$$

By comparing degrees, we see that Q has degree at most 2, say, $Q(z) = az^2 + bz + c$.

We have

$$P(z) = Q(z)(z^{6(k-1)} + z^{6(k-2)} + \cdots + z^6 + 1)(z^3 + 2z^2 + 2z + 1).$$

Because of the gaps in the middle factor, the set of numbers that occur as coefficients in $P(z)$ is the set of numbers that occur as coefficients in

$$\begin{aligned} R(z) = Q(z)(z^3 + 2z^2 + 2z + 1) &= \sum_{j=0}^{5} r_j z^j \\ &= az^5 + (2a+b)z^4 + (2a+2b+c)z^3 + (a+2b+2c)z^2 + (b+2c)z + c. \end{aligned}$$

If $a=0$, then (consider r_4) $b=0$ or 1. If $b=1$, then $r_3 - r_0 = 2$, a contradiction. If $b=0$, then $r_1 = 2c = 0$ so $c=0$ and $P(z) \equiv 0$. Hence we may assume $a=1$. Since $r_4 = 2+b$, we have $b=-2$ or -1. If $b=-2$, then $r_2 = 2c - 3 \leq -1$, which is impossible. Hence $b=-1$ and $r_2 = 2c-1 = 1$, so $c=1$. Hence $Q(z) = z^2 - z + 1$, and by (5) we have $P(z) = (x^q - 1)/(x-1)$. Hence all the coefficients δ_j are 1.

The remaining two cases are handled in a similar way, so we give only a sketch, leaving it to the reader to check the combinatorial details.

Case III. Set $q = 4k$, so that

$$\begin{aligned} P(z) &= \frac{z^{4k}-1}{(z-1)(z^2+1)}(az^2+bz+c) \\ &= (z^{4(k-1)} + z^{4(k-2)} + \cdots + z^4 + 1)(z+1)(az^2+bz+c). \end{aligned}$$

Upon requiring this polynomial to have only the coefficients 0 and 1, we find that the only possibilities for the product of the last two polynomials are 0, $z^3 + z^2 + z + 1$ and

(a) $1+z$ (b) z^2+z^3
(c) $1+z^3$ and (d) $z+z^2$.

These correspond to the cases in the statement of the Lemma.

Case IV. Set $q = 12k$, so

$$\begin{aligned} P(z) &= \frac{z^{12k}-1}{(z-1)(z^2-z+1)(z^2+1)}(az^4+bz^3+cz^2+dz+e) \\ &= (z^{12(k-1)} + z^{12(k-2)} + \cdots + z^{12} + 1)(z^7 + 2z^6 + z^5 - z^4 - z^3 + z^2 + 2z + 1)(az^4 + \cdots + dz + e). \end{aligned}$$

An analysis shows that the product of the last two polynomials must be either 0 or $1 + z + z^2 + \cdots + z^{10} + z^{11}$ or else one of

(a) $1+z+z^4+z^5+z^8+z^9$ (c) $1+z^3+z^4+z^7+z^8+z^{11}$
(b) $z^2+z^3+z^6+z^7+z^{10}+z^{11}$ (d) $z+z^2+z^5+z^6+z^9+z^{10}$,

and these reduce to the corresponding cases in Case III. The proof of the lemma is now complete.

It is widely recognized that the sixth cyclotomic polynomial often plays an exceptional role; see [**AR**, p. 358, Cor. 2], and [**GO**, p. 736, Theorem 6].

Proof of the Theorem. Let $F_1(z)$ be the Laplace transform of $f_1(-x)$. Because $f_1(-x)$ is bounded for $x>0$, we see that $F_1(z)$ extends analytically to the half-plane $\{\operatorname{Re} z>0\}$, since one can differentiate under the integral sign there. By Theorem A,

$$F_1(z)=\sum_{n=1}^{\infty}\frac{(-1)^n\varepsilon_n}{z^{n+1}}$$

where $\varepsilon_n=1$ if $n\in N_1$ and $\varepsilon_n=0$ otherwise. Clearly, F_1 is analytic for $|z|>1$. Let

$$G(z)=\sum_{n=1}^{\infty}\varepsilon_n z^n;$$

G is obtained from F_1 via the transformation $z\mapsto -1/z$. Now $G(z)$ is analytic in the union of the unit disc and $\{\operatorname{Re} z<0\}$. Since the coefficients of $G(z)$ are either 0 or 1, we see by Theorem B that

$$G(z)=\frac{P(z)}{1-z^q}.$$

In particular, the ε_n are periodic with period q, past some point. In other words, N_1 is, modulo a finite set, the finite union of disjoint arithmetic progressions $\{nq+r\}$. This says that

$$f_1(x)=p(x)+\sum_{r=0}^{q-1}\delta_r S_{r,q}(z),$$

where $p(x)$ is a polynomial, and each δ_r is either 0 or 1. Let

$$P(\xi)=\sum_{r=0}^{q-1}\delta_r\xi^r.$$

By using Lemma 1, we see that

$$f_1(x)=p(x)+\frac{1}{q}\left\{P(1)e^x+P\left(\frac{1}{\omega}\right)e^{\omega x}+\cdots+P\left(\frac{1}{\omega^{q-1}}\right)e^{\omega^{q-1}x}\right\}.$$

Hence $f_1(x)$ is an exponential polynomial; so we may apply Lemma 2 to see that $p(x)\equiv 0$ and that $P(\xi)=0$ for every qth root of unity ξ with $\operatorname{Re}\xi<0$. We now apply Lemma 3 to conclude that either $P(\xi)\equiv 0$, or $P(\xi)=(\xi^q-1)/(\xi-1)$, or else that one of the four cases (a), (b), (c), (d) of Lemma 3 holds. These correspond exactly to the sums (a), (b), (c), (d) of the theorem, and the proof is complete. The corollaries are immediate consequences of the theorem.

Finally, if $k(e^x-1)$ is the k-fold repeated power series for e^x-1, we denote by $b(k)$ the number of different subseries of $k(e^x-1)$ that are bounded for $x<0$. Our methods can be used to prove that $b(k)$ is finite for each k and is actually of polynomial growth; indeed $c_1k^5\leqslant b(k)\leqslant c_2k^5$. It seems unlikely that there is an elegant expression for $b(k)$. Similarly, one could consider $z(k)$, the number of different subseries of ke^x that tend to 0 as $x\to-\infty$. For example, $2e^x$ has the subseries

$$\begin{aligned} f_1(x)&=\left[\sum_{n\equiv 0,3 \bmod 6}+2\sum_{n\equiv 1,2 \bmod 6}\right]\frac{x^n}{n!}\\ &=\left[e^x+\frac{2}{\sqrt{3}}e^{x/2}\sin\frac{\sqrt{3}}{2}x\right].\end{aligned}$$

Here $f_1(x)\to 0$ as $x\to-\infty$, and yet $f_1(x)\neq le^x$ for any integer l.

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LETTER TO AN AUTHOR

Dear Author:

You wrote to me about a rejected article, "I only wonder why so many people to whom I showed 'my' proof have never seen it before." Perhaps you wonder too easily.

Have you ever reflected on how few people are aware of having seen *anything*—that is, anything outside their individual specialties? I first realized this just after taking my undergraduate degree in Mathematics. I wanted to know some fact about Lebesgue integration and turned for help to a Ph.D. mathematician who happened to be available; for I had inflated ideas, based on departmental propaganda, of the vast attainments a Ph.D. was supposed to have. However, this Ph.D. was a number-theorist, and didn't know the answer: another illusion shattered.

All my professional life I have had experiences like yours; so have friends of mine; I have always assumed that they happen to everybody. They involve (as yours did) rather minor points (neat remarks, clever little proofs). Whoever proves the Riemann Hypothesis is not going to have to ask around to see whether it has already been done. We are shocked that the idea was not as original as we thought; at the same time we are chagrined to find that, once thought of, it was not immediately taken up by the world at large. After all, Farey, Pell, and Waring are remembered for less. Sixteen centuries ago, Aelius Donatus had the same experience. "Confound the people," he remarked to a student (who was to become famous as St. Jerome) "who had our ideas first."

It is also true that every generation has to make its own discoveries (H. M. Cundy, *Math. Gaz.*, 43 (1959), 85). There are methods, theorems, teaching techniques, that turn up about every twenty years and then are forgotten. Once in a long while one of them happens to be adopted by one of the more successful textbooks, and then at least the generation that studied from that book will remember it. Indeed, what we remember from undergraduate mathematics is by and large what we learned as students. We don't have time to learn more; perhaps we are too busy writing.

Sincerely yours,

R. P. Boas

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A SHARPENING OF A PUTNAM CONGRUENCE ON BINOMIAL COEFFICIENTS

JOHN HOWARD SMITH

The congruence $\binom{pa}{pb} \equiv \binom{a}{b} \pmod p$ (where $a \geqslant b \geqslant 0$, $p > 0$, p prime), from the 1977 Putnam exam [1], can be improved by replacing the modulus p by p^2 as follows:

Let S be the disjoint union of a sets, $S_1, \ldots, S_a$, each with p objects. For each i take a cyclic permutation of the objects of S_i and let G be the permutation group these permutations generate. Clearly, $|G| = p^a$.

Let $\mathfrak{T}$ be the set of all pb-element subsets of S. $|\mathfrak{T}| = \binom{pa}{pb}$. Then G acts on $\mathfrak{T}$ in the obvious way, and the size of the G-orbit of $T \in \mathfrak{T}$ is p^k where k is the number of S_i for which $T \cap S_i \neq \varnothing$, S_i. Since k is 0 when T is the union of b of the S_i and $k \geqslant 2$ otherwise, one has $\binom{pa}{pb} \equiv \binom{a}{b} \pmod{p^2}$.

Reference

1. A. P. Hillman, G. L. Alexanderson, and L. F. Klosinski, The William Lowell Putnam Mathematical Competition, this MONTHLY, 86 (1979) 168–175.

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CORRECTION TO "A MINIMAL-PATH ALGORITHM FOR THE 'MONEY CHANGING PROBLEM'"

(this MONTHLY, 86 (1979) 832–834)

ALBERT NIJENHUIS

In order to be in precise accordance with its description, the BASIC program in the Appendix requires the insertion of one instruction

555 E(FNM(NO)) = −1.

Although there could conceivably be cases in which the original program would give incorrect results, the author has been unable to construct an example where this happens.

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MISCELLANEA

37. The method of "postulating" what we want has many advantages; they are the same as the advantages of theft over honest toil.

—Bertrand Russell, *Introduction to Mathematical Philosophy*, 2nd ed., 1920, p. 71.

EDITED BY DEBORAH TEPPER HAIMO AND FRANKLIN TEPPER HAIMO

Material for this department should be sent to Professor Deborah Tepper Haimo, Department of Mathematical Sciences, University of Missouri, St. Louis, MO 63121.

INTEGRATION ON $[0,1]^\omega$

ALAN DOW AND JOHN GINSBURG

In this note we consider real-valued functions $f:[0,1]^\omega \to R$ and describe the extent to which the integral of f over $[0,1]^\omega$ can be evaluated from iterated integrals over the coordinates according to the formula

$$\int_{[0,1]^\omega} f d\mu = \lim_{n\to\infty} \int_0^1 \cdots \int_0^1 \int_0^1 f(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots) dx_1 dx_2 \cdots dx_n.$$

Here $\int_{[0,1]^\omega} f d\mu$ denotes the integral of f with respect to the usual product measure μ on $[0,1]^\omega$ (see [1] for the elementary properties of product measure). A second glance at this statement reveals that some caution is necessary in interpreting the equality of the two sides. This is because the right-hand side

$$\lim_{n\to\infty} \int_0^1 \cdots \int_0^1 \int_0^1 f(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots) dx_1 dx_2 \cdots dx_n$$

must be interpreted as the limit of a sequence of functions.

DEFINITION. Let f be integrable on $[0,1]^\omega$. For each $n = 1,2,3,\ldots$ we define a function $f_n:[0,1]^\omega \to R$ by

$$f_n(y) = \int_0^1 \cdots \int_0^1 \int_0^1 f(x_1, x_2, \ldots, x_n, y_{n+1}, y_{n+2}, \ldots) dx_1 dx_2 \cdots dx_n$$

where $y = (y_1, y_2, \ldots)$ is an element of $[0,1]^\omega$. Then $I(f)$ is defined to be the function $I(f)(y) = \lim_{n\to\infty} f_n(y)$ where this is understood to be the pointwise limit where it exists.

The following example illustrates this definition and also shows that the result of this procedure may be rather curious.

EXAMPLE. Let $Z = \{(x_1, x_2, \ldots) \in [0,1]^\omega : x_n \neq 0 \text{ for at most finitely many } n\}$. The measure of Z is 0. Let f be the characteristic function of Z. If $y = (y_1, y_2, \ldots)$ is in Z, then, for all n and for all $x_1, x_2, \ldots, x_n$ in $[0,1]$, $(x_1, x_2, \ldots, x_n, y_{n+1}, y_{n+2}, \ldots)$ is also in Z, and so $f_n(y) = \int_0^1 \cdots \int_0^1 \int_0^1 1 dx_1 dx_2 \cdots dx_n = 1$ for all n. Therefore, $I(f)(y) = 1$ for all y in Z. Similarly, if $y \notin Z, f_n(y) = \int_0^1 \cdots \int_0^1 \int_0^1 0 dx_1 dx_2 \cdots dx_n = 0$ for all n and so $I(f)(y) = 0$ for all $y \notin Z$. We thus see that, in this case, $I(f) = f$.

The preceding example reveals the point of the above definition: The limit of the finite iterated integrals in general results in a function. However, we now proceed to show that, for a large class of functions f, $I(f)$ is a constant function and equal to $\int_{[0,1]^\omega} f d\mu$.

THEOREM. *Let f be a continuous real-valued function on $[0,1]^\omega$. Then $I(f)$ is constant and is equal to $\int_{[0,1]^\omega} f d\mu$.*

Proof. In the theorem, we are of course referring to the product topology on $[0,1]^\omega$. We first show

(*) for every $\epsilon > 0$ there exists an integer N such that if $x_i = y_i$ for $i \leq N$ and $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$ are in $[0,1]^\omega$ then $|f(x) - f(y)| < \epsilon$.

Let $x \in [0,1]^\omega$. Since f is continuous at x, there is an integer n_x and a $\delta_x > 0$ such that, if

$y\in[0,1]^\omega$ and $|y_i-x_i|<\delta_x$ for all $i\leqslant n_x$, then $|f(x)-f(y)|<\epsilon/2$. Define $G_x=\{y\in[0,1]^\omega:|y_i-x_i|<\delta_x$ for $i=1,2,\ldots,n_x\}$. The collection of G_x's for $x\in[0,1]^\omega$ forms an open cover of the compact space $[0,1]^\omega$. Hence we can choose a finite subcover $\{G_x:x\in F\}$ where F is a finite subset of $[0,1]^\omega$. Let $N=\max\{n_x:x\in F\}$. Then N is the required integer.

We now show that $I(f)(y)$ exists and is constant (i.e., $I(f)(y)=I(f)(z)$ for all $y,z\in[0,1]^\omega$). Let $\epsilon>0$ and let N be chosen as above and $m\geqslant n>N$. We show that, for $y,z\in[0,1]^\omega$, $|f_n(y)-f_m(z)|<\epsilon$ which shows simultaneously that $f_n(y)$ and $f_m(z)$ are Cauchy sequences which converge to the same limit. Since $m\geqslant n$

$$\int_0^1\cdots\int_0^1\int_0^1 f(x_1,\ldots x_n,y_{n+1},y_{n+2},\ldots)\,dx_1dx_2\cdots dx_n$$
$$=\int_0^1\cdots\int_0^1\int_0^1 f(x_1,\ldots,x_n,y_{n+1},y_{n+2},\ldots)\,dx_1dx_2\cdots dx_n dx_{n+1}\cdots dx_m.$$

Hence

$$|f_n(y)-f_m(z)|\leqslant\int_0^1\cdots\int_0^1\int_0^1|f(x_1,x_2,\ldots,x_n,y_{n+1},y_{n+2},\ldots)$$
$$-f(x_1,x_2,\ldots,x_m,z_{m+1},z_{m+2},\ldots)|\,dx_1dx_2\cdots dx_n dx_{n+1}\cdots dx_m$$
$$<\int_0^1\cdots\int_0^1\int_0^1\epsilon\,dx_1\cdots dx_m=\epsilon,$$

by (*).

We now show that $I(f)$ equals $\int_{[0,1]^\omega}fd_\mu$. Let μ_n be the usual product measure on $[0,1]^n$. Define h_n, a real valued function on $[0,1]^\omega$, by

$$h_n(x_1,x_2,\ldots,x_n,x_{n+1},\ldots)=f(x_1,\ldots,x_n,0,0,\ldots).$$

The condition (*) implies $\lim_{n\to\infty}\int_{[0,1]^\omega}h_n d\mu\to\int_{[0,1]^\omega}fd\mu$. Define g_n on $[0,1]^n$ by $g_n(x_1,\ldots,x_n)=f(x_1,x_2,\ldots,x_n,0,0,\ldots)$. First we show that $\int_{[0,1]^n}g_n d\mu_n=\int_{[0,1]^\omega}h_n d\mu$. Let ψ_k be a sequence of simple functions which converge monotonically to g_n. Let

$$\psi_k=\sum_{i=1}^{m_k}c_{k,i}\chi_{E_{k,i}}$$

where $E_{k,i}$ is a measurable subset of $[0,1]^n$. We can extend ψ_k to a simple function ψ_k' on $[0,1]^\omega$ by $\psi_k'(x_1,x_2,\ldots)=\psi_k(x_1,\ldots,x_n)$. Note that

$$\psi_k'=\sum_{i=1}^{m_k}c_{k,i}\chi_{E'_{k,i}}$$

where $E'_{k,i}=\{x\in[0,1]^\omega:(x_1,\ldots,x_n)\in E_{k,i}\}$. Since $E'_{k,i}=E_{k,i}\times[0,1]\times[0,1]\times\cdots$ it follows that $\mu(E'_{k,i})=\mu(E_{k,i})\times1\times1\times\cdots=\mu(E_{k,i})$. Hence

$$\int_{[0,1]^\omega}\psi_k'd\mu=\sum_{i=1}^{m_k}c_{k,i}\mu(E'_{k,i})=\sum_{i=1}^{m_k}c_{k,i}\mu(E_{k,i})=\int_{[0,1]^n}\psi_k d\mu_n.$$

Since ψ_k' converges monotonically to h_n, we observe that

$$\int_{[0,1]^\omega}h_n=\lim_{k\to\infty}\int_{[0,1]^\omega}\psi_k'd\mu=\lim_{k\to\infty}\int_{[0,1]^n}\psi_k d\mu_n=\int_{[0,1]^n}g_n d\mu_n.$$

And since

$$\int_{[0,1]^n}g_n d\mu_n=\int_0^1\cdots\int_0^1\int_0^1 g_n dx_1\cdots dx_n$$

by Fubini's theorem, we obtain

$$\int_{[0,1]^\omega} h_n d\mu = \int_0^1 \cdots \int_0^1 \int_0^1 g_n dx_1 \cdots dx_n$$
$$= \int_0^1 \cdots \int_0^1 \int_0^1 f(x_1,\dots,x_n,0,0,\dots) dx_1 \cdots dx_n.$$

Therefore

$$\int_{[0,1]^\omega} f d\mu = \lim_{n\to\infty} \int_{[0,1]^\omega} h_n d\mu$$
$$= \lim_{n\to\infty} \int_0^1 \cdots \int_0^1 \int_0^1 f(x_1,x_2,\dots,x_n,0,0,\dots) dx_1 dx_2 \cdots dx_n = I(f)(0).$$

REMARK: The class of functions for which the conclusion of the theorem is valid is somewhat broader than the class of continuous functions—as the proof shows, as long as f satisfies (*) the equality holds. The reader might find it amusing to compute the integral $\int_{[0,1]^\omega} f d\mu$ for some simple continuous functions like $f(x_1,x_2,\dots)=\sum_{k=1}^{\infty} x_k/2^k$, using this method.

The authors wish to thank Ulf Nilsson for his inspiration.

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ON nTH ROOTS OF POSITIVE OPERATORS

D. R. BROWN AND M. J. O'MALLEY

A bounded operator A on a Hilbert space H is positive provided $\langle Ax,x\rangle \geqslant 0$ for all $x\in H$. These operators are symmetric and, as such, constitute a natural generalization of nonnegative real diagonal matrices. The following result is thus both well known and not surprising:

THEOREM. *A positive operator has a unique positive square root (under operator composition).*

This may be established by integration of the correct function, invoking the spectral theorem for self-adjoint operators. A more accessible argument for those not acquainted with the mysteries of spectral measures may be found in [1, p. 317].

While square roots and their iterates seem to provide a sufficient analytic tool for most purposes, it is also a (folk) theorem that positive operators possess unique positive nth roots for every positive integer n. As in the $n=2$ case, existence follows from an application of the spectral theorem; however, we give an argument in the spirit of [1]. The purpose in so doing is not to exercise the reader's knowledge of induction but, rather, to illustrate another use of the Law of the Mean as a motivational instrument.

Let I be the identity operator on H, and let $B(H)$ denote the set of bounded operators on H. We shall need the following properties of positive operators:

(1) The relation on symmetric operators defined by $A \leqslant B$, if and only if $B-A$ is positive, is reflexive, transitive, and consistent with the notation $0 \leqslant A$ for any positive A; moreover, this relation is preserved by operator addition and positive real scalar multiplication and reversed by negative scalar multiplication.

(2) If A and B are positive and if $AB=BA$, then $A^{\frac{1}{2}}B=BA^{\frac{1}{2}}$, from which it follows that AB is positive.
(3) If $0 \leqslant A \leqslant I$, then $0 \leqslant I-A \leqslant I$.
(4) If $0 \leqslant A$, then $A \leqslant \|A\| I$, so that $(\|A\|)^{-1}A \leqslant I$ if $A \neq 0$.
(5) If $0 \leqslant A \leqslant I$, then $A^n \leqslant A$ for all positive integers n.

We also require:

Lemma. *If $\{S_n\}$ is a sequence in $B(H)$ such that $0 \leqslant S_n \leqslant S_{n+1} \leqslant I$, then there exists $S \in B(H)$ such that $\{S_n u\} \to Su$ for all $u \in H$.*

All of the conclusions above are verified by straightforward arguments in [**1**, pp. 317–320].

Theorem. *Let $A \in B(H)$, $0 \leqslant A$, and let k be a positive integer. Then there exists a unique positive operator B such that $B^k = A$.*

Proof. By (4) above, we need consider only the case in which $A \leqslant I$. We first prove the existence of B. Since the theorem is a tautology for all operators when $k=1$, we assume the existence of positive $(k-1)$st roots for all positive operators.

Under the momentary supposition that B exists, let $R=I-A$ and $S=I-B$. Then $(I-S)^k = I-R$, so that

$$S=(1/k)\left[R+\sum_{r=2}^{k}\binom{k}{r}(-1)^r S^r\right]. \tag{*}$$

Clearly the existence of a positive operator satisfying this implicit relation is necessary and sufficient to establish the existence of the desired operator B. To this end, we define a sequence of operators by $S_0=0, S_{n+1}=(1/k)[R+\sum_{r=2}^{k}\binom{k}{r}(-1)^r S_n^r]$. In order to show $S_n \leqslant S_{n+1}$ it suffices to show, under the assumption $0 \leqslant S_{n-1} \leqslant S_n \leqslant I$, that

$$0 \leqslant S_{n+1}-S_n=(1/k)\left[\sum_{r=2}^{k}\binom{k}{r}(-1)^r(S_n^r-S_{n-1}^r)\right].$$

To accomplish this, we digress to a consideration of the polynomial $f(x)=\sum_{r=2}^{k}\binom{k}{r}(-1)^r x^r = (1-x)^k+kx-1$. Since $f'(x)=k[1-(1-x)^{k-1}] \geqslant 0$ on $[0,1]$, clearly f is increasing on this interval. To translate this to operators, it is necessary to examine the situation more carefully. By the Mean Value Theorem, given $0 \leqslant y < z \leqslant 1$, there exists a (unique) number $c \in (y,z)$ such that

$$f(z)-f(y)=f'(c)(z-y). \tag{**}$$

Upon solving, $c=1-[(1/k)\sum_{r=0}^{k-1}(1-y)^{k-r-1}(1-z)^r]^{1/(k-1)}$.

Returning to our operator problem, we wish to apply this information to the sequence $\{S_n\}$. Since all members of this family are polynomials in $R=I-A$, any two of them commute. This is a property sufficient to permit imitation of equation (**) with operators; let $z=S_n$, $y=S_{n-1}$. In this format, we use C to represent the operator $I-J$, where J is (any) positive $(k-1)$st root of the operator $(1/k)\sum_{r=0}^{k-1}(I-S_{n-1})^{k-r-1}(I-S_n)^r$. The following chain of equalities is easily calculated:

$$\begin{aligned}
S_{n+1}-S_n &= (1/k)\cdot(f(S_n)-f(S_{n-1}))\\
&=(1/k)\left\{k\left[I-(I-C)^{k-1}\right]\right\}\cdot(S_n-S_{n-1})\\
&=\left[I-(I-C)^{k-1}\right]\cdot(S_n-S_{n-1})\\
&=[I-J^{k-1}]\cdot(S_n-S_{n-1})\\
&=\left[I-\left\{(1/k)\sum_{r=0}^{k-1}(I-S_{n-1})^{k-r-1}(I-S_n)^r\right\}\right]\cdot(S_n-S_{n-1}).
\end{aligned}$$

By application of remarks (2), (3), and (5), the assumption of existence of $(k-1)$st roots, and the inductive hypothesis $S_{n-1} \leqslant S_n$, the latter operator product exists and is positive. Hence $S_n \leqslant S_{n+1}$, and the sequence $\{S_n\}$ is increasing. Of course, the Law of the Mean is not applicable in this setting, nor is it used other than to motivate the choice of C. Indeed, the discerning reader will note that the extremes of the chain above may be shown to be equal without the introduction of C. However, the rather unusual factorization of $S_{n+1}-S_n$ would be more difficult to discover without the example furnished by the derivative in the real function situation.

To invoke the Lemma and complete the proof of existence of kth roots, it remains to show $S_n \leqslant I$ for all n. Assuming $0 \leqslant S_m \leqslant I$, we have

$$kS_{m+1} = R + \sum_{r=2}^{k} \binom{k}{r}(-1)^r S_m^r = R - I + kS_m + (I - S_m)^k.$$

By remark (5), $(I-S_m)^k \leqslant I - S_m$; therefore

$$\begin{aligned} R + kS_m - I + (I-S_m)^k &\leqslant R + kS_m - I + I - S_m \\ &\leqslant I + (k-1)S_m \leqslant kI. \end{aligned}$$

Hence $kS_{m+1} \leqslant kI$ and $S_{m+1} \leqslant I$, as desired. Thus, the Lemma gives an operator as in (*), and $I - S = B$ is a kth root of A.

To prove, inductively, the uniqueness of a positive kth root of A, we first observe that, if T is any positive kth root of A, then T must perforce commute with A, hence with $I-A=R$, hence with each S_n, and thus with S and $I-S=B$. Let $u \in H, v=(B-T)u$. Then

$$0 = \langle (B^k - T^k)u, v \rangle = \left\langle \left(\sum_{r=0}^{k-1} B^{k-r-1}T^r \right)(B-T)u, v \right\rangle = \sum_{r=0}^{k-1} \langle B^{k-r-1}T^r v, v \rangle.$$

Since B and T commute, $0 \leqslant B^{k-r-1}T^r$, whence $\langle B^{k-r-1}T^r v, v\rangle = 0$, $r = 0, 1, \ldots, k-1$. Let F_r be any positive (hence symmetric) square root of $B^{k-r-1}T^r$. Then

$$\|F_r v\|^2 = \langle F_r v, F_r v \rangle = \langle F_r^2 v, v \rangle = 0,$$

so that $F_r v = 0$ and $B^{k-r-1}T^r v = F_r^2 v = 0$. Therefore $B^{k-r-1}T^r(B-T)u = 0$, or $B^{k-r}T^r u = B^{k-r-1}T^{r+1}u$, $r = 0, 1, \ldots, k-1$. In particular, for $r = k-1$, $BT^{k-1} = T^k$. Multiplying by T, we have $B^{k+1} = BA = BT^k = T^{k+1}$.

If $k=2$, the argument above shows $Bv = 0 = Tv$, whence $\|(B-T)u\|^2 = \langle (B-T)^2 u, u\rangle = \langle (B-T)v, u \rangle = 0$. Hence $Bu = Tu$ for all $u \in H$, and B is thus unique. Now assume all positive roots, of order less than k, for positive operators are unique. If $k = 2j$, than $(B^j)^2 = B^{2j} = B^k = T^k = (T^j)^2$, whence $B^j = T^j$ and thus $B = T$. If k is odd, we have shown above that $B^{k+1} = T^{k+1}$; so, by the even exponent argument, again $B = T$. This completes the proof.

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ON THE VOLUME OF UNIONS OF TRANSLATES OF A CONVEX SET

WULF REHDER

1. Introduction. In [4], E. Th. Poulsen posed a seemingly simple question that, as F. A. Valentine [5, p. 189] puts it, has "baffled" many mathematicians—including this author.

Using the theorem for $s-1$, and the translation-invariance of V, we get at once the inequality

$$V\left(K \cup \cup_{j=2}^{s-1} K_j(\theta')\right) + V(K_s(\theta')) \geqslant V\left(K \cup \cup_{j=2}^{s-1} K_j(\theta)\right) + V(K_s(\theta)). \tag{2}$$

We still have to investigate the volume

$$V\left(\left(K \cup \cup_{j=2}^{s-1} K_j(\theta')\right) \cap K_s(\theta')\right).$$

First, write

$$\left(K \cup \cup_{j=2}^{s-1} K_j(\theta')\right) \cap K_s(\theta') = (K \cap K_s(\theta')) \cup \cup_{j=2}^{s-1} (K_j(\theta') \cap K_s(\theta')). \tag{3}$$

The volume of this union (3) remains unchanged if each of the sets undergoes a translation by the same vector $\theta'(-z_s)$. Thus

$$K \cap K_s(\theta') \quad \text{becomes} \quad (\theta'(-z_s) + K) \cap K;$$

$$K_j(\theta') \cap K_s(\theta') \quad \text{becomes} \quad (\theta'(z_j - z_s) + K) \cap K, \qquad j = 2, 3, \ldots, s-1.$$

Each set on the right is, by (1), contained in the corresponding sets

$$(\theta(-z_s) + K) \cap K,$$

$$(\theta(z_j - z_s) + K) \cap K, \qquad j = 2, 3, \ldots, s-1,$$

which, in turn, are translated by the same vector θz_s into the sets

$$K \cap K_s(\theta),$$

$$K_j(\theta) \cap K_s(\theta), \qquad j = 2, 3, \ldots, s-1.$$

Altogether, the volume of the union (3) is not bigger than the volume of the union

$$(K \cap K_s(\theta)) \cup \cup_{j=2}^{s-1} (K_j(\theta) \cap K_s(\theta)).$$

This fact, together with (2), proves the theorem.

As yet, Poulsen's problem (even for spheres) has not been solved in general.

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A METHOD FOR SOLVING DIOPHANTINE EQUATIONS

Péter Kiss

Introduction. The following problem was proposed in this Monthly by R. M. Hashway and solved by O. P. Lossers [7]: Solve the diophantine equation

$$a + b \cdot 10^k = (a+b)^2,$$

where $0 < a, b < 10^k$, and find all solutions when $k \leqslant 5$. The solutions yield some interesting numbers in the decimal system: for example, 2025 has the property $2025 = 20 \cdot 10^2 + 25 = (20+25)^2$.

The purpose of this note is to generalize this problem by giving a method for solving the diophantine equation

$$x_0+x_1A^s+x_2A^{2s}+\cdots+x_nA^{ns}=(x_0+x_1+\cdots+x_n)^k, \tag{1}$$

where A, s, and k are fixed positive integers. If we replace $x_0+x_1+\cdots+x_n$ by x, we then have

$$x^k=x+x_1(A^s-1)+x_2(A^{2s}-1)+\cdots+x_n(A^{ns}-1) \tag{2}$$

and, using the notation $A^s-1=B$, we get the congruence

$$x^k\equiv x \pmod{B}. \tag{3}$$

From the solutions of (3), we can determine the integer values of the variables $n,x_0,x_1,\dots,x_n$ in (2); thus it is enough to solve (3).

We shall solve (3) explicitly, apply the solutions to solve (1), and give some consequences. The case $A=10$ is especially interesting; we shall therefore deal in detail only with the case when A is an even integer.

Notations and a Theorem. Let A be an even integer so that $B=A^s-1=p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ is odd, where $p_1,p_2,\dots,p_r$ are distinct odd primes and $\alpha_i>0$ $(i=1,2,\dots,r)$. We introduce some notations. Let g_i be a primitive root $(\bmod\ p_i^{\alpha_i})$ and let $d_i=(k-1,\phi(p_i^{\alpha_i})),k-1=d_id_i',\phi(p_i^{\alpha_i})=d_ic_i$, where ϕ is Euler's function. Further notations are:

$$P_i=B/p_i^{\alpha_i}=p_1^{\alpha_1}\cdots p_{i-1}^{\alpha_{i-1}}\cdot p_{i+1}^{\alpha_{i+1}}\cdots p_r^{\alpha_r}$$

and $G_i=0$ or $G_i=g_i^{c_iq_i}$, where $q_i=0,1,\dots,d_i-1$ $(i=1,2,\dots,r)$.

Now we can prove the following theorem.

THEOREM. *All the incongruent solutions of the congruence*

$$x^k\equiv x \pmod{B} \tag{4}$$

are

$$x\equiv\sum_{i=1}^{r}G_iP_i^{\phi(p_i^{\alpha_i})}\pmod{B}.$$

Proof. Congruence (4) is equivalent to the congruence system

$$\begin{gathered} x^k\equiv x \pmod{p_1^{\alpha_1}}\\ \vdots\\ x^k\equiv x \pmod{p_r^{\alpha_r}}. \end{gathered} \tag{5}$$

First we solve the congruence

$$x^k\equiv x \pmod{p_i^{\alpha_i}} \qquad (1\leqslant i\leqslant r). \tag{6}$$

But $(x,x^{k-1}-1)=1$, so by (6) $x\equiv 0$ or

$$x^{k-1}\equiv 1 \pmod{p_i^{\alpha_i}}. \tag{7}$$

In (7) we can write x, using the primitive root g_i, in the form

$$x\equiv g_i^{\beta_i} \pmod{p_i^{\alpha_i}},$$

where $0\leqslant\beta_i<\phi(p_i^{\alpha_i})$. Substituting in (7), we get

$$g_i^{(k-1)\beta_i}\equiv 1\equiv g_i^0 \pmod{p_i^{\alpha_i}}.$$

Hence it follows that $(k-1)\beta_i\equiv 0 \pmod{\phi(p_i^{\alpha_i})}$, which means that $(k-1)\beta_i=d_id_i'\beta_i$ is divisible by $\phi(p_i^{\alpha_i})=d_ic_i$. Therefore $(d_i',c_i)=1$ implies $c_i|\beta_i$, i.e., $\beta_i=c_iq_i$ for some q_i. Here $0\leqslant q_i<d_i$ because of the bounds on β_i. Thus the solutions of (6) are

$$x \equiv 0 \pmod{p_i^{\alpha_i}},$$
$$x \equiv g_i^{c_i q_i} \pmod{p_i^{\alpha_i}},$$

the number of which is $d_i + 1$, taking into account the possible values of q_i, and all the solutions are incongruent $(\bmod\, p_i^{\alpha_i})$.

We have now obtained the result that we can decompose every congruence of system (5) into linear congruences and so we can write the system (5) in the form

$$\begin{aligned} x &\equiv G_1 \pmod{p_1^{\alpha_1}} \\ x &\equiv G_2 \pmod{p_2^{\alpha_2}} \\ &\vdots \\ x &\equiv G_r \pmod{p_r^{\alpha_r}} \end{aligned} \tag{8}$$

(G_i is defined above.) With respect to the values of the G_i, (8) supplies $(d_1+1)(d_2+1)\cdots(d_r+1)$ linear congruence-systems and each of them gives one solution (mod B).

If we multiply the ith line of (8) by $P_i^{\phi(p_i^{\alpha_i})}$, then the modulus $p_i^{\alpha_i} P_i^{\phi(p_i^{\alpha_i})}$ is divisible by B, so that

$$x \cdot P_i^{\phi(p_i^{\alpha_i})} \equiv G_i P_i^{\phi(p_i^{\alpha_i})} \pmod{B}.$$

Let us perform the multiplications in the cases $i = 1, 2, \ldots, r$, and add the results; we get

$$x \cdot \sum_{i=1}^{r} P_i^{\phi(p_i^{\alpha_i})} \equiv \sum_{i=1}^{r} G_i P_i^{\phi(p_i^{\alpha_i})} \pmod{B}.$$

We must show for the proof of the theorem that

$$\sum_{i=1}^{r} P_i^{\phi(p_i^{\alpha_i})} \equiv 1 \pmod{B}.$$

It is enough to show that $\sum_{i=1}^{r} P_i^{\phi(p_i^{\alpha_i})} - 1$ is divisible by $p_j^{\alpha_j}$ for $j = 1, 2, \ldots, r$. But that is obvious since $p_j^{\alpha_j} | P_i$ for $i \neq j$ by the definition of P_i and $p_j^{\alpha_j} | (P_j^{\phi(p_j^{\alpha_j})} - 1)$ by Euler's congruence theorem; thus the theorem is proved.

An Application of the Theorem. As an application of our theorem we solve the diophantine equation

$$x_0 + x_1 10^2 + x_2 10^4 + \cdots + x_n 10^{2n} = (x_0 + x_1 + \cdots + x_n)^4. \tag{9}$$

First solve the congruence

$$x^4 \equiv x \pmod{99}. \tag{10}$$

In our case $B = 99 = 3^2 \cdot 11$ so $p_1^{\alpha_1} = 3^2$, $\phi(3^2) = 6$, $g_1 = 2$, $d_1 = (3,6) = 3$, $c_1 = 2$, $P_1 = 11$, $G_1 = 0$ or $2^{2q_1}(q_1 = 0, 1, \text{or } 2)$; and $p_2^{\alpha_2} = 11$, $\phi(11) = 10$, $g_2 = 2$, $d_2 = (3, 10) = 1$, $c_2 = 10$, $P_2 = 9$, $G_2 = 0$ or $2^0 = 1$. By the theorem, one of the solutions of (10) (choosing the values $G_1 = 0, G_2 = 1$) is

$$x \equiv 9^{10} \equiv 45 \pmod{99}.$$

Every number of the form $x = x_0 + x_1 + \cdots + x_n = 45 + 99t$ $(t = 0, 1, 2, \ldots)$ determines an $(n, x_0, x_1, \ldots, x_n)$-integer solution of (9). For example, in case $x = 45$

$$45^4 - 45 = 4100580 = 4 \cdot 999999 + 10 \cdot 9999 + 6 \cdot 99,$$

and so $n = 3$, $x_3 = 4$, $x_2 = 10$, $x_1 = 6$, which imply $x_0 = x - (x_1 + x_2 + x_3) = 25$. This solution gives an interesting number in the decimal system:

$$45^4 = 4100625 = (4 + 10 + 06 + 25)^4.$$

Similarly, solving the congruence $x^4 \equiv x \pmod{9}$, we get, among others, the solutions $x = 22$ and $x = 28$, which have the property

$$22^4 = 234256 = (2 + 3 + 4 + 2 + 5 + 6)^4$$

and

$$28^4 = 614656 = (6+1+4+6+5+6)^4.$$

We note that not every solution of congruence (3) gives a solution for equation (1) which has a similar property in base A, since it is not certain that $0 \leqslant x_i < A^s$ for every $i (i=0,1,\ldots,n)$.

Consequences of the Theorem. It follows from the theorem that (4) has as many incongruent solutions as we can select values of $G_1, G_2, \ldots, G_r$. This verifies the following corollary.

COROLLARY 1. *The number of incongruent solutions of congruence* (4) *is* $(d_1+1)(d_2+1)\cdots(d_r+1)$.

We can see by the theorem that, by a change in the value of k, the solutions of (4) depend only on the values of d_i. But if $(k_1-1, \phi(B)) = (k_2-1, \phi(B))$ for integers k_1 and k_2, then the values of d_i are equal in both the cases $k=k_1$ and $k=k_2$. So we get:

COROLLARY 2. *If* $(k_1-1, \phi(B)) = (k_2-1, \phi(B))$ *for positive integers* k_1, k_2, B *and if* B *is odd, then the congruences*

$$x^{k_1} \equiv x \pmod{B} \quad \textit{and} \quad x^{k_2} \equiv x \pmod{B}$$

are equivalent.

REMARKS. 1. We note that the theorem and the corollaries can be extended to even B. Namely, if $p_1 = 2$ and $\alpha_1 < 3$, then there exists a primitive root $(\bmod\, p_1^{\alpha_1})$; and if $\alpha_1 \geqslant 3$, then for any odd integer x, we have $x \equiv (-1)^{\beta} 5^{\beta_1} (\bmod\, p_1^{\alpha_1})$, where $\beta = 0$ or $\beta = 1$ and $0 \leqslant \beta_1 < 2^{\alpha_1 - 2}$ (see, e.g., [4]).

2. Extended to even B, our theorem includes an earlier result [3] on the congruence $a^2 \equiv a$ $(\bmod\, n^k)$ and a result of C. P. Popovici [5] on the congruence $x^k \equiv x \pmod{10^n}$; furthermore Corollary 1 implies a theorem of E. Hewitt [2] on the identical congruence $x^k \equiv x \pmod{B}$, a theorem of R. D. Carmichael [1] on absolutely pseudoprime numbers, the solution of a problem [8] proposed by S. Collins, S. M. Reddy and N. J. A. Sloane, and the solution of a problem of *Elemente der Math.* [6].

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ON A SURPRISING INEQUALITY OF GOLDBERG AND STRAUS

RAY REDHEFFER AND CAREY SMITH

Let $c = \{c_j\}$ and $z = \{z_j\}$ be sequences of complex numbers for $j = 1, 2, \ldots, m$ with $m \geqslant 2$. Leaving c fixed but permuting z, we can form various sums of products, such as

$$P = c_1 z_1 + c_2 z_2 + \cdots + c_m z_m$$

$$Q = c_1 z_m + c_2 z_{m-1} + \cdots + c_m z_1$$
$$R = c_1 z_m + c_2 z_2 + \cdots + c_{m-1} z_{m-1} + c_m z_1.$$

It is desired to find at least one expression like this which is large. More specifically, one would like to find a positive constant C such that

$$\max_\pi \left| \sum_{i=1}^{m} c_i z_{\pi(i)} \right| \geqslant C \max_i |z_i|, \tag{1}$$

where π denotes a permutation, the sequence c is fixed, and z is arbitrary. Some interesting applications to matrix algebra are given in [1], [2]. Here we discuss the basic inequality (1).

One cannot have $C > 0$ in (1) if the sum of all the c_j is 0, since then we could take $z_j = 1$. If all the c_j are equal, the inequality fails again since we could choose the z_j to be any numbers whose sum if 0. Hence, the constant C in (1) ought to involve at least the two parameters

$$s = |c_1 + c_2 + \cdots + c_m|, \qquad d = \max_{i,j} |c_i - c_j|. \tag{2}$$

The following theorem is proved in [1]:

THEOREM 1 (Goldberg and Straus). *If $sd \neq 0$, inequality (1) holds with $C = sd/(2s+d)$.*

It seems to the authors somewhat surprising that one can get by with so little information about the sequence c, and more surprising that the constant is independent of m.

Goldberg has asked whether the constant C given by Theorem 1 is sharp. In answer to this question, we shall establish the following:

THEOREM 2. *If $sd \neq 0$, the best value of the constant C in (1) satisfies*

$$\frac{sd}{2s + d - 2s/m} \leqslant C \leqslant \min\left(s, \frac{sd}{2s + d - 2s/m - 2d/m}\right)$$

and the inequality on the right becomes an equality when c and z are real.

This shows that, while the Goldberg-Straus constant is not optimum for any m, it is the best that can be chosen independently of m even if the sequences are real. In the course of the proof we shall see that Theorem 2 remains valid when $c_j \in V$, $z_j \in \mathbb{C}$ or $c_j \in \mathbb{C}$, $z_j \in V$, where V is an arbitrary vector space over the complex field $\mathbb{C}$. The fact that the optimum constant (independent of m) is the same for this case as for the real case is another surprising aspect of Theorem 1.

Proof of the left-hand inequality. The following proof is shorter than the proof of Theorem 1 in [1] but uses similar ideas. Let the z_j be so numbered that $|z_m| = \max |z_j|$, and also so that

$$|z_m - z_1| = \max_j |z_m - z_j| = t|z_m|, \tag{3}$$

where t is defined by this equation. We suppose the c_j numbered so that $d = |c_m - c_1|$. Subtracting the above expressions P, R we then get

$$2\max(|P|, |R|) \geqslant |P - R| \geqslant dt|z_m|.$$

This shows that the constant C always satisfies $C \geqslant dt/2$.

Next let us make a cyclic permutation of the z_j to get

$$P_j = c_1 z_{1+j} + c_2 z_{2+j} + \cdots + c_m z_{m+j}.$$

(As a notational convenience, $z_{m+i} = z_i$.) Then

$$\begin{aligned} |P_1 + P_2 + \cdots + P_m| &= s|z_1 + z_2 + \cdots + z_m| \\ &= s|mz_m + (z_1 - z_m) + (z_2 - z_m) + \cdots + (z_{m-1} - z_m)| \\ &\geqslant s|z_m|[m - (m-1)t], \end{aligned}$$

where the last expression follows from (3). Since the largest $|P_j|$ is at least equal to the average, it

follows that

$$\max_j |P_j| \geqslant s|z_m|[1-(1-1/m)t]$$

and hence $C \geqslant s - st + st/m$. Thus,

$$C \geqslant \max(dt/2, s - st + st/m)$$

no matter what value $t \geqslant 0$ may have. The choice of t giving the poorest value of C is that for which the two expressions are equal. It will be found that the common value is the expression on the left in Theorem 2, and this completes the proof.

Proof of the right-hand inequality. In this discussion c_i and z_i are real. Since c_i can be multiplied by -1, we assume $s > 0$; and since the desired inequality is homogeneous, we let $\max|z_j| = 1$. Multiplying z_j by -1, if necessary, we see that there is no loss of generality in assuming

$$c_1 \leqslant c_2 \leqslant \cdots \leqslant c_m, \qquad -1 \leqslant z_1 \leqslant z_2 \leqslant \cdots \leqslant z_m = 1, \tag{4}$$

as well as

$$c_1 + c_2 + \cdots + c_m = s > 0, \qquad c_m - c_1 = d. \tag{5}$$

We refer to (4) and (5) as the *constraints*.

With P and Q as above, the value of the sum for any permutation satisfies

$$Q \leqslant \sum c_i z_{\pi(i)} \leqslant P.$$

(This well-known fact is trivial for $m=2$, and the general case follows from the case $m=2$.) Hence, the constant C for the given choice of c and z satisfies

$$C(c,z) = \max(P, -Q).$$

Our problem is to choose c and z, subject to the constraints, in such a way that this expression $C(c,z)$ is minimized.

Any alteration of c and z that decreases P, increases Q, and preserves the constraints leads to a smaller value of $C(c,z)$, and hence is permissible in the search for a minimum. A class of such alterations is suggested by the following lemma:

LEMMA 1. *Let a_j and b_j be increasing sequences of real numbers for $j = 1, 2, \ldots, n$ and let*

$$p = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n, \qquad q = a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1.$$

Then if all a_j are replaced by their arithmetic mean, the value of p is not increased, and the value of q is not diminished.

For proof let p_j denote the sums obtained by cyclic permutation of the a_j, add the inequalities $q \leqslant p_j \leqslant p$, and divide by n.

According to the lemma, we can replace the values $z_1, z_2, \ldots, z_{m-1}$ by their mean, which we denote by t. This change preserves the constraints and it does not increase P or diminish Q, as will now be shown. We have

$$P = (c_1 z_1 + \cdots + c_{m-1} z_{m-1}) + c_m z_m \geqslant t(c_1 + \cdots + c_{m-1}) + c_m z_m$$

$$Q = c_1 z_m + (c_2 z_{m-1} + \cdots + c_m z_1) \leqslant c_1 z_m + t(c_2 + \cdots + c_m),$$

where we have applied the lemma, with $n = m-1$, to the expressions in parentheses. In the first case, $a_j = z_j$, $b_j = c_j$; in the second case, $a_j = z_j$, $b_j = c_{j+1}$.

In view of the constraints, these expressions reduce to

$$P(t) = c_m + t(s - c_m), \qquad Q(t) = P(t) - d(1-t).$$

Evidently P and Q are linear and $P(1) = Q(1) = s$. The constant associated with these sequences is

$$C(t) = \max[P(t), -Q(t)]$$

and we want to choose t, $-1 \leqslant t \leqslant 1$, so that this is minimum. It is helpful to distinguish three cases.

Case 1. $s+d \geqslant ms$. In this case the constant on the right in Theorem 2 is s. Since $c_j \leqslant c_m$, we have $s+d \leqslant mc_m$; hence $ms \leqslant mc_m$ or, in other words, $s \leqslant c_m$. This shows that $P(t)$ is decreasing. The value of $C(t)$ is least when $t=1$, and in that case $C(t)=s$. This gives the desired result and shows that it is optimum. (We can never have $C>s$, as seen by the choice $z_j=1$).

Case 2. $s+d<ms$, $P(-1)+Q(-1)>0$. If $c_m \geqslant s$, we can reason as in Case 1 and get the value $C=P(1)=s$. Since this is larger than the constant in Theorem 2, we consider sequences with $c_m<s$. In this case $P(t)$ is increasing and, since $P(-1)+Q(-1)>0$, it is readily checked that the t for minimum $C(t)$ is $t=-1$. The value of the minimum is $C(-1)=P(-1)=2c_m-s$. On the other hand, the hypothesis $P(-1)+Q(-1)>0$ gives $2c_m>s+d$, and hence $C(-1)>d$. Now, the desired expression in Theorem 2 is at most d for $m \geqslant 2$, and hence the constant $C(-1)$ is too large. (Note, however, that all sequences considered in Case 2 satisfy the inequality of Theorem 2. They merely fail to show that the constant is sharp.)

Case 3. $s+d<ms$, $P(-1)+Q(-1) \leqslant 0$. In this case the value of t for minimum $C(t)$ satisfies

$$P(t)+Q(t)=0, \qquad -1 \leqslant t \leqslant 1.$$

By a short calculation we get t and then

$$C(t)=P(t)=-Q(t)=\frac{sd}{2s+d-2c_m}. \tag{6}$$

The inequality $-1 \leqslant t \leqslant 1$ is geometrically obvious and is readily checked algebraically under the hypothesis of Case 3.

We must now determine c_m so the above expression $C(t)$ is as small as possible, keeping the constraints. Evidently $C(t)$ is least when c_m is least, and since

$$c_2+c_3+\cdots+c_{m-1}+2c_m=d+s,$$

c_m is minimized by taking $c_j=c_m$ for $j \geqslant 2$. The value c_1 is then determined by $c_m-c_1=d$. Thus the critical sequences are

$$c=\{c_1,c_m,\ldots,c_m\}, \qquad z=\{t,t,\ldots,t,1\},$$

with $c_m-c_1=d$, $c_1+(m-1)c_m=s$. By addition of these equations, we get $mc_m=d+s$ and (6) reduces to the expression in Theorem 2. The hypothesis $P(-1)+Q(-1) \leqslant 0$ is equivalent to $d+s \geqslant 2c_m$, and it holds since $d+s=mc_m$ and $m \geqslant 2$. This completes the proof.

An open problem. In [3] it is seen that the best constant C for $m=2$, z and c complex, is $ds/(s^2+d^2)^{1/2}$, which is smaller than the value $C=\min(s,d)$ given by Theorem 2. Determination of the best constant for $m \geqslant 3$, z and c complex, is left as an open problem.

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AN INFINITE PRODUCT FOR e

NICHOLAS PIPPENGER

Wallis's infinite product [**1**, p. 180]

$$\frac{\pi}{2}=\frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\frac{8}{7}\cdots$$

was used by Stirling to determine the constant factor in his asymptotic formula [**2**, p. 137] $n!\sim(2\pi n)^{1/2}e^{-n}n^{n}$. A striking companion to Wallis's product is

$$\frac{e}{2}=\left(\frac{2}{1}\right)^{1/2}\left(\frac{2}{3}\frac{4}{3}\right)^{1/4}\left(\frac{4}{5}\frac{6}{5}\frac{6}{7}\frac{8}{7}\right)^{1/8}\cdots,$$

which is proved as follows. For $\nu\geqslant 2$, the νth factor is $[2^{\nu-1}\cdots 2^{\nu}/(2^{\nu-1}+1)\cdots(2^{\nu}-1)]^{1/2^{\nu}}=[(2^{\nu-1}-1)!!^{2}2^{\nu}!!^{2}/2\cdot 2^{\nu-1}!!^{2}(2^{\nu}-1)!!^{2}]^{1/2^{\nu}}$, where $n!!=n(n-2)\cdots 4\cdot 2$ if n is even, $n(n-2)\cdots 3\cdot 1$ if n is odd. Since $2^{\nu}!!=2^{2^{\nu-1}}2^{\nu-1}!$ and $(2^{\nu}-1)!!=2^{\nu}!/2^{\nu}!!=2^{\nu}!/2^{2^{\nu-1}}2^{\nu-1}!$, this expression becomes $[2^{2^{\nu}}2^{\nu-1}!^{6}/2\cdot 2^{\nu-2}!^{4}2^{\nu}!^{2}]^{1/2^{\nu}}$. By induction on ν, the product of the first ν factors is $[2\cdot 2^{\nu 2^{\nu}}2^{\nu-1}!^{2}/2^{\nu}!^{2}]^{1/2^{\nu}}$. Applying Stirling's formula and letting $\nu\rightarrow\infty$ completes the proof.

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RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4

HOW MANY PAIRS OF PRODUCTS OF CONSECUTIVE INTEGERS HAVE THE SAME PRIME FACTORS?

P. ERDŐS

Denote by $M(n,k)$ the least common multiple of the k consecutive integers $n+1,n+2,\ldots,n+k$. I conjectured that if $0<n<n+k\leqslant m$, then

$$M(n,k)\neq M(m,k),\tag{1}$$

and I thought that the following stronger result also holds.

If $k>2$, then $M(n,k)$ and $M(m,k)$ have the same prime factors on at most finitely many occasions.

For $k=2$, of course, $2n(n+1)=m(m+1)$ has infinitely many solutions, so that $n(n+1)$ and

$m(m+1)$ have the same prime factors. Several colleagues found examples with $k \geqslant 3$, but I know of no example for $k \geqslant 6$.

Are there infinitely many n, m, k with $1 \leqslant n$, $n+3 \leqslant n+k \leqslant m$ so that $M(n,k)$, $M(m,k)$ have the same prime factors?

Is there a k_0 such that for $k > k_0$ this never happens?

Finally, estimate the number of pairs (n,m) with $1 \leqslant n < m < x$ for which $n(n+1)$ and $m(m+1)$ have the same prime factors.

A well-known theorem of Størmer and Pólya states that if $a_1 < a_2 < a_3 < \cdots$ are all composed of the primes $p_1, p_2, \ldots, p_r$ then $a_{h+1} - a_h \to \infty$. Wintner conjectured more than 40 years ago that there is an infinite sequence of primes $p_1 < p_2 < p_3 < \cdots$ such that, if $b_1 < b_2 < b_3 < \cdots$ are the integers composed of the p_i, then $b_{h+1} - b_h \to \infty$. I conjecture that, if the sequence $p_1 < p_2 < p_3 < \cdots$ is sufficiently dense, then this can't happen, e.g., if $\Sigma 1/p_i$ diverges, then $b_{h+1} - b_h = 1$ infinitely often.

It is easy to see that for any function $f(k)$ tending to infinity as fast as we wish there is a sequence $\{p_k\}$ with $p_k > f(k)$ so that $b_{h+1} - b_h = 1$ has infinitely many solutions; I need Brun's method to prove this.

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HOW MANY i-j REDUCED LATIN RECTANGLES ARE THERE?

JOHN R. HAMILTON AND GARY L. MULLEN

There is a large literature [2] on **latin squares**, $n \times n$ squares with each of the numbers $1, 2, \ldots, n$ in each row and column. A **latin rectangle** is an array of m rows and n columns with $m \leqslant n$ in which each row is a permutation of $1, 2, \ldots, n$ and each column has distinct elements. A latin rectangle is **reduced** if the first row is in the standard order $1, 2, \ldots, n$ and we say that it is i-j **reduced** if the first i rows are cyclic permutations of $1, 2, \ldots, n$ and the first j columns are in the form $k, k+1, \ldots, k+m-1$ for $k = 1, \ldots, j$. Thus a latin rectangle of order $m \times n$ is i-j reduced if it has the following form.

$$\begin{array}{ccccccc}
1 & 2 & \cdots & j & \cdots & n-1 & n \\
2 & 3 & \cdots & j+1 & \cdots & n & 1 \\
\vdots & \vdots & & \vdots & & \vdots & \vdots \\
i & i+1 & \cdots & j+i-1 & \cdots & i-2 & i-1 \\
\vdots & \vdots & & \vdots & & & \\
m-1 & m & \cdots & m+j-2 & & & \\
m & m+1 & \cdots & m+j-1 & & &
\end{array}$$

It is understood that if a number in the rectangle exceeds n then it is reduced mod n. We can allow $i = 0$ or $j = 0$ so that a 1-0 reduced rectangle is reduced and a general rectangle is 0-0 reduced. As an illustration, the following rectangle is a 2-1 reduced latin rectangle of order 4×5.

$$\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
2 & 3 & 4 & 5 & 1 \\
3 & 1 & 5 & 2 & 4 \\
4 & 5 & 2 & 1 & 3
\end{array}$$

In [5] the second author considered the case $m = n$ and studied some elementary properties of i-j reduced latin squares of order n. In particular, the number $L(i,j,n)$ of i-j reduced latin

squares of order n was determined if $n \leqslant 8$. We thus restrict our attention to the cases where $m<n$.

Let $L(i,j,m,n)$ denote the number of i-j reduced latin rectangles of order $m \times n$. If $m=2$, the values of $L(1,0,2,n)$ are the "rencontres numbers," while if $m=3$ then $L(2,0,3,n)$ is a "ménages number." Formulas, in terms of n, are given for these numbers in [7]. If $m=4$, $L(3,0,4,n)$ is the number of three-discordant permutations of an n set. This problem has been studied in [4] and [9], while, if $m=5$, $L(4,0,5,n)$ is the number of four-discordant permutations of an n set, which has recently been studied by Whitehead in [10]. In [2] discordant permutations were called very reduced latin rectangles.

We mention several properties of i-j reduced latin rectangles that can be proved by elementary combinatorial arguments.

Table 1

2 × 4	2 × 5	2 × 6	2 × 7
L(1,0)=9	L(1,0)=44	L(1,0)=265	L(1,0)=1,854
L(1,1)=3	L(1,1)=11	L(1,1)= 53	L(1,1)= 309
	L(1,2)= 3	L(1,2)= 11	L(1,2)= 53
		L(1,3)= 3	L(1,3)= 11
			L(1,4)= 3

3 × 5	3 × 6	3 × 7
L(1,0)=552	L(1,0)=21,280	L(1,0)=1,073,760
L(2,0)= 13	L(2,0)= 80	L(2,0)= 579
L(1,1)= 46	L(1,1)= 1,064	L(1,1)= 35,792
L(1,2)= 5	L(1,2)= 58	L(1,2)= 1,274
L(2,1)= 4	L(1,3)= 5	L(1,3)= 58
	L(2,1)= 20	L(1,4)= 5
	L(2,2)= 5	L(2,1)= 115
		L(2,2)= 23
		L(2,3)= 5

4 × 6	4 × 7	5 × 7
L(1,0)=393,120	L(2,0)= 83,600	L(2,0)= 2,185,152
L(2,0)= 1,462	L(3,0)= 144	L(3,0)= 3,722
L(3,0)= 20	L(1,1)= 1,293,216	L(4,0)= 31
L(1,1)= 6,552	L(1,2)= 12,112	L(1,1)=11,270,400
L(1,2)= 130	L(1,3)= 158	L(1,2)= 39,004
L(1,3)= 9	L(1,4)= 9	L(1,3)= 354
L(2,1)= 120	L(2,1)= 4,110	L(1,4)= 17
L(2,2)= 10	L(2,2)= 193	L(2,1)= 36,128
L(3,1)= 6	L(2,3)= 11	L(2,2)= 638
	L(3,1)= 35	L(2,3)= 26
	L(3,2)= 8	L(3,1)= 303
		L(3,2)= 24
		L(4,1)= 9

THEOREM 1.
(a) $L(0,0,m,n)=n!\,L(1,0,m,n,)$
(b) $L(0,j,m,n)=(n-j)!\,L(1,j,m,n)$ *for* $0\leqslant j\leqslant n$
(c) $L(i,j,n-1,n)=L(i,j,n,n)=L(i,j,n)$
(d) *If* $i+j\geqslant n-1$, *then* $L(i,j,m,n)=1$.
(e) $L(1,1,2,n)=\sum_{j=0}^{n-2}(-1)^j\binom{n-2}{j}(n-1-j)!$
(f) *For fixed* $k, L(1,j,2,j+k)=L(1,j+1,2,j+k+1)$ *for all* j.

These results were used to determine $L(i,j,m,n)$ for $m,n\leqslant 7$ on an IBM 370/3033 computer by filling cells sequentially by the usual backtrack method. (See Table 1.) If $m=n-1$, the values of $L(i,j,n-1,n)=L(i,j,n)$ have been omitted since they appear in [5, p. 752]. Any value not listed is understood to be 1. For brevity, we list the value of $L(i,j,m,n)$ as $L(i,j)$ under the column headed $m\times n$. The values of $L(1,0,4,7)$ and $L(1,0,5,7)$ are omitted, as they would have required too much time to compute.

We note that for small m and n and for $m=n-1$ or n, it appears that $(n-m)!\,L(1,0,m,n)=(n-1)!\,L(1,1,m,n)$. Is this true generally?

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CLASSROOM NOTES

EDITED BY DEBORAH TEPPER HAIMO AND FRANKLIN TEPPER HAIMO

Material for this department should be sent to Professor Deborah Tepper Haimo, Department of Mathematical Sciences, University of Missouri, St. Louis, MO 63121.

PROOFS THAT $\Sigma 1/p$ DIVERGES

CHARLES VANDEN EYNDEN

The theorem of the title, in which p runs through the primes, was deduced by Euler in 1737 from the "equation"

$$\sum_{n=1}^{\infty}\frac{1}{n}=\prod\left(1-\frac{1}{p}\right)^{-1}.$$

Kronecker later fixed up this argument by replacing n and p by n^A and p^A, $A>1$, and letting $A \to 1$ [1].

It is my purpose in this paper to examine some fairly recent (compared to the age of the theorem, anyway) easy proofs of this theorem, where the word "easy" needs some definition. Of course any theorem may be proved very compactly if enough preliminary knowledge is assumed, and many number theory texts derive Euler's theorem from much stronger results. Here we are concerned with direct proofs appropriate for an undergraduate number theory class.

Our presumed audience will know something about infinite series (otherwise the statement of the theorem will not even make sense), but only from calculus, along with some elements of number theory.

We may not assume, for example, familiarity with the connection between the convergence of the series Σa_n and the infinite product $\Pi(1 \pm a_n)$. In fact, although our audience knows what it means for a series to converge absolutely, it does not know that rearranging such a series is justified. This means that some arguments must be complicated by replacing series with finite sums.

The proofs below have been left in more or less their original form, except that the notation is standardized. The letter p always represents a prime, and all lower-case letters except e represent positive integers. For given positive integers a and b with $a \leqslant b$, let P denote the set of primes p satisfying $a \leqslant p \leqslant b$, let M be those integers n all of whose prime divisors are in P, and for a given integer x let M_x be all elements of M not exceeding x. Let $|S|$ denote the number of elements in the set S.

Erdős's Proof. Paul Erdős published the following proof in 1938 [2]. If $\Sigma 1/p$ converges we can choose b so that $\Sigma_{p>b} 1/p < \frac{1}{2}$. Take $a=1$. Suppose $n \in M_x$, and write $n = k^2 m$, where m is square-free. Since $m = \Pi_S p$, where S is some subset of P, m can assume at most $2^{|P|}$ values. Also $k \leqslant \sqrt{n} \leqslant \sqrt{x}$. Thus $|M_x| \leqslant 2^{|P|}\sqrt{x}$.

Now the number of positive integers $\leqslant x$ divisible by a fixed p does not exceed x/p. Thus $x - |M_x|$, the number of such integers divisible by some prime greater than b, satisfies

$$x - |M_x| \leqslant \sum_{p>b} \frac{x}{p} < \frac{x}{2}.$$

We see

$$\frac{x}{2} < |M_x| \leqslant 2^{|P|}\sqrt{x},$$

or $\sqrt{x} < 2^{|P|+1}$, which is clearly false for x sufficiently large.

Comments. The proof above, which is notable for its lack of series manipulations, is given in the classic book by Hardy and Wright [3], as well as by Calvin Long's text [4].

Bellman's and Moser's Proofs. The details of Richard Bellman's 1943 proof [5] and Leo Moser's 1958 proof [6] will be omitted, since both appeared in this MONTHLY. Bellman assumed an a sufficiently large so that $\Sigma_P 1/p < 1$ with $b = \infty$, and from this derived the convergence of first $\Sigma_M 1/n$ and then the harmonic series.

Moser derived from the false assumption that $\Sigma 1/p$ converges the true conclusion that $\pi(x)/x \to 0$ as $x \to \infty$, where $\pi(x)$ denotes (as usual) the number of primes $\leqslant x$. A contradiction was then produced from this and the assumption that $\Sigma_P 1/p < \frac{1}{2}$ for large enough a.

Bellman used the rearrangement of positive series several times in the proof, and Moser used the result that a convergent series is Cesàro summable to the same limit [7], which our hypothetical audience is unlikely to have seen.

Dux's Proof. In 1956 Erich Dux [8] gave a proof that began and ended similarly to Bellman's but also made use of the rearrangement of positive series. Let $a=1$ and, assuming $\Sigma 1/p$ converges, choose b so that $\Sigma_{p>b} 1/p = A < 1$. Define M' to be all $n' > 1$ divisible by primes only

exceeding b, and M'' to be all positive integers not in M or M'. (Note that 1 is in M.)

Then, since P is finite,

$$\sum_M \frac{1}{n} = \prod_P \left(1+\frac{1}{p}+\frac{1}{p^2}+\cdots\right) = \prod_P \left(1-\frac{1}{p}\right)^{-1} < \infty,$$

and

$$\sum_{M'} \frac{1}{n'} < \sum_{p>b} \frac{1}{p} + \left(\sum_{p>b} \frac{1}{p}\right)^2 + \cdots = \frac{A}{1-A} < \infty;$$

so

$$\sum_{M''} \frac{1}{n''} = \left(-1+\sum_M \frac{1}{n}\right)\sum_{M'} \frac{1}{n'} < \infty.$$

This contradicts the divergence of the harmonic series.

Clarkson's Proof. This 1966 proof by James A. Clarkson [9] calls to mind Euclid's proof that the number of primes is infinite. If $\sum 1/p$ converges, we can choose a so that $\sum_P 1/p < \frac{1}{2}$ for all b. Let $Q = \prod_{p<a} p$. For fixed r, it is possible to choose b large enough so that all the factors of the numbers $1+iQ$, $1 \leqslant i \leqslant r$, are in P, since if $p<a$ then $p \nmid 1+iQ$.

Now each term of the sum $\sum_{i=1}^r 1/(1+iQ)$ whose denominator is a product of j primes (not necessarily distinct) occurs at least once in the expansion of

$$\left(\sum_P \frac{1}{p}\right)^j < 2^{-j}. \tag{1}$$

Thus

$$\sum_{i=1}^r \frac{1}{1+iQ} < \sum_{j \geqslant 1} 2^{-j} \leqslant 1.$$

But, since r was arbitrary, this implies that the harmonic series converges.

Comments. The expansion of $(\sum_P 1/p)^j$ recalls the proofs of Bellman and Dux. Getting the inequality after (1) requires rearrangement of infinite series. This proof (with $b=\infty$) is reproduced in Apostol's *Introduction to Analytic Number Theory* [10]. The last sentence of the proof is most easily justified by comparison with $\sum 1/2Qi$.

Two More Proofs. A very simple identity forms the basis for two more proofs of my own design, namely,

$$\left(1+\frac{1}{p}\right)\left(1+\frac{1}{p^2}+\frac{1}{p^4}+\cdots+\frac{1}{p^{2k}}\right) = 1+\frac{1}{p}+\frac{1}{p^2}+\cdots+\frac{1}{p^{2k+1}}.$$

Taking the product over P and letting $k \to \infty$ yields

$$\prod_P \left(1+\frac{1}{p}\right)\sum_M \frac{1}{n^2} = \sum_M \frac{1}{n}. \tag{2}$$

Since $\sum 1/n^2$ converges and $\sum 1/n$ diverges, this means,

$$\text{for } a=1, \prod_P \left(1+\frac{1}{p}\right) \to \infty \quad \text{as } b \to \infty. \tag{3}$$

Continuation A. Since for $C>0, e^C = 1+C+C^2/2!+\cdots > 1+C$, we have

$$\prod_P \left(1+\frac{1}{p}\right) < \prod_P e^{1/p} = \exp\left(\sum_P \frac{1}{p}\right).$$

This, with (3), shows, that $\sum 1/p$ diverges.

Continuation B. By (3) $\Pi_P(1+1/p)\to\infty$ as $b\to\infty$ for any fixed a, and so the same is true for $\Sigma_M 1/n$ by (2). If $\Sigma 1/p$ converges, we can choose a so that $\Sigma_P 1/p<\frac{1}{2}$ for all b, then choose b and x large enough so that $\Sigma_{M_x}1/n>2$. Since every n in M_x except 1 is of the form pn for $p\in P$ and $n\in M_x$, we have

$$\sum_P \frac{1}{p}\sum_{M_x}\frac{1}{n}\geqslant \sum_{M_x}\frac{1}{n}-1.$$

Then

$$\frac{1}{2}>\sum_P\frac{1}{p}\geqslant 1-\left(\sum_{M_x}\frac{1}{n}\right)^{-1}>1-\frac{1}{2},$$

a contradiction.

Comments. The proof using Continuation A is so simple (and the theorem is so old) that it would be foolhardy to call it new, although I have not found the arrangement anywhere. Of course $e^C>1+C$ may also be proved without recourse to the series for e^C, but I believe most of our hypothetical audience will remember this expansion. Continuation B has a better chance of being a novelty.

The reader should note a very recent proof by Frank Gilfeather and Gary Meisters [**11**].

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COUNTING MAGIC SQUARES

JOEL SPENCER

A magic square of size n and order r is defined to be an $n\times n$ matrix A with nonnegative integral coefficients such that every row and column sums to r (diagonals are not examined). Let $H_n(r)$ denote the number of such squares. Richard Stanley [3] has shown

(1) $H_n(r)$ is a polynomial in r,

(2) $H_n(r)$ has degree $(n-1)^2$.

Our main object is to give an elementary proof of (1). First, we require a combinatorial result.

LEMMA 1. *Let $A=[a_{ij}]$ be an $n\times n$ magic square of order $r>0$. There exists a permutation σ on $\{1,\dots,n\}$ so that $a_{i,\sigma i}>0$ for $1\leqslant i\leqslant n$.*

This lemma is a consequence of Birkhoff's Theorem that the doubly stochastic matrices are the convex hull of the permutation matrices, and it may be found in most texts on combinatorial theory.

Second, we require a result from linear algebra.

LEMMA 2. *Let $(\mathbb{S}, <)$ be a finite partial order. Let sequences $U_s(r)$ be defined for all $s \in S$ satisfying a linear system*

$$U_s(r) = U_s(r-1) + \sum_{v<s} \alpha_{sv} U_v(r-1)$$

for some constants α_{sv}. Then for all $s \in \mathbb{S}$, $U_s(r)$ is a polynomial in r.

For arbitrary linear systems of difference equations, each sequence may be expressed as the sum of polynomials times λ^r where λ is an eigenvalue of the coefficient matrix. In this case the coefficient matrix is lower triangular with ones in the main diagonal, where the indexing of the rows and columns of the matrix is obtained from a total order on S consistent with the given partial order on S. Thus $\lambda = 1$ is the only eigenvalue and all $U_s(r)$ are polynomial.

Define, for any magic square $A = (a_{ij})$,

$$nz(A) = \{(i,j) : a_{ij} > 0\}.$$

We let $[n]$ denote $\{1,\ldots,n\}$ so that $[n]\times[n]$ denotes the positions of the matrix. Call $T \subseteq [n]\times[n]$ a transversal if

$$T = \{(i,\sigma i) : 1 \leqslant i \leqslant n\}$$

for some permutation σ. Call $S \subseteq [n]\times[n]$ full if $S \supseteq T$ for some transversal T. For each full S let $T(S)$ be a particular transversal such that $S \supseteq T(S)$. Let $\mathbb{S}$ denote the family of full S. For $S \in \mathbb{S}, r \geqslant 1$, set $U_S(r)$ equal to the number of magic squares A of order r with $nz(A) = S$. Now we may write our system of difference equations.

For $S \in \mathbb{S}, r \geqslant 2$,

$$U_S(r) = \sum_{S - T(S) \subseteq V \subseteq S} U_V(r-1).$$

Let A be an $n \times n$ magic square of order r with $nz(A) = S$. Let I^* be the matrix with ones in positions $T(S)$ and zeros elsewhere. Set $A^* = A - I^*$. Clearly A^* is a magic square of order $r-1$ with $S - T(S) \subseteq nz(A^*) \subseteq S$. Conversely, given such $A^*, A = A^* + I^*$ is a magic square of order r with $nz(A) = S$. This bijection establishes the equality.

By Lemma 2 (with containment the partial order) each $U_S(r)$ is polynomial. By Lemma 1

$$H_n(r) = \sum_{S \in \mathbb{S}} U_S(r)$$

so that $H_n(r)$ is polynomial.

We show (2) by bounding $H_n(r)$. The values a_{ij}, $1 \leqslant i,j \leqslant n-1$, determine at most one magic square. As a partial converse, if

$$\left| a_{ij} - \frac{r}{n} \right| < \frac{1}{n(n-1)^2} r \quad \text{for } 1 \leqslant i,j \leqslant n-1$$

the a_{ij} do extend to magic squares. Thus

$$\left[\frac{2r}{n(n-1)^2} \right]^{(n-1)^2} \leqslant H_n(r) \leqslant (r+1)^{(n-1)^2}.$$

These bounds are crude, but they suffice to determine $\deg(H_n)$.

As H_n is polynomial, we may naturally define $H_n(\alpha)$ for any complex α. Stanley has shown

(3) $H_n(-1) = \cdots = H_n(-n+1) = 0$,

(4) $H_n(r) = (-1)^{n-1} H_n(-n+r)$.

It is not clear if we may derive (3), (4) by elementary means.

These results were conjectured by Anand, Dumir, and Gupta [1]. A different proof is given by Ehrhart [2].

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2. E. Ehrhart, Sur les carrés magiques, C. R. Acad. Sci. Paris Ser. A, 277 (1973) 651–654.
3. R. Stanley, Linear homogenous diophantine equations and magic labelings of graphs, Duke Math. J., 40 (1974) 607–632.

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POINTWISE CONVERGENCE OF FOURIER SERIES

PAUL R. CHERNOFF

The purpose of this note is to show how the usual proofs of convergence of Fourier series [1], [2] may be considerably shortened and the underlying mechanism clarified. In fact, one can get stronger conclusions with less effort.

It is best to work with complex exponentials rather than sines and cosines. Then, for a Lebesgue integrable 2π-periodic function f, the nth Fourier coefficient is defined by

$$\hat{f}(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx.$$

The (unsymmetric) partial sums of the Fourier series are defined by

$$S_{m,n}(x)=\sum_{k=-m}^{n}\hat{f}(k)e^{ikx}.$$

Usually one considers only the symmetric partial sums $S_n(x)=S_{n,n}(x)$.

Just as with the standard proofs, everything turns upon one essential fact: the Riemann-Lebesgue lemma. This states that $\hat{f}(n)$ tends to 0 as $|n|$ becomes infinite. For f square-integrable (e.g., piecewise continuous), Riemann-Lebesgue is an immediate consequence of the very easy Bessel inequality. The case of merely integrable f follows from the square-integrable case by an approximation argument (cf. [2, Chapter 1]).

THEOREM. *Let f be integrable and suppose that f is differentiable at the point x_0. Then the Fourier partial sums $S_{m,n}(x_0)$ converge to $f(x_0)$ as $m,n\to\infty$.*

Proof. As usual we may suppose that $x_0=0$ and $f(x_0)=0$: just subtract a constant from f and shift the origin. (The following argument would work without this preliminary reduction, of course, but the formulas would be a bit more complicated.) Since $f(0)=0$ and $f'(0)$ exists, the function $g(x)=f(x)/[e^{ix}-1]$ is bounded near 0 and thus is integrable because f is integrable. Now we have

$$f(x)=(e^{ix}-1)g(x)$$

so that the Fourier coefficients satisfy

$$\hat{f}(k)=\hat{g}(k-1)-\hat{g}(k).$$

The Fourier series is a telescoping series! Indeed,

$$S_{m,n}(0)=\sum_{k=-m}^{n}\hat{f}(k)=\hat{g}(-m-1)-\hat{g}(n),$$

and this tends to 0 $(=f(0))$ by Riemann-Lebesgue. ∎

REMARKS: (1) If we assume that f is piecewise continuous (the usual classroom hypothesis), then g is also piecewise continuous. So one need not venture outside the piecewise continuous realm.

Similarly, if f is square-integrable, so is g. So we need only the simplest version of the Riemann-Lebesgue lemma if we are content to work with a narrower class of functions f.

(2) The differentiability hypothesis is much stronger than necessary. All that is really needed for our argument to work is that $(f(x)-f(x_0))/(x-x_0)$ be Lebesgue integrable in a neighborhood of x_0. This is certainly the case if f satisfies a Lipschitz or Hölder condition at x_0, e.g., if f has one-sided derivatives at x_0.

(3) We showed that the *unsymmetric* partial sums of the Fourier series converge to $f(x_0)$. This means that the positive and negative halves of the series converge separately, a conclusion that does not follow from the usual proof.

(4) We can also deal with a jump discontinuity. Thus, suppose that f has left-hand and right-hand limits at 0, which we denote $f(0^-)$ and $f(0^+)$. Suppose also that f has one-sided slopes at 0, i.e., $(f(h)-f(0^+))/h$ and $(f(-h)-f(0^-))/h$ converge to limits as h decreases to 0. Then the *symmetric* partial sum $S_n(0)$ converges to $[f(0^+)+f(0^-)]/2$.

Proof. Subtract a constant so that $f(0^+)=-f(0^-)$. We then must show that $S_n(0)$ converges to 0. Now

$$S_n(0)=\int_{-\pi}^{\pi} f(x)D_n(x)\,dx.$$

$D_n(x)=(1/2\pi)\Sigma_{-n}^{n}e^{ikx}$ is the famous "Dirichlet kernel," but all we need observe is that it is an even function. Hence

$$S_n(0)=\int_{-\pi}^{\pi}\tfrac{1}{2}[f(x)+f(-x)]D_n(x)\,dx.$$

Now we simply apply the previous theorem to the function $\frac{1}{2}[f(x)+f(-x)]$.

(5) Again the differentiability hypothesis can be weakened considerably. To deduce that $S_n(0)\to 0$ we need only the integrability of the function $[f(x)+f(-x)]/x$. It is not even necessary to assume that $f(0^+)$ and $f(0^-)$ exist.

Incidentally, the restriction to symmetric partial sums in (4) is really necessary, as one can see by considering the simplest example, that is, $f(x)=+1$ for $x>0$, -1 for $x<0$.

The author's research was partially supported by the National Science Foundation.

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MATHEMATICAL EDUCATION

EDITED BY W. E. MASTROCOLA

Material for this department should be sent to W. E. Mastrocola, Department of Mathematics, Colgate University, Hamilton, NY 13346.

MAA SECTION SEMINARS

A primary purpose of the Mathematical Association of America is to further the mathematical education of its members. On the national level, the MAA accomplishes this goal in several ways—through its many publications, its meetings, and its various institutes. On the sectional

level, MAA activities have often been confined to Section meetings. Recently a few Sections of the Association have acted to broaden the mathematical background of their members by conducting week-long seminars on important mathematical topics. These seminars have been highly successful and have been enthusiastically received by participants. We describe here some of the formats and logistics of such seminars. Our aim is to encourage more Sections of the Association, working either singly or jointly with other Sections, to sponsor such seminars.

Formats. Most seminars have been held during the summer and have run for five days. There are at least three workable formats for a five-day seminar:

(1) One principal speaker giving five (or more) one-hour lectures on a single topic.
(2) Two principal speakers, each giving five one-hour lectures. The speakers need not coordinate their lectures, but their topics should be closely related.
(3) A large number (say 8 to 10) of invited speakers, each giving one or two one-hour lectures.

In all cases, additional time can be reserved for single lectures by seminar participants and for informal working groups.

To have a successful short course, attention should be given to providing an atmosphere that stimulates group activity.

Seminar Expenses. The primary expense for the Section is the fee(s) for the principal speaker(s). A frequent honorarium is $500 for five one-hour lectures (with perhaps one or two supplementary lectures). Other costs include the printing and mailing of advertisements, correspondence, and other incidentals. However, these expenditures are minor and can perhaps be shared with the host institution. N.B.: If format (3) is used, then perhaps speaker expense can be greatly reduced. For example, at the 1977 seminar sponsored by the North Central Section, none of the fourteen invited speakers accepted an honorarium.

Financial Resources for the Section. The expenses for a Section seminar can run as high as $600–$700 for format (1) and $1100–$1200 for format (2). How can these costs be met by the Section? First of all, the MAA has granted $300 to Sections wishing to conduct a seminar. However, a larger sum (up to $600) can be obtained by a Section under certain circumstances, e.g., the use of format (2). A second source of income is a seminar registration fee. (A fee of $20 is considered reasonable.) Third, if the Section sponsors the seminar jointly with the host institution, then a portion of the expenses can be borne by the host. Finally, if appropriate, notes from the various lectures can be reproduced and sold (with the permission of the speakers, of course). Also, it may be that the MAA would be willing to contribute more than $300 toward a Section seminar if a sufficiently good case existed.

Location. A university or college in the Section, preferably one that can offer low rates for room and board (approximately $55–$75 for five days) and that is located in an attractive region, is the ideal location.

Participant Costs. Aside from travel, participant expenses are confined mainly to room and board and registration. A total of less than $100 for these expenses can be considered reasonable.

What to Do. Should your Section decide to conduct a seminar, what should you do? Briefly, here are some of the important steps and considerations. At least one year prior to beginning the seminar, its theme and site should be chosen by the Executive Committee of the Section. In choosing a topic, the Executive Committee should survey the members of the Section (via a newsletter and at Section meetings). To determine the site, the Executive Committee should solicit bids in which facilities and costs would be detailed. At the same time, a committee to organize the seminar should be appointed. This committee should include a member of the Executive Committee and the person in charge of arrangements at the host institution. The

organizing committee should choose and obtain firm commitments from all invited speakers. The seminar should be advertised nationally (in MAA publications, for example) and locally (by a mailing to each member of the host Section and contiguous Sections). If appropriate, plans should be made for the publication of notes from the seminar. Of course many other details may also require attention. Upon request, the Committee on Continuing Education will provide suggestions for handling these details.

Section Seminars (Examples).

Section/Year: North Central/1977
Topic: Mathematical Modeling
Length/Place: 5 days/Bemidji State University
Format: No principal speaker(s); 14 speakers, each giving one or two lectures
Chief Organizer: Gerald Bergum, South Dakota State University, Brookings, SD 57001
Attendance: 46

Section/Year: Michigan/1978
Topic: Coding and Combination
Length/Place: 5 days/Northern Michigan University
Format: Two principal speakers: Vera Pless, University of Illinois at Chicago Circle, "Coding Theory"; Chester Salwach, Lafayette College, "Planes and Biplanes"

Section/Year: Ohio/1975
Topic: Mathematical Programming and Operations Research
Length/Place: 5 days/Youngstown State University
Format: One speaker, Gerald L. Thompson, Carnegie-Mellon University; 2 sessions per day
Attendance: 80

Sanford L. Segal, Chairman
MAA Committee on Continuing Education

PROBLEMS AND SOLUTIONS

EDITED BY VLADIMIR DROBOT

Send all proposed problems, in duplicate if possible, to Professor Vladimir Drobot, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053. Please include solutions, relevant references, etc.

An asterisk (*) indicates that neither the proposer nor the editors supplied a solution.

Solutions should be sent to the addresses given at the head of each problem set.

A publishable solution must, above all, be correct. Given correctness, elegance and conciseness are preferred. The answer to the problem should appear right at the beginning. If your method yields a more general result, so much the better. If you discover that a MONTHLY *problem has already been solved in the literature, you should of course tell the editors; include a copy of the solution if you can.*

PROBLEMS DEDICATED TO EMORY P. STARKE

Solutions of these problems dedicated to Emory P. Starke should be mailed to Prof. A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131 (U.S.A.), by September 30, 1980. To facilitate consideration, type with double spacing in the format used below. If acknowledgment is desired, include a self-addressed card, label, or envelope.

S 30. *Proposed by Solomon W. Golomb, University of Southern California.*

(i) There is a random group of n people in a room. What is the probability that there is a 73-day period (beginning *anywhere* in the year) which contains all their birthdays? (We assume that the birthdays are independent and uniformly distributed modulo 365, we ignore leap-days, and of course the *year* of birth is not counted as part of the birthday.)

(ii) More generally, if n points are placed, independently and at random, on a circle of radius $1/2\pi$ (hence, circumference 1), what is the probability that all these points can be covered by a (movable) arc of length α, $0 \leqslant \alpha \leqslant 1$? Express the answer as a function $P_n(\alpha)$.

S 31. *Proposed by Leo J. Alex, SUNY College at Oneonta.*

In each of the following, find all solutions in nonnegative integers a, b, and c:
(i) $1+5^a=2\cdot 3^b+3\cdot 2^c$;
(ii) $1+2^a=4\cdot 3^b+5^c$.

SOLUTIONS OF PROBLEMS DEDICATED TO EMORY P. STARKE

Linearization of Product of q-Appell Polynomials

S 7 [1979, 222]. *Proposed by George E. Andrews, Pennsylvania State University, and Richard Askey, University of Wisconsin, Madison.*

Let $p_n(x)=(x+1)(x+q)\cdots(x+q^{n-1})$, $n=1,2,\ldots,p_0(x)=1$. Find the coefficients $a(k,m,n)$ defined by

$$p_n(x)\cdot p_m(x)=\sum_{k=0}^{m+n} a(k,m,n)\cdot p_k(x). \tag{1}$$

Solution by Harley Flanders, Florida Atlantic University, Boca Raton. Obviously $a(k,m,0)=\delta_{km}$ and $a(k,m,n)=a(k,n,m)$. Note that $p_n(x)(x+q^n)=p_{n+1}(x)$. Multiply (1) by $x+q^n=(x+q^k)+(q^n-q^k)$ to obtain

$$\sum_0^{m+n+1} a(k,m,n+1)p_k=\sum_0^{m+n} a(k,m,n)\left[p_{k+1}+(q^n-q^k)p_k\right].$$

Equate coefficients of the linearly independent p_k:

$$a(0,m,n+1)=(q^n-1)a(0,m,n),$$

$$a(m+n+1,m,n+1)=a(m+n,m,n),$$

$$a(k,m,n+1)=a(k-1,m,n)+(q^n-q^k)a(k,m,n)$$

for $1\leqslant k\leqslant m+n$. From the first two relations follow

$$a(0,m,n)=0 \quad \text{if} \quad m>0 \quad \text{or} \quad n>0,$$

$$a(m+n,m,n)=1.$$

By computing the a's for several small n, one is led to guess

$$a(k,m,n)=\frac{p_r(-q^m)p_r(-q^n)}{p_r(-q^r)}, \quad r=m+n-k.$$

A tedious calculation shows that this indeed solves the difference equation and the boundary conditions.

Also solved by W. A. Al-Salam (Canada), L. Carlitz, L. Kuipers (Switzerland), Blagoj S. Popov (Yugoslavia), Otto G. Ruehr, and the proposers.

ELEMENTARY PROBLEMS

Solutions of these Elementary Problems should be mailed in duplicate to Dr. J. L. Brenner, 10 Phillips Road, Palo Alto, CA 94303 (U.S.A.), by September 30, 1980. Please place the solver's name and mailing address on each (double-spaced) sheet. Include a self-addressed card or label (for acknowledgment).

E 2830. *Proposed by R. P. Boas, Northwestern University.*

Let a point on a straight line with steadily increasing acceleration from time $t=0$ to $t=T$. Show that its velocity at mid-time ($t=T/2$) cannot exceed its average velocity (meaning, as usual, total distance divided by total time). This statement is not necessarily true if $1/2$ is replaced by any other number between 0 and 1.

E 2831. *Proposed by M. Cavachi, University of Bucharest, Rumania.*

Prove that a convex hexagon with no side longer than 1 unit must have at least one main diagonal not longer than 2 units.

E 2832. *Proposed by David Merriell, Vassar College.*

Take a dish containing n strands of cooked spaghetti where n is large enough and each strand is long enough so that it is not obvious whether two ends belong to the same strand. Reach in, select two ends at random (independently) and join them with edible paste. Continue to select two unjoined ends and join them until there are no more ends. One will then have a certain number of loops of spaghetti. What is the expected total number of loops, and the expected number of 1-loops, i.e., loops of length one?

E 2833*. *Proposed by Anders Bager, Denmark.*

Call an integer $n \geqslant 3$ a Phibonacci number if

$$\phi(n)=\phi(n-1)+\phi(n-2),$$

where ϕ denotes the Euler ϕ-function. Is there any composite Phibonacci number?

E 2834. *Proposed by James W. Fickett, Texas A & M University.*

The width of a compact subset S of $\mathbb{R}^3$ is the least $d \geqslant 0$ such that S lies between some two parallel planes a distance d apart. (i) Prove or disprove: There is a constant $\gamma>0$ such that any compact set of unit width contains a four-point subset of width at least γ. (ii)* Generalize to $\mathbb{R}^n$.

SOLUTIONS OF ELEMENTARY PROBLEMS

A Sum of 1's and -1's

E 2758 [1979, 128]. *Proposed by Bruce C. Berndt, University of Illinois, Urbana, and Ronald J. Evans, University of California, San Diego.*

Let c and d be relatively prime positive integers of opposite parity and define

$$F(d,c)=\sum_{j=1}^{c-1}(-1)^{j+1+[dj/c]},$$

where $[x]$ denotes the integer with $[x]\leqslant x<[x]+1$. Prove that $F(d,c)+F(c,d)=1$.

Solution and a generalization based on independent solutions of Duane M. Broline, Auburn University; F. S. Cater, Portland State University; L. Carlitz, Duke University; Lorraine L. Foster, California State University, Northridge; F. D. Hammer, Los Gatos, California; L. E. Mattics, University of South Alabama; and Joseph Silverman, Harvard University. Let c,d,r be positive integers with $(c,d)=1$ and let λ be an indeterminate $\neq 0,1$. Set $F(d,c;\lambda)=\Sigma_{i=1}^{c-1}\lambda^{i-1+[di/c]}$. It will be shown that

$$F(dr,cr;\lambda)+F(cr,dr;\lambda)=A(1-\lambda^{s-2})/(1-\lambda)+2(A-1)/\lambda,$$

where $s=c+d$ and $A=\Sigma_{m=0}^{r-1}\lambda^{ms}$. The desired result is obtained by taking $r=1,\lambda=-1$.

Clearly $F(dr,cr;\lambda)=A\,F(d,c;\lambda)+(A-1)/\lambda$, so it suffices to show that

$$F(d,c;\lambda)+F(c,d;\lambda)=(1-\lambda^{s-2})/(1-\lambda). \tag{*}$$

Observe that $i-1+[di/c]\neq j-1+[cj/d]$ for integers i, j with $0<i<c$, $0<j<d$. Otherwise, there would exist integers i,j,t,r_1,r_2 with $(c+d)i=tc+r_1$ $(0<r_1<c)$ and $(c+d)j=td+r_2$ $(0<r_2<d)$; but addition of these two equalities leads to a contradiction. Thus $F(d,c;\lambda)+F(c,d;\lambda)$ is a sum of $s-2$ distinct powers of λ. The smallest exponent that occurs is 0, and the largest is (if, say, $c<d$) $d-2+[c(d-1)/d]=s-2+[-c/d]=s-3$. Thus $F(d,c;\lambda)+F(c,d;\lambda)=\Sigma_0^{s-3}\lambda^m$, which yields (*).

Also solved by J. Suck (Germany) and the proposers.

Suck obtained another generalized version. Supposing that every lattice point (i,j), $0<i<c$, $0<j<d$, is assigned a real value $v(i,j)$ that depends only on $i+j$, he proved that

$$\Sigma_{i=0}^{c-1}v(i,[f(i)])+\Sigma_{j=0}^{d-1}v(j,[f^{-1}(j)])=\Sigma_{i=0}^{c-1}v(i,0)+\Sigma_{j=0}^{d-1}v(j,c-1)$$

for any f that is a continuous increasing function on $[0,c]$ with $f(0)=0$, $f(c)=d$.

A Block Matrix Equal to a Kronecker Product

E 2762 [1979, 223]. *Proposed by Peter Hoffman, University of Waterloo, Canada.*

Let $A_1,\ldots,A_n$ be $k\times k$ matrices over a field F, such that $A=A_1+\cdots A_n$ is invertible. Show that the block-matrix

$$B=\begin{bmatrix} A_1 & A_2 & A_3 & \cdots & & A_n & 0 & \cdots & 0\\ 0 & A_1 & A_2 & \cdots & & A_{n-1} & A_n & \cdots & 0\\ \vdots & \vdots & \vdots & & & & & \vdots & \vdots\\ 0 & 0 & \cdots & A_1 & A_2 & & \cdots & \cdots & A_n \end{bmatrix}$$

has full rank, i.e., rank $(B)=mk$ where m is the number of block-rows.

Solution by Gerald S. Rogers, New Mexico State University. Let I_m denote the $m\times m$ identity matrix; let 1_n be the $n\times 1$ matrix (=vector $1,1,\ldots,1$) with all 1's. Let C be the partitioned $k\times kn$ matrix $(A_1\ A_2\ \cdots\ A_n)$. Then B is the Kronecker product $I_m\otimes C$, so rank $B=m(\text{rank } C)=m$ (rank C') (notation for transposed matrix). In the equation $J=XA'=XA_1'+\cdots+XA_n'=(1_n'\otimes X)C'$, take $J=I_k$. Then $X=(A')^{-1}$ is a solution. It follows that rank C' must be k.

Also solved by F. S. Cater, M. L. J. Hautus (Netherlands), Thomas Jager, A. A. Jagers (Netherlands), N. Miku (Netherlands), University of South Alabama Problem Group, J. Suck (Germany), Hann Tzong Wang (Taiwan), Buck Ware, and the proposer.

F. S. Cater remarks that if rank $A=r<k$, then rank $B>mr$.

Primes in an Arithmetic Progression

E 2766 [1979, 223]. *Proposed by I. Borosh and D. Hensley, Texas A & M University.*

Let r be a positive rational number but not an integer. Prove that there are infinitely many positive integers n such that $[nr]$ is prime. (Here $[x]$ is the greatest integer in x.)

Solution by W. L. Abbott (Canada), D. M. Bloom, Robert Breusch, F. S. Cater, John Christopher, Randall J. Covill, Milton Eisner, James Fickett, Michael Filesta, Lorraine L. Foster, Nick Franceschine III, Thomas Jager, L. Kuipers (Switzerland), M. K. Kwong, Gordon S. Lessells (Nigeria), J. Metzger, N. Miku (Netherlands), Santa Clara Problem Solving Ring, Jeffrey Shallitt, George Shulman, Joseph Silverman, University of South Alabama Problem Group, Blair Spearman, E. Trost (Switzerland), Joseph Wiener & John Spellman, Ye Yangbo (P. R. China), and J. Benjamin Zipperer. Write $r=a/b$, $(a,b)=1$, $b>1$. Choose integers x,y with $ax-by=1$. Note $[(bm+x)a/b]=[am+y+1/b]=am+y$; $(a,y)=1$. Thus there are infinitely many positive integers $n=bm+x$ for which $[nr]$ is prime.

Decomposing an Interval into Homeomorphic Subsets

E 2768 [1979, 307]. *Proposed by Jim Fickett, University of Colorado, Boulder.*

Is there a subset E of $[0,1]$ such that E and $[0,1]\setminus E$ are homeomorphic?

I. *Solution by G. A. Edgar, The Ohio State University, Columbus.* The extended real line $[-\infty,\infty]$ is homeomorphic to $[0,1]$. Write $[x]$ for the greatest integer $\leqslant x$. Set

$$E=\{x\in(-\infty,\infty)|[x] \text{ is an even integer}\}\cup\{\infty\}.$$

Then E is clearly homeomorphic to its complement

$$E'=\{x\in(-\infty,\infty)|[x] \text{ is an odd integer}\}\cup\{-\infty\}.$$

II. *Solution by Leroy F. Meyers, The Ohio State University, Columbus.* A more general result is proved: if n is a positive finite cardinal number or $\aleph_0$ or c, then any nondegenerate interval can be partitioned into n homeomorphic sets. The partition is trivial if n is 1 (one set) or c (c singletons). If $2\leqslant n\leqslant\aleph_0$, the following table gives a partition $\{A_k:0\leqslant k\leqslant n\}$. All cases are covered by the table since every nondegenerate interval is homeomorphic to $(0,1]$, to $(0,1)$, or to $[-2,2]$.

Interval	A_k if $2\leqslant n<\aleph_0$	A_k if $n=\aleph_0$
$(0,1]$	$\left(\frac{k}{n},\frac{k+1}{n}\right]$	$(2^{-k-1},2^{-k}]$
$(0,1)$	$\bigcup_{j=0}^{\infty}\left[\frac{1}{nj+k+2},\frac{1}{nj+k+1}\right)$	$[2^{-k-1},2^{-k})$
$[-2,2]$	$\bigcup_{j=0}^{\infty}I_{k,2j}\cup\bigcup_{j=0}^{\infty}I_{k+1,2j+1}$	$I_{0,2k-1}\cup\bigcup_{j=0}^{\infty}I_{k,2j}\cup\bigcup_{j=0}^{\infty}I_{k+1,2j+1}$

where

$$I_{0,-1}=\varnothing,$$

$$I_{0,0}=[-2,2^{-n+1}] \text{ if } 2\leqslant n<\aleph_0,$$

$$I_{0,0}=[-2,0] \text{ if } n=\aleph_0,$$

$$I_{0,j}=(1+2^{-j},1+2^{-j+1}] \text{ if } j>0,$$

$$I_{k,0}=[2^{-k}+2^{-k-1},2\cdot 2^{-k}] \text{ if } 0<k<n,$$
$$I_{k,j}=[2^{-k}+2^{-k-j-1},2^{-k}+2^{-k-j}) \text{ if } 0<k<n \text{ and } j>0, \text{ and}$$
$$I_{n,j}=I_{0,j} \text{ if } 2\leqslant n<\aleph_0 \text{ and } j\geqslant 0.$$

Each of the sets A_k is a half-open interval, or each is a denumerable disjoint union of half-open intervals (and one closed interval, for $[-2,2]$). A sufficiently small neighborhood of a point in A_k meets at most one of these intervals (except when $n=2$ for $[-2,2]$, in which case every neighborhood of the point 2^{-k} meets denumerably many intervals of A_k). In all cases, the required homeomorphisms are easy to construct.

Also solved by Steven F. Bellenot, Stephan C. Carlson, J. H. Carruth, Robert Connelly, Eric K. van Douwen, Lee Erlebach, Jerrold W. Grossman, A. A. Jagers (Netherlands), F. Burton Jones, Man Kam Kwong, K. M. Levassar, O. P. Lossers (Netherlands), Robert A. McCoy, Mark Merriman, Mark D. Meyerson, Edward T. Ordman, M. J. Reed, and John C. Tripp.

Fermat's Last Theorem for Even Exponents

E 2771 [1979, 308]. *Proposed by R. Breusch, Amherst, Massachusetts.*

Let p be a prime and $p\not\equiv 1 \pmod 8$. Prove that the equation $x^{2p}+y^{2p}=z^{2p}$ has no solution in positive integers x,y,z with $xyz\not\equiv 0 \pmod p$.

Editorial Comment. The article by H. W. Becker on this subject (*Math. Mag.* 1955, 297–298) seems to be erroneous. L. L. Foster and W. M. Blundon called attention to Dickson's *History* [vol. 2, p. 736, line (-6)], and to F. Niedermeier, *J. Reine Angew. Math.*, 185 (1943), 111–112, and an earlier article of Kummer (1837).

Solution by Barry Powell, Kirkland, Washington. A more general result is: Let p be a prime, $p\not\equiv 1 \pmod 8$. Then the equation
$$x^{2p}+y^{2p}=z^{p} \tag{*}$$
has no solution in positive integers x,y,z with $p\nmid xyz$.

Proof. The case $p=2$ is well known. Let p be odd. Suppose x,y,z have no common factor. Consideration of (*) mod 8 shows that (i) $2\nmid z$. Hence $2|x$ or $2|y$. Without loss of generality, assume (ii) $2|y$. From the relation (iii) $y^{2p}=z^p-x^{2p}=(z-x^2)\theta$, $\theta=\sum_{i=1}^{p}z^{p-i}(x^2)^{i-1}$, it can be shown that (iv) $(z-x^2,\theta)=1$, because by assumption, $(z-x^2,\theta)\neq p$. Thus there is an odd positive integer a such that (v) $\theta=a^2\equiv 1 \pmod 8$. Further, since $x^2\equiv 1 \pmod 8$, relations (ii), (iii) show that $z^p\equiv 1$ or $5 \pmod 8$. If $z\equiv 1 \pmod 8$, then $\theta=\sum_1^p\equiv p \pmod 8$, so that $p\equiv 1 \pmod 8$. If $z\equiv 5 \pmod 8$, then $\theta\equiv 3p-2 \pmod 8$, so that again, $p\equiv 1 \pmod 8$.

A similar argument gives the following related results: (a) The equation $(2x)^{2p}+y^p\equiv z^p$ has no solutions in positive integers x,y,z where $p\nmid x$ and $2<p\not\equiv 1 \pmod 4$. (b) The equation $x^{2p}+y^{2p}=5^m z^{2p}$ has no solutions in positive integers x,y,z if $p\equiv 3$ or $7 \pmod{20}$, m any positive integer, $2p\nmid m$.

Also solved by L. E. Mattics and the proposer.

ADVANCED PROBLEMS

Solutions of Advanced Problems should be sent to Professor Roger C. Lyndon, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109 (U.S.A.). To facilitate their consideration, solutions of Advanced Problems in this issue should be typed (in duplicate, with double spacing) and should be mailed before September 30, 1980. If acknowledgment is desired, include a self-addressed card.

6296. *Proposed by Melvin Hochster, University of Michigan.*

For what commutative rings with identity is it true that every square matrix is similar to its transpose?

6297*. *Proposed by Eric Chandler, North Carolina State University.*

By how much is a rope shortened when an overhand knot is tied in it?

The problem has two parts:

(a) Formulate the problem in precise mathematical language. The formulation should presumably be a reasonable approximation to physical reality under some hypothesis on the nature of the rope.

(b) Obtain a solution.

6298. *Proposed by J. L. Brenner, Palo Alto, California.*

If an arbitrary set of 19 lattice points (with integral coordinates) is given in euclidean 3-space, prove that some three have a centroid with integral coordinates. (This assertion is false if 19 is replaced by 18.)

6299. *Proposed by Roger L. Cooke, University of Vermont.*

(a) The functions

$$f(x)=\sum_{n=1}^{\infty} n^{-1}\sin(2^{-n}x) \quad \text{and} \quad g(x)=\sum_{n=1}^{\infty}(-1)^n n^{-1}\sin(2^{-n}x)$$

are obviously uniformly continuous, real-analytic functions on the line. Is either function bounded?

(b) What can be said about $h(x)=\sum_{n=1}^{\infty}(-1)^n n^{-1}\cos(2^{-n}x)$?

SOLUTIONS OF ADVANCED PROBLEMS

Irrationality of an Infinite Product

6233 [1978, 686]. *Proposed by James Lynch and Jan Mycielski, University of Colorado.*

Prove that $\prod_{n=1}^{\infty}(1-a^{-n})$ is irrational for every integer a with $|a|>1$.

Editor's Note: All solutions except one were based ultimately on a formula of Euler. We give this one solution, together with a second solution accompanied by further remarks and references.

Solution I by Paul A. Vojta, graduate student, Harvard University. The nth partial product is

$$(1)+(-a^{-1})+(-a^{-2}+a^{-1}a^{-2})+\cdots+(-a^{-n}+a^{-1}a^{-n}+\cdots+(-1)^n a^{-1}a^{-2}\cdots a^{-n}).$$

This is just the 2^{n-1}th partial sum of an infinite series of terms of the form $\pm a^{-n_1}a^{-n_2}\cdots a^{-n_r}$. This latter series converges absolutely since the sum of the absolute values of its terms is $\prod_{n=1}^{\infty}(1+|a|^{-n})<\infty$. Hence the series can be rearranged. Write it as $\sum_{m=0}^{\infty}(-1)^m s_m$ where s_m is the (infinite) sum of all possible products of m distinct negative integral powers of a. But $s_m=\sum_{n=1}^{\infty}a^{-nm}s_{m-1}=s_{m-1}/(a^m-1)$. Hence, by induction, $s_m=1/(a-1)(a^2-1)\cdots(a^m-1)$. Thus, for $m>1$,

$$\left|\sum_{n=m}^{\infty}(-1)^n s_n\right|<s_m\sum_{n=m}^{\infty}(a^m-1)^{m-n}<2s_m.$$

Also, for $m>2$,

$$\left|\sum_{n=m}^{\infty}(-1)^n s_n\right|\geqslant s_m-\left|\sum_{n=m+1}^{\infty}(-1)^n s_n\right|>s_m-2s_{m+1}>0.$$

Now assume $\prod_{n=1}^{\infty}(1-a^{-n})$ is rational. Then there exists a positive integer q such that $q\sum_{n=0}^{\infty}(-1)^n s_n$ is an integer. Choose $m>2$ such that $a^{m+1}-1>2q$. Then $(q/s_m)\sum_{n=0}^{\infty}(-1)^n s_n$ is

an integer. If $n \leqslant m, (q/s_m)(-1)^n s_n$ is an integer; hence $(q/s_m)\Sigma_{n=m+1}^{\infty}(-1)^n s_n$ is an integer. But

$$0<\left|\frac{q}{s_m}\sum_{n=m+1}^{\infty}(-1)^n s_n\right|<2s_{m+1}\left(\frac{q}{s_m}\right);$$

$$0<\left|\frac{q}{s_m}\sum_{n=m+1}^{\infty}(-1)^n s_n\right|<\frac{2q}{a^{m+1}-1};$$

$$0<\left|\frac{q}{s_m}\sum_{n=m+1}^{\infty}(-1)^n s_n\right|<1.$$

This contradicts the fact that the above is an integer. Therefore the infinite product is irrational.

Solution II by Peter Bundschuh, University of Köln, West Germany.

By Euler's well-known identity we have

$$P(a):=\prod_{n=1}^{\infty}(1-a^{-n})=\sum_{n=-\infty}^{\infty}(-1)^n a^{-(3n^2+n)/2} \qquad (*)$$

for all complex a with $|a|>1$. Assume now that $P(a)$ is rational for a certain rational integer $a\neq 0,1,-1$, say, $P(a)=p/q$ with integers $p,q\neq 0$. Then we get from (*) for all $N\geqslant 0$

$$\frac{p}{q}-\sum_{|n|<N}(-1)^n a^{-(3n^2+n)/2}=\sum_{|n|>N}\cdots.$$

Multiplying this equation by $q\cdot a^{(3N^2+N)/2}$ we find that the numbers

$$A_N:=q\cdot a^{(3N^2+N)/2}\sum_{|n|>N}(-1)^n a^{-(3n^2+n)/2} \qquad (N=0,1,\ldots)$$

are rational integers satisfying $|A_N|<|q||a|^{-2N}$ for all $N\geqslant 0$. This can be easily seen by taking $n=N+t$ or $n=-N-t$, $t=1,2,\ldots$, and by using first the formula for the geometric series and then the fact $|a|\geqslant 2$. So we have $A_N=0$ for all $N\geqslant N_0$ which means $a^{-(3N^2+N)/2}+a^{-(3N^2-N)/2}=0$ for all $N>N_0$ and this is impossible.

Remark: The formula $(\Pi_{n=1}^{\infty}(1-a^{-n}))^{-1}=1+\Sigma_{n=1}^{\infty}p(n)a^{-n}$, where $p(n)$ denotes the number of partitions of n, shows that $\Sigma_{n=1}^{\infty}p(n)a^{-n}$ is irrational for the same values of a. Using analytic means one can also show that $P(a)$ is not a Liouville number, more precisely: For any real $\varepsilon>0$ the inequality $|P(a)-p/q|\leqslant|q|^{-(7/3+\varepsilon)}$ has at most finitely many solutions p/q with rational integers $p,q\neq 0$. This and similar results can be found in two papers of the solver: Arithmetische Untersuchungen unendlicher Produkte, Inventiones Math., 6 (1969) 275–295; Ein Satz über ganze Funktionen und Irrationalitätsaussagen, Inventiones Math., 9 (1970) 175–184.

Also solved by Paul Braden, Paul F. Byrd, L. L. Foster, M. J. Knight, L. Kuipers (Switzerland), O. P. Lossers (Netherlands), L. E. Mattics, E. Trost (Switzerland), Peter Ungar, Ken Yocum, and the proposers. Kuipers and the proposers gave references to the work of Bundschuh.

Verifying Associativity

6238* [1978, 770]. *Proposed by F. David Hammer, Santa Cruz, California*

To see if a binary operation on a set with n elements is associative, one might think it necessary to verify directly n^3 instances of the associative law. Often, however, for instance if the operation is commutative and has an identity, considerably fewer need be verified. Is there a set of n elements and an operation on them for which all n^3 verifications are necessary?

Solution by Henry Borenson, Stuyvesant High School, New York, N.Y. with his 12th year honor classes M82H, in particular Irwin Jungreis, Leonid Fridman, and Chris Slawinski. It will be shown that at most n^3-n verifications are needed. Precisely, it is shown that there is a set S of n

triples abc such that, if the operation is associative on all triples not in S, it is associative on S as well. If $xx=x$ for all x, then $(xx)x=xx=x(xx)$ for all x, and we take S to be the set of triples xxx. If $xx\neq x$ for some x, invoke Fridman's Lemma: *If the triples abc, $(ab)cd$, $ab(cd)$, and bcd are associative, then the triple $a(bc)d$ is associative.* To prove this, using the hypotheses in order, we find that $(a(bc))d=((ab)c)d=(ab)(cd)=a(b(cd))=a((bc)d)$. Now, for $xx\neq x$, take $a=b=c=x$. Then $a(bc)d=x(xx)d$ is distinct from abc, $(ab)cd$, $ab(cd)$, and bcd. Thus we may take S to be the set of all triples xxd for arbitrary d.

Also solved by Duane M. Broline and Norman G. Gunderson. Borenson indicates that the result given is probably not best possible. Possibly there exist generalizations of Fridman's Lemma.

For further discussion of this problem see the following:

1. G. Szász, Die Unabhängigkeit der Assoziativitätsbedingungen, Acta. Sci. Math. Szeged, 15 (1953) 20–28.

2. ______, Über die Unabhängigkeit der Assoziativitätsbedingungen kommutativer multiplikativer Strukture, Acta. Sci. Math. Szeged, 15 (1954) 130–142.

3. D. Tamari, The associativity problem for monoids and the word problem for semigroups and groups, in Word Problems, North-Holland, 1973, pp. 591–607.

The paper by Tamari contains an extensive bibliography.

Growth of $x^y - y^x$

6239 [1978, 770]. *Proposed by F. David Hammer, Santa Cruz, California*

Is the following conjecture true? Let $p(x,y)$ be any polynomial in x and y; then $|x^y-y^x|\leqslant |p(x,y)|$ has only finitely many solutions (x,y) in unequal integers $\geqslant 2$.

Solution by Robert Breusch, Amherst College. If $1<y<x$ and p has degree m, then $|p(x,y)|\leqslant Ax^m$ for A the sum of the absolute values of the coefficients. We show that $|y^x-x^y|\leqslant Ax^m$ for only finitely many integers x, and $2\leqslant y\leqslant x-1$. If $g(y)=y^x x^{-y}$, then $g'(y)=0$ only for $y=x/\log x$, whence the minimum of $g(y)$ on the interval $2\leqslant y\leqslant x-1$ is either $g(2)$ or $g(x-1)$. If $x\geqslant 7$, then $g(2),g(x-1)>2$. Thus $y^x>2x^y$ and $|y^x-x^y|>\frac{1}{2}y^x\geqslant 2^{x-1}$. But $2^{x-1}>Ax^m$ for all sufficiently large x.

Also solved by Boris Datskovsky (Columbia University Problem Group), T. E. Elsner, Daniel Goldstein, Gustaf Gripenberg (Finland), O. P. Lossers (Netherlands), and Adam Riese.

Approximation by Terms of a Null Sequence

6240 [1978, 828]. *Proposed by Mihai Eşanu, Bucharest University, Romania*

Let $a_n\neq 0, \lim_{n\to\infty} a_n=0$. Prove that for every real number x there exist sequences $(\lambda_n),(\mu_n)$ of integers such that

$$x=\sum_{n=1}^{\infty}\lambda_n a_n=\prod_{n=1}^{\infty}\mu_n a_n.$$

Solution by Le Baron O. Ferguson, University of California, Riverside. Because we are free to change the signs of the integers at will, we can assume without loss of generality that $x>0$ and $a_n>0$ for all n. For the case of the sum, we simply choose, by induction, integers λ_n such that

$$\lambda_n a_n\leqslant x-\sum_{j=1}^{n-1}\lambda_j a_j<(\lambda_n+1)a_n \qquad n=1,2,\dots .$$

Then

$$0 \leqslant x - \sum_{j=1}^{n} \lambda_j a_j < a_n$$

and convergence follows from the hypothesis $a_n \to 0$. For the case of the product, first note that we could first divide x by a large integer and also pass to a subsequence of the a_i's; hence, without loss of generality, we assume that

$$1 > x > a_n \qquad n = 1, 2, \ldots.$$

Then there exists a positive integer μ_1 such that

$$\mu_1 a_1 \leqslant x \leqslant (\mu_1 + 1) a_1;$$

hence

$$1 \leqslant \frac{x}{\mu_1 a_1} < 1 + \frac{1}{\mu_1}.$$

We proceed by induction and pick positive integral μ_n's such that

$$\mu_n a_n \leqslant \frac{x}{\prod_{j=1}^{n-1} \mu_j a_j} < (\mu_n + 1) a_n \qquad n = 1, 2, \ldots \tag{1}$$

whence

$$1 \leqslant \frac{x}{\prod_{j=1}^{n} \mu_j a_j} \leqslant 1 + \frac{1}{\mu_n} \qquad n = 1, 2, \ldots. \tag{2}$$

It is possible to do this at each step in view of the left inequality in (2). From this same inequality and the right-hand inequality in (1), we have

$$(\mu_n + 1) a_n > 1$$

which forces $\mu_n \to \infty$. This together with (2) shows that the product converges to x.

Also solved by Kenneth Alexander, K. F. Andersen, J. M. Ash, John A. Baker, Robert C. Carson, F. S. Cater, L. E. Clarke (England), Michael W. Ecker, Leon Gerber, C. T. Giel, L.-S. Hahn, Steven Hubbard, Eli L. Isaacson, Thomas Jager, O. P. Lossers (Netherlands), Russell Lyons, J. G. Mauldon, Zane C. Motteler, Uri N. Peled, Steven Ricci, Adam Riese, Joseph Silverman, Alan H. Stein, University of South Alabama Problem Group, David Weisser, David Witte, and the proposer.

Equality of Measures

6242 [1978, 828]. *Proposed by Jan Mycielski, University of Colorado*

Let I be the interval $[0,1]$, λ the Lebesgue measure in I and μ a Borel measure in I. Suppose that $\lambda(A) = \frac{1}{2}$ implies $\mu(A) = \frac{1}{2}$ for every Borel set $A \subseteq I$. Prove the $\mu(B) = \lambda(B)$ for every Borel set $B \subseteq I$.

Editor's note: The problem seems to have three solutions. We print an example of each.

Solution I by Ethan Bolker, University of Massachusetts, Boston. Consider the vector valued measure (λ, μ) mapping Borel sets in I to the $x-y$ plane. It is atom free since λ is, hence its range is convex (Liapounoff's theorem). That range R contains $(0,0)$ and $(1, \mu(1))$ so it meets both open half planes determined by the line $x = 1/2$. But the hypothesis says R meets that line at the single point $(1/2, 1/2)$, so R must lie entirely on the line $y = x$. That is, for all B, $\lambda(B) = \mu(B)$. The hypothesis can be weakened: if for any α with $0 < \alpha < 1$ there is a β such that $\lambda(A) = \alpha$ implies $\mu(A) = \beta$ then $\mu = (\beta/\alpha)\lambda$.

The last result stated is Theorem 1.22 of [1]. The problem posed is Corollary 1.23.

References

1. E. D. Bolker, Functions resembling quotients of measures, Trans. Amer. Math. Soc., 124 (1966) 292–312.
2. P. R. Halmos, The range of a vector measure, Bull. Amer. Math. Soc., 54 (1948) 416–421.

Solution II by F. S. Cater, Portland State University. Let $\lambda E=0$. Let A be a set disjoint from E with $\lambda A=\frac{1}{2}$; then $\lambda(E\cup A)=\frac{1}{2}$ and

$$\tfrac{1}{2}=\mu(E\cup A)=\mu E+\mu A=\mu E+\tfrac{1}{2}$$

and $\mu E=0$. So μ is absolutely continuous with respect to λ. By Radon-Nikodým, there is a Lebesgue measurable function f on $[0,1]$ such that $\mu A=\int_A f$ for any Borel set A. Let $B=\{x: f(x)\geqslant 1\}$ and $C=\{x: f(x)<1\}$. Then $\lambda C<\frac{1}{2}$; otherwise any subset $A\subset C$ with $\lambda A=\frac{1}{2}$ provides $\mu A=\int_A f<\frac{1}{2}$ which is impossible. So $\lambda B\geqslant\frac{1}{2}$. Since $\mu A=\int_A f=\frac{1}{2}$ for any subset $A\subset B$ with $\lambda A=\frac{1}{2}$, it follows that $f=1$ a.e. on B and $f\leqslant 1$ a.e. on $[0,1]$. Since $\frac{1}{2}=\int_A f$ for any set A with $\lambda A=\frac{1}{2}$, it follows that $f=1$ a.e. on $[0,1]$. Finally,

$$\mu S=\int_S f=\lambda S$$

for any Borel set $S\subset[0,1]$.

Solution III by Ray C. Shiflett, California State University, Fullerton. If $b-a=\frac{1}{2}$, then any interval in I with endpoints a and b has equal λ and μ measure. Let $A_1,\ldots,A_{2^n}$ be the diadic intervals in I with length $1/2^n$. The union of 2^{n-1} of these intervals has length $\frac{1}{2}$. Choose two collections of 2^{n-1} of these diadic intervals so that exactly $2^{n-1}-1$ of the intervals are the same in both collections. It follows that the two intervals which are different have equal μ measure. Then all the diadic intervals of length $1/2^n$ have equal μ measure and that measure must be $1/2^n$. This shows that λ and μ are equal on a collection of sets which generates the σ-algebra of Borel sets and must therefore be equal on all the Borel sets.

Also solved by Kenneth Alexander, Alan Armstrong, J. M. Ash, C. W. Austin, John A. Baker, D. W. Brown, A. Brunnschweiler (Switzerland), Barry W. Brunson, L. E. Clarke (England), Robert D. Foley, Thomas H. Foregger, Daryl George, M. B. Gregory, James Hagler, Eli L. Isaacson, Anzelm Iwanik, David Jakel, Arnold Janssen (West Germany), Ole Jørshoe (Denmark), Ivan Netuka (Czechoslovakia), Gene Ortner, D. A. Overdijk (Netherlands), Michael Skalsky, W. Taylor, Eric K. van Douwen, Stanley Wagon, J. G. Wendel, David Witte, and the proposer.

Several solvers commented that (1) the countable additivity is not needed for μ; finite additivity is enough; and (2) $\frac{1}{2}$ can be replaced by any c, $0<c<1$.

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR., AND LYNN A. STEEN
with the assistance of the mathematics departments of St. Olaf, Carleton, and Macalester Colleges
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We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

An Introduction to Mathematical Models in the Social and Life Sciences. By Michael Olinick. Addison-Wesley, Reading, Mass., 1978. xiii+466 pp. $18.95. (Telegraphic Review, June-July 1978.)

An Introduction to Mathematical Modeling. By Edward A. Bender. John Wiley & Sons, New York, 1978. x+256 pp. $16.95. (Telegraphic Review, August-September 1978.)

Both books are useful additions to the literature on mathematical modeling. They will be useful for undergraduate and graduate courses, although at different levels. Olinick states in his preface that "the aim of this book is to encourage the teaching of mathematical model building early in the undergraduate program," whereas Bender writes that his book "is intended . . . for use primarily at an upper division or beginning graduate level." Both authors have succeeded.

What is the purpose of a course on mathematical models? Is it to teach a body of material on mathematical models that have been developed in the past, or to teach mathematical technique, or to teach the process of building a mathematical model? Of course, all of these are part of such a course, but which is most emphasized can be critical to the form of the course. The primary purpose might be better phrased in terms of what the student gets out of such a course: it should prepare the student to develop models for future applications and enable the student to understand and appreciate models developed by others. Consequently, there is no body of material on specific models which the student must master. However, a course on modeling can succeed only if the student is presented with a range of problems which are realistic, useful, and interesting. Models which have been developed and used for real applications are necessary. Neither simplistic "toy" models nor the latest and most complex models are appropriate. An application or model used for a first course on modeling should be carefully selected to provide a clear application of a particular mathematical technique with a minimum of prerequisite specialized knowledge and without sacrificing realism.

Both Olinick and Bender provide a diversity of such applications. Bender, particularly, has a large selection of different problems which are useful for a course. He includes an extensive bibliography of the original sources for his models. Olinick does not include as many models as Bender, but each one is explained in more detail. This enables him to include historical notes about the models and the scientists and mathematicians who developed them. These notes will help the student appreciate the significance and usefulness of the techniques. Olinick also gives references to the original work from which he takes his models. In addition, he includes a useful listing of books on applications of mathematics to a variety of social and life sciences. Both books can be useful supplementary sources for courses on mathematical modeling and will be invaluable sources of ideas for those who teach these courses.

A course on mathematical modeling must achieve a balance between the mathematical techniques which are taught and the modeling process itself. Both books start by assuming at least a year of calculus as a prerequisite. They both have an introductory chapter on the ideas of mathematical modeling. This chapter serves as an appropriate beginning. After that point, however, the authors diverge.

Olinick combines a detailed treatment of mathematical technique with a thorough presentation of the modeling process for each topic. This necessarily means that he covers relatively few models in depth. In order to maintain a balance among different mathematical techniques, Olinick divides the chapters in his book into three broad categories: deterministic models, axiomatic models, probabilistic models. This is a very useful way to structure a course on mathematical modeling, as well as a text. It suggests to the student how mathematical techniques which are not part of the course may fit in. Within this broad structure, Olinick treats one problem in each chapter and has series of chapters which develop coherently. For example, the sequence of Chapter 2, "Stable and Unstable Arms Races," Chapter 3, "Ecological Models: Single Species," and Chapter 4, "Ecological Models: Interacting Species," leads from fundamen-

tal ideas about deterministic models using differential equations through single differential equations to qualitative analysis of systems of differential equations. This sequence can be used as a three-to-four-week module for a course. The sequence of Chapter 10, "Markov Processes," Chapter 11, "Two Models of Cultural Stability," and Chapter 12, "Paired-Associate Learning," is another sequence developing applications of Markov chains. Chapter 5, on linear programming, does not fit this pattern and seems somewhat out of place. It does not develop models for a particular problem but discusses the fundamental ideas of a single technique. Its appearance at this point looks like a compromise by Olinick with the suggestions of a publisher as to what topics should be included in a particular kind of book. Certainly the structure and development of this book would be improved if Chapter 5 were left out. In contrast, Chapter 13, on epidemics, breaks with the overall pattern in a way which adds to the balance of the book. By bringing together both a deterministic approach using differential equations and a probabilistic approach, this chapter serves to unite two of the main themes of the book in an appropriate conclusion. It might have been better placed after Chapter 14, on simulation methods, especially if a simulation approach to the study of epidemics could have been included. Overall, the structure of Olinick's book makes it particularly appropriate for an introductory course on mathematical modeling.

Bender has not put so much structure into his book. He does divide it into two sections: "Elementary Methods" and "More Advanced Methods." Even in the section on elementary methods, however, the approach is one which demands a high level of mathematical sophistication. Bender's chapters are each organized around one type of mathematical technique, but each contains a number of different applications: Chapter 3, "Graphical Methods," contains sections on "the nuclear missile arms race," "biogeography," "diversity of species on islands," "theory of the firm," "cobweb models in economics," and "small-group dynamics." Chapter 9, "Local Stability Theory," includes "frictional damping of a pendulum," "species interaction and population size" and "Keynesian economics." This approach enables Bender to treat only briefly the main ideas of each model. The reader must be able to develop the mathematical details in order to understand these topics. Bender is really an "Introduction to Mathematical Modeling" only for someone who is already comfortable with the mathematical tools. A small advanced group of undergraduates or a group of graduate students would be an appropriate audience for Bender's approach. Since even the first part of the book will demand a high level of sophistication from the reader, Bender's division of the book into two parts loses its significance. The organization would be better if the corresponding chapters in the first and second parts of the book were either combined or put into a coherent sequence. For example, the two chapters cited above, Chapters 3 and 9, could be coordinated. Indeed, we see in Chapter 3 of Bender a section on the arms race which bears some similarities to the single topic of Chapter 2 in Olinick, and in Chapter 9 of Bender the section "species interaction" is exactly the topic treated in Olinick's Chapter 4. Bender has not knit his book into a cohesive whole.

The absence of structure in Bender's book is evident at the broader level of overall balance as well. This book has a preponderance of examples using continuous models. Much of Chapter 2, "Arguments from Scale"; Chapter 3, "Graphical Methods"; much of Chapter 4, "Basic Optimization"; and Chapters 7, 8, and 9 on differential equations demonstrate this imbalance. Whereas Olinick has three chapters which treat axiomatic methods, including all of Chapter 6, on the problem of social choice and Arrow's impossibility theorem, Bender devotes only three pages to axiomatic methods—a brief presentation of Arrow's theorem in Chapter 6, "Potpourri." Bender has two chapters and an appendix on probability methods, while Olinick develops a succession of five chapters primarily using such methods. In addition, with three chapters about Markov chains and one on linear programming, Olinick has a reasonable portion of his book treating discrete rather than continuous methods. Olinick achieves some overall balance among the mathematical techniques he presents. Bender's reader is warned in the preface that he has "selected the models, [and] they reflect [his] interests and knowledge . . . *caveat emptor*."

Both Bender and Olinick are excellent expositors. Olinick clearly explains the details of each model, leading the reader through the mathematical analysis. In addition, he discusses the way in which a model is developed, evaluated, modified, and developed further. His detailed examples provide excellent illustrations of the modeling process. His problems and models are accurately explained, except for one example of an application of Markov chains. In Example 3 of Chapter 10, Olinick describes the model of the flow of faculty through a college tenure system. In the model the professor once promoted without receiving tenure must next be promoted to tenure or not at all. However, in the Markov chain model he uses to describe this situation, it is possible to cycle between the states of "retained in untenured rank" and "promoted without tenure." The difficulty clearly arises because the description of states is confused with a description of the transitions. This example, with the error, is continued (p. 295) where calculations for specific probabilities are made. With the exception of this error, Olinick has written a very clear, accurate, and useful exposition.

Bender's exposition is characterized by clear, concise descriptions of problems and the models used to solve them. The reader must develop the details. Bender's fast-paced style and his use of many applications in each chapter are appropriate for challenging the students in an advanced undergraduate seminar or a graduate class to develop and explicate each model in more detail.

The problems included in each book reinforce their other aspects. Bender's problems are challenging questions which involve considerable extension of the models and examples which he has presented in the text or, more often, new problems for which the techniques in the chapter are appropriate. The ability of the student to understand the underlying mathematical techniques must be assumed when presenting problems of this sort. Some problems amount to what would be fully developed examples in Olinick's book. The tenure model mentioned above appears in a slightly different form (avoiding explicit use of Markov chains) as Problem 4 in Bender's Chapter 5.

Olinick provides exercises appropriate for students who will need practice with the mathematical techniques involved in each chapter. Chapter 4 includes, for example, exercises on finding the linearized Taylor series expansion for particular functions. Some of the exercises develop the fine detail of the mathematical technique used in the chapters, others apply the technique to variations of the problems presented. In addition to the exercises, Olinick includes a short list of "suggested projects." This is at the point where the student is invited to extend what was learned in the chapter to other applications or major extensions of those already covered. Olinick's projects are at the level of Bender's problems. One can find examples, such as the development of a model of symbiosis, Problem 3 in Chapter 9 of Bender and Project 6 of Chapter 4 of Olinick, where the same topics appear as problems for Bender and projects for Olinick. This again demonstrates the different levels of these two texts.

We can summarize the comparison of these two useful texts by looking back at the classic text in this field, by Kemeny and Snell [1]. Olinick, who owes a considerable and acknowledged debt to Kemeny and Snell, has written a book which develops mathematical models in a spirit very close to theirs, but with the more detailed explanation and more gradual development necessary to make his text accessible to students earlier in their college careers. Bender, on the other hand, has written a text which is more brief and concise about each model than Kemeny and Snell. This enables him to provide the wealth of different examples which are a rich field for exploration by a mathematically sophisticated student. Instructors of courses on mathematical modeling should consult both texts for ideas and would find each a very good text at the appropriate level.

Reference

1. John Kemeny and J. Laurie Snell, Mathematical Models in the Social Sciences. Boston, Ginn, 1962.

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TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook — P = professional reading
S = supplementary reading — L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, P*, *Mathematicians in Academia: 1975-2000*. Charlotte V. Kuh, Roy Radner. CBMS, 1980, v + 109 pp, $3 (P). A detailed demographic and econometric projection of mathematics faculty in four year colleges and universities. Among the conclusions: new hiring in 1991-92 may be only 8-24% of what it is currently; the median age of tenured faculty will rise from 43 in 1975 to over 57 in 2000; the number of mathematics faculty granted tenure in 1990-1995 will be about one-fourth of the number receiving tenure in 1970-1975. Careful caveats and numerous appendices permit skeptics to analyse the model and its assumptions, as well as its conclusions. LAS

GENERAL, S(13-18), P, L, *Advances in Fuzzy Set Theory and Applications*. Ed: Madan M. Gupta. North-Holland, 1979, xv + 753 pp, $73.25. [ISBN: 0-444-85372-3] Attempts to present "a coherent review of this growing field." Papers on theory and on applications in mathematics, engineering, social sciences, statistics, and medicine. Extensive bibliography. Note the high price. FLW

GENERAL, S*(13-15), L, *One Hundred Problems in Elementary Mathematics*. Hugo Steinhaus. Dover, 1979, 174 pp, $2.75 (P). [ISBN: 0-486-23875-X] An unaltered republication of a classic first published in English in 1964 by Basic Books. The problems are analogous in style and difficulty to the "elementary" problems in this *Monthly*. Complete solutions are provided. LAS

GENERAL,? *Condensation of Decimal and Metrical Systems: Jumelex Method*. Marius F. Gagnadre. Vantage Pr, 1980, 40 pp, $5.95. The author sees great advantage in his system of numeration which replaces pairs of adjacent digits (in the ordinary decimal system) with a single digit and attached dash (the position and direction of the dash indicates the other digit). LCL

GENERAL, P, *Transactions of the Twenty-Fifth Conference of Army Mathematicians*. US Army Research, Durham, NC, 1979, xxi + 792 pp, (P). Proceedings of a June 1979 conference held at The Johns Hopkins University. LAS

GENERAL, *Fundamentals of Mathematics and Statistics for Students of Chemistry and Allied Subjects*. C.J. Brookes, I.G. Betteley, S.M. Loxston. Wiley, 1979, vii + 496 pp, $20.25 (P). [ISBN: 0-471-99732-3] A cursory treatment of calculus, Fourier series, groups, matrices, probability, and statistics. The examples are primarily from chemistry. FLW

BASIC, T(13), *Intermediate Algebra for College Students, Fifth Edition*. Thurman S. Peterson, Charles R. Hobby. Har-Row, 1980, xii + 420 pp, $14.95. [ISBN: 0-06-045184-X] Undistinguished text for students with only ninth grade algebra background. This edition eliminates most theory, concentrates on explanation by example, and adds material on inequalities. (*Fourth Edition*, TR, March 1975.) MW

PRECALCULUS, T(13), L, *Precalculus Mathematics: A Study of Functions*. George W. Polites. Har-Row, 1980, xii + 497 pp, $15.95. [ISBN: 0-06-045259-5] Begins at a faster pace than many books at this level; Chapter 1 includes discussion of complex numbers, quadratic formula. Develops ideas for functions in general, then fairly standard applications to exponential, logarithmic functions and trigonometry (angles before circles). Concludes with conic sections. Exercises, answers, tables, index. JS

EDUCATION, S, L, *Cryptarithms*. Josephine Andree, Richard V. Andree. Crypto Project (Room 423, 601 Elm, Norman, OK 73019), 1978, 178 pp, *Instructor's Manual*, 82 pp, $4.50 set (P); *Logic Unlocks*, 1979 90 pp, *Teacher's Handbook*, 34 pp, $4.50 set (P); *Secret Ciphers*, 1979, 137 pp, *Teacher's Handbook*, $3.50 set (P); *Solving Ciphers*, 1979, *Instructor's Manual*, 46 pp, $3.50 set (P); *Sophisticated Ciphers*, 1978, 149 pp, *Instructor's Manual*, 21 pp, $3.50 set (P). Five-book series on problem solving and logical thinking aimed at students in grades 6 and up. Each book is a mini-course intended to provide "the joy of successfully solving difficult problems by use of logical thinking." In casual cartoon format, *Secret Ciphers* and *Solving Ciphers* provide an introduction to ciphers and their solution. *Sophisticated Ciphers* includes important new breakthroughs made in the past ten years. Contents of *Logic Unlocks* range from riddles and well-known games such as Hex, Mill and Reversi to problems posed as dilemmas in human relations and contributed non-trivial research problems. Hints and answers are provided where appropriate. *Cryptarithms* contains over 200 challenges and a carefully written description of the reasoning used to solve them. Each book is accompanied by a handbook for teachers. Innovative. JK

HISTORY, S(17-18), P, *Wolfgang Pauli*. Ed: A. Hermann, K. v. Meyenn, V.F. Weisskopf. Springer-Verlag, 1979, xlvii + 577 pp, $80. [ISBN: 0-387-08962-4] First volume of a scholarly edition of Pauli's scientific correspondence, from 1919 to 1929, with most of the celebrated theoretical physicists of the time. Will fascinate those interested in the development of theoretical physics who read German. JD-B

FOUNDATIONS, T(16-17: 1, 2), S, *Modal Logic, An Introduction*. Brian F. Chellas. Cambridge U Pr, 1980, xii + 295 pp, $14.95 (P); $42.50. [ISBN: 0-521-29515-7; 0-521-22476-4] Basic text in the logic of necessity and possibility intended for readers with a background of one course in formal logic. Three parts: Introduction and preliminaries, standard models and normal systems, minimal models and classical modal logics. Excellent set of exercises follow each section. LCL

FOUNDATIONS, S(15-16), *Grundbegriffe der Mathematik.* Edmund Hlawka, Christa Binder, Peter Schmitt. Prugg Verlag Wien, 1979, 195 pp, (P). [ISBN: 3-85385-038-3] A brief, modern and sophisticated treatment of some basic concepts of mathematics. Chapters on sets and functions, algebraic systems, order relations, convergence, axiom of choice. Assumes little previous knowledge but gets to categories, Hausdorff spaces, Moore-Smith convergence, among other things. JD-B

COMBINATORICS, P, L*, *The Four-Color Problem: Assaults and Conquest.* Thomas L. Saaty, Paul C. Kainen. McGraw, 1977, ix + 217 pp, $22. [ISBN: 0-07-054382-8] A book-length treatment of Saaty's award-winning 1972 *Monthly* paper "13 Colorful Variations on Guthrie's Four Color Conjecture," supplemented by extensive discussion of Appel and Haken's 1976 computer-aided solution. Contains an extensive collection of theorems and problems relating graph theory to various parts of mathematics, and a comprehensive bibliography. LAS

COMBINATORICS, S(14), L, *Excursions in Graph Theory.* Gary Haggard. U of Maine at Orono, 1980, 211 pp, $7.95 (P). [ISBN: 0-89101-040-8] An introduction to graphs by looking at a few of the classical problems, e.g., Euler graphs, Ringel's Problem, Hamiltonian Cycles, Steiner Triples, planar graphs and chromatic polynomials. The author's goal is to give non-mathematicians the opportunity to be creatively involved in graph theory. No exercises. CEC

COMBINATORICS, S(17), P, *Graph Theory and Combinatorics.* R.J. Wilson. Pitman, 1979, 148 pp, $15.95 (P). [ISBN: 0-273-08435-6] The proceedings of a conference in Combinatorics and Graph Theory held at The Open University, England on May 12, 1978. Includes ten papers. CEC

COMBINATORICS, T(16-17: 1, 2), S, P, L*, *Graph Algorithms.* Shimon Even. Computer Sci Pr, 1979, ix + 249 pp, $17.95. [ISBN: 0-914894-21-8] An algorithmic approach to graph theory. Intended for computer science students. Develops efficient algorithms and the appropriate theory for paths, spanning trees, Huffman trees, max flow, depth first search and planarity. Discusses NP-completeness and presents the main results on NP-complete graph problems. RWN

NUMBER THEORY, S(18), P, *Analytic and Combinatorial Generalizations of the Rogers-Ramanujan Identities.* David M. Bressoud. Memoirs No. 227. AMS, 1980, 54 pp, $6 (P). [ISBN: 0-8218-2227-6] In this monograph two very general theorems are given. The first is an analytic statement which contains many known analytic generalizations of the Rogers-Ramanujan Identities. The second is a combinatorial statement which contains as special cases many of the combinatorial generalizations. The connection between the two results is also discussed. CEC

NUMBER THEORY, T*(16: 1), S, L*, *Algebraic Number Theory.* Ian Stewart, David Tall. Chapman and Hall, 1979, xviii + 257 pp, $19.95. [ISBN: 0-470-26660-0] An introduction to the subject which assumes the reader has had some previous exposure to rings and fields. Culminates with proofs of Kummer's Theorem and Dirichlet's Units Theorem. Includes lots of historical remarks and exercises. Very readable. CEC

NUMBER THEORY, T(14-15: 1), *Elemente der Zahlentheorie.* H. Freund. Teubner, Stuttgart, 1979, 118 pp, (P). [ISBN: 3-519-02707-0] Elementary topics in number theory (divisibility, number systems, congruences) for teaching candidates. Emphasis is on topics relevant to school mathematics. Each chapter (there are only three) begins with a series of concrete observations and experiments which lead the student to formulate conjectures. These serve as motivation and models for the definitions and theorems which follow. Chapters close with a brief discussion of pedagogical applications. Exercises scattered throughout, with selected solutions in back. GHM

LINEAR ALGEBRA, T(14: 1), *Matrix Theory and Applications for Engineers and Mathematicians.* Alexander Graham. Ellis Horwood, 1979, 295 pp, $47.50. [ISBN: 0-85312-129-X] Good strong, middle-of-the-road linear algebra text. Last chapter deals with inverting matrices and includes the Vandermonde matrix and Faddeeva's Method. LLK

LINEAR ALGEBRA, T(13-14: 1), *Elementary Linear Algebra, Third Edition.* Paul C. Shields. Worth, 1980, x + 510 pp, $16.95. [ISBN: 0-87901-121-1] Linear transformations introduced very late. Good applications throughout. (*First Edition*, TR, August-September 1968; *Second Edition,* TR, March 1974.) LLK

ALGEBRA, T(16-17: 1), *Matrix Groups.* Morton L. Curtis. Springer-Verlag, 1979, xii + 191 pp, $12 (P). [ISBN: 0-387-90462-X] Introduces students to concepts of Lie group theory using a concrete approach based on matrix groups. Assumes an introductory course in linear algebra and some knowledge of differentiation of vector-valued functions. Twelve chapters, each with problem sets. AWR

ALGEBRA, S(18), P, *Lecture Notes in Mathematics-766: Transformation Groups and Representation Theory.* Tammo tom Dieck. Springer-Verlag, 1979, viii + 308 pp, $18 (P). [ISBN: 0-387-09720-1] Extended lecture notes, aimed at a select audience, assuming a substantial background in topology and representation theory for transformation groups. Uses Burnside ring as a recurrent theme, discusses such topics as compact Lie groups, equivariant homology, cohomology, homotopy. Bibliography, no index. JS

ALGEBRA, T(18: 1), S, P, L, *Symmetric Functions and Hall Polynomials.* I.G. Macdonald. Clarendon Pr, 1979, viii + 180 pp, $34.95. [ISBN: 0-19-853530-9] Contemporary, generalized development of Hall polynomials and algebras and the relation to Hall-Littlewood symmetric functions. Applications are made to the general linear group over finite fields and over non-archimedean local fields. Bibliography, index. JS

ALGEBRA, P, *Homological Group Theory.* Ed: C.T.C. Wall. London Math. Soc. Lect. Note Ser., No. 36. Cambridge U Pr, 1979, ix + 394 pp, $34.50 (P). [ISBN: 0-521-22729-1] Proceedings of the symposium held at Durham, England in September 1977. JAS

ALGEBRA, P. *Algebraic Theory of Semigroups.* Ed: G. Pollák. North-Holland, 1979, 753 pp, $117. [ISBN: 0-444-85282-4] Proceedings of the conference held in Szeged, Hungary, August 23-26, 1976. Contains a list of research problems and a few papers which were not able to be presented at the conference. JAS

ALGEBRA, P. *Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups.* A. Borel, N. Wallach. Annals of Math. Stud., No. 94. Princeton U Pr, 1980, xvii + 387 pp, $22.50; $10 (P). A treatment of the two types of cohomology spaces pertaining to a reductive Lie group which grew out of a seminar on the cohomology of discrete subgroups of semi-simple Lie groups at the Institute for Advanced Study in 1976-77. JAS

ALGEBRA, S(13), P, L. *The Finite Simple Groups and Their Classification.* Michael Aschbacher. Yale U Pr, 1980, ix + 61 pp, $6.95 (P). [ISBN: 0-300-02449-5] A brief description, for mathematicians who are not experts in group theory, of the current state of research concerning the classification of the finite simple groups. Based on four Whittemore Lectures given at Yale in September 1978. LAS

ALGEBRA, P. *Lecture Notes in Mathematics-745: Equational Compactness in Rings, With Applications to the Theory of Topological Rings.* David K. Haley. Springer-Verlag, 1979, iii + 167 pp, $9.80 (P). [ISBN: 0-387-09548-9] A conspectus of equational compactness in rings; the focus is on the importance of the model-theoretic setting of equational compactness (via the ultraproduct construction) for obtaining results of a purely algebraic or topological nature. LCL

ALGEBRA, P. *Ring Theory: Proceedings of the 1978 Antwerp Conference.* Ed: F. Van Oystaeyen. Lect. Notes in Pure and Appl. Math., V. 51. Dekker, 1979, 801 pp, $45 (P). [ISBN: 0-8247-6854-X] Up-to-date results in PI theory, fundamental papers on representation of artinian algebras over fields, papers concerning localization theory and module theory, by leading ring theorists from around the world. LCL

ALGEBRA, P. *Lecture Notes in Mathematics-746: τ-Rings and Wreath Product Representations.* Peter Hoffman. Springer-Verlag, 1979, v + 148 pp, $9.80 (P). [ISBN: 0-387-09551-9] A formal treatment of τ-rings which is entirely algebraic. τ-rings are an axiomatization of the tensor power construction used by Atiyah to construct K-theory operations, and are mathematically equivalent to the more familiar λ-rings. LCL

ALGEBRA, P. *Lecture Notes in Mathematics-717: Bidualräume und Vervollständigungen von Banachmoduln.* Michael Grosser. Springer-Verlag, 1979, 209 pp, $13.80 (P). [ISBN: 0-387-09257-9] Systematic study of second dual spaces and of certain topological and algebraic completions of Banach modules and Banach algebras. GHM

ALGEBRA, S(18), P. *Lecture Notes in Mathematics-744: Integral Representations.* Irving Reiner, Klaus W. Roggenkamp. Springer-Verlag, 1979, 275 pp, $14.30 (P). [ISBN: 0-387-09546-2] These notes arose from a series of lectures delivered by the authors at the Fourth School of Algebra, São Paulo, July 1976. The first half is entitled "Topics in integral representation theory" by the first author and the second half is called "Integral representations and presentations of finite groups" written by the second author. Includes lists of references and indexes. CEC

FINITE MATHEMATICS, T(13: 1). *Finite Mathematics for Business, Social Sciences, and Liberal Arts.* Louis M. Rotando. D. van Nostrand, 1980, viii + 519 pp, $16.95. [ISBN: 0-442-25760-0] A good balance of topics with a variety of applications. LLK

CALCULUS, T**(13-14: 3, 4). *Calculus.* Jerrold Marsden, Alan Weinstein. Benjamin/Cummings, 1980, xxvii + 1012 pp, $26.95. [ISBN: 0-8053-6930-9] One and several-variable calculus through the divergence theorem including a mid-text chapter on differential equations. Attractive. Impressive in several ways. Page size is 8 1/2" by 11" and weight is over five pounds. Student oriented. Flexible. Worked examples and solved exercises. Applications used to introduce new concepts. Exceptionally fresh applied problems. Striking computer-generated graphics. Numerous exercises, many challenging, several calculator-slanted. Answers to odd-numbered exercises. Prerequisite material in unnumbered initial chapter. Should be a winner. JK

COMPLEX ANALYSIS, T(16-18: 1, 2), S, P, L. *An Introduction to Classical Complex Analysis, Volume 1.* Robert B. Burckel. Pure and Appl. Math., V. 82-I. Acad Pr, 1979, 570 pp, $45. [ISBN: 3-7643-0989-X] Self-contained introduction in the "Satz-Beweis" style; theorems are precisely stated and the proofs are unusually detailed. Exercises scattered throughout represent continuations of the text. Twelve chapters cover basics including chapters on the winding number and its topological implications, polynomial approximation, the general Dirichlet problem, Riemann mapping theorem, normal families. Large bibliography (many entries from the *Monthly* and *Magazine*) and chapter notes keyed to it. LCL

DIFFERENTIAL EQUATIONS, P. *Local Jet Bundle Formulation of Bäcklund Transformations with Applications to Non-Linear Evolution Equations.* F.A.E. Pirani, D.C. Robinson, W.F. Shadwick. Reidel, 1979, viii + 132 pp, $11.85 (P). [ISBN: 90-277-1036-8] The authors introduce a formulation of Bäcklund transformations in the language of jets. This establishes a framework (not yet fully developed) in which more general problems can be studied. JAS

DIFFERENTIAL EQUATIONS, P. *The Tricomi Equation with Applications to the Theory of Plane Transonic Flow.* A.R. Manwell. Pitman, 1979, 185 pp, $16.95 (P). [ISBN: 0-273-08428-3] "An up-to-date account of both the achievements and limitations of transonic flow theory when based on the linear holograph equations, i.e., on the generalised Tricomi equation." RBK

DIFFERENTIAL EQUATIONS, P. *Singularités Des Systèmes Différentiels De Gauss-Manin.* Frederic Pham. Progress in Math., No. 2. Birkhäuser Boston, 1979, 339 pp, $16 (P). [ISBN: 3-7643-3002-3]

Notes from a course given at Nice in 1977-78 augmented with chapters by Lo Kam Chan and Ph. Maisonobe and J.-E. Rombaldi. JAS

NUMERICAL ANALYSIS, P, *Colloquium Numerical Treatment of Integral Equations.* Ed: H.J.J. te Riele. MC Syllabus, No. 41. Math Centrum, 1979, vii + 259 pp, Dfl. 31 (P). [ISBN: 90-6196-189-0] Extended versions of 11 of the 14 talks presented at the colloquium series held at the Mathematisch Centrum, Amsterdam from October 1978 to May 1979. The remaining papers have appeared elsewhere. Starts with a survey lecture by the editor on numerical methods for integral equations. JAS

NUMERICAL ANALYSIS, T*, S, P, L*, *Introduction to Numerical Analysis.* J. Stoer, R. Bulirsch. Trans: R. Bartels, W. Gautschi, C. Witzgall. Springer-Verlag, 1980, ix + 609 pp, $24. [ISBN: 0-387-90420-4] A very substantial introduction to numerical analysis which includes all standard topics and much more (e.g., Thiele's continued fraction, fast Fourier transform, Cholesky decomposition, simplex method, QR method, shooting methods). Some algorithms are given in Algol 60, but text is largely independent of computer language. Many exercises. TRS

NUMERICAL ANALYSIS, S(17-18), P, *Methode der finiten Elemente.* Hans Rudolf Schwarz. Teubner Stuttgart, 1980, 320 pp, (P). [ISBN: 3-519-02349-0] An introduction to the method of finite elements, written for mathematicians, scientists and engineers. Special consideration of problems of computation. Some applications worked out, no exercises. JD-B

NUMERICAL ANALYSIS, P, *Numerical Mathematics.* Ed: R. Ansorge, K. Glashoff, B. Werner. Birkhäuser Boston, 1979, 207 pp, $28. [ISBN: 3-7643-1099-5] Proceedings of a symposium honoring Lothar Collatz on his retirement. Contains an appreciation of his work, a survey of the very early history of numerical mathematics, 12 technical papers. JD-B

NUMERICAL ANALYSIS, S(13-18), P, *Advanced Analysis with the Sharp 5100 Scientific Calculator.* J.M. Smith. Wiley, 1979, 132 pp, $6.95 (P). [ISBN: 0-471-07753-4] Adaptation of the author's 1977 book, *Scientific Analysis on the Pocket Calculator, Second Edition* (TR, February 1978). Emphasizes numerical methods that are particularly appropriate for the Sharp 5100, which has an alphanumeric entry capability. RSK

NUMERICAL ANALYSIS, P, *A Survey of Numerical Methods for Partial Differential Equations.* Ed: I. Gladwell, R. Wait. Clarendon Pr, 1979, xiii + 425 pp, $34.95. [ISBN: 0-19-853351-9] A useful survey. Organized by type of equation: elliptic, parabolic, hyperbolic. Includes biharmonic, diffusion-convention, boundary layer, moving boundary, mixed and non-linear problems. Focuses on the more important finite difference, finite element and transformation methods. RWN

FUNCTIONAL ANALYSIS, S(18), P, *Essays in Commutative Harmonic Analysis.* Colin C. Graham, O. Carruth McGehee. Grund. der math. Wissenschaften, B. 238. Springer-Verlag, 1979, xxi + 464 pp, $42. [ISBN: 0-387-90426-3] Assumes a solid background in functional analysis, measure theory, Fourier analysis on locally compact Abelian groups; goes on from there to study such topics as idempotents, Helson sets, Silov boundary, Wiener-Lévy theorem, Tilde algebras; concludes with a chapter on unsolved problems. Appendices, bibliography, index. JS

FUNCTIONAL ANALYSIS, T(18), P, *Topological Vector Spaces II.* Gottfried Köthe. Grund. der math. Wissenschaften, B. 237. Springer-Verlag, 1979, xii + 331 pp, $39.80. [ISBN: 0-387-90440-9] The sequel to the author's well-known first volume presents a substantial chapter on linear mappings and duality (e.g., Pták's theory, DeWilde's theory, graph topology) and another on spaces of linear mappings (e.g., compactness, nuclearity, approximation theory, projective and injective tensor products). TRS

FUNCTIONAL ANALYSIS, P, *Functional Analysis: Surveys and Recent Results II.* Ed: Klaus-Dieter Bierstedt, Benno Fuchssteiner. Math. Stud., V. 38. North-Holland, 1980, xii + 342 pp, $41.50. [ISBN: 0-444-85403-7] Proceedings of the second Paderborn conference on functional analysis which was held in February 1979. Contains 18 articles, all in English, on a wide variety of topics in functional analysis and its applications. (The proceedings of the first Paderborn conference (November 1976) were published in 1977 as Volume 27 in this same series.) LAS

FUNCTIONAL ANALYSIS, P, *Fractional Calculus and Integral Transforms of Generalized Functions.* A.C. McBride. Pitman, 1979, 179 pp, $16.95 (P). [ISBN: 0-273-08415-1] A unifying approach to the use of generalized functions to study integral operators on $(0,\infty)$. Fractional calculus, hypergeometric integral equations, Hankel transforms, and dual integral equations of Titchmarsh type are studied in detail. RBK

OPTIMIZATION, T(18: 1), S, P, *Optimization and Approximation.* W. Krabs. Wiley, 1979, xii + 220 pp, $28.95. [ISBN: 0-471-99741-2] The relationship between continuous approximation problems and infinite optimization. Considers linear, convex and nonlinear problems. Includes motivating examples, existence theory, generalizations. Presumes functional analysis. RWN

OPTIMIZATION, T(17-18: 1), S, P, *Finite Dynamic Programming: An Approach to Finite Markov Decision Processes.* D.J. White. Wiley, 1978, xiii + 204 pp, $39.95. [ISBN: 0-471-99629-7] Presents a classification scheme for dynamic programming problems according to the form of the recurrence relation and the constraints, and outlines the fundamental theory for each form. Includes discounting and non-discounting, single-period and multi-period, and deterministic and probabilistic problems. Exercises. RWN

ANALYSIS, P, *Algebraic Potential Theory.* Maynard Arsove, Heinz Leutwiler. Memoirs No. 226. AMS, 1980, v + 130 pp, $7.60 (P). [ISBN: 0-8218-2226-8] "Global aspects of classical potential theory are developed in a purely algebraic way in terms of a new algebraic structure called a mixed lattice semigroup [which] generalizes the notion of a Riesz space... ." TAV

ANALYSIS, T*(16-18: 1, 2), S*, P*, L**, *Inequalities: Theory of Majorization and Its Applications.* Albert W. Marshall, Ingram Olkin. Math. in Sci. and Eng., V. 143. Acad Pr, 1979, xx + 569 pp, $49.50. [ISBN: 0-12-473750-1] The theory of majorization is fundamental to a number of applications and provides a natural context for proving a number of inequalities. The first half covers basic theory and applications to combinatorics, geometry, matrices, and numerical analysis. The second half, on stochastic applications, generalizations, and complementary topics, is more advanced. This well-written, highly motivated book contains a wealth of good mathematics. LCL

ANALYSIS, T*(18: 1, 2), S, L, *Interpolation of Operators and Singular Integrals: An Introduction to Harmonic Analysis.* Cora Sadosky. Pure and Appl. Math., V. 53. Dekker, 1979, xi + 375 pp, $35. [ISBN: 0-8247-6883-3] A clearly written introduction to harmonic analysis with a central theme of singular integrals. Presumes familiarity with integration theory, normed linear spaces and elementary complex analysis. A very nice treatment of Fourier analysis is followed by discussions of interpolation, bounded mean oscillation, and transform theory. Exercises, glossary of symbols and notation.TAV

ALGEBRAIC GEOMETRY, S(18), P, *Étale Cohomology.* J.S. Milne. Math. Series, No. 33. Princeton U Pr, 1980, xiii + 323 pp, $26.50. A careful and readable presentation of étale topology, sheaf theory, and cohomology with an application to the proof of the rationality of some very general L-series. This book makes available a unified exposition of a number of massive original papers. JAS

ALGEBRAIC GEOMETRY, P, *Partially Ordered Rings and Semi-Algebraic Geometry.* Gregory W. Brumfiel. London Math. Soc. Lect. Notes Ser., No. 37. Cambridge U Pr, 1979, 280 pp, $19.95 (P). [ISBN: 0-521-22845-X] In an attempt to understand more fully the dividing line between algebra and topology, the author attempts to formulate geometric and topological notions in purely algebraic terms (beginning with algebra and working toward geometry). Toward this end, he develops an abstract theory of partially ordered rings, and proceeds to use this language to introduce real semi-algebraic geometry via Artin-Schreier theory. LCL

DIFFERENTIAL GEOMETRY, P, *Pseudo-Riemannian Symmetric Spaces.* M. Cahen, M. Parker. Memoirs No. 229. AMS, 1980, iv + 108 pp, $6.80 (P). [ISBN: 0-8218-2229-2] A partial classification of pseudo-Riemannian symmetric manifolds analogous to E. Cartan's classification for the positive definite case. JAS

DIFFERENTIAL GEOMETRY, T(17-18: 1, 2), P, *Differential Analysis on Complex Manifolds.* R.O. Wells, Jr. Grad. Texts in Math., V. 65. Springer-Verlag, 1980, x + 260 pp, $19.80. [ISBN: 0-387-90419-0] A corrected *Second Edition* with a completely revised chapter on compact complex manifolds using classical finite-dimensional representation theory for SL(2,c) to give a "natural proof of the Lefschetz decomposition theorem." (*First Edition,* TR, October 1973.) JAS

ALGEBRAIC TOPOLOGY, S(18), P, *Adams Completion and its Applications.* Sribatsa Nanda. Pure and Appl. Math., No. 51. Queen's U, 1979, 57 pp, (P). A reasonably self-contained and very readable exposition of the Adams completion and its applications based on two seminars given at the Memorial University at St. John's. JAS

TOPOLOGY, P, *Surveys in General Topology.* Ed: George M. Reed. Acad Pr, 1980, xiii + 557 pp, $35. [ISBN: 0-12-584960-5] Invited survey articles on various aspects of point set topology mostly contributed by the participants in the 1978-79 special "Year of Topology" at Ohio University. JAS

TOPOLOGY, P, *Topological Structures II: Proceedings of the Symposium in Amsterdam, October 31-November 2, 1978.* Ed: P.C. Baayen, J. van Mill. Math Centrum, 1979. *Part 1,* Math. Centre Tracts, No. 115, x + 189 pp, Dfl. 23 (P) [ISBN: 90-6196-185-8]; *Part 2,* Math. Centre Tracts, No. 116, vii + 202 pp, Dfl. 25 (P). [ISBN: 90-6196-186-6] Papers from, inspired by, and solicited as a result of the symposium in Amsterdam, October 31 to November 2, 1978. Includes more than the title promises, e.g., some geometry. JAS

PROBABILITY, T*(16-17: 2), P, L, *Diffusions, Markov Processes, and Martingales, Volume 1: Foundations.* David Williams. Wiley, 1979, xiii + 237 pp, $33.50. [ISBN: 0-471-99705-6] In this fine detailed treatment (marred only slightly by an inequality printed backwards in the definition of a stopping time), the author's enthusiasm and wit come through to urge the reader onward to more new delights--a fine book, well conceived and executed. TAV

PROBABILITY, P, *Controlled Stochastic Processes.* I.I. Gihman, A.V. Skorohod. Trans: Samuel Kotz. Springer-Verlag, 1979, vii + 237 pp, $32. [ISBN: 0-387-90410-7] A very thorough, precise, mathematical treatment of controlled processes. Presupposes measure theory, functional analysis and a background in (abstract) stochastic processes. The exposition is clear, but the level of rigor makes for tough reading. TAV

PROBABILITY, P, *The Theory of Stochastic Processes III.* I.I. Gihman, A.V. Skorohod. Trans: S. Kotz. Grund. math. Wissenschaften, B. 232. Springer-Verlag, 1979, 387 pp, $47.80. [ISBN: 0-387-90375-5] The third and final volume of the authors' work covers the theory of martingales, stochastic integrals and differential equations, diffusion and continuous Markov processes. The translation is smooth, and the treatment very readable. Contains a (surprisingly) short bibliography and errata for *Volumes I* and *II*. TAV

PROBABILITY, P, *Branching Processes with Continuous State Space.* P.J.M. Kallenberg. Math. Centre Tracts No. 117. Math Centrum, 1979, ii + 122 pp, Dfl. 15 (P). [ISBN: 90-6196-188-2] The author develops the theory of branching processes with continuous values. The development parallels that of the Galton-Watson process in terms of the expected size of the offspring distribution. Applicable to models where volume or weight of population is of greater interest than numerical size. TAV

PROBABILITY, P. *Application of the Wigner Distribution to Harmonic Analysis of Generalized Stochastic Processes.* A.J.E.M. Janssen. Math. Centre Tracts, No. 114. Math Centrum, 1979, vii + 169 pp, Dfl. 21 (P). [ISBN: 90-6196-184-X] An abstract treatment of generalized stochastic processes, with an emphasis on the role of the Wigner distribution in the spectral analyses of such processes. Appendices, index of symbols, bibliography. TAV

PROBABILITY, P. *Random Walks with Stationary Increments and Renewal Theory.* H.C.P. Berbee. Math. Centre Tracts, No. 112. Math Centrum, 1979, iii + 223 pp, Dfl. 27 (P). [ISBN: 90-6196-182-3] Renewal theory is a central idea for random walks with stationary independent increments. Removing the independence condition changes the theory considerably. Palm theory and coupling are used to develop renewal theory in this setting. TAV

STATISTICS, T(16-18: 1, 2), S, P, L. *Introduction to Bivariate and Multivariate Analysis.* Richard H. Lindeman, Peter F. Merenda, Ruth Z. Gold. Scott F, 1980, 444 pp, $15.95. [ISBN: 0-673-15099-2] Assumes some background in statistics and matrix algebra. A thorough treatment of the bivariate case followed by multiple regression, canonical correlation, discriminant analysis, factor analysis, and analysis of categorical data. FLW

STATISTICS, T(15: 2). *Basic Business Statistics, Concepts and Applications.* Mark K. Berenson, David M. Levine. P-H, 1979, xv + 719 pp, $17.95. [ISBN: 0-13-057596-8] Applied descriptive text suitable for business students who take only one or two courses. Emphasis on simple examples; uses actual data from a large survey and other sources. Limited mathematical development. Appeals to intuitive development and understanding of procedures. Covers suitable set of topics. WC

STATISTICS, T*(15-16: 2). *Mathematical Statistics, Third Edition.* John E. Freund, Ronald E. Walpole. P-H, 1980, xii + 548 pp, $18.95. [ISBN: 0-13-562066-X] Revision of Freund's well-known 1971 *Second Edition* (TR, January 1972), with an improved format. Content remains basically the same, but has been reorganized and expanded, particularly in the areas of distribution theory, estimation and nonparametric statistics. RSK

STATISTICS, T(17-18: 1, 2), P, L. *Theory of Games and Statistical Decisions.* David Blackwell, M.A. Girshick. Dover, 1979, xi + 355 pp, $5 (P). [ISBN: 0-486-63831-6] An unaltered republication of the 1954 Wiley original. A classic treatment of decision theory that synthesizes two innovative theories that emerged in post-WWII publications: von Neumann and Morgenstern's theory of games and Wald's statistical decision theory. LAS

STATISTICS, T(14-16: 1), L. *Introduction to Statistics: A Nonparametric Approach for the Social Sciences.* Chris Leach. Wiley, 1979, xv + 339 pp, $37.50. [ISBN: 0-471-99743-9] Introductory textbook and manual for use by social scientists with no background in statistics. The focus on nonparametric methods allows student to grasp the standard statistical ideas without requiring a sophisticated knowledge of mathematics. To achieve a coherent, unified approach, the author concentrates on tests falling within framework of Fisher's method of randomization, emphasizing similarities between the tests presented. Chapter summaries and endpapers aid in selection of appropriate tests. There are only 82 exercises in entire text. GHM

STATISTICS, T(18: 2), P*. *Multiple Decision Procedures: Theory and Methodology of Selecting and Ranking Populations.* Shanti S. Gupta, S. Panchapakesan. Wiley, 1979, xxv + 573 pp, $34.50. [ISBN: 0-471-05177-2] A volume in the Wiley Series in Probability and Mathematical Statistics. Comprehensive up-to-date survey of how to determine the best population(s) when sampling from k populations. Gives equal emphasis to the indifference zone and subset selection formulations. Extensive bibliography.RSK

STATISTICS, S(13-17). *Statistical Exercises in Medical Research.* John F. Osborn. Halsted Pr, 1979, xii + 154 pp, $14.95 (P). [ISBN: 0-470-26744-5] Companion volume to P. Armitage's 1971 text, *Statistical Methods in Medical Research.* Practical exercises based on real data. Includes answers.RSK

STATISTICS, T(14-16: 1), S. *Multiple Regression and the Analysis of Variance and Covariance.* Allen L. Edwards. Freeman, 1979, xv + 212 pp, $13.50; $8.50 (P). [ISBN: 0-7167-1080-3; 0-7167-1081-1] One of their Series of Books in Psychology. Designed to show the correspondence between analysis of variance and multiple regression analysis. Presumes a course in elementary (non-calculus) statistics and previous or concurrent work in analysis of variance. RSK

COMPUTER PROGRAMMING, T(13: 1), S, P, L. *COBOL, Programming and Applications.* C. Joseph Sass. Allyn, 1979, xv + 431 pp, $13.95 (P). [ISBN: 0-205-06550-3] An introduction to Cobol and data processing which assumes no previous experience. It features lots of detailed examples, an ample supply of exercises and an emphasis on top-down programming. The major portion of the text is machine and vendor independent. CEC

COMPUTER PROGRAMMING. *Microsoft Basic.* Ken Knecht. Dilithium Pr, 1979, 158 pp, $8.95 (P). [ISBN: 0-918398-23-1] A description of MITS Basic. Not a very good introduction to programming in Basic, but if one is familiar with some version of Basic, this book does a good job of describing the eccentricities of this version (which is similar to Radio Shack Basic). CEC

COMPUTER PROGRAMMING, S(13). *The A to Z Book of Computer Games.* Thomas C. McIntire. TAB Books, 1979, 308 pp, $7.95 (P). [ISBN: 0-8306-1062-6] Complete listings of twenty-six computer games along with game and program descriptions. A wide variety of games and relatively short programs. All in Basic. CEC

Reviewers Whose Initials Appear Above

William Carlson, St. Olaf; Clifton E. Corzatt, St. Olaf; John Dyer-Bennet, Carleton; Lorraine L. Keller, St. Olaf; Roger B. Kirchner, Carleton; Richard S. Kleber, St. Olaf; Joseph Konhauser, Macalester; Loren C. Larson, St. Olaf; George H. Mills, St. Olaf; R.W. Nau, Carleton; A. Wayne Roberts, Macalester; Thomas R. Savage, St. Olaf; John Schue, Macalester; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; T.A. Vessey, St. Olaf; Martha Wallace, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY PAUL A. HAEDER, University of Nebraska at Omaha

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036

PERSONAL ITEMS

Associate Professor *Charles H. Jepsen*, Grinnell College, Grinnell, Iowa, has been promoted to Professor.

Associate Professors *W.E.L. Clarke* and *Clifford Pope* are recent appointments to the staff of Atlantic Union College, South Lancaster, Massachusetts.

Assistant Professor *Isom Harson*, Howard University, Washington, D.C., has been promoted to Associate Professor.

Dr. *Stephan Carlson*, formerly at Baylor University, has been appointed Assistant Professor at Grinnell College.

Assistant Professor *J. Winston Crawley*, Shippensburg State College, Shippensburg, Pennsylvania, has been promoted to Associate Professor.

The following items have been reported from the University of Hawaii:

Associate Professor *Dorothy K. Reed* has retired. *Jerrold R. Griggs*, Ph.D., California Institute of Technology, and Dr. *James B. Nation*, University of Utah, have been appointed Assistant Professors. Dr. *Michael S. Waterman*, Los Alamos Scientific Laboratory, has been appointed Professor. Assistant Professor *Everett L. Lady* has been promoted to Associate Professor. Associate Professor *Ralph S. Freese* has been promoted to Professor.

Professor *Richard M. Karp*, University of California, Berkeley, has been awarded the $1,000 Lanchester prize for "The best English-language published contribution in Operations Research for 1977."

Assistant Professor *Cherrie J. Webster*, Northwestern State University of Louisiana, has been promoted to Associate Professor.

Annabel Santana has been appointed Instructor at Hunter College of CUNY.

Professor *Paul M. Young*, Kansas State University, has been appointed Vice President for University Facilities.

Professor *Siegfried K. Grosser* of the University of Vienna has been re-elected Vice-President of Austrian Mathematical Society and chairs the Committee on Instruction.

Dr. *Jack Alanen* has been appointed Chairman and Professor, Department of Computer Sciences, at California State University, Northridge.

Dr. *Baldev Sachdeva*, Pennsylvania State University, has been appointed Visiting Assistant Professor of Mathematics at the University of New Haven, West Haven, Connecticut.

Dr. *Efim Khalimsky*, a recent refugee from the U.S.S.R., has been appointed Visiting Associate Professor at the City College of New York.

Dr. *Jeffrey Nunemacher*, formerly at the University of Texas, Austin, has been appointed Visiting Assistant Professor at Kenyon College, Gambier, Ohio.

Assistant Professor *R. Richard Summerhill*, Kansas State University, has been promoted to Associate Professor.

Instructor *William Navidi*, County College of Morris, Dover, New Jersey, has been promoted to Assistant Professor.

Instructor *Sandra Clarkson*, Hunter College of CUNY, has been promoted to Assistant Professor.

Debra Gutridge Davis has been appointed Instructor at Denison University, Granville, Ohio.

Dr. *Donald B. Tillotson* has retired from Northwest Nazarene College, Nampa, Idaho, after 36 years of service and has been given the title of Professor Emeritus.

Professor *John Emerson* has returned to the Mathematics Department, Middlebury College, Middlebury, Vermont, after a year of research and study at the Harvard Biostatistics Department and the Sidney Farber Cancer Institute in Boston.

Dr. *Daniel J. Fitzgerald*, formerly at Oklahoma State University, has been appointed Assistant Professor at Kansas Newman College, Wichita, Kansas.

Dr. *Elmer L. Peterson*, formerly Professor at Northwestern University, has been appointed Professor of Mathematics and Operations Research at North Carolina State University, Raleigh.

Dr. *Edwin H. Mookini*, Honolulu, Hawaii, died on November 4, 1979. He was a member of the Association for fifteen years.

William C. Taylor, M.A., Ft. Walton Beach, Florida, died on August 2, 1979. He was a member of the Association for thirty years.

NEW EDITOR FOR NEWS AND NOTICES

My plans for retirement from my career as an educator call for some adjustments in the editorship of this section of the *Monthly*. I shall continue to supply copy through the August-September issue. Thereafter your new editor will be Professor Frank Kocher, Pennsylvania State University. Because of required lead-time, it is suggested that all news items mailed after July 1, 1980, be sent to him at University Park, Pennsylvania 16802.

It has been a pleasure working with the many fine members of the M.A.A. in a number of roles, especially in the capacity as *News and Notices* Editor.

Paul A. Haeder
University of Nebraska at Omaha

THIRD INTERNATIONAL TIME SERIES MEETING

The Third International Time Series Meeting (First American Conference) will be held August 14-15, 1980 at the Shamrock Hilton Hotel in Houston, Texas. Oliver D. Anderson and M. Ray Perryman are the program directors. The conference will convene immediately following the American Statistical Association meetings at the same location. Persons desiring to present a paper, serve on the program committee, serve as a session chairperson or discussant, or organize a session, should contact M. Ray Perryman, Director, Center for the Advancement of Economic Analysis, Hankamer School of Business, Baylor University, Waco, Texas 76706. Persons desiring to present a paper should send a two page abstract to Dr. Perryman by May 1, 1980 (papers will be published in a proceedings volume).

NATO ADVANCED STUDY INSTITUTE ON STATISTICAL DISTRIBUTIONS IN SCIENTIFIC WORK

A NATO Advanced Study Institute on statistical distributions in scientific work will be held at the University of Trieste, Trieste, Italy during July 14-July 25, 1980. The Institute will be preceded by a Short Intensive Preparation Course during July 10-July 15 and followed by a Research Conference and Workshop during July 23-August 1.

The emphasis of the program will be on the theory and applications of statistical distributions with reference to multidimensional random variables. Relevant significant developments with respect to single dimension random variables will be included. Both quantitative methodology and scientific significance will be emphasized. Both instrumentation-related and computer-related problems will be discussed.

The purpose of the activity is to give a unified and integrated view of different classes of distributions and to describe novel methodologies related to statistical distributions and/or their applications. Alternatively, contributions on the description and characterization of distributions which are useful in a particular area of application, e.g., medicine, engineering, forestry, etc., will be welcomed.

Participants in full attendance will receive partial support as per need and supply. Young predoctoral and postdoctoral statisticians, scientists, engineers, managers, and faculty are encouraged to apply. Suggestions for topics and speakers are also invited.

The program committee consists of: G.P. Patil, USA (Chairman); B. Baldessari, Italy (Director); T. Cacoullos, Greece; S. Engen, Norway; S. Kotz, USA; J.E. Mosimann, USA; J.K. Ord, England; C.R. Rao, USA; L. Rondini, Italy; C. Taillie, USA (Co-Director); J. Tiago de Oliveira, Portugal; W.G. Warren, Canada; and M.E. Wise, Netherlands.

For further details write to: Professor G. P. Patil, Department of Statistics, 318 Pond Laboratory, The Pennsylvania State University, University Park, PA 16802 USA.

MEASURE THEORY CONFERENCE

The Department of Mathematical Sciences at Northern Illinois University will host a Conference on Measure Theory on October 24-25, 1980. The purpose of the conference is to promote a higher degree of professional contact among mathematicians who use measure theory as a major tool in their work. The topics of Baire and Borel measures on topological spaces, vector measures, and measures on topological spaces, vector measures, and measures on topological vector spaces (with applications to physics) will be emphasized. Invited speakers are Joseph Diestel, William Graves, Charles Newman, Mary Ellen Rudin, Dennis Sentilles, and Dorothy Maharam Stone. There will also be sessions for contributed papers. Gerald Goldin and Robert Wheeler are the organizers for the conference. For additional information, contact Ms. Arlene Neher, Continuing Education, Northern Illinois University, DeKalb, Illinois 60115 or the Department of Mathematical Sciences, N.I.U.

INTEREST IN MAA EXPRESSED BY PEOPLE'S REPUBLIC OF CHINA

Daniel K. C. Chen, member of the Faculty of Function Theory, the Normal Institute of Shanghai, is the first citizen from the People's Republic of China to become a member of the Mathematical Association of America. He has been active in "recruiting new MAA members, directing the purchase of MAA publications, and distributing MAA news." As a result, the Normal Institute of Shanghai has sponsored a program for the translation of MAA publications. Mr. Chen is working toward participation of the People's Republic in the International Mathematical Olympiad to be held in Washington, D.C. in 1981. He has further pledged his efforts to promote good relations between the MAA and the People's Republic of China.

Members wishing to help Mr. Chen in this international effort may contact him at:

Apartment 16, House 3, Friendship 1st Village
Guangling 1st Road, Shanghai 200081
People's Republic of China

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

FEBRUARY 1980 MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The fifty-sixth annual meeting of the Louisiana-Mississippi Section of MAA was held at the Louisiana Tech University, Ruston, Louisiana, on February 15 and 16, 1980. Lousiana Tech University served as host for the meeting. There wer 125 registered participants for the Section meeting which was held jointly with the Louisiana-Mississippi Branch of the NCTM.

Professor Jack B. Garner of Louisiana Tech University presided as chairman. Dr. Dorothy Bernstein, President of MAA, gave two invited addresses: *Mathematical Models and Existence Theorems* and *Who is the MAA?* Three panel discussions were held as follows: *Remedial Mathematics,* moderator Professor James D. Abbot of University of New Orleans; *Ways to Encourage Superior Students,* moderator Professor Temple Fay of University of Southern Mississippi; and *Uses of Minicomputers and Handheld Calculators in the Classroom,* moderator Professor James D. Abbot of University of New Orleans.

There were four sessions of contributed papers, plus two sessions of student papers. Papers presented were:

An Operator Equation Characterizing Two BMO (T) Functions with Difference in $L^{\infty}(T)$, R.B. Tucker, University of Mississippi

A Fixed Point Theorem of Nonexpansive Mappings, H. Kaneko, Louisiana Tech University

Numerical Approximation of Limits of Indeterminate Forms, W. Watson, Louisiana Tech University

Divergence Can Be Contagious - A Continued Fraction Convergence Theorem, R. Heller, Mississippi State University

Singular Perturbation Solutions of a Nonlinear Differential System, J.B. Garner, Louisiana Tech U

Primary Extensor Formulation for the Undamped Harmonic Oscillator, G. Okhuysen, Mississippi State U

Existence and Uniqueness Results for a Diffusion-Reaction System, R.E. Hanna, Louisiana Tech Univ.

On Independent Sets of Cubic Graphs, G. Hopkins, University of Mississippi

Recent Developments in Mechanical Theorem Proving, G.M. Butler (Speaker) and D.S. Lankford, Louisiana Tech University

On the Application of Approximation Theory to Control Problems, P.Z. Daffer (Speaker) and H. Kaneko, Louisiana Tech University

A Characterization of Minimal Regular Spaces, L. L. Herrington, Louisiana State University-Alexandria

A Probabilistic Representation of an Instantaneous Unit Hydrograph, V.K. Gupta, J.R. Hudgins (Speaker), R. Wang, and E. Waymire, University of Mississippi

On the Factorization of the Binomial Coefficient $\binom{2n}{n}$, W. Staton, University of Mississippi

Maschke's Theorem and Commutative Semigroup Rings, B. Glastad, University of Mississippi

A Kirkman Type Counting Problem, P.G. Webster, University of Southern Mississippi

Student papers presented were:

Solvability Results for a Nonlinear Boundary Value Problem, G.J. Buffington, Louisiana Tech University

An Analysis of a Boundary Value Problem by Reduction to an Initial Value Problem, M.K. Dunn, Louisiana Tech University

Numerical Solutions for Axial Symmetric Temperative Field Problems, R.S. Colvin, Louisiana Tech U

The Core of an n-Person Game, G. Robichaux, Nicholls State University

A Note on Graph Theory, F. Portier, Nicholls State University

An Example Using the New Finite Pivoting Rules for the Simplex Algorithm, T. Williams, Nicholls State University

The Implementation of Computer Movies in the Model of a Steel Rotor, S. Haw (Speaker) and R.E. Funderlic, Nicholls State University

A Monte Carlo Simulation of Hardy-Weinberg Law, T. Williams, Nicholls State University

On Friday evening a cash bar followed by a buffet banquet was held at the Holiday Inn, Ruston. Musical entertainment provided by Glynn Harris and Steve Lawson was greatly enjoyed by those attending. Chairman Garner recognized retired section members present at the banquet.

The following persons were elected officers of the 1980-81 Executive Committee: Edwin P. Oxford, University of Southern Mississippi, Chairman; James H. Abbott, University of New Orleans, Louisiana Vice-Chairman; and Daniel A. Hogan, Hinds Junior College, Mississippi Vice-Chairman.

FEBRUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The Annual Meeting of the Northern California Section was held on February 23, 1980, jointly with the Northern California Section of SIAM, at the Naval Postgraduate School, Monterey.

The membership elected Hans Samelson from Stanford as vice-chairman. Succeeding Carroll Wilde as Chairman is Sister Madeleine Rose Ashton and succeeding William Chinn as Program Chairman is Carroll Wilde.

The following invited papers were presented:

Borel-Programmable Functions, David Blackwell, University of California Berkeley

Isoperimetric Results, Ivan Niven, University of Oregon

Mathematical Models and Existence Theorems, Dorothy Bernstein, President, MAA

Solutions of Boundary Value Problems via Singular Integral Equations, Gordon E. Lotta, University of Virginia.

A luncheon was held at the old Del Monte Hotel, now Hermann Hall, and featured a talk by Angus E. Taylor on *The Historical Aspects of the Beginnings of Point-Set Topology in Abstract Spaces.* The day's program was concluded with a meeting for department heads and MAA representatives.

UPDATE ON MAA LIFE INSURANCE PLAN

There have been some changes in the MAA insurance plan, changes that may not be known by all of the membership.

Members who were insured under the MAA approved Life Insurance Plan as of September 30, 1979, received a credit equal to 65% of the semiannual payment due April 1, 1980. Credits of this type make the Plan very attractive and it is hoped that they will continue to be available in future years. They cannot, of course, be promised or guaranteed.

As of October 1, 1979, members have been able to purchase up to $120,000 coverage in $12,000 units. The maximum benefit amount is now $35,000 for a spouse (not to exceed 50% of the member benefit) and $2500 for each dependent child.

All inquiries regarding the MAA Life Insurance Plan or any other plan in the Group Insurance Program should be directed to the MAA Insurance Administrator, 1707 L Street, N.W., Washington, D.C., 20036.

REPORTS ON SPRING MEETINGS

Although many of the spring Section meetings will have taken place when this information is published, reports of the meetings have not been received by the editor at the time the material is due at the publishers. The following is a resumé of the dates and places of scheduled spring meetings and the invited speakers.

NEW JERSEY

Hyatt House Hotel, Cherry Hill, New Jersey, March 15, 1980.
Invited speakers: John M. Cozzolino, Wharton School of the University of Pennsylvania, *The Mathematics of Risk Analysis.*
Kenneth C. Wolff, Montclair State College, *Weaving Applications into the Content of Traditional Courses.*

FLORIDA

University of Jacksonville, March 7-8, 1980.
Invited speakers: Dorothy Bernstein, Brown University (President of MAA).
Ernst Snapper, Dartmouth College.

OKLAHOMA-ARKANSAS

Westmark Community College, Fort Smith, Arkansas, March 28-29, 1980
Invited speakers: R.D. Anderson, Louisiana State University, *Algorithmically Defined Functions.*
Walter Rudin, University of Wisconsin-Madison, *An Offspring of Harmonic Analysis.*

KENTUCKY

Western Kentucky University, Bowling Green, Kentucky, April 11-12, 1980.
Invited speaker: Doris Schattschneider, Moravian College, *Mathematical Symmetry and Art* and *Will it Tile? Try the Conway Criterion!*

MARYLAND-D.C.-VIRGINIA

University of Richmond, Richmond, Virginia, April 12, 1980.
Invited speaker: Leonard Gillman, University of Texas, *Optimal Strategies for Sports.*

NEBRASKA

Doane College, Crete, Nebraska, April 18-19, 1980.
Invited speaker: Dorothy L. Bernstein, Brown University, *A Small College's Experiences with Applications in the Mathematics Curriculum* and *Mathematical Modeling and Existence Theorems.*

WISCONSIN

University of Wisconsin-Milwaukee, March 28-29, 1980.
Invited speakers: Peter Hilton, Batelle Research Institute, Seattle, Washington, *Modeling Within Mathematics.*
Neal J. A. Sloane, Bell Telephone Laboratories, *Hadamard, Hotelling, Harwit: A New Application of Some Old Mathematics.*

KANSAS

Joint meeting with Kansas Association of Teachers of Mathematics, Kansas State University, Manhattan, April 11-12, 1980.
Invited speakers: Richard V. Andree, University of Oklahoma, *Some Problems Neither My Computer Nor I Can Solve. . .Yet!*
Dorothy L. Bernstein, Brown University, *Preparation for Careers in Applied Mathematics.*

ILLINOIS

John A. Logan College, Carterville, Illinois, April 25-26, 1980.
Invited speaker: Paul R. Halmos, Indiana University, *How to be a Mathematician.*

OHIO

Wittenberg University, April 25-26, 1980.
Invited speaker: Gail Young, University of Rochester, Rochester, New York, *Heroes of Mathematics (on R.L. Moore and his method).*

EASTERN PENNSYLVANIA and DELAWARE

Cedar Crest College, Allentown, Pennsylvania, April 26, 1980.
Invited speakers: N.J.A. Sloane, Bell Labs, *Hadamard, Hotelling, and Harwit: A New Application of Some Old Mathematics.*
Jessie MacWilliams, Bell Labs, *A Survey of Coding Theory.*
Curtis Greene, Haverford College.

CALENDAR OF FUTURE MEETINGS

Sixtieth Summer Meeting, University of Michigan, Ann Arbor, August 18–20, 1980.

Sixty-fourth Annual Meeting, San Francisco, California, January 9–11, 1981.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, last weekend in April or first weekend in May. Deadline for papers six weeks before meeting.

EASTERN PENNSYLVANIA AND DELAWARE, Saturday before Thanksgiving.

FLORIDA, early March. Deadline for paper titles two weeks before meeting.

ILLINOIS, first Friday/Saturday in May.

INDIANA

INTERMOUNTAIN, Utah State University, Logan, late April or early May 1980.

IOWA, third weekend in April. Deadline for papers February 1.

KANSAS, March or April. Deadline for papers January 1.

KENTUCKY, early April. Deadline for papers six weeks before meeting.

LOUISIANA–MISSISSIPPI, Friday-Saturday before February 20. Deadline for papers three months before meeting.

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Goucher College, Towson, Maryland, November 14–15, 1980.

METROPOLITAN NEW YORK, Mercy College, Dobbs Ferry, May 4, 1980.

MICHIGAN, Hope College, Holland, May 2–3, 1980.

MISSOURI, late March/early April. Deadline for papers January 31.

NEBRASKA, April.

NEW JERSEY, Union College, Cranford, October 25, 1980.

NORTH CENTRAL, end of October and April. Deadline for papers October 1 and April 1.

NORTHEASTERN, Saturday after Thanksgiving, and third week in June in odd-numbered years.

NORTHERN CALIFORNIA, first or second Saturday in February.

OHIO

OKLAHOMA–ARKANSAS, (approx.) Friday and Saturday of first weekend in April. Deadline for papers three weeks before meeting.

PACIFIC NORTHWEST, Central Washington University, Ellensburg, Washington, June 20–21, 1980.

ROCKY MOUNTAIN, last weekend in April or first in May. Deadline for papers eight weeks before meeting.

SEAWAY, Herkimer County Community College, Herkimer, New York, May 2–3, 1980.

SOUTHEASTERN

SOUTHERN CALIFORNIA, first or second Saturday in March.

SOUTHWESTERN, Northern Arizona University, Flagstaff, spring 1980.

TEXAS, Friday and Saturday in early April. Deadline for papers March 1.

WISCONSIN, Friday and Saturday between mid-April and first week in May. Deadline for papers six weeks before meeting.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Toronto, January 3–8, 1981.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES, Sheraton National Hotel, Arlington, Virginia, October 9–13, 1980.

AMERICAN MATHEMATICAL SOCIETY, University of Michigan, Ann Arbor, August 19–22, 1980.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Amherst, Massachusetts, June 23–26, 1980.

ASSOCIATION FOR COMPUTING MACHINERY, Nashville, Tennessee, October 27–29, 1980.

ASSOCIATION FOR SYMBOLIC LOGIC

ASSOCIATION FOR WOMEN IN MATHEMATICS, University of Michigan, Ann Arbor, August 18–22, 1980.

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHÉMATIQUES, Montreal, June 3–5, 1980.

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS, University of Michigan, Ann Arbor, August 18–21, 1980.

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, St. Louis, Missouri, April 22–25, 1981.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Shoreham Hotel, Washington, D.C., May 5–7, 1980.

PI MU EPSILON, University of Michigan, Ann Arbor, August 18–20, 1980.

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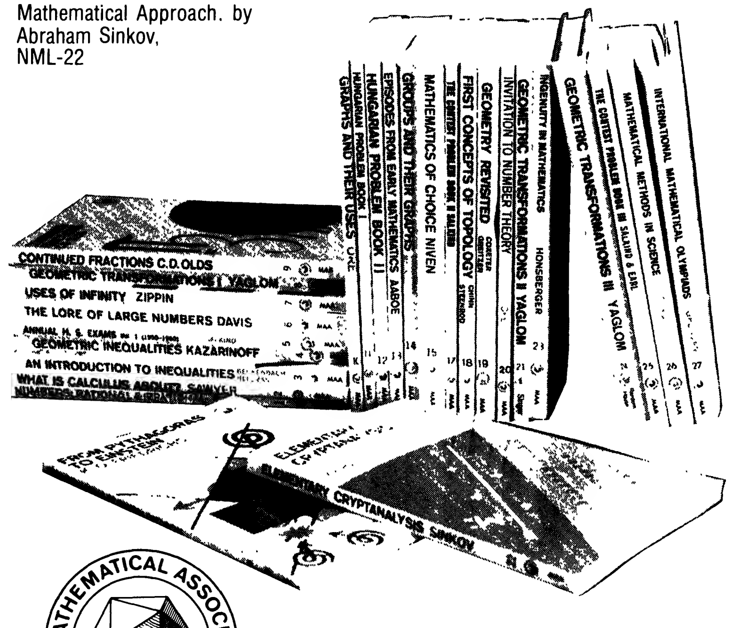

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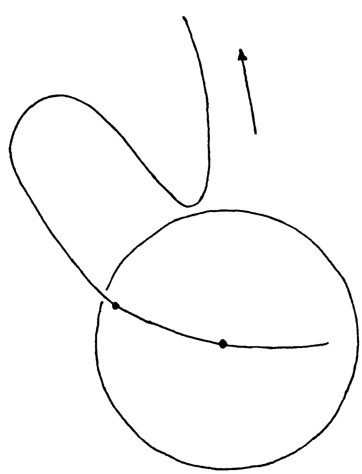

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THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

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THE MATHEMATICAL ASSOCIATION OF AMERICA

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Reports of surveys by the National Assessment of Educational Progress (1860 Lincoln St., Suite 300, Denver, CO 80295), issued last September, have shown a serious decline in the mathematical problem-solving abilities of this country's 9-, 13-, and 17-year-olds. The implications of the NAEP findings should be of concern to teachers of mathematics at all levels. In a news release of September 13, 1979, Shirley A. Hill, president of the National Council of Teachers of Mathematics, commented on the results of the NAEP surveys. Her statement (slightly revised) appears below, followed by a commentary on the state of mathematics in our schools by Martha Zelinka.

Readers who feel uncomfortable with an issue of the MONTHLY that devotes so much space to the high schools should ask themselves where the college students come from in the first place.

—R.P.B.

NATIONAL ASSESSMENT OF EDUCATIONAL PROGRESS

SHIRLEY A. HILL

The results of the second national assessment of mathematics reinforce the warnings that professional organizations concerned with mathematics education have been stressing in recent years. An excessive narrowing of the mathematics curricula—in the name of "Back to the Basics"—to the mechanistic learning of computational skills is detrimental to the development of problem solving.

While students displayed a fairly high level of skill in whole-number computation in the assessment, their abilities to apply these skills to the solution of realistic problems were significantly lower. It is obvious that there is little benefit to be gained by concentrating extraordinary efforts on computing skills and minimal competencies if our graduates cannot effectively *apply* mathematics in the real world. The assessment results present convincing evidence that we cannot simply assume that if students perform well on tests of lower-order skills then they can consequently use those skills in solving real-life problems. And, surely, the latter is our primary objective.

Throughout the NAEP reports, there is evidence that students proceed mechanically and thoughtlessly through problems, seeking a familiar routine or a rigid rule to apply. In many instances a careless reading of what is called for is apparent; in others, one finds a common failure to note that some answers are not realistic or even reasonable. Students often appear to lack a basic sense of quantitative relationships. While a reliance on drill and rote memorization of rules will produce a good showing on tests of short-term retention, this reliance also creates a mind set that is antithetical to insight into the essence of a problem.

The inescapable conclusion to be derived from the results of the second national assessment of mathematics is that there is a critical need for attention to higher-order cognitive skills. Reasoning, analyzing, estimating, selecting appropriate information, and inferring—these are basic skills that are essential to the effective application of mathematics.

The NAEP mathematics report should be invaluable to teachers, mathematics educators, curriculum developers, and school policy-makers in identifying other areas where increased efforts are needed. Foremost among these areas are decimals, ratios, and percents. All are increasingly important to the development of knowledgeable consumers, as is the ability to interpret quantitative data. As the calculator becomes an indispensable tool, the understanding and use of decimals assumes a more prominent place in problem solving. Furthermore, the disappointing assessment results on algebraic items suggest the need to reexamine instructional methods, content, and the present placement of algebra courses in the curriculum for some students.

Responsible readers of the NAEP reports will recognize that simplistic judgments about the overall mathematics success or failure of students, schools, teachers, or society cannot be made from such assessments. Nor was the assessment program designed to elicit such judgments. It is the identification of strengths and weaknesses that will provide guidance to mathematics teaching—not condemnation or praise.

But the challenge of the assessment results goes beyond the education community. During much of this decade, public demand and pressure—and, often, resulting legislation—have placed predominant attention on minimal skills. Tests have been developed to measure the "accountability" of the schools to this mandate. Consequently, mathematics textbooks and tests consisting of computation and routine word problems have dominated the market.

Many schools and teachers have responded to public pressure by focusing mainly upon the materials to be tested or on areas most easily and quickly affected by concentrated classroom effort. Less classroom time has been devoted to the processes of problem solving than to routine processes of paper-and-pencil calculation.

Public opinion should have its place in decisions about educational goals and objectives. But the clear message of the NAEP mathematics reports is a challenge to the public at large. The public must reexamine its present priorities and weigh the results of a mechanistic rote-skill curriculum against the need—now and in the future—for problem solvers with the flexibility to apply their knowledge in unexpected as well as routine ways.

DEPARTMENT OF CURRICULUM AND INSTRUCTION, UNIVERSITY OF MISSOURI, KANSAS CITY, MO 64110.

THE STATE OF MATHEMATICS IN OUR SCHOOLS

MARTHA ZELINKA

September 13, 1979, was a jubilant day for the Mathematical Association of America: the formal opening of the Headquarters, the Dedication of the Edgar H. Vaughn Building and the Dolciani Mathematical Center in Washington, D.C.

Was it coincidence that the same day saw the meeting of the members of the Committee on National Assessment of Educational Progress? They analyzed the findings of mathematics tests administered in 1978, involving 71,000 youngsters of ages 9, 13, and 17; this set of tests had last been given in 1973. (NAEP is a long-term, government-sponsored program, charged with periodic evaluation of the learning of American children in ten subject areas.)

September 14, 1979, the *Washington Post*: "According to a survey released today, mathematical ability of pupils declines." Is it the ability that declined?

October 10, 1979, the *Christian Science Monitor*: "High school science in decline, . . . alarming decline in high school math and science teaching. . . . They could use another sputnik."

These articles are both thoughtful and, no doubt, alarming. Let us hope that the reader goes beyond the headlines. The concerns expressed by Shirley A. Hill, President of NCTM, and Roy H. Forbes, Director of NAEP, about possible reasons for the decline in performance are those anticipated years ago by the mathematics community, when the layman's vocabulary was first enriched from the monotonous "declining SAT scores" to include "accountability," "minimum competency," "back-to-basics," "mainstreaming." Since then, these dangerous trends have become reality in many areas. They were offered as "solutions" by outsiders far removed from the classroom scene, mostly politically inspired nonscholars. Yet, and this is most deplorable, criticisms were also voiced by great minds, playing directly into the hands of those responsible for outside pressures. A federal law (PL94-142) now requires that students with special needs be taught in the least restrictive environment—in the classroom, whenever possible. The idea of

Martha Zelinka retired from Weston High School in June 1979 but plans to return to the classroom in the near future. She has been very active in the Association of Advanced Placement Mathematics Teachers. See this MONTHLY, 85 (1978) 629, for more biographical detail.—*Editors*.

"mainstreaming" embodied in this statute has implications, never thought through, that drain the energy of teachers.

We have been subjected to headlines and attacks for years, interwoven with such refreshing apparitions as, for example, the reviews in this MONTHLY, May 1979, pp. 401–412, and many excellent articles in the MAA and NCTM journals. These and other professional magazines provide much-needed help, professional strength, and moral support to mathematics teachers who are honest in their efforts and anxious to improve themselves and their work. These are troubled times, and we must face the problems that go far beyond the classroom yet directly affect the very functioning of even the best teachers. We can and we must improve the conditions for learning; we must be willing to learn from our mistakes and find strength in our successes. So let's turn from the occasional newspaper account and findings of committees to a survey of the school scene, where by no means all is bad, as I see it.

What is the state of mathematics in our schools, especially in the high schools? I want, first, to emphasize the positive—it gets little coverage by the news media; second, to turn to the already much-publicized and often justified criticisms; and, third, to express my concerns and hopes.

Let me quote part of the News Release of the Mathematical Association of America, July 12, 1977: "A team of eight U.S. High School Students won First Place in the 19th International Mathematical Olympiad, held in Belgrade, Yugoslavia, July 3 and 4, 1977." The team is chosen by a two-stage national competition. On March 8, 1977, 341,473 secondary school students from 6,137 schools in the USA and Canada participated in the twenty-eighth Annual High School Examination. About 100 students who earned top scores in this examination were invited to take part in the 6th USA Mathematical Olympiad, May 3, 1977. The eight winners of this examination, "one of the most difficult mathematical challenges given to high school students anywhere," made up the U.S. team. This was the fourth U.S. team to compete in the International Mathematical Olympiad (IMO), the first to win top honors.

Yes, the training for this team was probably decisive in its success. But this period of intensive study could only deepen and widen the competence of the team members and sharpen their minds. The foundation was there—it had been laid long ago, systematically and consistently. The MAA Examination is limited to precalculus mathematics with emphasis on intermediate algebra and geometry; it requires the students' ability and ingenuity to use what has been learned. For each of the participants the Examination is a tremendous learning experience. Those who are less well prepared have a chance to see what it takes to perform well, a little better, or up the ladder and all the way to an honor score and beyond to the rarely achieved maximum score. Each student leaves the examination perhaps frustrated, satisfied, or elated, but each with respect for those who did a superior job individually and for their school and thus respect for academic work itself. And if the student is not a senior, there is another year of learning and more success ahead, provided there is a desire to do hard work.

Hence, this examination *does reflect* what is going on in a great many classrooms in the United States. When over 340,000 students participate and we can boast of such results, nationally and internationally, we can be justly proud. The influence of family and school, but, above all, the self-discipline of the individual and a commitment to hard work, are the ingredients for success. To those who feel that the top performers do well in spite of outside influences—above all, in spite of us teachers—let me say that this is true in rare cases only. There are few geniuses; however, there are many others, to whom classwork does make a difference. They can be helped, inspired, *taught*, and led to success. In 1978 the USA team earned second place in the IMO, and, in 1979, fifth place. However, the participation increased: in 1978, 6,342 schools registered 370,414 students, and in 1979, 6,425 schools registered 377,764 students.

Mathematics competition has also been on the increase on the local, regional, and state levels. It has contributed to a good atmosphere in the classroom. The day after a meet one can

hear, "How did *we* do?" from boys and girls whose awareness is surprising and very reassuring.

Let me point out another program, an educational phenomenon, that came into its own in 1955 with the "Kenyon Plan" as its ancestor: the Advanced Placement Program. It has not only survived the trying sixties, but has grown steadily without compromising its standards. The single most important contribution, I believe, is that the AP Program can set the tone in a school. If properly conceived, it begins with the mapping out of a meaningful program in the elementary school grades and reaches all the way into high school, with its climax the Advanced Placement Examination in grade 12. If, as is often the case, the K-8 program is not designed to give the student the width and depth in preparation, it is possible for able students to accelerate at some point in high school. The precalculus courses, however, must not be narrow; acceleration must not be vertical only.

In June 1979, there were over 24,700 AP-Calculus Exams, of a total of 139,000 examinations in 21 disciplines, with 750 faculty members grading the essay part of the tests. Secondary school and college teachers work together, exchanging ideas, problems, hopes, and fears; it is like a training session at its best, a bonus of the AP Program, a boost to academic work. Added experience and enthusiasm is brought back to the classroom for another year.

The number of AP tests is staggering. Maybe one-fifth or one-sixth of the participants should not have attempted such a course. The opponents of this program, however, forget the remaining four-fifths or five-sixths and the fact that for years the Advanced Placement Program has saved many potential high school dropouts and helped to develop great talents.

What about the large, solid middle group of our school population, called the under-achievers? Have we ever succeeded in showing them how to work up to capacity? Are we demanding enough of our children, at every level? Do we give them a chance to experience well-earned success, the joy of a clear view after a steep climb?

Just think for a moment of the hours of practice and effort expended by the musician—do all of them reach Carnegie Hall? By the ice skater, skier, swimmer—can all of them qualify for the Olympic Games? The tennis or football pro—how many are frustrated by not "making it"? Yet how we do enjoy, admire, and respect the performances of those who belong to the elite! How much recognition did the IMO team get? Even in the mathematics community, how many are aware of this competition? Another thought: when the hoped-for success, in spite of the good efforts of the music mentor, tennis teacher, etc., does not materialize, can we change the outcomes by changing the keys of the piano, the strings of the violin, the scoring of the judges, or the rules of the game?

I have taken a long time to come to my second point: the criticisms hinted at earlier. Take the last remarks above and compare them to the situation in the schools. We are surrounded by "experts," judges, and critics: the test was too hard, the homework assignment too long, the textbook not readable, and *the teacher* . . . , and the *New Math* . . . ! (What an unfortunate choice the term "New Math" was!) There are, of course, justified criticisms—but we teachers, and I am proud to be one, and our professional organizations, the NCTM and the MAA, are our own severe, serious, and concerned critics. We are very much aware that a good deal in our classrooms is in dire need of review and change. I have been disturbed when some students, for whom general high school courses in low-ability groups were a struggle, reach college and come back and tell me that they are Education Majors! I shudder. They are purposeful, they have developed personalities and attitudes that would make them valuable for many jobs—but let it not be teaching! In the classroom they are unequal to the task. And so we hear demands for minimum competency tests not only for high school graduation but also for teaching certification. Long before certification it should be decided whether a person has what it takes to be a good teacher, and the decision must come from those in charge of teacher training; it is their grave responsibility.

In a series of articles (for example, see the *Mathematics Teacher*, October 1976), distinguished mathematicians have attempted to establish the direction that mathematics education will take

in the future. There is no easy answer; there is no *one* answer. The previously mentioned articles in this MONTHLY and the need for remedial mathematics courses at the college level show either that the high school diploma does not mean much or that college admission policies ignore formerly required qualifications. These, no doubt, are critical facts. We are also aware that, for more than ten years, SAT scores have gone down. However, to blame the "New Math" for declining scores is entirely wrong. In this vast country many areas unfortunately were not touched by the "New Math"; names like Max Beberman and E. G. Begle and the significance of SMSG are unknowns.

The goals for the reforms, starting before 1960, are as valid today as they were then: to bring about a change in content for grades K-12, emphasizing student understanding and participation, and, above all, the training of teachers. Teachers attended well-conducted summer workshops, NSF Institutes, in-service training sessions. They brought back to the classroom enthusiasm and good up-to-date mathematics and dedication to the job. What a tremendous *positive* influence this period had on the good students involved in this reform. The students were challenged, they learned to read mathematics, they were excellent critics of the SMSG Sample Textbooks, and they found errors and so participated in revisions. In the hands of a good teacher, the content and the vocabulary could be adjusted—even the average student could be included. Of course, it takes a good teacher and good teaching too; I have never underestimated their importance on any level from kindergarten to college. I wonder how many of the loudest critics of the "New Math" have ever examined the SMSG *Statistics for Elementary and Intermediate Grades*? What a chance to learn new ideas and arithmetic skills! Or the SMSG *Elementary Functions* text for an 11th or 12th grade course?

Alas, the training of teachers for this task, limited to begin with, soon left much to be desired and finally was eliminated. At a 1970 Conference on the Goals for Mathematics Education in the Seventies, Dr. Begle remarked, "Our 'New Math' experience of the 1960's has taught us a great deal about how to teach better mathematics, but very little about how to teach mathematics better."

Instead of the lengthy story above, to "stand up" for the 1960 reforms I could have just given the test scores of my students, going back to the academic year 1959–60, when I first taught an experimental SMSG Algebra I course, up to June 1979; the current scores concern me too—they are too high—yet they speak for the SMSG texts! So I do think that it is high time to stop acid public attacks on the "New Math" and, rather, honor the memory of Professor Begle and thank him for his outstanding contributions. It is too bad that the tremendous effort that went into the creation of sample textbooks could not have been followed by an equal and equally important effort of teacher training.

Costly studies have been conducted to find the cause for the decline of SAT scores. In my humble opinion, but firm conviction, these low scores are a direct consequence of the school scene created some 15 years ago by various pressures, in the face of weak leadership and loss of our sense of responsibility. From college level down to the junior high school, and by no means involving students only, an anti-academic, "anti-everything" atmosphere was fast spreading. Colleges were dropping CEEB score requirements; they substituted other "credentials," all the way to the art of baking cookies—the cookies crumbled! In the high schools the concept of open campus was introduced; required supervised study hall was eliminated; smoking areas were set aside; arena scheduling was adopted; it was more important to entertain the students than to teach them. How can we expect school work, SAT scores, or Achievement Test scores to remain on the same level? No school was exempt from the negative influence of this social upheaval. It is alarming when the mathematics text is too difficult for the nonreader, the nonmotivated, the child who has been passed from grade to grade and has learned nothing. Such children appeared next in algebra and geometry classes and, by now, all the way into college and beyond. The top sections are shrinking; the number of average, low, and newly created remedial sections are taking over. How easy the solution of these important problems if it were the "New Math" that was the culprit!

Third, and finally, my concerns and hopes; I have referred to them along the way. Emphatically, I am concerned that, when we come up with good ideas and the success is not immediate, we lose interest and give up. It would take too long to enumerate the innovations and give an account of the possible implications, starting with the teaching machine, programmed learning, modular scheduling, team teaching, etc. We built lecture halls for large-group instruction only to argue next that *the* way of teaching is Individualized Instruction in its many versions and modifications.

We must learn that what is good for one teacher may not work for another in a different situation. It is not one method versus another, but possibly a combination of approaches. We, as teachers, must be convinced of what we are doing and of its importance in terms of what we are trying to accomplish. We have a responsibility to the sequential nature of mathematics courses, at least in the present setup, and we must not let our own likes and dislikes dictate.

We must know that arithmetic is important, that everybody must develop a feeling for number, magnitude, the facility for computation, with the degree of difficulty appropriate for the grade level. But we must *not* give in to meaningless pressures and train Living Computers. The decision when and how to use the calculator—provided it is available—must be left to the responsible, well-informed teacher. This important gadget, a book, a teaching aid, the computer are all important and as effective as the teacher who uses them. There are excellent films and other aids, and we use mimeographed worksheets by the ton to reach down to the low reading level. All is done in an effort to keep going or to make up for lost time. One of my deep concerns is that too much is done *for* the students and little *with* or *by* them. Some classrooms or "learning centers" have turned into toyshops, with the teacher the toyshop operator—an expensive waste of time. The greatest danger in this nurturing of student apathy is the prefabricated packet and, to some degree, the prefabricated "beautifully planned and written or drawn" overhead projector materials—no student participation! The students remain passive, follow a step-by-step procedure, perpetuate their laziness; there is no thinking, hence no learning, going on.

Now comes the public cry for more school structure, accountability, back-to-basics—concerns that are an outgrowth of our economic and social and political state, by no means a sudden love for learning or academics. Our response to these demands has not been forceful enough, and we have lost dangerous ground. The answers are not simple, but they must come from the School Mathematics Community, where all levels work together as they did so successfully during the SMSG period. Our children must be prepared to handle an uncertain future in an unstable economic period. When we can do so well for some, as the MAA Examination, the Olympiad Team, and the AP program indicate, we can and we *must* do a lot better for all who *can* be taught.

One final concern: a careful reexamination of federal law PL94–142 is an absolute must, before it tears the school work apart. "Special needs" suggests special training for those in charge. We do not expect the oculist to perform heart surgery! Is the teacher suddenly an expert in medicine, psychology, etc.? The law requires periodic core evaluation of each child in a class. Its demands on teachers, away from the already short-changed average student, have proved a disaster. I do not believe that we needed more government involvement, a Department of Education. It may bring votes; will it help us in our endeavors?

There is hope for the school situation if we give the necessary attention to *teacher training*. Teaching is hard work, probably one of the most difficult jobs, but also one of the most rewarding. We must try to attract the most able young people who are willing to work hard, who are enthusiastic, who can inspire the students, and who can take them as far as their ability and desire to learn will allow. We can, we *must*, create the right learning atmosphere, or "minimum competency" will be our maximum output.

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MODERN MULTIPLIER RULES

B. H. POURCIAU

I. Multiplier Rules and Separation

0. Scenic Overlook. In undergraduate advanced calculus the study of constrained optimization is ruled by a tradition just begging to be broken: the constraints are always equations, never *inequalities*. In particular, while the Euler-Lagrange Multiplier Rule for equality constraints is a nearly universal topic, multiplier rules admitting inequality constraints are ignored. Twenty-five years ago, when results concerning inequality constrained optima were fairly new, when the interested audience was small because the importance of these results was unclear, and when the existing proofs were tedious and opaque, this tradition was fully justified. But not today. The theory of inequality constrained optima is now an extensive, expanding field with its own name —nonlinear programming. Multiplier rules for inequality constraints (especially the famous Karush-Kuhn-Tucker Theorems) are fundamental to modern mathematical economics; in fact, according to David Gale [**15**, p. 22], they provide the "single most important tool in modern economic analysis both from the theoretical and computational point of view." Extended to Banach spaces, versions of these rules are equivalent to basic necessary conditions, like the Pontryagin Maximum Principle, in optimal control theory. Moreover, the proofs of these multiplier rules, *if the geometry is viewed in the right space*, are both transparent and reasonably short.

What is this "right" space? Well, suppose, for example, that U is an open subset of real euclidean n-space R^n, p and q are nonnegative integers, the $p+q+1$ real functions $\phi_0,\phi_1,\ldots,\phi_p,\phi_{p+1},\ldots,\phi_{p+q}$ are defined on U, and we wish to find some $a\in U$ minimizing ϕ_0 subject to the p **inequality constraints** $\phi_1\leqslant 0,\ldots,\phi_p\leqslant 0$ and the q **equality constraints** $\phi_{p+1}=0,\ldots,\phi_{p+q}=0$. By this we mean $a\in S$ and $\phi_0(a)\leqslant\phi_0(x)$ for every x in S, where S stands for the set $\{x\in U:\phi_i(x)\leqslant 0 \text{ when } i=1,\ldots,p \text{ and } \phi_i(x)=0 \text{ when } i=p+1,\ldots,p+q\}$ of **feasible points**. *If the geometry of this constrained optimization problem is viewed in the image space R^{p+q+1} of the mapping $\Phi=(\phi_0,\phi_1,\ldots,\phi_p,\phi_{p+1},\ldots,\phi_{p+q})$ taking x in U to $(\phi_0(x),\phi_1(x),\ldots,\phi_p(x),\phi_{p+1}(x),\ldots,\phi_{p+q}(x))$, then the existence of such an optimal solution $a\in U$ is characterized by the empty intersection of two certain sets, and the associated multiplier rule follows from separating with a hyperplane two related convex sets.* Besides their geometric appeal and comparative brevity, derivations based on this scheme apply with only minor alterations when the domain of the functions ϕ_i is infinite-dimensional, because the analysis is not done in the domain, but in the image space R^{p+q+1} of the mapping Φ. Thus, with no difficulty, the multiplier rules proved in Parts II and IV of this paper become fundamental necessary conditions for the existence of constrained optima in the calculus of variations and optimal control theory.

In this article we motivate and clarify this approach to necessary conditions in optimization theory before using it to establish four basic multiplier rules. In Part I, after we look at a few interesting applications, some pictures in the plane illustrate the geometry behind our later demonstrations and suggest the subsequent precise statements of the multiplier rules. Then, after stopping to record and prove a separation theorem for convex subsets of R^n, we look at the connection between multiplier rules and separation and sketch the common scheme of the demonstrations. Part II contains the proofs of the Convex Multiplier Rule and the John Multiplier Rule, necessary conditions for the existence of an optimal solution when no equality

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constraints are present (the case $q=0$). Of these, the first assumes $\phi_0,\phi_1,\ldots,\phi_p$ are convex but not necessarily differentiable, while the second assumes $\phi_0,\phi_1,\ldots,\phi_p$ are differentiable but not necessarily convex.

We mentioned that multiplier rules follow from separating two particular convex sets. When no equality constraints are present, this separation is easy to establish. To prove the sets are separated when equality constraints are permitted, the key idea is the preservation of "interiority," discussed in Part III. The proofs of the Carathéodory Multiplier Rule and the Carathéodory-John Multiplier Rule are then recorded in Part IV. Only equality constraints are allowed (the case $p=0$) in the Carathéodory Multiplier Rule, while the Carathéodory-John Multiplier Rule admits both inequality and equality constraints. A trivial corollary of the Carathéodory Multiplier Rule is the Euler-Lagrange Multiplier Rule; an easy corollary of the Carathéodory-John Multiplier Rule is the famous Karush-Kuhn-Tucker Theorem, well known especially to mathematical economists.

While multiplier rules for inequality constrained optimization problems are the principal theme of this article, a leitmotif involves two other topics our undergraduates sadly fail to see. One is the separation of convex subsets of R^n, which is fundamental in nonlinear programming, the calculus of variations, optimal control theory, game theory, and mathematical economics and which provides a first (finite-dimensional) glimpse of the Hahn-Banach Theorem of functional analysis. The other is strong differentiability, a smoothness condition lying between differentiability and continuous differentiability and enjoying certain technical advantages over both.

1. Applications.

Here is one of the multiplier rules proved later in this article.

JOHN MULTIPLIER RULE. *Suppose U is an open subset of R^n, and $\phi_0,\phi_1,\ldots,\phi_p$ are $p+1$ real functions on U, each differentiable at $a\in U$. Whenever $a\in U$ minimizes ϕ_0 subject to the inequality constraints $\phi_1\leqslant 0,\ldots,\phi_p\leqslant 0$, some nonzero $l=(\lambda_0,\lambda_1,\ldots,\lambda_p)$ in R^{p+1} satisfies*

(i) *if $\phi=\sum_{i=0}^{p}\lambda_i\phi_i$, then $\phi'(a)=0$,*
(ii) *$\lambda_i\geqslant 0$ when $i=0,1,\ldots,p$, and*
(iii) *$\lambda_i\phi_i(a)=0$ when $i=1,\ldots,p$.*

Before we come to the use of such a result and while we have it here before us for reference, we should comment on some associated words and phrases. As we mentioned in Section 0, to say the point $a\in U$ minimizes ϕ_0 subject to the inequality constraints $\phi_1\leqslant 0,\ldots,\phi_p\leqslant 0$ is to mean $\phi_0(a)\leqslant\phi_0(x)$ for every x in U which satisfies the p constraints, that is, for every x in the feasible set $S=\{x\in U:\phi_i(x)\leqslant 0 \text{ when } i=1,\ldots,p\}$. For short we sometimes say $a\in U$ is an optimal point or a constrained minimum point. In any case, notice this is a *local*, not a global, property of the function ϕ_0. According to the John Multiplier Rule, this local property leads to a list of necessary conditions. The numbers λ_i are called Lagrange multipliers. The function $\phi=\sum_{i=0}^{p}\lambda_i\phi_i$ is called the Lagrangian. Conclusion (i) tells us the constrained minimum point is a critical point of the Lagrangian. (By $\phi'(a)$, we mean the derivative or gradient of ϕ at $a\in U$.) Conclusion (iii), known as the complementary slackness condition, says that inactive constraints (those with $\phi_i(a)<0$) have zero multipliers.

Plainly, then, a multiplier rule records certain properties that necessarily occur at any constrained minimum point. Of what use is such a theorem? Well, suppose we call a point in U a *candidate* if it satisfies the p inequality constraints and admits a nonzero $l=(\lambda_0,\lambda_1,\ldots,\lambda_p)$ fulfilling the requirements (i), (ii), and (iii). Computationally, the value of a multiplier rule is that it cuts the search for constrained minimum points down to the candidates in U. Once identified, the candidates can be examined for other (possibly sufficient) signs of optimality. Sometimes geometric, physical, or numerical evidence puts forward a certain point as a possible constrained minimum point, and a multiplier rule is used as an elimination test: if it is not a candidate, it is not optimal. But the value of multiplier rules is clearest, perhaps, in nonnumerical situations

where the candidacy conditions may imply a significant general relationship that must be obeyed at any optimal point.

Let's look at two examples.

A. Consider this famous inequality for finite sums:

HÖLDER'S INEQUALITY. *Suppose* $\xi_1,\ldots,\xi_n$, $\eta_1,\ldots,\eta_n$ *are positive numbers*, $p>1$, *and* $q=p/(p-1)$. *Then*

$$\sum_{i=1}^{n}\xi_i\eta_i \leqslant \left(\sum_{i=1}^{n}\xi_i^p\right)^{1/p}\left(\sum_{i=1}^{n}\eta_i^q\right)^{1/q}.$$

We can derive this inequality from the John Multiplier Rule using the following strategy. Let U be the collection of all $x=(\xi_1,\ldots,\xi_n)$ with every coordinate ξ_i positive, and choose any $y=(\eta_1,\ldots,\eta_n)$ in U. On U define the functions ϕ_0 and ϕ_1 by

$$\phi_0(x)=\sum_{i=1}^{n}\xi_i\eta_i \quad\text{and}\quad \phi_1(x)=\sum_{i=1}^{n}\xi_i^p-1.$$

If the maximum value of ϕ_0 subject to the constraint $\phi_1\leqslant 0$ is $(\Sigma_{i=1}^n\eta_i^q)^{1/q}$, then Hölder's inequality follows easily. For in this case, whenever $x=(\xi_1,\ldots,\xi_n)$ belongs to U, we can set $\bar{x}=x/(\Sigma_{i=1}^n\xi_i^p)^{1/p}$, so that $\phi_1(\bar{x})\leqslant 0$, and obtain

$$\phi_0(\bar{x})\leqslant\left(\sum_{i=1}^{n}\eta_i^q\right)^{1/q},$$

which, written out broadly, is Hölder's inequality.

The John Multiplier Rule will aid us in computing the maximum value of ϕ_0 subject to the constraint $\phi_1\leqslant 0$. Since the feasible set $S=\{x\in U:\phi_1(x)\leqslant 0\}$ is compact and ϕ_0 is continuous, ϕ_0 does achieve a maximum value on S, say, at $a\in S$. At this point the function $-\phi_0$ is minimized subject to the constraint $\phi_1\leqslant 0$, and the John Multiplier Rule guarantees the existence of multipliers λ_0 and λ_1, not both zero, such that

(i) if $\phi=-\lambda_0\phi_0+\lambda_1\phi_1$, then $\phi'(a)=0$

(ii) $\lambda_0\geqslant 0$, $\lambda_1\geqslant 0$ and

(iii) $\lambda_1\phi_1(a)=0$.

From (i) the partial derivative of ϕ with respect to each coordinate ξ_i must vanish at $a=(\alpha_1,\ldots,\alpha_n)$:

(a) $-\lambda_0\eta_i+\lambda_1 p\alpha_i^{p-1}=0 \qquad (i=1,\ldots,n)$.

By (ii) both λ_0 and λ_1 are nonnegative. In fact, though, both are positive. For if $\lambda_0=0$, then (a) implies the contradiction $a=(\alpha_1,\ldots,\alpha_n)=0$, and if $\lambda_1=0$, then (a) implies the contradiction $y=(\eta_1,\ldots,\eta_n)=0$. Since $\lambda_0>0$, we may assume $\lambda_0=1$ with no loss of generality. Since $\lambda_1>0$, condition (iii) tells us $\phi_1(a)=0$, that is,

(b) $\Sigma_{i=1}^n\alpha_i^p=1$.

Our object now is to compute the maximum value $\phi_0(a)$ in terms of y. From (a) and (b),

$$\begin{aligned}\phi_0(a)&=\sum_{i=1}^{n}\alpha_i\eta_i\\&=\lambda_1 p\sum_{i=1}^{n}\alpha_i^p\\&=\lambda_1 p.\end{aligned}$$

But if $q=p/(p-1)$, then (a) implies

$$\eta_i^q=(\lambda_1 p)^q\alpha_i^p,\qquad (i=1,\ldots,n);$$

and summing over i with a subsequent application of (b), we find

$$\sum_{i=1}^{n} \eta_i^q = (\lambda_1 p)^q.$$

The maximum value of ϕ_0 subject to the constraint $\phi_1 \leqslant 0$ is therefore $(\sum_{i=1}^{n}\eta_i^q)^{1/q}$, and this, as we have said, establishes Hölder's inequality.

B. (Adapted from Baumol [3].) During certain hours of the day (the peak periods) assume the demand on a perfectly competitive firm exhausts its capacity, while in other hours (the off-peak periods) excess capacity remains. If the firm is maximizing its profit, how is the marginal operating cost related to the unit price?

For each $i = 1,2,\ldots,24$, let $\xi_i > 0$ be the quantity demanded in hour i and π_i the unit price in hour i. (Perfect competition implies $\partial\pi_i/\partial\xi_i = 0$.) Suppose $C(x)$ is the daily cost of producing according to the demand schedule $x = (\xi_1,\ldots,\xi_{24})$, and suppose $c(\mu)$ is the daily cost of holding the hourly output capacity at $\mu > 0$. If the firm is maximizing its profit, then (x,μ) must minimize the negative profit,

$$\phi_0(x,\mu) = -\sum_{i=1}^{24} \pi_i\xi_i + C(x) + c(\mu),$$

subject to the twenty-four capacity constraints,

$$\phi_i(x,\mu) = \xi_i - \mu \leqslant 0 \qquad (i = 1,\ldots,24).$$

Assuming ϕ_0 and each constraint ϕ_i are differentiable at (x,μ), the John Multiplier Rule guarantees the existence of multipliers $\lambda_0,\lambda_1,\ldots,\lambda_p$ not all zero satisfying

(i) if $\phi = \sum_{i=0}^{24}\lambda_i\phi_i$, then $\phi'(x,\mu) = 0$

(ii) $\lambda_i \geqslant 0$ for $i = 0,1,\ldots,24$ and

(iii) $\lambda_i\phi_i(x,\mu) = 0$ for $i = 1,\ldots,24$.

By (i) the partial derivative of ϕ with respect to each ξ_i must vanish at (x,μ):

(a) $-\lambda_0\pi_i + \lambda_0 \dfrac{\partial C}{\partial \xi_i}(x) + \lambda_i = 0;$

as must the partial derivative of ϕ with respect to μ at (x,μ):

(b) $\lambda_0 \dfrac{\partial c}{\partial \mu}(\mu) - \sum_{i=1}^{24}\lambda_i = 0.$

The equations in (iii) are

(c) $\lambda_i(\xi_i - \mu) = 0 \qquad (i = 1,\ldots,24).$

Notice $\lambda_0 \geqslant 0$ from (ii); yet if $\lambda_0 = 0$ then (a) contradicts the guarantee that not all the λ_i are zero, so $\lambda_0 > 0$, and in this case we may assume with no loss that $\lambda_0 = 1$. In an off-peak hour i, we have $\xi_i < \mu$; hence $\lambda_i = 0$ by (c), and thus (a) becomes

$$\pi_i = \frac{\partial C}{\partial \xi_i}(x).$$

On the other hand, in a peak hour i we have $\xi_i = \mu$, so (a) is

$$\pi_i = \frac{\partial C}{\partial \xi_i}(x) + \lambda_i,$$

where $\lambda_i \geqslant 0$ from (ii) and $\sum_{i=1}^{24}\lambda_i = (\partial c/\partial\mu)(\mu)$ from (b). Summing up, the John Multiplier Rule had led us to this economic law: *in an off-peak period, a profit maximizing firm in a perfectly competitive industry experiences a marginal operating cost equal to the unit price*; *in a peak period, the marginal operating cost is below or equal to the unit price, with the excesses over the marginal operating cost summing to the marginal cost of capacity.*

2. Pictures. As we mentioned in Section 0, an intrinsic geometric consequence of constrained optimality, occurring in the image space R^{p+q+1} of the mapping $\Phi=(\phi_0,\phi_1,\ldots,\phi_p,\phi_{p+1},\ldots,\phi_{p+q})$, underlies all our later multiplier rule derivations. If we wish to minimize ϕ_0 subject to the single inequality constraint $\phi_1\leqslant 0$ or the single equality constraint $\phi_1=0$, pictures of the geometry are easy to draw, because the image space of the mapping $\Phi=(\phi_0,\phi_1)$ is the plane. For two such examples, we shall sketch the geometric configuration in the image space R^2 and see how it leads to an appropriate multiplier rule. For more sketches similar to those below, see Hestenes [27, Chapter 6].

Let $\cdot$ denote the dot product in R^2. Then, given any real number γ and any nonzero l in R^2, the subset $H=\{y\in R^2: l\cdot y=\gamma\}$ is a line. We abbreviate the closed half-plane $\{y\in R^2: l\cdot y\geqslant\gamma\}$ by H^+ and the closed half-plane $\{y\in R^2: l\cdot y\leqslant\gamma\}$ by H^-. Two subsets C_1 and C_2 of R^2 are said to be **separated** if for some line H we have $C_1\subset H^+$ and $C_2\subset H^-$.

EXAMPLE A. Consider the problem of minimizing in R^2 the function ϕ_0 subject to the inequality constraint $\phi_1\leqslant 0$, where $\phi_0(\xi_1,\xi_2)=|\xi_1+\xi_2+2|$ and $\phi_1(\xi_1,\xi_2)=\max\{-\xi_1,-\xi_2\}$. Clearly the optimal solution is $(\xi_1,\xi_2)=(0,0)$. Let us sketch the image $\Phi(R^2)$ of R^2 under the mapping $\Phi=(\phi_0,\phi_1)$, paying close attention to the relative location of the image $\Phi(0,0)=(2,0)$ of the optimal solution. From the identity

$$\xi_1+\xi_2+2\max\{-\xi_1,-\xi_2\}=|\xi_1-\xi_2|,$$

we infer that $\eta_0=\phi_0(\xi_1,\xi_2)$ and $\eta_1=\phi_1(\xi_1,\xi_2)$ satisfy $\eta_0+2\eta_1-2=|\xi_1-\xi_2|\geqslant 0$ whenever $\xi_1+\xi_2+2\geqslant 0$. On the other hand, if $\xi_1+\xi_2+2\leqslant 0$, then $-\eta_0+2\eta_1-2\geqslant 0$. From these inequalities it

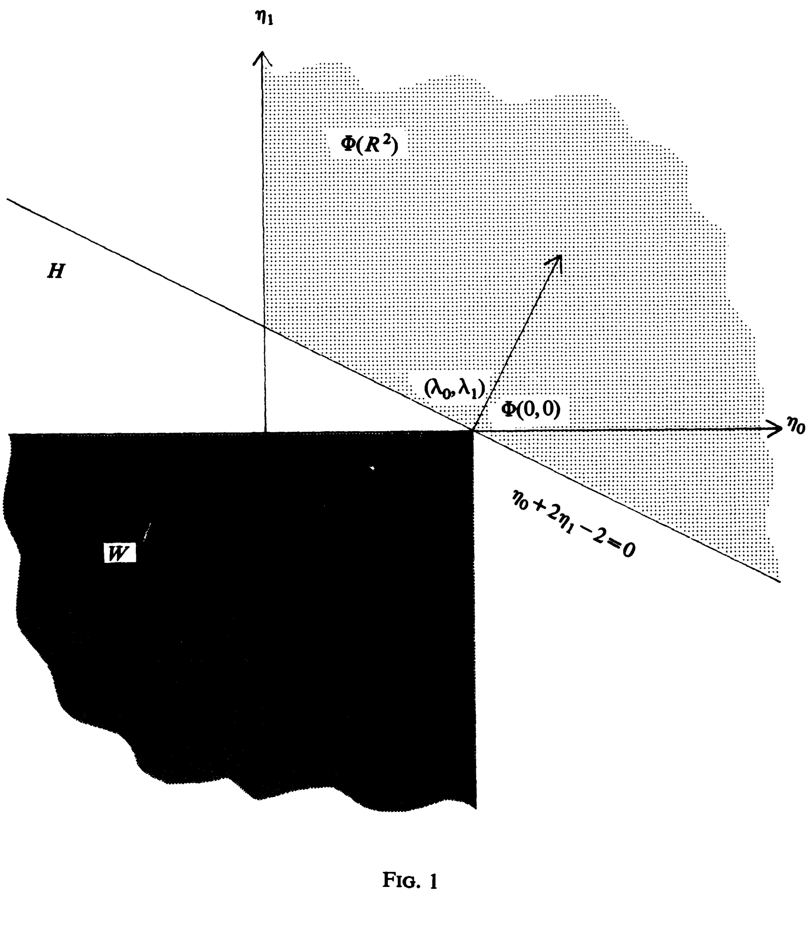

FIG. 1

follows that the image of the plane under Φ is the subset

$$\Phi(R^2)=\{(\eta_0,\eta_1)\in R^2:\eta_0\geqslant 0,\eta_0+2\eta_1-2\geqslant 0\}.$$

(See Fig. 1.)

Denote the subset $\{(\eta_0,\eta_1)\in R^2:\eta_0<2,\eta_1\leqslant 0\}$ by W, and observe that $\Phi(R^2)\cap W=\varnothing$ and $(2,0)=\Phi(0,0)\in\Phi(R^2)\cap\overline{W}$. (Here $\overline{W}$ abbreviates the topological closure of W.) Moreover, the line

$$H=\{(\eta_0,\eta_1)\in R^2:\eta_0+2\eta_1-2=0\}$$
$$=\{(\eta_0,\eta_1)\in R^2:(1,2)\cdot(\eta_0,\eta_1)=2\}$$

passes through the image $(2,0)$ of the optimal solution $(0,0)$ and separates the sets $\Phi(R^2)$ and W. In fact $\Phi(R^2)\subset H^+$ and $W\subset H^-$. Letting $(\lambda_0,\lambda_1)=(1,2)$, the first inclusion implies the function $\phi=\lambda_0\phi_0+\lambda_1\phi_1$ satisfies $\phi(\xi_1,\xi_2)\geqslant 2$ for every point (ξ_1,ξ_2) in R^2. But $\Phi(0,0)\in H$, so $\phi(0,0)=2$, and we infer

(i) if $\phi=\lambda_0\phi_0+\lambda_1\phi_1$, then $\phi(0,0)\leqslant\phi(\xi_1,\xi_2)$ for every (ξ_1,ξ_2) in R^2.

In other words, the optimal solution $(0,0)$ to the constrained problem is an *unconstrained* minimum point for the function ϕ. Also observe

(ii) $\lambda_0\geqslant 0$, $\lambda_1\geqslant 0$, and

(iii) $\lambda_1\phi_1(0,0)=0$.

In Section 3 we state the Convex Multiplier Rule for minimizing a function ϕ_0 subject to p inequality constraints $\phi_1\leqslant 0,\ldots,\phi_p\leqslant 0$, where $\phi_0,\phi_1,\ldots,\phi_p$ are convex functions defined on a convex subset U of R^n. (We define convex sets and functions in Section 3.) The conditions (i), (ii), and (iii) obtained geometrically above are the conclusions of the Convex Multiplier Rule for $p=1$. Notice the function $\phi=\lambda_0\phi_0+\lambda_1\phi_1$ in this example is not differentiable at the optimal solution $(0,0)$, so condition (i) cannot be replaced with the statement $\phi'(0,0)=(0,0)$.

EXAMPLE B. On the open disk $U=\{(\xi_1\xi_2)\in R^2:\xi_1^2+\xi_2^2<1\}$, suppose we minimize ϕ_0 subject to the inequality constraint $\phi_1\leqslant 0$, where $\phi_0(\xi_1,\xi_2)=\xi_2+\xi_1^2+\xi_2^3$ and $\phi_1(\xi_1,\xi_2)=-\xi_2$. The optimal solution is, of course, $(0,0)$. Let Φ be the mapping (ϕ_0,ϕ_1) from U into R^2. Then $\Phi(0,0)=(0,0)$. Since $\eta_0=\phi_0(\xi_1,\xi_2)$ and $\eta_1=\phi_1(\xi_1,\xi_2)$ satisfy $\eta_0+\eta_1+\eta_1^3=\xi_1^2\geqslant 0$, the image of U under Φ is the subset

$$\Phi(U)=\{(\eta_0,\eta_1)\in R^2:\eta_0+\eta_1+\eta_1^3\geqslant 0\}.$$

(See Fig. 2.)

Let W be the collection of all points (η_0,η_1) with $\eta_0<0$ and $\eta_1\leqslant 0$ and notice, as in Example A, that $\Phi(U)\cap W=\varnothing$ and $(0,0)=\Phi(0,0)\in\Phi(U)\cap\overline{W}$. But this time there is no line separating $\Phi(U)$ and W. The culprit, of course, is $\Phi(U)$, which does not stay on its own side of the line. We therefore "linearize $\Phi(U)$ about the point $\Phi(0,0)=(0,0)$" by forming the set $A(U)$, where A is the affine mapping from R^2 into R^2 "tangent to Φ at $(0,0)$."

To be more precise, let L be the derivative $\Phi'(0,0)=((\partial\phi_i/\partial\xi_j)(0,0))$ of Φ at $(0,0)$, compute

$$L=\begin{pmatrix}0 & 1\\ 0 & -1\end{pmatrix},$$

and define A on R^2 by $A(\xi_1,\xi_2)=\Phi(0,0)+L(\xi_1,\xi_2)=L(\xi_1,\xi_2)$. (Here $L(\xi_1,\xi_2)$ stands for the matrix product.

$$L\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}$$

written as a row.) Then $A(\xi_1,\xi_2)=\xi_2(1,-1)$, implying $A(U)$ is the line segment

$$A(U)=\{(\eta_0,\eta_1)\in R^2:\eta_0+\eta_1=0,-1<\eta_0<1\}.$$

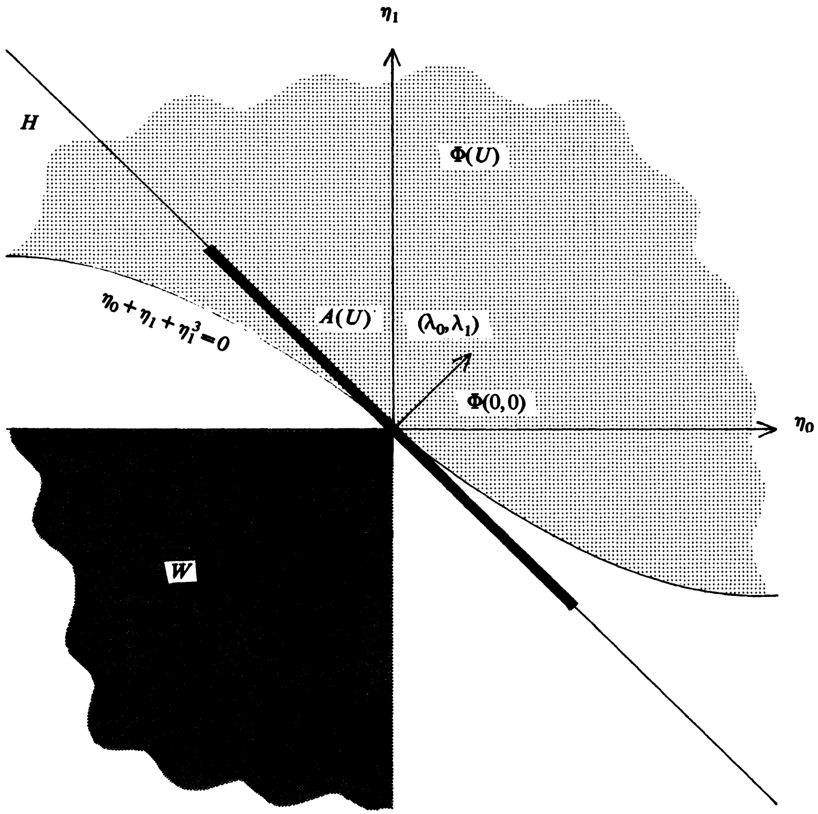

FIG. 2

Observe that $A(U)$ and W are separated by the line

$$H=\{(\eta_0,\eta_1)\in R^2:\quad \eta_0+\eta_1=0\}$$
$$=\{(\eta_0,\eta_1)\in R^2:\quad (1,1)\cdot(\eta_0,\eta_1)=0\}.$$

In fact, $A(U)\subset H\subset H^+$ and $W\subset H^-$. Letting $(\lambda_0,\lambda_1)=(1,1)$, we find $\lambda_0\phi_0'(0,0)+\lambda_1\phi_1'(0,0)=(0,0)$. (This is no surprise since $A(U)\subset H$, and $\phi_i'(0,0)$ is the ith row of A.) In other words:

(i) if $\phi=\lambda_0\phi_0+\lambda_1\phi_1$, then $\phi'(0,0)=(0,0)$.

Said differently, the optimal solution $(0,0)$ to the constrained problem is a critical point of the function ϕ. Also notice

(ii) $\lambda_0\geqslant 0$, $\lambda_1\geqslant 0$ and

(iii) $\lambda_1\phi_1(0,0)=0$.

In Section 3 we record the John Multiplier Rule for minimizing a function ϕ_0 subject to p inequality constraints $\phi_1\leqslant 0,\dots,\phi_p\leqslant 0$, where $\phi_0,\phi_1,\dots,\phi_p$ are defined on an open subset U of R^n and differentiable at the optimal solution. The conditions (i), (ii), and (iii) obtained geometrically above are the conclusions of the John Multiplier Rule for $p=1$.

3. Four Multiplier Rules. In this section we record those multiplier rules proved in Parts II and IV. After the geometric examples above, the statements should seem natural.

For any points x and z in R^n, let $[x,z]$ denote the segment $\{(1-\alpha)x+\alpha z: 0\leqslant\alpha\leqslant 1\}$ joining x to z. Then a subset C of R^n is said to be **convex** provided $[x,z]\subset C$ whenever x and z belong to C. If C is a convex subset of R^n and ϕ is a real function on C, then ϕ is said to be a **convex function** provided

$$\phi[(1-\alpha)x+\alpha z]\leqslant(1-\alpha)\phi(x)+\alpha\phi(z)$$

whenever x and z belong to C and $0\leqslant\alpha\leqslant 1$. Equivalently, ϕ is convex if its "epigraph" $\{(x,\eta)\in R^{n+1}: x\in C, \phi(x)\leqslant\eta\}$ is a convex subset of R^{n+1}.

Recall from Section 1 that the statement, $a\in U$ minimizes ϕ_0 subject to the p inequality constraints $\phi_1\leqslant 0,\ldots,\phi_p\leqslant 0$, means $a\in S$ and $\phi_0(a)\leqslant\phi_0(x)$ for every $x\in S$, where S is the collection $\{x\in U:\phi_i(x)\leqslant 0$ when $i=1,\ldots,p\}$ of feasible points in U.

CONVEX MULTIPLIER RULE. *Suppose U is a convex subset of R^n, and $\phi_0,\phi_1,\ldots,\phi_p$ are $p+1$ convex functions on U. Whenever $a\in U$ minimizes ϕ_0 subject to the inequality constraints $\phi_1\leqslant 0,\ldots,\phi_p\leqslant 0$, some nonzero $l=(\lambda_0,\lambda_1,\ldots,\lambda_p)$ in R^{p+1} satisfies*

(i) *if $\phi=\Sigma_{i=0}^p\lambda_i\phi_i$ and $x\in U$, then $\lambda_0\phi_0(a)=\phi(a)\leqslant\phi(x)$,*
(ii) *$\lambda_i\geqslant 0$ for $i=0,1,\ldots,p$ and*
(iii) *$\lambda_i\phi_i(a)=0$ for $i=i,\ldots,p$.*

COROLLARY. *If for each $i=1,\ldots,p$, ϕ_i is not identically zero on the feasible set $S=\{x\in U:\phi_1(x)\leqslant 0$ when $i=1,\ldots,p\}$, then the conclusions of the Convex Multiplier Rule hold with a positive λ_0. Conversely, if the conclusions of the Convex Multiplier Rule hold with a positive λ_0, then $a\in U$ minimizes ϕ_0 subject to the inequality constraints $\phi_1\leqslant 0,\ldots,\phi_p\leqslant 0$.*

The Convex Multiplier Rule was first proved by Uzawa **[54]** and Karlin **[32]** in 1958. Supposing the functions ϕ_i to be differentiable as well as convex, Kuhn and Tucker **[35]** established the corollary in 1951, and it is now widely known as the Kuhn-Tucker Theorem. In the form above, with only convexity assumed, the corollary is due to Uzawa **[54]** and Karlin **[32]**.

Any condition on the constraint functions of a multiplier rule ensuring a positive λ_0—the restriction in the corollary above is our first example—is called a **constraint qualification**. Let us call a multiplier rule restricted if it contains a constraint qualification and intrinsic if it does not. Commonly, restricted multiplier rules are presented as the fundamental necessary conditions of constrained optimization; intrinsic multiplier rules, when mentioned at all, are often derided as useless theoretical niceties. This point of view, pervasive in the literature, is most unfortunate, for the theory and for applications. On the theoretical side, intrinsic forms apply to any constrained optimization problem of the proper class, not just to those satisfying some ad hoc constraint qualification, and in those cases where a restricted multiplier rule fails to apply, the multiplier λ_0 can still be positive. Intrinsic multiplier rules are also *deeper* than their restricted counterparts: the assumptions of the intrinsic form are weaker, yet, from it, restricted versions follow (usually with ease) as *corollaries*. On the practical side, one often reads in the literature that an intrinsic multiplier rule, because it admits the possibility $\lambda_0=0$, can only be applied in the guise of one of its corollaries, a restricted multiplier rule. But this is false, for it often happens, in cases when a given constraint qualification is either difficult to validate or actually invalid, that the assertion $\lambda_0>0$ is a straightforward consequence of the conclusions of the intrinsic version and the special conditions of the application. A variation of this can occur in applied areas like mathematical economics, where the validity of a constraint qualification is sometimes made an (unnecessary) assumption of a particular theory. For these theoretical and practical reasons, the four basic multiplier rules proved in this article are intrinsic, and multiplier rules with constraint qualifications are obtained as corollaries.

Exchanging now the supposition of convexity for that of differentiability, we have the result below, established (in more generality) by Fritz John [31] in 1948.

JOHN MULTIPLIER RULE. *Suppose U is an open subset of R^n, and $\phi_0,\phi_1,\ldots,\phi_p$ are $p+1$ real functions on U, each differentiable at $a\in U$. Whenever $a\in U$ minimizes ϕ_0 subject to the inequality constraints $\phi_1\leqslant 0,\ldots,\phi_p\leqslant 0$, some nonzero $l=(\lambda_0,\lambda_1,\ldots,\lambda_p)$ in R^{p+1} satisfies*

(i) *if $\phi=\Sigma_{i=0}^p\lambda_i\phi_i$, then $\phi'(a)=0$,*
(ii) *$\lambda_i\geqslant 0$ when $i=0,1,\ldots,p$ and*
(iii) *$\lambda_i\phi_i(a)=0$ when $i=1,\ldots,p$.*

COROLLARY (KARUSH-KUHN-TUCKER THEOREM). *If some h in R^n satisfies $\phi_i'(a)h<0$ for all those $i=1,\ldots,p$ such that $\phi_i(a)=0$, then the conclusions of the John Multiplier Rule hold with a positive λ_0.*

Using a different constraint qualification, Karush [33] proved this corollary in his 1939 master's dissertation. Independently, in 1951, it was obtained by Kuhn and Tucker [35] with the same constraint qualification employed by Karush, and today it is generally known as the Kuhn-Tucker Theorem. Karush-Kuhn-Tucker Theorem is a more accurate title. Kuhn presents an interesting history of this and other nonlinear programming results in [34].

Turning now to equality constrained optimization problems, we have the necessary condition below of Carathéodory [7] and Bliss [5]. When equality constraints are admitted, we suppose the functions involved in the problem are strongly differentiable. For the definition of strong differentiability and an explanation of its presence in equality constrained multiplier rules, see Sections 5 and 8. One can always read "continuously differentiable on U" for the words "strongly differentiable at $a\in U$," since the first condition implies the second.

CARATHÉODORY MULTIPLIER RULE. *Suppose U is an open subset of R^n, and $\phi_0,\phi_1,\ldots,\phi_q$ are $q+1$ real functions on U, each strongly differentiable at $a\in U$. Whenever $a\in U$ minimizes ϕ_0 subject to the equality constraints $\phi_1=0,\ldots,\phi_q=0$, some nonzero $l=(\lambda_0,\lambda_1,\ldots,\lambda_q)$ in R^{q+1} satisfies*

(i) *if $\phi=\sum_{i=0}^{q}\lambda_i\phi_i$, then $\phi'(a)=0$ and*

(ii) $\lambda_0\geq 0$.

COROLLARY (EULER-LAGRANGE MULTIPLIER RULE). *If the derivatives $\phi_1'(a),\ldots,\phi_q'(a)$ are linearly independent, then the conclusions of the Carathéodory Multiplier Rule hold with a positive λ_0.*

I know of only two advanced calculus textbooks that establish the Carathéodory Multiplier Rule (at least in the continuously differentiable case) and obtain the Euler-Lagrange Multiplier Rule as a corollary—Bartle [2] and Sagan [51]—even though the intrinsic Carathéodory form is no harder to prove. Known generally as the Lagrange Multiplier Rule, the corollary was actually used first by Leonhard Euler [14], as Lagrange himself acknowledged in [36]. Happily, the traditional use of the Greek letter lambda for the multipliers λ_i (often called Lagrange multipliers) is consonant with this history: λ for Lagrange, λ for Leonhard.

When both inequality and equality constraints are present, the theorems of Carathéodory and John can be blended into the general multiplier rule below, hidden in the more abstract necessary conditions of Halkin and Neustadt [25], but first explicitly stated and proved for the continuously differentiable case in 1967 by Mangasarian and Fromovitz [39]. Demonstrations of this basic result, different from that in [39], can be found in Halkin [18], Hestenes [27], McShane [41], and Pourciau [47].

CARATHÉODORY-JOHN MULTIPLIER RULE. *Suppose U is an open subset of R^n, and $\phi_0,\phi_1,\ldots,\phi_p$, $\phi_{p+1},\ldots,\phi_{p+q}$ are $p+q+1$ real functions on U, each strongly differentiable at $a\in U$. Whenever $a\in U$ minimizes ϕ_0 subject to the p inequality constraints $\phi_1\leq 0,\ldots,\phi_p\leq 0$ and the q equality constraints $\phi_{p+1}=0,\ldots,\phi_{p+q}=0$, some nonzero $l=(\lambda_0,\lambda_1,\ldots,\lambda_p,\lambda_{p+1},\ldots,\lambda_{p+q})$ in R^{p+q+1} satisfies*

(i) *if $\phi=\sum_{i=0}^{p+q}\lambda_i\phi_i$, then $\phi'(a)=0$,*

(ii) *$\lambda_i\geq 0$ when $i=0,1,\ldots,p$ and*

(iii) *$\lambda_i\phi_i(a)=0$ when $i=1,\ldots,p$.*

COROLLARY (KARUSH-KUHN-TUCKER THEOREM). *If the derivatives $\phi_{p+1}'(a),\ldots,\phi_{p+q}'(a)$ are linearly independent, and there is some h in R^n such that*

(a) *$\phi_i'(a)h<0$ for all those $i=1,\ldots,p$ with $\phi_i(a)=0$, and*

(b) *$\phi_i'(a)h=0$ for $i=p+1,\ldots,p+q$,*

then the conclusions of the Carathéodory-John Multiplier Rule hold with a positive λ_0.

Like the corollary to the John Multiplier Rule, the corollary above is popularly known as the

Kuhn-Tucker Theorem. Proved by Karush [33] in 1939 using the same constraint qualification (the Kuhn-Tucker constraint qualification) later used by Kuhn and Tucker [35] in 1951, it appears in the literature now with various constraint qualifications attached. Reflecting the much longer history of interest in infinite (versus finite) dimensional optimization problems, Bolza [6], in the calculus of variations, had obtained a multiplier rule containing the finite-dimensional Karush-Kuhn-Tucker Theorem as early as 1914! (See Hestenes [27, p. 178].)

4. Separation of Convex Sets. Though fundamental in several areas of modern pure and applied mathematics, convexity theory (the study of convex sets and functions) is neglected in undergraduate analysis. This is difficult to understand, for the subject is not only important but geometrically appealing and accessible as well. The present section is reserved for a discussion of the separation of convex subsets of R^k, a basic element in our derivations of multiplier rules with inequality constraints. For more information on convexity, good sources are Berge [4], Eggleston [13], Mangasarian [38], Roberts and Varberg [49], Rockafellar [50], Stoer and Witzgall [52], and Valentine [55].

Let S be a subset of R^k. A point y in S is said to be an **interior point** of S provided $\delta > 0$ satisfies $N_\delta(y) \subset S$, where $N_\delta(y)$ stands for the open δ-ball $\{w \in R^k : |w-y| < \delta\}$ centered at y. The set of all interior points of S is denoted by $\operatorname{int} S$ and called the **interior** of S. The collection

$$\left\{ \sum_{j=1}^{m} \alpha_j y_j : y_j \in S, \alpha_j \geqslant 0, \quad \sum_{j=1}^{m} \alpha_j = 1, \quad m \geqslant 1 \right\}$$

is abbreviated by $\operatorname{co} S$ and called the **convex hull** of S. It is the intersection of all convex sets containing S.

Given $l = (\lambda_1, \ldots, \lambda_k)$ and $y = (\eta_1, \ldots, \eta_k)$ in R^k, let $l \cdot y$ stand for the euclidean inner product $\sum_{i=1}^k \lambda_i \eta_i$. A subset H of R^k is called a **hyperplane** provided, for some nonzero l in R^k and some real number γ, H has the form $\{y \in R^k : l \cdot y = \gamma\}$. Nonempty subsets C_1 and C_2 of R^k are said to be **separated** if there is a hyperplane H satisfying $C_1 \subset H^+$ and $C_2 \subset H^-$, where $H^+ = \{y \in R^k : l \cdot y \geqslant \gamma\}$ and $H^- = \{y \in R^k : l \cdot y \leqslant \gamma\}$ are the **closed half-spaces** of H. When C_1 and C_2 are separated, and both C_1 and C_2 have nonempty intersections with H, then C_1 and C_2 are said to be **strictly separated.**

LEMMA A. *If C is a nonempty, closed, convex subset of R^k with $0 \notin C$, then C and $\{0\}$ are strictly separated.*

Proof. If d maps R^k into R by $d(y) = |y|^2$, where $|y| = \sqrt{y \cdot y}$, then d assumes a minimum value on the closed set C, say at $l \in C$. (For the *minimum* value it is not necessary to have C compact.) Fix $y \in C$. Then $\alpha \in [0,1]$ implies $l + \alpha(y-l) \in C$ by the convexity of C. Since the minimum value of d on C is assumed at l, we know

$$|l + \alpha(y-l)|^2 \geqslant |l|^2$$

or

$$2\alpha l \cdot y - 2\alpha l \cdot l + \alpha^2 y \cdot y - 2\alpha^2 l \cdot y + \alpha^2 l \cdot l \geqslant 0.$$

Let $\alpha \in (0,1]$, divide by α, and suppose $\alpha \to 0$. Then

$$l \cdot y \geqslant l \cdot l = |l|^2.$$

Because $l \in C$ implies $l \neq 0$, setting $\gamma = \frac{1}{2}|l|^2 > 0$ yields $l \cdot y > \gamma > 0$, exhibiting the desired separation.

LEMMA B. *If C is a nonempty, convex subset of R^k with $0 \notin C$, then C and $\{0\}$ are separated.*

Proof (Berge [4]). For each $y \in C$ let $K(y)$ denote the subset $\{l \in R^k : |l| = 1 \text{ and } l \cdot y \geqslant 0\}$ of the unit sphere in R^k. If $\bigcap_{y \in C} K(y) \neq \varnothing$, then any $l \in \bigcap_{y \in C} K(y)$ satisfies $l \cdot y \geqslant 0$ for all $y \in C$, and

the proof will be complete. Because the sets $K(y)$ are closed subsets of the (compact) unit sphere, it suffices to show that any finite intersection of the sets $K(y)$ is nonempty. Let $y_1,\ldots,y_p$ belong to C. Then the convex hull $\operatorname{co}\{y_1,\ldots,y_p\}$ of the finite set $\{y_1,\ldots,y_p\}$ is a closed, convex subset of R^k, with $0\notin\operatorname{co}\{y_1,\ldots,y_p\}$. By Lemma A, some nonzero $l\in R^k$ satisfies $l\cdot y\geqslant 0$ for each $y\in\operatorname{co}\{y_1,\ldots,y_p\}$ and *a fortiori* for $y_1,\ldots,y_p$, proving that $l\in\bigcap_{j=1}^{p}K(y)$.

LEMMA C. *If C is a nonempty, convex subset of R^k with $0\notin\operatorname{int}C$, then C and $\{0\}$ are separated.*

Proof. Because $0\notin\operatorname{int}C$, there is a sequence $\{y_n\}$ in R^k such that $y_n\notin C$ for all n and $y_n\to 0$. By Lemma B, for each n some l_n in R^k with $|l_n|=1$ satisfies $l_n\cdot y\geqslant l_n\cdot y_n$ for every y in C. A subset of the unit sphere in R^k, the sequence $\{l_n\}$ must possess a subsequence $\{l_{n_j}\}$ converging to some l with $|l|=1$. Then, given any y in C, letting $j\to\infty$ in the inequality $l_{n_j}\cdot y\geqslant l_{n_j}\cdot y_{n_j}$ yields $l\cdot y\geqslant 0$, as desired.

Given subsets C_1 and C_2 of R^k, we use the notation C_1-C_2 for the set $\{y_1-y_2 : y_1\in C_1, y_2\in C_2\}$. Notice C_1 and C_2 are disjoint when and only when $0\notin C_1-C_2$.

SEPARATION THEOREM. *Suppose C_1 and C_2 are nonempty, convex subsets of R^k. Then each of the conditions below ensures that C_1 and C_2 are separated*:

(1) $0\notin C_1-C_2$,
(2) $0\notin\operatorname{int}(C_1-C_2)$,
(3) $\operatorname{int}C_2\neq\varnothing$ *and* $0\notin C_1-\operatorname{int}C_2$

Proof. (1) Let $C=C_1-C_2$. Then C is a nonempty, convex subset of R^k with $0\notin C$. By Lemma B, C and $\{0\}$ are separated; that is, some nonzero $l\in R^k$ satisfies $l\cdot y\leqslant 0$ for all $y\in C$. This means that for all $y_1\in C_1$ and all $y_2\in C_2$, we have $l\cdot y_1\leqslant l\cdot y_2$. Hence, if $\gamma_1=\sup\{l\cdot y_1 : y_1\in C_1\}$ and $\gamma_2=\inf\{l\cdot y_2 : y_2\in C_2\}$, then $\gamma_1\leqslant\gamma_2$. Put $\gamma=(\gamma_1+\gamma_2)/2$, and observe that every $y_1\in C_1$ and every $y_2\in C_2$ satisfy

$$l\cdot y_1\leqslant\gamma\leqslant l\cdot y_2.$$

(2) Let $C=C_1-C_2$. Then C is a nonempty, convex subset of R^k with $0\notin\operatorname{int}C$. By Lemma C, C and $\{0\}$ are separated, and the proof is completed as in (1).

(3) Applying condition (1), we infer C_1 and $\operatorname{int}C_2$ are separated. But this means C_1 and $\overline{\operatorname{int}C_2}=C_2$ are separated.

5. Structure of the Proofs: Multiplier Rules and Separation. Suppose $\phi_0,\phi_1,\ldots,\phi_p$, $\phi_{p+1},\ldots,\phi_{p+q}$ are $p+q+1$ real functions on an open subset U of R^n, and assume $a\in U$ minimizes ϕ_0 subject to the p inequality constraints $\phi_1\leqslant 0,\ldots,\phi_p\leqslant 0$ and the q equality constraints $\phi_{p+1}=0,\ldots,\phi_{p+q}=0$. Before we begin proving our four multiplier rules, we anticipate the common underlying structure of the demonstrations by elaborating on this assertion made in Section 0: if the geometry of the constrained optimization problem is viewed in the image space R^{p+q+1} of the mapping $\Phi=(\phi_0,\phi_1,\ldots,\phi_{p+q})$, then the existence of the optimal solution is characterized by the empty intersection of two certain sets, and the associated multiplier rule follows from separating two related *convex* sets.

Prodded by the pictures in Section 2, we look at the subset W consisting of those points $y=(\eta_0,\eta_1,\ldots,\eta_{p+q})$ in R^{p+q+1} such that

$$\begin{aligned}&\eta_0<\phi_0(a)\\ &\eta_i\leqslant 0\quad\text{for}\quad i=1,\ldots,p\quad\text{and}\\ &\eta_i=0\quad\text{for}\quad i=p+1,\ldots,p+q.\end{aligned}$$

Observe that $\Phi(x)\in W$ for some $x\in U$ when and only when x satisfies the inequality $\phi_0(x)<\phi_0(a)$ and each of the $p+q$ constraints. Therefore $a\in U$ is an optimal solution if and only if $\Phi(U)\cap W=\varnothing$. This is our characterization of optimality. Where is the image $\Phi(a)$ of the optimal solution?

Since $\Phi(a)$ belongs to $\Phi(U)\cap\overline{W}$ (here $\overline{W}$ stands for the closure of W), yet $\Phi(U)\cap W$ is empty, $\Phi(a)$ lies on the boundary of $\Phi(U)$ and on the boundary of W. It is the point where the two sets "touch." The pictures in Section 2 provide examples of this configuration.

As we see in the next paragraph, when the image $\Phi(U)$ happens to be convex, the associated multiplier rule is an easy consequence of separating with a hyperplane the disjoint convex sets $\Phi(U)$ and W. Sadly, $\Phi(U)$ is not generally convex. Even when U is a convex set and $\phi_0,\phi_1,\ldots,\phi_{p+q}$ are convex functions, $\Phi(U)$ is not necessarily convex. To deal with this misfortune, we replace the image $\Phi(U)$ with a "convex approximation" K of $\Phi(U)$. Yet it is just this replacement that creates the only true knot in the derivations: proving K and W are separated. Once we know K and W are separated, the multiplier rule follows with no difficulty. Variations in the proofs of our four basic multiplier rules come from two sources: changes in the choice of the convex approximation K and shifts in the difficulty of proving K and W are separated.

If the image $\Phi(U)$ happens to be convex, these two sources of variation and complication do not exist, and the derivation of the associated multiplier rule is particularly pleasant. As we know, an optimal solution $a\in U$ is characterized by the empty intersection $\Phi(U)\cap W$. Suppose the image $\Phi(U)$ is convex. By the Separation Theorem of Section 4, the disjoint convex sets $\Phi(U)$ and W can be separated. Let H denote a hyperplane in R^{p+q+1} separating $\Phi(U)$ and W, say $\Phi(U)\subset H^+$ and $W\subset H^-$. For simplicity, assume $\Phi(a)=0$. Since $0=\Phi(a)\in\Phi(U)\cap\overline{W}$, H must pass through 0, indicating that H has the form $\{y\in R^{p+q+1}: l\cdot y=0\}$ for some nonzero $l=(\lambda_0,\lambda_1,\ldots,\lambda_{p+q})$ in R^{p+q+1}. From $\Phi(U)\subset H^+$ and $W\subset H^-$ we have

(a) $l\cdot\Phi(x)\geqslant 0$ for every x in U and

(b) $l\cdot y\leqslant 0$ for every y in W.

Since $\Phi(a)=0$, inequality (a) says $l\cdot\Phi(x)\geqslant l\cdot\Phi(a)$ for every $x\in U$, which translates into the first conclusion of our multiplier rule:

(i) if $\phi=\sum_{i=0}^{p+q}\lambda_i\phi_i$, then $\phi(a)\leqslant\phi(x)$ for every $x\in U$.

When the functions $\phi_0,\phi_1,\ldots,\phi_{p+q}$ are differentiable at $a\in U$, we can even conclude $\phi'(a)=0$, since (i) indicates $a\in U$ is an unconstrained minimum point of ϕ. Two other conclusions follow easily from inequality (b):

(ii) $\lambda_i\geqslant 0$ whenever $i=0,1,\ldots,p$ and

(iii) $\lambda_i\phi_i(a)=0$ whenever $i=1,\ldots,p$.

The details are included in the proofs of the next section, but let us see how condition (ii), say, follows from inequality (b): if λ_j is negative for some $j=0,1,\ldots,p$, then the term $\lambda_j\eta_j$ forces the sum $l\cdot y=\sum_{i=0}^{p}\lambda_i\eta_i$ for y in W to have no upper bound, contradicting (b).

To prove the four basic multiplier rules, we replace the not necessarily convex image $\Phi(U)$ with a convex approximation K and try to mimic the argument described above. Variations in this mimicry stem, as we have said, from changes in the choice of K and shifts in the difficulty of proving K and W are separated. We examine these variations briefly for the three multiplier rules that admit inequality constraints.

In the Convex Multiplier Rule, the set U and the functions $\phi_0,\phi_1,\ldots,\phi_p$ are supposed convex. The image $\Phi(U)$ is not necessarily convex, but these convexity assumptions permit the construction of a convex set K which contains $\Phi(U)$, yet still remains disjoint from W. Separating the disjoint convex sets K and W provides a separation of $\Phi(U)$ and W, which is all we need to derive the conclusions of this multiplier rule.

Consider now the John Multiplier Rule, where the functions $\phi_0,\phi_1,\ldots,\phi_p$ are assumed differentiable at the optimal solution. Without convexity hypotheses, we cannot construct the convex set K used in proving the Convex Multiplier Rule, but the differentiability assumption allows an alternate way of "convexifying" the image $\Phi(U)$: we "linearize" $\Phi(U)$ about the point $\Phi(a)$ by forming the set $A(U)$, where A denotes the affine mapping $A(x)=\Phi(a)+L(x-a)$ and L denotes the derivative $\Phi'(a)=((\partial\phi_i/\partial\xi_j)(a))$ of Φ at the optimal solution $a\in U$. Now the image $A(U)$ is convex, if we assume (without loss) that U is convex, and our guess is that $A(U)$ and W are separated. Nothing deeper than the differentiability of the functions ϕ_i at the optimal

solution is needed to confirm this guess, and the resulting separation yields the three conclusions of the John Multiplier Rule.

The Carathéodory-John Multiplier Rule is proved with the same scheme, but the presence of equality constraints makes the separation of $A(U)$ and W more difficult to establish. Help is provided by the Convex Interior Mapping Theorem, recorded and proved in Section 10. This result, involving a mapping Φ from an open subset U of R^n into R^k and a point $a \in U$, is true when Φ is supposed continuous on U and differentiable at $a \in U$, yet the proof requires the Brouwer Fixed-Point Theorem (or an equivalently deep result), which is uncommon fare for undergraduate analysis courses. (See, however, Milnor's surprisingly elementary demonstration of the Brouwer Fixed-Point Theorem in **[42]**.) When Φ is strongly differentiable at $a \in U$ (Section 8), the Convex Interior Mapping Theorem admits a proof based on the simpler, constructive Lipschitz Fixed-Point Theorem. This fact motivates the assumption of strong differentiability in the Convex Interior Mapping Theorem and hence in the Carathéodory-John Multiplier Rule as well.

II. Proofs: Multiplier Rules With No Equality Constraints

6. Convex Multiplier Rule.

CONVEX MULTIPLIER RULE. *Suppose U is a convex subset of R^n, and $\phi_0, \phi_1, \ldots, \phi_p$ are $p+1$ convex functions on U. Whenever $a \in U$ minimizes ϕ_0 subject to the inequality constraints $\phi_1 \leq 0, \ldots, \phi_p \leq 0$, some nonzero $l = (\lambda_0, \lambda_1, \ldots, \lambda_p)$ in R^{p+1} satisfies*

(i) *if $\phi = \sum_{i=0}^{p} \lambda_i \phi_i$ and $x \in U$, then $\lambda_0 \phi_0(a) = \phi(a) \leq \phi(x)$,*

(ii) *$\lambda_i \geq 0$ for $i = 0, 1, \ldots, p$ and*

(iii) *$\lambda_i \phi_i(a) = 0$ for $i = 1, \ldots, p$.*

Proof. We may assume without loss of generality that $\phi_0(a) = 0$. If Φ denotes the mapping $(\phi_0, \phi_1, \ldots, \phi_p)$ from U into R^{p+1}, and W stands for the subset $\{y = (\eta_0, \eta_1, \ldots, \eta_p) \in R^{p+1} : \eta_0 < \phi_0(a)$ and $\eta_i \leq 0$ for $i = 1, \ldots, p\}$ of R^{p+1}, then the optimality of $a \in U$ implies $\Phi(U) \cap W = \varnothing$ and $\Phi(a) \in \Phi(U) \cap \overline{W}$. Now $\Phi(U)$ is not necessarily convex, but for the moment suppose we can construct a convex set K such that $\Phi(U) \subset K$ and $K \cap W = \varnothing$. Then K and W are disjoint, nonempty, and convex. Moreover $\Phi(a) \in K \cap \overline{W}$. By the Separation Theorem of Section 4, K and W are separated by a hyperplane passing through $\Phi(a)$: some nonzero $l = (\lambda_0, \lambda_1, \ldots, \lambda_p)$ in R^{p+1} satisfies

(a) $l \cdot [y - \Phi(a)] \geq 0$ for all $y \in K$ and

(b) $l \cdot [y - \Phi(a)] \leq 0$ for all $y \in W$.

Because K contains $\Phi(U)$, condition (a) implies $\sum_{i=0}^{p} \lambda_i \phi_i(a) \leq \sum_{i=0}^{p} \lambda_i \phi_i(x)$ for every $x \in U$, and this is the inequality in conclusion (i). The equality $\lambda_0 \phi_0(a) = \phi(a)$ is an obvious consequence of conclusion (iii), proved below. To prove (ii), suppose $\lambda_j < 0$ for some $j = 0, \ldots, p$, and observe the term $\lambda_j \eta_j$ forces the sum $l \cdot y = \sum_{i=0}^{p} \lambda_i \eta_i$ to have no upper bound on W since the restrictions on $y = (\eta_0, \eta_1, \ldots, \eta_p) \in W$ are $\eta_0 < \phi_0(a)$ $\eta_i \leq 0$ for $i = 1, \ldots, p$. This contradicts inequality (b). To establish (iii), choose any $j = 1, \ldots, p$ and notice $\bar{y} = (\phi_0(a), \ldots, \frac{1}{2}\phi_j(a), \ldots, \phi_p(a))$ belongs to $\overline{W}$. Because inequality (b) holds on $\overline{W}$ if it holds on W, we have $l \cdot [\bar{y} - \Phi(a)] \leq 0$, which is the same as $-\frac{1}{2}\lambda_j \phi_j(a) \leq 0$ or $\lambda_j \phi_j(a) \geq 0$. But on the other hand, $\lambda_j \phi_j(a) \leq 0$ since $\lambda_j \geq 0$ and $\phi_j(a) \leq 0$. Therefore $\lambda_j \phi_j(a) = 0$.

To complete the proof, we must show how to construct a convex set K satisfying $\Phi(U) \subset K$ and $K \cap W = \varnothing$. For each $x \in U$, let $K(x)$ denote the set $\{y = (\eta_0, \eta_1, \ldots, \eta_p) \in R^{p+1} : \phi_i(x) \leq \eta_i$ if $i = 0, 1, \ldots, p\}$. Set $K = \bigcup_{x \in U} K(x)$. From $\Phi(x) \in K(x)$ it follows that $\Phi(U) \subset K$. Suppose $\bar{y} = (\eta_0, \eta_1, \ldots, \eta_p) \in K \cap W$. Then there exists some $\bar{x} \in U$ such that $\phi_0(\bar{x}) \leq \eta_0 < \phi_0(a)$ and $\phi_i(\bar{x}) \leq \eta_i \leq 0$ for $i = 1, \ldots, p$; the existence of such an $\bar{x}$ contradicts the optimality of $a \in U$. Hence $K \cap W = \varnothing$. It remains to show K is convex. Choose any two points $y = (\eta_i)$ and $w = (\omega_i)$ in K and fix $\alpha \in [0, 1]$. For some x and z in U, we have $y \in K(x)$ and $w \in K(z)$. When $i = 0, 1, \ldots, p$ it

follows that $\phi_i[(1-\alpha)x+\alpha z] \leqslant (1-\alpha)\phi_i(x)+\alpha\phi_i(z) \leqslant (1-\alpha)\eta_i+\alpha\omega_i$, which implies $(1-\alpha)y+\alpha w \in K((1-\alpha)x+\alpha z) \subset K$. Thus K is convex, and the proof is complete.

COROLLARY (KARUSH-KUHN-TUCKER THEOREM). *If no ϕ_i for $i=1,\ldots,p$ is identically zero on the feasible set $S=\{x \in U: \phi_i(x) \leqslant 0$ for $i=1,\ldots,p\}$, then the conclusions of the Convex Multiplier Rule hold with a positive λ_0. Conversely, if the conclusions of the Convex Multiplier Rule hold with a positive λ_0, then $a \in U$ minimizes ϕ_0 subject to the inequality constraints $\phi_1 \leqslant 0,\ldots,\phi_p \leqslant 0$.*

Proof. Assume the conclusions of the Convex Multiplier Rule hold only when $\lambda_0=0$. Since $\lambda_0=0$, from (i) we have $0=\phi(a) \leqslant \phi(x)$ for all $x \in U$. On the other hand, if $i=1,\ldots,p$ and $x \in S$, then $\phi_i(x) \leqslant 0$ and $\lambda_i \geqslant 0$, so $\phi(x) \leqslant 0$ for all $x \in S$. It follows that ϕ is identically zero on S. Since no term $\lambda_i\phi_i$ of ϕ is positive on S, each term must be identically zero on S. Yet by assumption, no ϕ_i is identically zero on S. Hence $\lambda_i=0$ for $i=1,\ldots,p$, contradicting $l=(\lambda_0,\lambda_1,\ldots,\lambda_p)\neq 0$. Now we reverse directions and assume conditions (i), (ii), and (iii) are true with $\lambda_0>0$. Suppose x lies in the feasible set S. Observe (i) says $\lambda_0\phi_0(a) \leqslant \phi(x)=\lambda_0\phi_0(x)+\sum_{i=1}^p\lambda_i\phi_i(x)$. But since each λ_i is nonnegative, we know $\sum_{i=1}^p\lambda_i\phi_i(x) \leqslant 0$, and therefore $\lambda_0\phi_0(a) \leqslant \lambda_0\phi_0(x)$. Dividing by $\lambda_0>0$ yields $\phi_0(a) \leqslant \phi_0(x)$.

7. John Multiplier Rule.

JOHN MULTIPLIER RULE. *Suppose U is an open subset of R^n, and $\phi_0,\phi_1,\ldots,\phi_p$ are $p+1$ real functions on U, each differentiable at $a \in U$. Whenever $a \in U$ minimizes ϕ_0 subject to the inequality constraints $\phi_1 \leqslant 0,\ldots,\phi_p \leqslant 0$, some nonzero $l=(\lambda_0,\lambda_1,\ldots,\lambda_p)$ in R^{p+1} satisfies*

(i) *if $\phi=\sum_{i=0}^p\lambda_i\phi_i$, then $\phi'(a)=0$,*
(ii) *$\lambda_i \geqslant 0$ for $i=0,1,\ldots,p$ and*
(iii) *$\lambda_i\phi_i(a)=0$ for $i=1,\ldots,p$.*

Proof. Without any loss of generality, we may assume U is convex, $a=0$, and $\phi_0(a)=0$. Let Φ be the mapping $(\phi_0,\phi_1,\ldots,\phi_p)$ from U into R^{p+1}, and let W be the subset $\{y=(\eta_0,\eta_1,\ldots,\eta_p) \in R^{p+1}: \eta_0<0$ and $\eta_i \leqslant 0$ if $i=1,\ldots,p\}$. The optimality of $0 \in U$ entails $\Phi(U) \cap W=\varnothing$ and $\Phi(0) \in \Phi(U) \cap \overline{W}$. Suppose L denotes the derivative of Φ at $0 \in U$, define the affine mapping A by $A(x)=\Phi(0)+Lx$, and assume, for the moment, that $A(U)$ and W are separated. Since $\Phi(0)=A(0)$ belongs to $A(U) \cap \overline{W}$, the separating hyperplane passes through $\Phi(0)$. Therefore some nonzero $l=(\lambda_0,\lambda_1,\ldots,\lambda_p)$ in R^{p+1} satisfies

(a) $l\cdot[y-\Phi(0)] \geqslant 0$ for all $y \in A(U)$ and
(b) $l\cdot[y-\Phi(0)] \leqslant 0$ for all $y \in W$.

From (a) we have, for every $x \in U$, the inequality $l\cdot[Ax-\Phi(0)]=l\cdot Lx \geqslant 0$. This implies $l\cdot Lx=0$ for every $x \in R^n$ because U contains a neighborhood of 0. Equivalently, $l\cdot(\phi_0'(0)x,\ldots,\phi_p'(0)x)=0$ on R^n, since the derivatives $\phi_i'(0)$ are the rows of L, and written out this is $\sum_{i=0}^p\lambda_i\phi_i'(0)x=0$ for all $x \in R^n$, which is conclusion (i). Conclusions (ii) and (iii) follow from inequality (b) exactly as they did in the proof of the Convex Multiplier Rule.

We now verify that $A(U)$ and W are indeed separated. Suppose $A(U)$ and W are not separated. Then, because $\operatorname{int} W=\{y=(\eta_0,\eta_1,\ldots,\eta_p) \in R^{p+1}: \eta_i<0$ for $i=0,1,\ldots,p\}$ is nonempty, condition (3) of the Separation Theorem in Section 4 implies $0 \in A(U)-\operatorname{int} W$. Hence for some $\bar{x} \in U$, we have $A\bar{x} \in \operatorname{int} W$. Since Φ is differentiable at 0, for every $\alpha>0$ such that $\alpha\bar{x} \in U$ we can write $\Phi(\alpha\bar{x})=A(\alpha\bar{x})+r(\alpha\bar{x})|\alpha\bar{x}|$, where $r(\alpha\bar{x})\to 0$ as $\alpha\to 0$. The set $\operatorname{int} W$ being open, we can fix $\alpha \in (0,1)$ so that $\alpha\bar{x} \in U$ and $A\bar{x}+r(\alpha\bar{x})|\bar{x}| \in \operatorname{int} W$. But $\alpha y \in \operatorname{int} W$ whenever $\alpha>0$ and $y \in \operatorname{int} W$. Thus $\alpha[A\bar{x}+r(\alpha\bar{x})|\bar{x}|] \in \operatorname{int} W$, that is, $\alpha\Phi(0)+L(\alpha\bar{x})+r(\alpha\bar{x})|\alpha\bar{x}| \in \operatorname{int} W$. For each $i=0,1,\ldots,p$ we have $\phi_i(0) \leqslant \alpha\phi_i(0)$ because $\alpha \in (0,1)$ and $\phi_i(0) \leqslant 0$. This implies $\Phi(\alpha\bar{x})=\Phi(0)+L(\alpha\bar{x})+r(\alpha\bar{x})|\alpha\bar{x}| \in \operatorname{int} W \subset W$, indicating that $\Phi(U) \cap W \neq \varnothing$, a contradiction. Consequently, $A(U)$ and W must indeed be separated, and the proof is finished.

COROLLARY (KARUSH-KUHN-TUCKER THEOREM). *If some h in R^n satisfies $\phi_i'(a)h<0$ for every $i=1,\ldots,p$ such that $\phi_i(a)=0$, then the conclusions of the John Multiplier Rule hold with a positive λ_0.*

Proof. Choose $h \in R^n$ satisfying the condition above, and suppose the conclusions of the John Multiplier Rule hold, but only when $\lambda_0=0$. From conclusion (i) we have $\Sigma_{i=1}^p \lambda_i \phi_i'(a)h=0$. Let I and J be the subsets of $\{1,2,\ldots,p\}$ such that $\phi_i(a)<0$ when $i \in I$ and $\phi_i(a)=0$ when $i \in J$. Then $I \cup J=\{1,2,\ldots,p\}$. From (iii) we know $\lambda_i=0$ if $i \in I$. Therefore $\Sigma_{i \in J}\lambda_i\phi_i'(a)h=\Sigma_{i=1}^p \lambda_i\phi_i'(a)h=0$. But no term of this first sum is positive because $\lambda_i \geqslant 0$ and $\phi_i'(a)h<0$ if $i \in J$. This means each term is zero, forcing $\lambda_i=0$ for $i \in J$. Now we have $\lambda_0=0$ and $\lambda_i=0$ for i in $I \cup J=\{1,2,\ldots,p\}$, contradicting $l=(\lambda_0,\lambda_1,\ldots,\lambda_p)\neq 0$.

III. Interiority

8. Strong Derivatives. There are several assertions to establish in proving the classical Implicit Mapping Theorem, but leaving the easy and less central parts aside, there remains the crux of the proof, and that is to establish Theorem A below.

THEOREM A. *If U is an open subset of R^n, if Φ is a continuously differentiable mapping from U into R^k, and if the derivative L of Φ at $a \in U$ maps R^n onto R^k, then $\Phi(a) \in \operatorname{int} \Phi(U)$.*

It was Carathéodory who first understood the relevance for equality constrained optimization of results—let us call them interior mapping theorems—which give conditions ensuring that $\Phi(a)$ belongs to the interior of the image $\Phi(U)$. Briefly (we have seen this before), his idea for the continuously differentiable case was this: if $a \in U$ minimizes ϕ_0 subject to the equality constraints $\phi_1=0,\ldots,\phi_q=0$, then $\Phi(a)$ cannot lie in the interior of the image $\Phi(U)$, where $\Phi=(\phi_0,\phi_1,\ldots,\phi_q)$ maps U into R^{q+1}. By Theorem A, $L=\Phi'(a)$ must then fail to map onto R^{q+1}, and the desired multiplier rule follows easily. (Section 9 contains the details of this argument.)

The common first step in the proof of Theorem A is to show that Φ satisfies a certain approximation condition—which we shall call strong differentiability—at the point $a \in U$. With strong differentiability at $a \in U$ in the hypothesis (rather than continuous differentiability on U) we obtain an interior mapping theorem with a one-point derivative condition and a proof identical to that of Theorem A:

THEOREM B. *If U is an open subset of R^n, if Φ maps U into R^k, and if the strong derivative L of Φ at $a \in U$ exists and maps R^n onto R^k, then $\Phi(a) \in \operatorname{int} \Phi(U)$.*

We need some definitions. Let U be an open subset of R^n. A mapping Φ from U into R^k is said to be **differentiable** at $a \in U$ if there is a linear mapping L from R^n into R^k with the property that for every $\epsilon>0$, there is some $\delta>0$ such that $|\Phi(x)-\Phi(a)-L(x-a)|\leqslant\epsilon|x-a|$ whenever $x \in U$ and $|x-a|\leqslant\delta$. The linear mapping L, often denoted by $\Phi'(a)$, is called the **derivative** of Φ at $a \in U$. (We shall not discriminate between the linear mapping L and its matrix representation with respect to the standard bases in R^n and R^k.) When a norm, say $\|L\|=\max\{|Lx|:|x|=1\}$, has been introduced on the collection of all linear mappings from R^n into R^k, we say Φ is **continuously differentiable** on U provided the assignment $x \to \Phi'(x)$ is continuous on U. A linear mapping L is called the **strong derivative** of Φ at $a \in U$, and Φ is said to be **strongly differentiable** at $a \in U$ if the following condition is satisfied: for every $\epsilon>0$, there is some $\delta>0$ such that

$$|\Phi(x_2)-\Phi(x_1)-L(x_2-x_1)|\leqslant\varepsilon|x_2-x_1|$$

wherever x_1 and x_2 lie in U, $|x_1-a|\leqslant\delta$, and $|x_2-a|\leqslant\delta$.

For us the important properties of a strongly differentiable map are these:

(1) Φ is differentiable with derivative L wherever Φ is strongly differentiable with strong derivative L.

(2) Φ is strongly differentiable at any point on a neighborhood of which Φ is continuously differentiable.

The first property is obvious from the definitions; the second is an easy consequence of the Mean Value Theorem.

For more on strong derivatives, especially their relation to inverse and implicit mapping

theorems, see Cartan [8], Gleason [16], Leach [37], Nijenhuis [43], and Ortega and Rheinboldt [44]. For their relevance to optimization theory, examine Clarke [10], Halkin [19], [20], and Pourciau [46].

9. Interior Mapping Theorems. As the key to establishing the Carathéodory Multiplier Rule for strongly differentiable functions, Theorem B of the previous section deserves a less prosaic title. We shall call it the Interior Mapping Theorem and set aside the present section for its proof. The Interior Mapping Theorem would follow easily from an implicit mapping theorem for strongly differentiable mappings, but the argument below is direct and based on the Lipschitz Fixed-Point Theorem, which is proved in most modern advanced calculus texts.

A mapping Ψ from a subset K of R^k into R^k is said to be **contractive** provided for some $\epsilon\in[0,1)$ and every y_1 and y_2 in K we have $|\Psi(y_2)-\Psi(y_1)|\leqslant\epsilon|y_2-y_1|$.

LIPSCHITZ FIXED-POINT THEOREM. *A contractive mapping Ψ from a closed subset K of R^k into itself has a fixed point: $\Psi(y)=y$ for some $y\in K$.*

INTERIOR MAPPING THEOREM. *If U is an open subset of R^n, if Φ maps U into R^k, and if the strong derivative L of Φ at $a\in U$ exists and maps R^n onto R^k, then $\Phi(a)\in\operatorname{int}\Phi(U)$.*

Proof. For simplicity and without loss of generality, assume $a=0$ and $\Phi(a)=0$. Because L maps R^n onto R^k, some linear mapping M from R^k into R^n makes LM the identity mapping on R^k. Fix $\alpha>0$ so $M(y)$ lies in U whenever y is in K_α, where K_α stands for the closed ball $\overline{N_\alpha(0)}$ in R^k. If Ψ maps K_α into R^k by $\Psi(y)=y-\Phi M(y)$, then $\Psi'(0)=0$ is a strong derivative, so there is a $\delta\in(0,\alpha]$ such that

$$y_1 \text{ and } y_2\in K_\delta \text{ implies } |\Psi(y_2)-\Psi(y_1)|\leqslant\tfrac{1}{2}|y_2-y_1|. \qquad (*)$$

Now choose any $\bar{y}$ in $K_{\delta/2}$, and let $\overline{\Psi}$ be the assignment $y\to\bar{y}+\Psi(y)$ from K_δ into R^k. That $\overline{\Psi}$ is a contractive mapping from K_δ into itself is an easy consequence of (*). If $w\in K_\delta$ is the fixed point of $\overline{\Psi}$, then $\bar{y}+\Psi(w)=w$, or $\bar{y}=\Phi M(w)$. But $M(w)\in U$ because $w\in K_\delta$, so $\bar{y}=\Phi(x)$ for some x in U, and we have proved $K_{\delta/2}\subset\Phi(U)$.

From the Interior Mapping Theorem, the Carathéodory Multiplier Rule follows easily, as we shall see, but the deeper Carathéodory-John Multiplier Rule, which permits both equality and inequality constraints, hinges on a more subtle result:

CONVEX INTERIOR MAPPING THEOREM. *If U is an open subset of R^n, if C is a convex subset of U, if Φ maps U into R^k, and if the strong derivative L of Φ at $a\in U\cap\overline{C}$ exists and satisfies $L(a)\in\operatorname{int}L(C)$, then $\Phi(a)\in\operatorname{int}\Phi(C)$.*

For a proof of this theorem, based on the separation of convex sets and the Lipschitz Fixed-Point Theorem, read Pourciau [47].

IV. Proofs: Multiplier Rules With Equality Constraints

10. Carathéodory Multiplier Rule.

CARATHÉODORY MULTIPLIER RULE. *Suppose U is an open subset of R^n, and $\phi_0,\phi_1,\ldots,\phi_q$ are $q+1$ real functions on U, each strongly differentiable at $a\in U$. Whenever $a\in U$ minimizes ϕ_0 subject to the equality constraints $\phi_1=0,\ldots,\phi_q=0$, some nonzero $l=(\lambda_0,\lambda_1,\ldots,\lambda_q)$ in R^{q+1} satisfies*

(i) *if $\phi=\sum_{i=0}^{q}\lambda_i\phi_i$, then $\phi'(a)=0$ and*

(ii) $\lambda_0\geqslant 0$.

Proof. Let Φ be the mapping $(\phi_0,\phi_1,\ldots,\phi_q)$ from U into R^{q+1}, and let W be the set of all $y=(\eta_0,\eta_1,\ldots,\eta_q)$ in R^{q+1} such that $\eta_0<\phi_0(a)$ and $\eta_i=0$ if $i=1,\ldots,q$. The optimality of $a\in U$ implies $\Phi(U)\cap W=\varnothing$. Since $\Phi(a)\in\Phi(U)\cap\overline{W}$, we must have $\Phi(a)\notin\operatorname{int}\Phi(U)$, for otherwise

points of W would lie in $\Phi(U)$. It follows from the Interior Mapping Theorem (Section 9) that the strong derivative L of Φ at $a \in U$ fails to map R^n onto R^{q+1}, which means the $q+1$ rows $\phi_0'(a),\ldots,\phi_q'(a)$ of L are linearly dependent, proving (i). If need be, (ii) can be ensured by replacing each λ_i with $-\lambda_i$.

COROLLARY (EULER-LAGRANGE MULTIPLIER RULE). *If the derivatives $\phi_1'(a),\ldots,\phi_q'(a)$ are linearly independent, then the conclusions of the Carathéodory Multiplier Rule hold with a positive λ_0.*

Proof. From $l=(0,\lambda_1,\ldots,\lambda_q)\neq 0$ and $0=\phi'(a)=\sum_{i=1}^q\lambda_i\phi_i'(a)$, it follows that the derivatives $\phi_1'(a),\ldots,\phi_q'(a)$ are linearly dependent.

11. Carathéodory-John Multiplier Rule.

CARATHÉODORY-JOHN MULTIPLIER RULE. *Suppose U is an open subset of R^n, and $\phi_0,\phi_1,\ldots,\phi_p$, $\phi_{p+1},\ldots,\phi_{p+q}$ are $p+q+1$ real functions on U, each strongly differentiable at $a\in U$. Whenever $a\in U$ minimizes ϕ_0 subject to the p inequality constraints $\phi_1\leqslant 0,\ldots,\phi_p\leqslant 0$ and the q equality constraints $\phi_{p+1}=0,\ldots,\phi_{p+q}=0$, some nonzero $l=(\lambda_0,\lambda_1,\ldots,\lambda_p,\ \lambda_{p+1},\ldots,\lambda_{p+q})$ in R^{p+q+1} satisfies*
(i) *if $\phi=\sum_{i=0}^{p+q}\lambda_i\phi_i$, then $\phi'(a)=0$,*
(ii) *$\lambda_i\geqslant 0$ for $i=0,\ldots,p$ and*
(iii) *$\lambda_i\phi_i(a)=0$ for $i=1,\ldots,p$.*

Proof. Without loss of generality, we suppose $a=0$, U is convex, and $\Phi(a)=0$, where Φ is the mapping $(\phi_0,\phi_1,\ldots,\phi_{p+q})$ from U into R^{p+q+1}. Let W be the collection of all $y=(\eta_0,\eta_1,\ldots,\eta_{p+q})$ in R^{p+q+1} such that

$$\eta_0<0,$$
$$\eta_i\leqslant 0 \quad \text{for} \quad i=1,\ldots,p \text{ and}$$
$$\eta_i=0 \quad \text{for} \quad i=p+1,\ldots,p+q.$$

The optimality of $0\in U$ guarantees $\Phi(U)\cap W=\varnothing$. We also have $\Phi(0)\in\Phi(U)\cap\overline{W}$. Let L be the strong derivative of Φ at $0\in U$. (Since $\Phi(0)=0$, the affine mapping A given by $Ax=\Phi(0)+Lx$ is just L.) If $L(U)$ and W are separated, the conclusions (i), (ii) and (iii) follow exactly as they did in the proof of the John Multiplier Rule.

Suppose $L(U)$ and W are not separated. Then condition (2) of the Separation Theorem in Section 4 ensures $0\in\operatorname{int}[L(U)-W]$. This is the same as writing $M(0,0)\in\operatorname{int}M(U\times W)$ if M is the derivative of Ψ at $(0,0)$ and Ψ is the mapping from $U\times R^{p+q+1}$ into R^{p+q+1} given by $\Psi(x,y)=\Phi(x)-y$. From the Convex Interior Mapping Theorem (Section 9), we infer

$$0=\Psi(0,0)\in\operatorname{int}\Psi(U\times W)\subset\Psi(U\times W),$$

which implies $0=\Phi(x)-y$ for some x in U and some y in W, contradicting the condition $\Phi(U)\cap W=\varnothing$. Hence $L(U)$ and W must indeed be separated, completing the proof.

COROLLARY (KARUSH-KUHN-TUCKER THEOREM). *If $\phi_{p+1}'(a),\ldots,\phi_{p+q}'(a)$ are linearly independent, and some $h\in R^n$ satisfies*
(a) *$\phi_i'(a)h<0$ for those $i=1,\ldots,p$ such that $\phi_i(a)=0$ and*
(b) *$\phi_i'(a)h=0$ for $i=p+1,\ldots,p+q$,*
then the conclusions of the Carathéodory-John Multiplier Rule hold with a positive λ_0.

Proof. Suppose, on the contrary, that the conditions of this corollary are met, yet the conclusions (i), (ii), and (iii) of the Carathéodory-John Multiplier Rule hold only when $\lambda_0=0$. Choose $h\in R^n$ satisfying properties (a) and (b). Using conclusion (i), property (b), and the assumption $\lambda_0=0$, we have $\sum_{i=1}^p\lambda_i\phi_i'(a)h=0$. Just as in the proof of the corollary to the John Multiplier Rule, property (a) now implies $\lambda_i=0$ if $i=1,\ldots,p$. This means conclusion (i) reduces

to $\Sigma_{i=p+1}^{p+q}\lambda_i\phi_i'(a)=0$, and we infer that $\lambda_{p+j}=0$ for $j=1,\ldots,q$ from the linear independence of $\phi_{p+1}'(a),\ldots,\phi_{p+q}'(a)$. We have shown that $l=(\lambda_0,\lambda_1,\ldots,\lambda_{p+q})=0$, an impossibility. Hence some nonzero $l=(\lambda_0,\lambda_1,\ldots,\lambda_{p+q})$ satisfying the conclusions of the Carathéodory-John Multiplier Rule must enjoy $\lambda_0>0$.

12. History and Current Events. To establish our four basic multiplier rules, we have made specific applications of a scheme for deriving necessary conditions in constrained optimization problems which involves separating two certain convex subsets of the image space R^{p+q+1} of the mapping $\Phi=(\phi_0,\phi_1,\ldots,\phi_{p+q+1})$. Briefly tracing the history of this method, we begin in 1935 with C. Carathéodory **[7**, p. 177], who proved the multiplier rule named after him and recognized the relevance to optimization theory of images and interiority. E. J. McShane **[40]** in 1939 was the first to establish necessary conditions by separating subsets of the image space. (L. Pontryagin and his colleagues later proved the basic necessary condition of optimal control theory, the Maximum Principle, using this technique of McShane's.) For the case of inequality constraints alone, F. John **[31]** separated subsets of the image space to obtain his multiplier rule without implicit (or interior) mapping theorems. McShane, who was dealing with inequality and equality constraints in the calculus of variations, required *implicit* mapping theorems to separate certain convex *cones* in the image space. M. Hestenes has developed this approach into a cohesive theory for the calculus of variations, optimal control theory, and, more recently, finite-dimensional constrained optimization. See his books, **[26]** and **[27]**. Over the past fifteen years, H. Halkin, the leading modern proponent of the scheme used in the present article, has employed *interior* mapping theorems and the separation of certain convex *sets* in the image space to prove elegant and deep results concerning necessary conditions in optimal control theory and general nonlinear optimization. Consult, for example, **[19]**, **[20]**, **[21]**, and **[22]**.

Over the past ten years, much work has been directed toward weakening the smoothness assumption on the functions ϕ_i in the basic Carathéodory-John Multiplier Rule. For example, Halkin **[18]** and Hestenes **[27]**, with rather intricate proofs based on the Brouwer Fixed-Point Theorem, show that conclusions (i), (ii), and (iii) of the Carathéodory-John Multiplier Rule remain in force whenever $\phi_0,\phi_1,\ldots,\phi_{p+q}$ are differentiable at $a\in U$ and $\phi_{p+1},\ldots,\phi_{p+q}$ are continuous on U. Some recent papers have presented natural analogs of the Carathéodory-John Multiplier Rule (or its corollary, the Karush-Kuhn-Tucker Theorem) which apply when $\phi_0,\phi_1,\ldots,\phi_{p+q}$ are Lipschitz continuous. In this setting the functions ϕ_i may not admit derivatives at the optimal solution, a malady that has prompted the introduction of set-valued derivatives. See, for instance, Auslander **[1]**, Clarke **[9]**, **[10]**, **[11]**, **[12]**, Goldstine **[17]**, Halkin **[21]**, **[22]**, **[24]**, Hestenes **[27]**, Hiriart-Urruty **[28]**, **[29]**, **[30]**, Pourciau **[45]**, **[46]**, **[48]**, Sweetzer **[53]**, and Warga **[56]**, **[57]**.

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TOTAL TORSION

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In elementary differential geometry it sometimes happens that the geometry of a proof can become obscured by analysis. It is truly unfortunate when this happens since the main object of the subject is geometry. One obvious example of this situation is the classical proof of the Gauss-Bonnet Theorem for surfaces in R^3 (see [4], for example). Luckily this particular situation is remedied by the introduction of piecewise linear methods (see [1]). Although these piecewise linear methods are not strictly true to the differential part of differential geometry, they are closely related and they do make the geometry of the proof completely transparent.

The object of this paper is to show how piecewise linear methods can be used to give a similar illustrative geometric proof of the following total torsion result (see [5] or, more recently, [4]):

THEOREM 1. *The total torsion $\int \tilde{\tau}\, ds$ of a closed unit speed regular curve $\tilde{\alpha}: R \to S^2$ on the unit 2-sphere S^2 is zero.*

(Note: Throughout this paper tildes denote smooth concepts and lack of tildes denotes polygonal concepts. For example, a smooth curve is denoted by $\tilde{\alpha}$ and a polygonal curve by α; or the torsion of a smooth curve is denoted by $\tilde{\tau}$ and the torsion of a polygonal curve—which will be defined—by τ.)

Thanks to R. Patterson and R. Rigdon, with whom I had several useful conversations concerning this topic.

It will, in fact, turn out that, viewed geometrically, Theorem 1 is closely related to the following total curvature result for planar curves. (Neither the result nor its proof will be used in the sequel; they are presented simply to draw an analogy.)

THEOREM 2. *If $\tilde{\alpha}$ is a closed unit speed regular planar curve which is regularly homotopic to a circle, then the total curvature $\int \tilde{\kappa}\, ds$ of $\tilde{\alpha}$ is $\pm 2\pi$, the sign depending on orientation.*

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(A *regular curve* is a curve with a continuously turning non-trivial tangent vector. A *regular homotopy* is a homotopy which at every stage is a regular curve and such that the tangent vectors move continuously with the homotopy.)

Proof (of Theorem 2). Having chosen a preferred normal direction to the plane of $\tilde{\alpha}$ there is a well-determined right-handed orthonormal coordinate system at each point $\tilde{\alpha}(s_0)$ of the graph of $\tilde{\alpha}$ consisting of the unit tangent vector $\tilde{t}(s_0)$ to $\tilde{\alpha}$ at s_0 and a unit normal vector $\tilde{n}(s_0)$ to $\tilde{\alpha}$ at s_0. The curvature of $\tilde{\alpha}$ from s_0 to s_1, $\int_{s_0}^{s_1} \tilde{\kappa}\, ds$, is the length of the normal circular image of $\tilde{\alpha}$ from s_0 to s_1: for each value of s between s_0 and s_1, parallel translate $\tilde{n}(s)$ to the origin; $\tilde{n}(s)$ will lie on the unit circle S^1, and as s varies from s_0 to s_1, $\tilde{n}(s)$ will vary over S^1; $\int_{s_0}^{s_1} \tilde{\kappa}\, ds$ is the cumulative arclength traced out on S^1 by $\tilde{n}(s)$ from $\tilde{n}(s_0)$ to $\tilde{n}(s_1)$, counterclockwise arclengths considered as positive and clockwise arclengths considered as negative (see Fig. 1). The total curvature $\int \tilde{\kappa}\, ds$ of $\tilde{\alpha}$ is an integral multiple $2\pi n$ of 2π: if we start tallying curvature on S^1 with $\tilde{n}(s_0)$, we will also end tallying at $\tilde{n}(s_0)$; the integer n is the *rotation number* of $\tilde{\alpha}$. The Whitney-Graustein Theorem (see [6]) states that two regularly homotopic unit speed regular planar curves have the same rotation number. Since $\tilde{\alpha}$ is regularly homotopic to a circle, its rotation number is ± 1 (since the rotation number for a circle is ± 1), and consequently the total curvature of $\tilde{\alpha}$ is $\pm 2\pi$. End of proof.

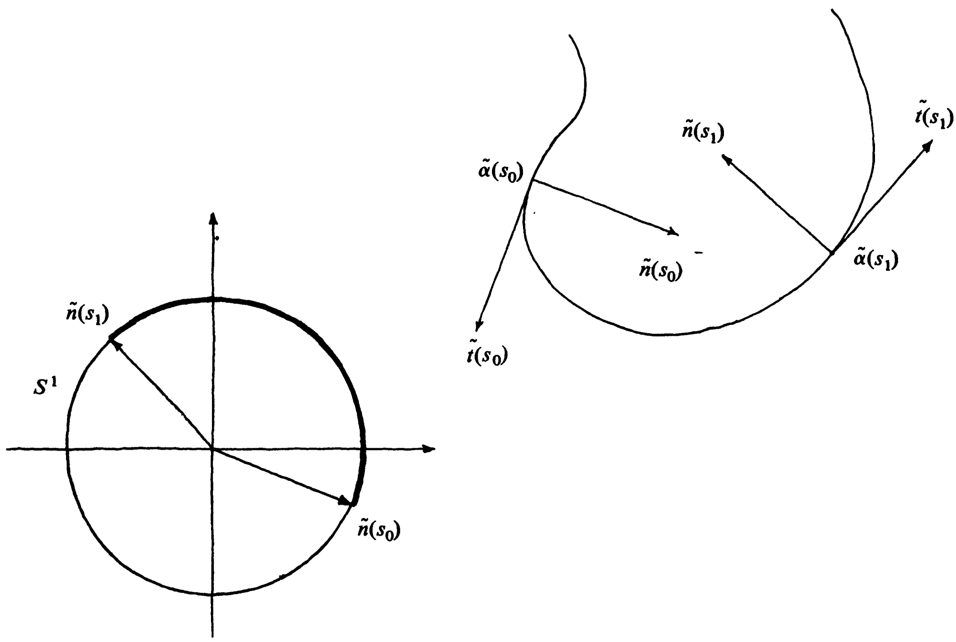

FIG. 1

The two key factors of this (total *curvature*) result, which will appear later in the proof of Theorem 1 (a total *torsion* result), are first the use of the unit circle as a tallying device, and second the use of invariance under (regular) homotopy to find the integral multiple of 2π.

Now back to torsion.

Recall (see [3], [4], and [2]) that (smooth) torsion can be described as follows (see Fig. 2): Let $\tilde{\alpha}$ denote a unit speed regular curve in R^3, and let s_0 be a fixed parameter value for which the curvature of $\tilde{\alpha}$ is non-zero. The normal plane to $\tilde{\alpha}$ at s_0 divides R^3 into two half-spaces, and the unit tangent vector $\tilde{t}(s_0)$ at s_0 lies in one of these half-spaces. For parameter values s' and s'' close to $s_0, s' < s_0 < s''$, let θ denote the angle between $-\pi/2$ and $\pi/2$ whose magnitude is the

(undirected) angle between the binormals $\tilde{b}(s')$ and $\tilde{b}(s'')$ to $\tilde{\alpha}$ at s' and s'', and whose sign is $\pm$ as $\pm\,\tilde{b}(s')\times\tilde{b}(s'')$ points into the half-space determined by the unit tangent vector at s_0. The torsion of $\tilde{\alpha}$ at s_0 is

$$\tilde{\tau}(s_0)=\lim_{s',s''\to s_0}\frac{\theta}{s''-s'};$$

$\tilde{\tau}(s_0)$ measures the extent to which $\tilde{\alpha}$ fails to lie in its osculating plane at s_0.

Now suppose α is a polygonal curve in R^3 with vertices v_i; for simplicity we will assume that α is closed and that the number of vertices is finite. With the above description of smooth torsion in mind, we now define the polygonal torsion of α. In contrast to smooth torsion, which is a function of points, the polygonal torsion of a polygonal curve α will be a function of the segments of α. In other words, the polygonal torsion τ of α will associate to each segment

$$\sigma_i=\{(1-t)v_i+tv_{i+1}:t\in[0,1]\}$$

of α a real number $\tau(\sigma_i)=\tau_i$.

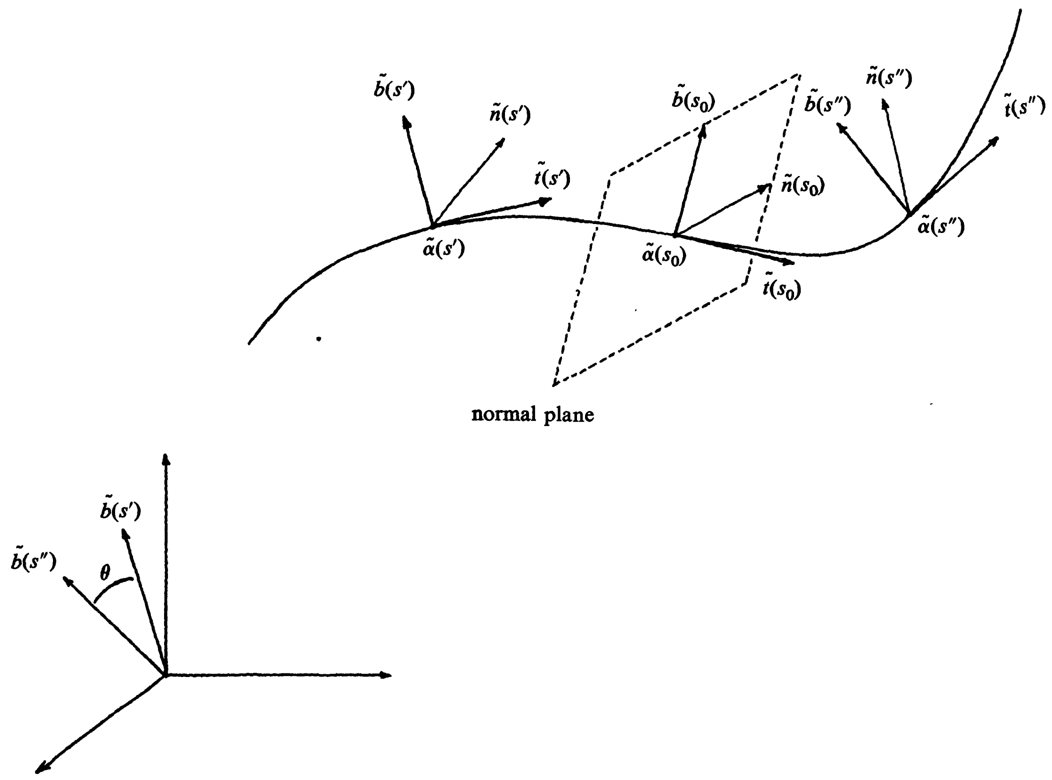

FIG. 2

DEFINITION 3. If σ_{i-1}, σ_i, and σ_{i+1} are coplanar, $\tau(\sigma_i)=0$. If σ_{i-1}, σ_i, and σ_{i+1} are not coplanar, observe that the normal plane to $v_{i+1}-v_i$ divides R^3 into two half-spaces, and $v_{i+1}-v_i$ lies in exactly one of these half-spaces; let θ_i denote the angle between $-\pi$ and π whose magnitude is the (undirected) angle between the "binormals"

$$b_i=\frac{(v_i-v_{i-1})\times(v_{i+1}-v_i)}{|(v_i-v_{i-1})\times(v_{i+1}-v_i)|}\quad\text{and}\quad b_{i+1}=\frac{(v_{i+1}-v_i)\times(v_{i+2}-v_{i+1})}{|(v_{i+1}-v_i)\times(v_{i+2}-v_{i+1})|},$$

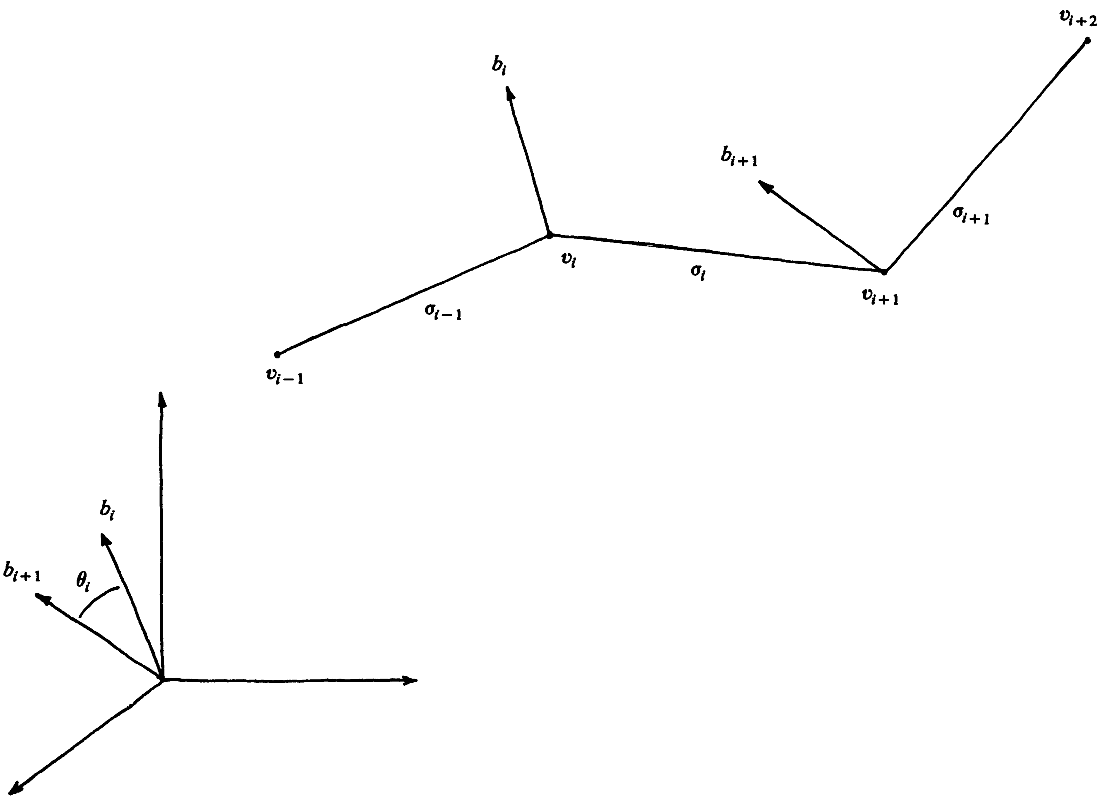

FIG. 3

and whose sign is $\pm$ as $\pm b_i \times b_{i+1}$ points into the half-space determined by $v_{i+1} - v_i$ (see Fig. 3); then

$$\tau(\sigma_i) = \frac{\theta_i}{|v_{i+1} - v_i|}.$$

In making this definition we have forced the following result.

PROPOSITION 4. *Let $\{\alpha_i\}$ be a sequence of polygonal secant approximations to the closed regular curve $\tilde{\alpha}$ such that the vertices of α_i are vertices of α_j for $i<j$, and such that α_i approaches $\tilde{\alpha}$ uniformly as i tends to ∞. Then for each s_0 the torsion $\tilde{\tau}(s_0) = \lim_{i\to\infty} \tau(\sigma_i)$ where, for each i, σ_i is the segment of α_i for which $\tilde{\alpha}(s_0)$ lies between v_i and v_{i+1}.*

Now if α is a sufficiently close polygonal secant approximation to the closed unit speed regular curve $\tilde{\alpha}$ then we may approximate the total torsion of $\tilde{\alpha}$:

$$\int \tilde{\tau} ds \cong \sum_i \tau_i |v_{i+1} - v_i| = \sum_i \theta_i, \tag{1}$$

the summation taken over all segments σ_i of α. The next result is the key to the proof of Theorem 1; it is, in fact, the polygonal analog of Theorem 1.

PROPOSITION 5. *Let α be a closed polygonal curve in R^3 whose vertices all lie on S^2 and for which the lengths of all the segments σ_i are equal (this latter condition is the polygonal analog of unit speed). Then $\Sigma_i \theta_i$, the summation taken over all segments σ_i of α, is an integral multiple of 2π.*

In analogy to the proof of Theorem 2, Theorem 1 will follow from (1), Proposition 5, and arguing that one can deform α to a curve whose total torsion is zero in such a manner that the

total torsion remains unchanged throughout the deformation. Not only are the proof of Theorem 1 and the proof of Theorem 2 similar in this respect, but they are also similar in that Proposition 5 is proved by using a circle as a tallying device.

Proof (of Proposition 5). For each segment σ_i of α we define a map T_i from the unit circle $S^1 \subset R^2 \subset R^3$ (the second inclusion being into the e_1e_2-coordinate plane) to S^2 as follows: Let Π_i denote the perpendicular bisecting plane of σ_i, and let p_i denote the point of intersection of $\Pi_i \cap S^2$ and the ray emanating from the origin which passes through the midpoint of σ_i. There is a unique rotation of R^3 taking e_1 to p_i and e_3 to $(v_{i+1}-v_i)/|v_{i+1}-v_i|$, and T_i is the restriction of this map to S^1.

Since b_i and b_{i+1} are perpendicular to σ_i, b_i and b_{i+1} are in $\Pi_i \cap S^2$, and, by construction, θ_i is the (directed) angle from $T_i^{-1}(b_i)$ to $T_i^{-1}(b_{i+1})$ (directed angles in R^2 being measured in the usual fashion). Furthermore $T_i^{-1}(b_{i+1}) = T_{i+1}^{-1}(b_{i+1})$: this is simply the fact that $T_{i+1} \circ T_i^{-1} : \Pi_i \to \Pi_{i+1}$ is rotation about the line containing b_{i+1} (see Fig. 4). (This is where the assumption that the lengths of the σ_i are all the same is necessary; if the lengths of σ_i and σ_{i+1} are *not* the same, then $T_{i+1} \circ T_i^{-1}$ is *not* this rotation.) Thus θ_0 is the angle from $T_0^{-1}(b_0)$ to $T_0^{-1}(b_1) = T_1^{-1}(b_1), \theta_1$ is the angle from $T_1^{-1}(b_1)$ to $T_1^{-1}(b_2) = T_2^{-1}(b_2)$, etc.; since the terminal side of the last angle θ_i is again $T_0^{-1}(b_0)$, it follows that $\Sigma_i \theta_i$ is an integral multiple of 2π. End of proof.

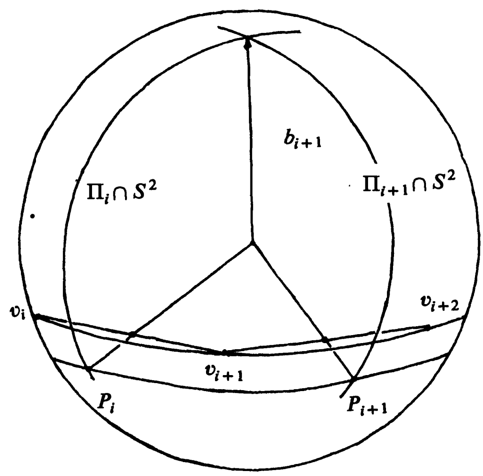

FIG. 4

COROLLARY 6. *If $\alpha_u, u \in [0,1]$, is a continuous deformation from α_0 to α_1, and if at each stage of the deformation α_u satisfies the assumptions of Proposition* 5 (*the common length of the segments is allowed to vary from one stage of the deformation to another*), *then $\Sigma_i \theta_i$ is the same for both α_0 and α_1.*

Proof. This follows immediately from the continuity of the construction of Proposition 5 and the fact that the only continuous functions from [0, 1] to the integers are the constant functions. End of proof.

Our proof of Theorem 1 now follows from showing that for every closed unit speed regular curve $\tilde{\alpha}$ on S^2 there is a large integer N, a (close) polygonal secant approximation α of $\tilde{\alpha}$ consisting of N segments for which the vertices of α all lie on S^2 and for which the lengths of the segments of α are all equal, and which is trivially homotopic via a homotopy as described in Corollary 6. For then, using (1),

$$\int \tilde{\tau}\, ds \cong \sum_i \theta_i = 0,$$

the equality following since the total torsion of trivial paths is clearly zero; so Corollary 6 implies that the integral multiple of 2π arising in Proposition 5 is zero.

Rigorous proofs of the existence of a polygonal secant approximation to $\tilde{\alpha}$ and a homotopy as described above are given in the Appendix. We simply sketch the main ideas here:

First we show that, if $\tilde{\alpha}: R \to R^3$ is any closed unit speed regular curve, then for every $\varepsilon > 0$ there is an integer N and a polygonal secant approximation α to $\tilde{\alpha}$ such that the length of α is within ε of the length of $\tilde{\alpha}$, and α is composed of precisely N segments, all of which have the same length: For a given $\varepsilon > 0$, pick any polygonal secant approximation β to $\tilde{\alpha}$ whose length is within ε of the length of $\tilde{\alpha}$; let δ denote the length of the shortest segment of β. Since the graph of $\tilde{\alpha}$ is compact, it can be covered by, say, N open spherical balls of radius δ.

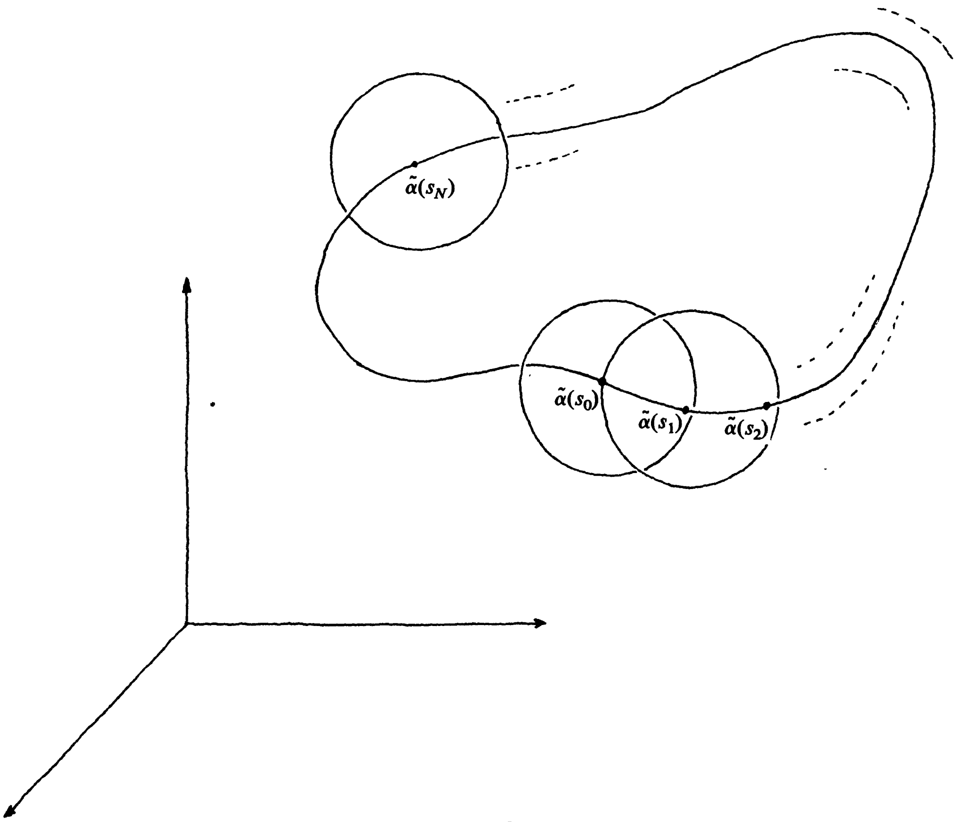

FIG. 5

Now pick an arbitrary parameter value s_0, and let r be a positive number much smaller than δ. Construct a 2-sphere with center $\tilde{\alpha}(s_0)$ and radius r; this sphere will intersect the graph of $\tilde{\alpha}$ at a unique point $\tilde{\alpha}(s_1)$ where $s_1 > s_0$. Construct a 2-sphere with center $\tilde{\alpha}(s_1)$ and radius r; this sphere will intersect the graph of $\tilde{\alpha}$ at a unique point $\tilde{\alpha}(s_2)$ where $s_2 > s_1$. Continue in this manner until N 2-spheres have been constructed (see Fig. 5). By connecting $\tilde{\alpha}(s_0), \tilde{\alpha}(s_1), \ldots, \tilde{\alpha}(s_N)$ with line segments, we obtain a polygonal secant approximation to that part of $\tilde{\alpha}$ starting at $\tilde{\alpha}(s_0)$ and ending at $\tilde{\alpha}(s_N)$ whose segments are all of equal length. As long as r is much smaller than δ, the directed arclength $h(r)$ from $\tilde{\alpha}(s_N)$ along $\tilde{\alpha}$ and back to $\tilde{\alpha}(s_0)$ (i.e., the length of $\tilde{\alpha}$ less s_N) will clearly be positive. If r gets close to δ, however, then (since $\tilde{\alpha}$ can be covered by N open

spherical balls of radius δ), the directed arclength $h(r)$ will be negative. As a function, $h:[0,\delta]\to R$ is continuous, so there must be some $r_0\in[0,\delta]$ for which $h(r_0)=0$; performing our construction with $r=r_0$, we obtain the desired approximation.

A rigorous proof, again, is given in the Appendix. Such a proof is necessary to handle problems which arise when the graph of $\tilde{\alpha}$ is not as nice as the curve illustrated in Fig. 5. For example, $\tilde{\alpha}$ might intersect itself (we avoid this problem by working in the parameter space instead of the image space). Also unless δ is chosen more carefully, h may not be continuous (consider the situation illustrated in Fig. 6; observe that as the radius of the sphere increases—the center remaining fixed—the point in the direction of increasing parameter at which the sphere meets the curve jumps discontinuously). The rigorous proof avoids these (and other) problems with our sketchy "proof."

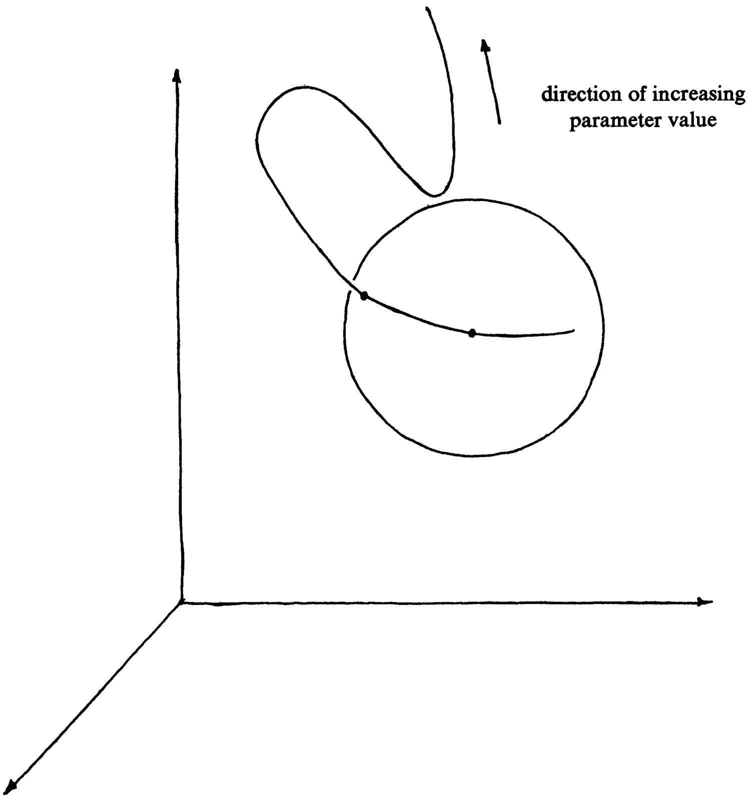

FIG. 6

The existence of the desired homotopy follows from the continuity of the construction of α.

Appendix. We now give rigorous proofs for the two results whose proofs have just been sketched.

PROPOSITION 7. *Suppose $\tilde{\alpha}:R\to R^3$ is a closed unit speed regular curve. Then*

(1) *for every $\varepsilon>0$ there is an integer N and a polygonal secant approximation α to $\tilde{\alpha}$ such that the length of α is within ε of the length of $\tilde{\alpha}$, and α is composed of precisely N segments all of which have the same length, and*

(2) *furthermore given any parameter value s_0 there is a polygonal secant approximation α as above for which $\tilde{\alpha}(s_0)$ is a vertex of α.*

Proof. Suppose s_0 and ε are given.

As a consequence of the definition of arclength there is a polygonal secant approximation β to $\tilde{\alpha}$ whose length is within ε of the length L of $\tilde{\alpha}$.

LEMMA 8. *There is a small number $\delta_0>0$ such that, for each parameter value s, the minimal parameter value $t\geqslant s$ for which $|\tilde{\alpha}(s)-\tilde{\alpha}(t)|=r$ varies continuously with $r\in[0,\delta_0]$.*

Proof (of Lemma 8). First observe that there is an open neighborhood U_0 of the diagonal in $R\times R$ such that if $(s,t)\in U_0$ and $\tilde{\alpha}(s)=\tilde{\alpha}(t)$ then $s=t$: Since $\tilde{\alpha}$ is locally one-to-one, for each parameter value s_0 there is an open interval neighborhood $I(s_0)$ of s_0 such that if $s,t\in I(s_0)$ and $\tilde{\alpha}(s)=\tilde{\alpha}(t)$ then $s=t$. A finite set of squares $I(s)\times I(s)$ for $s\in[s_0,s_0+L]$ covers $\{(s,s):s\in[s_0,s_0+L]\}$, and there is a number $\gamma>0$ which is so small that

$$\{(s,t)\in R\times R:s\in[s_0,s_0+L],\ t\in(s-\gamma,s+\gamma)\}$$

is contained in the union of the squares in this finite cover. Using the fact that $\tilde{\alpha}$ has period L, it is easy to show that the desired condition is satisfied for

$$U_0=\{(s,t)\in R\times R:t\in(s-\gamma,s+\gamma)\}. \tag{2}$$

Now define $f:R\times R\to R$ by $f(s,t)=|\tilde{\alpha}(t)-\tilde{\alpha}(s)|$; f is clearly continuous. Moreover as long as $\tilde{\alpha}(s)\neq\tilde{\alpha}(t)$, f is differentiable: for example, $(\partial f/\partial t)(s,t)$ is the cosine of the angle between $\tilde{\alpha}'(t)$ and $\tilde{\alpha}(t)-\tilde{\alpha}(s)$. In particular, if

$$U=U_0\cap\{(s,t)\in R\times R:t>s\}$$

and if we define $\partial f/\partial t$ to be 1 along the diagonal, then $\partial f/\partial t:U\to[-1,1]$ is continuous (continuity along the diagonal follows since, as long as $t-s>0$, the angle between $\tilde{\alpha}'(t)$ and $\tilde{\alpha}(t)-\tilde{\alpha}(s)$ is the same as the angle between $\tilde{\alpha}'(t)$ and $(\tilde{\alpha}(t)-\tilde{\alpha}(s))/(t-s)$. We can, and do, assume that γ was chosen so small that $\partial f/\partial t:U\to(1/2,1]$.

Let $\delta_0=\min_{s\in R}f(s,s+\gamma)$ (since, as a function of s, $f(s,s+\gamma)$ has period L, $\delta_0=\min_{s\in[s_0,s_0+L]}f(s,s+\gamma)$ so δ_0 exists; and since, for each s, $f(s,s)=0$ and $f(s,-):[s,s+\gamma]\to R$ is increasing, $\delta_0>0$). The condition of the Lemma now follows since for each s,$f(s,-):[s,s+\gamma]\to[0,\delta_0]$ is continuous and increasing. End of proof (of Lemma).

Let δ denote the smaller of δ_0 and the length of the shortest segment of β. Choose a finite cover of the graph of $\tilde{\alpha}$ by open spherical balls whose centers are all on the graph of $\tilde{\alpha}$ and whose radii are all δ. There is a (finite) cover of $[s_0,s_0+L]$ by N open subintervals such that the image under $\tilde{\alpha}$ of each open subinterval in the cover is contained in one of the finite open spherical balls in the cover of the graph of $\tilde{\alpha}$; at the minimum N may be the number of open spherical balls in the cover of $\tilde{\alpha}$, but N is not necessarily chosen to be minimal in any sense.

We define a function $h:[0,\delta]\to R$ as follows: Given $r\in[0,\delta]$, construct an open spherical ball with center $\tilde{\alpha}(s_0)$ and radius r; let s_1 denote the minimal parameter value larger than s_0 for which $|\tilde{\alpha}(s_1)-\tilde{\alpha}(s_0)|=r$. Now construct an open spherical ball with center $\tilde{\alpha}(s_1)$ and radius r; let s_2 denote the minimal parameter value larger than s_1 for which $|\tilde{\alpha}(s_2)-\tilde{\alpha}(s_1)|=r$. By continuing in this manner we can find s_N, and $h(r)=L-s_N$ (i.e., $h(r)$ is the "directed" arclength from $\tilde{\alpha}(s_N)$ back to $\tilde{\alpha}(s_0+L)=\tilde{\alpha}(s_0)$.

Now h is continuous: Define g: $[s_0,\infty)\times[0,\delta]\to[s_0,\infty)$ by letting $g(s,r)$ denote the smallest parameter value $t\geqslant s$ for which $f(s,t)=r$; g is differentiable (if F: $[s_0,\infty)\times[0,\delta]\times[s_0,\infty)\to R$ is given by $F(s,r,t)=f(s,t)-r$, then at each point where $F(s,r,t)=0,\partial F/\partial t=\partial f/\partial t$ is positive so the Implicit Function Theorem (see [7]) implies that for each (s,r) there is a unique $g(s,r)$ for which $F(s,r,g(s,r))=0$, i.e., $f(s,g(s,r))=r$, and that g is differentiable in s and r). In particular, g is continuous, and thus h is continuous: h may be written as a composite of continuous functions—namely, $h(r)$ is L less the N-fold composite of $g(-,r)$ evaluated at s_0.

Since $h(0)=L>0$ and $h(\delta)<0$ (since the graph of $\tilde{\alpha}$ has a cover by "N" open spherical balls of radius δ), there is an $r_0\in[0,\delta]$ for which $h(r_0)=0$. This value of r_0 corresponds to a polygonal secant approximation α to $\tilde{\alpha}$ which has N segments. And since δ is smaller than the length of the

smallest segment of β, the length of α is within ε of the length of $\tilde{\alpha}$. End of proof (of Proposition).

COROLLARY 9. *Suppose $\tilde{\alpha}: R \to S^2$ is a closed unit speed regular curve. Then there is a polygonal secant approximation α to $\tilde{\alpha}$ as in part 1 of Proposition 7 which is trivially homotopic via a homotopy as described in Corollary 6.*

Proof. Let $\tilde{\alpha}_u : R \to S^2, u \in [0,1]$, be a smooth homotopy from $\tilde{\alpha}_0 = \tilde{\alpha}$ to a point at $\tilde{\alpha}_1$.

First observe that we can modify the argument given in Proposition 7 to show that there is a small number $\delta_0 > 0$ such that for each $u \in [0,1]$ and for each parameter value s, the minimal parameter value $t \geqslant s$ for which $|\tilde{\alpha}_u(s) - \tilde{\alpha}_u(t)| = r$ varies continuously with $r \in [0, \delta_0]$: For each $u_0 \in [0,1]$ there is an open interval neighborhood $I(u_0)$ of u_0 and a neighborhood $U(u_0)$ of the diagonal in $R \times R$ such that for each $u \in I(u_0)$ and $(s,t) \in U(u_0)$, $\tilde{\alpha}_u(s) = \tilde{\alpha}_u(t)$ implies $s = t$. If we choose a finite cover of $[0,1]$ by such intervals $I(u_0)$, and if we let U_0 be the intersection of the $U(u_0)$'s for $I(u_0)$ in this open cover, then U_0 is a neighborhood of the diagonal in $R \times R$ such that for each $u \in [0,1]$ and $(s,t) \in U_0, \tilde{\alpha}_u(s) = \tilde{\alpha}_u(t)$ implies $s = t$. We can, in fact, find a number $\gamma > 0$ which is so small that U_0 may be assumed to be of the form (2). Now let $f_u(s,t) = |\tilde{\alpha}_u(t) - \tilde{\alpha}_u(s)|$, and

$$U = U_0 \cap \{(s,t) \in R \times R : t > s\}.$$

Since $\tilde{\alpha}_u$ is a smooth homotopy, $f_u(s,t)$ is differentiable (in s, t, and u) wherever $\tilde{\alpha}_u(s) \neq \tilde{\alpha}_u(t)$, and since $[0,1]$ is compact we can again choose γ so small that $\partial f_u / \partial t : U \to (1/2, 1]$. If we then let

$$\delta_0 = \min_{\substack{s \in R \\ u \in [0,1]}} f_u(s, s+\gamma)$$

the desired conclusion is met.

Now let β denote a polygonal secant approximation to $\tilde{\alpha}$ whose length is within ε of the length of $\tilde{\alpha}$, and let δ denote the smaller of δ_0 and the length of the shortest segment of β. For each $u_0 \in [0,1]$ there is a finite cover of the graph of $\tilde{\alpha}_{u_0}$ by open spherical balls whose centers are all on the graph of $\tilde{\alpha}_{u_0}$ and whose radii are all δ; in fact there is an open interval neighborhood $J(u_0)$ of u_0 such that for every $u \in J(u_0)$ this open cover covers $\tilde{\alpha}_{u_0}$. There is a (finite) cover of $[s_0, s_0 + L_{u_0}]$ by $N(u_0)$ open subintervals such that the image under $\tilde{\alpha}_{u_0}$ of each open subinterval in the cover is contained in one of the (finite) open spherical balls in the cover of $\tilde{\alpha}_{u_0}$; in fact the same is true for any $u \in J(u_0)$. Choose a finite open cover of $[0,1]$ by such intervals $J(u_0)$, and let N be the maximum of $N(u_0)$ for $J(u_0)$ in this finite open cover.

Now pick an arbitrary parameter value s_0. Proposition 7 ensures us that for every $u \in [0,1]$ we can obtain a polygonal secant approximation α_u to $\tilde{\alpha}_u$ which has $\tilde{\alpha}_u(s_0)$ as a vertex and which is composed of N segments, all of which have the same length. The fact that this collection of polygonal secant approximations forms a homotopy as described in Corollary 6 is a simple consequence of the fact that the lengths $r_0 = r_0(u)$ arising in the construction of α_u vary continuously with u.

The continuity of $r_0 = r_0(u)$ can be seen as follows: The function h_u arising in the construction of α_u is given by associating, to each value of $r \in [0, \delta]$, L_u less the N-fold composite of $g_u(-,r)$ evaluated at s_0. Clearly $h_u(r)$ is differentiable (in u and r), and we can use the Implicit Function Theorem to show that $r_0 = r_0(u)$ is a continuous (in fact differentiable) function of u if we can show $\partial h_u / \partial r$ is never zero. But this is equivalent to showing that the partial with respect to r of the N-fold composite of $g_u(-,r)$ evaluated at s_0 is never zero, and this is clear. The partial of $h_u(r)$ with respect to r can be written as the partial of g_u with respect to r evaluated at some (s,r) plus a summation of terms of the form

$$\frac{\partial g_u}{\partial s}(g_u(s,r),r) \cdot \frac{\partial g_u}{\partial r}(s,r) \tag{3}$$

for various values of s (the value of s is the same in both factors of each term of the summation,

but the value of s may vary from term to term). By implicitly differentiating the equation

$$f_u(s,g_u(s,r))=r \tag{4}$$

with respect to r we find that

$$\frac{\partial g_u}{\partial r}(s,r)=\left(\frac{\partial f_u}{\partial t}(s,g_u(s,r))\right)^{-1}$$

and since $g_u(s,r)\geqslant s, (\partial g_u/\partial r)(s,r)$ is positive. Since the first term in the partial of $h_u(r)$ with respect to r is thus positive, it suffices to show that each succeeding term (of the form (3)) is also positive. Since the second factor of each such term (of the form (3)) is always positive, it only remains to show that the first factor is also positive. Again implicitly differentiating equation (4) with respect to s we find that

$$\frac{\partial g_u}{\partial s}(s,r)=-\frac{\dfrac{\partial f_u}{\partial s}(s,g_u(s,r))}{\dfrac{\partial f_u}{\partial t}(s,g_u(s,r))};$$

the denominator is always positive (by construction), and the numerator is always negative (since $f_u(s,t)=f_u(t,s)$,

$$\frac{\partial f_u}{\partial s}(s,g_u(s,r))=\frac{\partial f_u}{\partial t}(g_u(s,r),s),$$

and for $t=g_u(s,r)>s, (\partial f_u/\partial s)(s,g_u(s,r))=(\partial f_u/\partial t)(t,s)$ is the cosine of the angle between $\tilde{\alpha}_u'(s)$ and $\tilde{\alpha}_u(s)-\tilde{\alpha}_u(t)$; since $t>s, (\partial f_u/\partial s)(s,g_u(s,r))$ is the cosine of π less the angle between $\tilde{\alpha}_u'(s)$ and $(\tilde{\alpha}_u(s)-\tilde{\alpha}_u(t))/(s-t)$, and since $(s,t)\in U, (\partial f_u/\partial s)(s,g_u(s,r))\in[-1,-1/2))$, so $(\partial g_u/\partial s)(s,r)$ is always positive. End of proof (of Corollary).

This completes the proof of Theorem 1.

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MISCELLANEA

38. It may be treading on dangerous ground to suggest that there is such a thing as excessive generality, though even so convinced an analyst as Picard is not without misgivings on the subject. Yet it can sometimes be wished that writers who develop general theories at great length would pause to enquire how far they are available for the solution of special problems.

— A. E. H. Love, Presidential Address (1914) to the London Mathematical Society: *Proc. London Math. Soc.* (2) 14 (1915) 178–188.

ANALYSIS OF MODE: MATH OPENS DOORS EVERYWHERE

EILEEN L. POIANI

Since 1975, the Mathematical Association of America has been sponsoring the secondary school lectureship program WAM—Women and Mathematics—designed to encourage young women to seek adequate preparation in mathematics. Throughout its five consecutive years of operation, WAM has received primary funding support from the International Business Machines Corporation (IBM). By conservative estimates as of January 1980, WAM visits had reached more than 500 schools, 42,000 students, and 4,300 parents, teachers, and counselors. Requests thus far have exceeded 650 from secondary schools alone.

The idea for WAM was sparked when IBM representatives held a reception for top scorers in the U.S.A. Mathematical Olympiad and noticed that no women were among the winners. The U.S.A. Olympiad is an annual mathematics contest begun in 1972 for invited high school students who have excelled in previous mathematics competitions. From among the high achievers in the U.S.A. Olympiad, a special team is selected for participation in the International Olympiad held each summer. Since 1974, American teams have entered the international competition, which started in 1959. No women have yet been members of the American team, although two participated for the first time in the 1978 training sessions, and, in all, it is believed that since 1967 not more than four or five women from other countries have participated each year in the international contest [7, p. 807].

The absence of women from the Olympiad symbolized their absence from a host of fields which today require sound mathematical preparation—traditional fields like engineering and the physical sciences, and areas like business, the social and biological sciences, psychology, and medicine, which increasingly involve mathematical applications. To encourage 9th and 10th graders to keep career doors open by electing to take more than the minimum school mathematics requirement and to acquaint teachers, counselors, parents, and other influential individuals with the importance of mastering mathematics, WAM was created.

Visits to schools for a half to one full day are designed to enable WAM speakers to converse formally and informally with students, teachers, and counselors, on why mathematics is needed for both college and non-college bound students regardless of sex. In addition to such visits, WAM also arranges presentations to professional societies, elementary school teachers, civic organizations, parents groups, and legislative leaders. Programs are held within WAM regions, which currently include New York/New Jersey, Connecticut, Boston, Chicago, the San Francisco Bay area, Southern California, South Florida, and Oregon. Plans are under way to open new regions.

Since guidance and career counselors exert such a profound influence on students, as documented by studies in [1] and [4], WAM speakers endeavored to meet with the counselors during the regular school visits. Unfortunately such discussion sessions rarely took place. Counselors were often unavailable because of heavy demands on their time for student interviews and administrative responsibilities. WAM was also unsuccessful in repeated attempts to seek an opportunity to explain the "women and mathematics" issues at regional and national meetings of the American Personnel and Guidance Association, the professional society for counselors.

Nevertheless, after those few school meetings which did take place, the evaluations by both

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WAM speakers and counselors were positive. A need for continued dialogue was expressed. Hence there arose the idea of developing a model for reaching guidance counselors by holding a one-day conference specifically for them.

New Jersey was the natural site for the conference since the national WAM director, Eileen L. Poiani, was located there, as were the New York/New Jersey WAM Coordinator, Lorraine Denby, and the WAM Special Materials Coordinator, Susan J. Devlin. In addition, special interest in such a project was expressed by Patricia Griffith, Program Development Specialist for the Division on Women, New Jersey Department of Community Affairs, and Adele Kaplan, Women's Division Director of the Rutgers Small Business Development Center. These five individuals served as the conference planning committee.

The committee named the conference MODE: Math Opens Doors Everywhere, to emphasize the key reason for the event. As listed in the program brochure, the four goals of MODE were to:

- disseminate information and materials on the usefulness of mathematics in specific careers;
- provide a forum for counselors to share their experiences, problems, and effective techniques in career counseling;
- discuss the special problems of combating the masculine image of mathematics;
- explore ways of overcoming math anxiety.

Foremost among the many technical concerns in planning the conference was who would fund it. School districts vary in ability and willingness to pay for such conferences, so MODE was designed with no cost to participants. The Prudential Foundation kindly agreed to support MODE with a grant of $2,250 and to provide facilities and lunch at their Newark, New Jersey, Corporate Office. Facilities and resources were limited to 100 participants.

Although it would be too lengthy to enumerate all technical concerns here, the following suggestions may be helpful to others when planning similar conferences:

1. Keep the time of the meeting consistent with the normal day of the audience and avoid conflicts with competing events.

2. Contact leaders of appropriate professional groups to determine what topics and speakers interest prospective participants and what has already been done in the area.

3. Involve counselors in planning programs for counselors.

4. Be in touch with school authorities who must approve released time for attendance at conferences and who are influential in encouraging attendance.

5. Try to get the conference announcement to as many counselors as possible.

6. Provide certificates of attendance, continuing-education units, resource materials, or similar incentives.

7. Build in a plan for action and meaningful follow-up to the program.

8. Ask participants to evaluate the conference.

9. Be prepared for "no-shows" and last minute registrants.

Since no single comprehensive list of all active New Jersey school counselors could be found, MODE was publicized by direct mailings to guidance department directors at all New Jersey high schools and some junior and middle schools, to leaders of counselors organizations, and to other appropriate school personnel; by distribution of material at the New Jersey Convention for teachers and counselors, at curriculum directors meetings, and at mathematics teachers gatherings; by articles in public and New Jersey Education Department newspapers; by phone calls to influential school leaders. The number of requests for participation reached 120, with half as many men as women, but was not as large as the conference planners had hoped.

Of the 94 who actually attended MODE on Tuesday, December 5, 1978, 38 percent were men and 62 percent were women. Both public and private urban, suburban, and rural schools were represented. Counselors as well as school and district guidance directors participated.

The MODE program highlighted morning and afternoon plenary sessions: "Mind Over Math," by Stanley Kogelman (director of a consulting service and author of [5] with the same

name), described what math anxiety is, how it affects career choices, and how it can be overcome; and "Keeping Doors Open for Young Women," by Patricia Casserly (Educational Testing Service research associate), reported on a study of how well-intentioned teachers and counselors influence young women to persist in or drop out of the school mathematics sequence. Five workshops, each repeated twice, covered the establishment of an effective student career center, a panel discussion about how math is needed for both college and non-college bound students, consideration of open and equal access to jobs, dialogue on career and math-related materials in all media, and a demonstration session on counseling the math avoider or math-anxious student. Participants indicated three workshop preferences when they originally registered for MODE, and workshop assignments adhered to them as closely as possible.

At the end of the day, MODE participants assembled to recapitulate the conference sessions and to outline a plan for action. They were given certificates and resource packets containing a comprehensive annotated bibliography and 38 reference booklets and brochures. The materials covered career and educational resources; a historical perspective on women and mathematics; sex differences and anxiety relative to mastering mathematics; and miscellaneous inspirational references.

Among the noteworthy observations and suggestions made by the counselors during the course of the day are the following:

1. The authoritarian nature of mathematics is reinforced by the teacher's demand that a problem be solved only by the method taught in class. This tends to inhibit creativity and be anxiety-producing. More than one correct approach to problem-solving should be encouraged by the teacher.

2. Written descriptions of mathematics courses should be understandable by nonmathematicians. Foreign language course descriptions are not written in the foreign language, so mathematics courses should not be described in very technical language. Students, parents, and counselors should be able to obtain some sense of why the course is useful from its description and a companion glossary of common mathematical terms.

3. Geometry and algebra require rather different skills and perceptions. Students who do not comprehend the interrelationships of those subjects and who excel in one but not the other often opt out of mathematics after satisfying the school's minimum requirement. Teachers need to clarify and emphasize how the spatial relations of geometry can be used in algebra.

4. Limited time to search the literature and limited school budgets often impede the acquisition of needed career materials. Schools definitely want and need bibliographies and materials like those distributed at MODE, as well as information about how to obtain free brochures for students.

5. Students today place high priority on the monetary rewards of jobs or careers. Counselors would benefit from knowing up-to-date salary information about various careers at both entry and advanced levels.

6. College admissions officers, in their zeal to recruit students, often reinforce the "you don't need math" attitude by saying that "you don't need math to enter College X." By downplaying the hard courses in exchange for wooing a shrinking pool of applicants, the admissions officer closes career doors for many students. Labeled the "critical filter" by Lucy Sells [**2**, p. 601], mathematics will undoubtedly continue to be required preparation for entrance into all but a handful of traditionally female-dominated major fields.

Participants were asked to evaluate MODE at the conclusion of the conference. Based on a 76 percent response, all found that the MODE brochure clearly explained the objectives of the conference and 83 percent felt the workshops met their expectations. On a five-point rating scale of 5 (Excellent) to 1 (Poor), the plenary and workshop sessions overwhelmingly received 5's, 4's, and 3's, in that order. In terms of achieving the MODE goals, 92 percent felt they had learned applicable techniques to reduce math anxiety and 94 percent had benefited educationally from

the topics discussed and materials distributed. About 96 percent of the respondents found the contacts with others to share experiences, problems, and techniques and the opportunity to identify potential resources among conference speakers and sponsors to be valuable outcomes of MODE.

The momentum of the conference has continued in some very tangible ways. Having attended MODE, the Chicago WAM coordinator, Sister Kathleen Sullivan, applied the model to host the second MODE at Barat College in Lake Forest, Illinois, March 20–21, 1979. Sixty-five participants (high school counselors, math teachers, and some college counselors and teachers) and 75 Barat College students attended. Evaluation results were also gratifying.

Furthermore, one junior high school and one high school hosted special days in Spring 1979 devoted to the issues raised at New Jersey MODE and featured MODE speakers at these events. An in-service day for teachers in a large central New Jersey school district and several lectures by MODE presenters also took place in the Spring and Fall of 1979. Some participants in MODE have requested a follow-up conference to continue the dialogue. Networks have been initiated among the participating counselors themselves and between speakers and participants. Phone inquiries in response to a newspaper article about MODE continue to be received, the caller often seeking a remedy for her (more often than his) math anxiety.

MODE was the culmination of a unique cooperative endeavor among diverse New Jersey organizations represented on the planning committee. It sought to underscore why *M*ath *O*pens *D*oors *E*verywhere and why strenuous efforts are needed to reverse the persistent trend that student enrollment in upper-level high school mathematics courses runs to nearly twice as many men as women [3], while the population remains about equally divided. Programs like MODE can be instrumental in improving the situation which Plato so poignantly described over 2,400 years ago [6]:

> Nothing can be more absurd than the practice which prevails in our country of men and women not following the same pursuits with all their strength and with one mind, for thus the state, instead of being a whole, is reduced to a half.

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A SIMPLE APPLICATION OF THE THEOREM OF COMPOSITION OF ACCELERATIONS

S. N. MAITRA

Let us consider two reference frames $OXYZ$ and $OX'Y'Z'$, regarding the former as fixed and the latter as rotating. Then the absolute velocity and acceleration of a moving particle P, which are derived in all textbooks on vectorial mechanics, are given by

$$\vec{V} = \vec{V}_r + \vec{\omega} \times \overrightarrow{OP} \tag{1}$$

$$\vec{a} = \vec{a}_r + 2\vec{\omega} \times \vec{V}_r + \vec{\omega} \times (\vec{\omega} \times \overrightarrow{OP}) + \dot{\vec{\omega}} \times \overrightarrow{OP} \tag{2}$$

where $\overrightarrow{OP} = \vec{r}$, and the velocity and acceleration relative to the moving frame $OX'Y'Z'$, along with the transport velocity, acceleration, and Coriolis acceleration, are defined as

$$\vec{V}_r = \dot{x}'i' + \dot{y}'j' + \dot{z}'k'$$
$$\vec{a}_r = \ddot{x}'i' + \ddot{y}'j' + \ddot{z}'k'$$
$$\vec{V}_t = \vec{\omega} \times \overrightarrow{OP}$$
$$\vec{a}_t = \vec{\omega} \times (\vec{\omega} \times \overrightarrow{OP}) + \dot{\vec{\omega}} \times \overrightarrow{OP}$$
$$\vec{a}_c = 2\vec{\omega} \times \vec{V}_r.$$

(The dot indicates a derivative with respect to time.) (i', j', k') are the unit vectors associated with the moving trihedral $OX'Y'Z'$, and (i, j, k) are the unit vectors associated with the fixed trihedral $OXYZ$. The position of the particle P at any instant is defined by the coordinates (x, y, z) with respect to the fixed frame and by the coordinates (x', y', z') with respect to the rotating frame. $\vec{\omega}$ is the absolute angular velocity of the rotating frame. In this note, we have applied the *law of composition of accelerations* (given by Equation (2)) to discuss the motion of a particle P that is moving with a *uniform* velocity V_0 in a straight line on the $X'Y'$ plane relative to the rotating frame $OX'Y'Z'$. Here we may think of P as an insect walking on a rotating sheet of cardboard, described as the $X'Y'$ plane. For simplicity, let us assume that the frame $OX'Y'Z'$ is rotating about the Z'-axis with a *constant angular velocity* ω and that the Z'-axis is coincident with the Z-axis.

Hence, in this case,

$$\vec{\omega} = \omega k, \quad \vec{V}_r = \vec{V}_0, \quad \vec{a}_r = 0.$$

Let the rectilinear path of the particle P on the $X'Y'$ plane relative to the frame $OX'Y'Z'$ make an angle of α with the X'-axis.

Now,

$$\begin{aligned} i' &= i\cos(\omega t) + j\sin(\omega t) \\ j' &= -i\sin(\omega t) + j\cos(\omega t) \\ \overrightarrow{OP} &= i'x' + j'y' = ix + jy \\ \vec{a} &= i\ddot{x} + j\ddot{y} \\ \vec{V}_r &= \vec{V}_0 = (i'\cos\alpha + j'\sin\alpha)V_0, \end{aligned} \tag{3}$$

so that Equation (2) reduces to the form

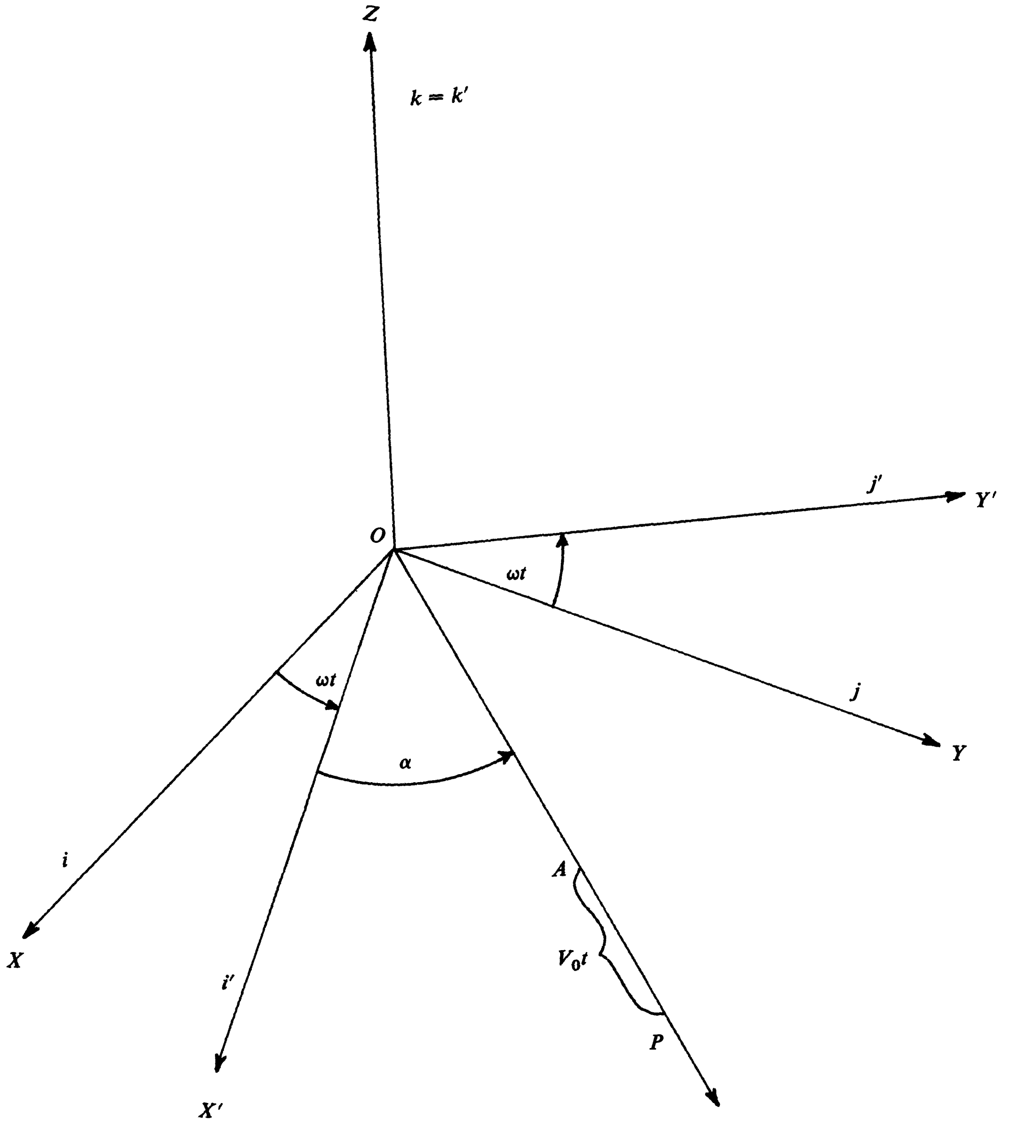

FIG. 1. Rotation of the Frame $OX'Y'Z'$

$$\begin{aligned} i\ddot{x}+j\ddot{y} &= 2\omega k'\times(i'V_0\cos\alpha+j'V_0\sin\alpha)+k\omega(k\omega\cdot\overrightarrow{OP})-\omega^2\overrightarrow{OP} \\ &= 2(-i'\sin\alpha+j'\cos\alpha)V_0\omega-\omega^2(xi+yj) \\ &= 2\{-i\sin(\omega t+\alpha)+j\cos(\omega t+\alpha)\}V_0\omega-\omega^2(xi+yj), \end{aligned}$$

which give the scalar relationships

$$\begin{aligned} \ddot{x}+\omega^2x &= -2V_0\omega\sin(\omega t+\alpha) \\ \ddot{y}+\omega^2y &= 2V_0\omega\cos(\omega t+\alpha). \end{aligned} \tag{4}$$

Either of Equations (4) is analogous to that of a harmonic oscillator disturbed by a periodic force while the period of free oscillation is the same as that of the disturbing force. Hence the motion of the particle P with reference to the fixed frame $OXYZ$ is compounded of these two disturbed harmonic oscillations along the X- and Y-axes, respectively.

The general solutions of Equations (4) can be given by

$$\begin{aligned} x &= A\cos(\omega t+\mu)+V_0 t\cos(\omega t+\alpha) \\ y &= B\sin(\omega t+\lambda)+V_0 t\sin(\omega t+\alpha) \end{aligned} \tag{5}$$

where A, B, μ, and λ are constants to be evaluated from the initial conditions. Let us suppose that initially at $t=0$ the two frames are coincident with each other and that the particle is at a distance r_0 from the origin O.

Differentiating (5) with respect to the time t, the absolute velocity components are obtained as

$$\begin{aligned} \frac{dx}{dt} &= -A\omega\sin(\omega t+\mu)+V_0\cos(\omega t+\alpha)-V_0 t\omega\sin(\omega t+\alpha) \\ \frac{dy}{dt} &= B\omega\cos(\omega t+\lambda)+V_0\sin(\omega t+\alpha)+V_0 t\omega\cos(\omega t+\alpha). \end{aligned} \tag{6}$$

Initially at $t=0$, $x=r_0\cos\alpha$, $y=r_0\sin\alpha$ and, because of Equation (1),

$$\frac{dx}{dt}=V_0\cos\alpha-r_0\omega\sin\alpha, \qquad \frac{dy}{dt}=V_0\sin\alpha+r_0\omega\cos\alpha$$

as a consequence of which, Equations (5) and (6) give

$$\begin{gathered} A=r_0,\ B=r_0 \\ \mu=\alpha,\ \lambda=\alpha. \end{gathered}$$

Hence the position of the particle at a subsequent time t, with respect to the fixed frame $OXYZ$, is given by the coordinates

$$\begin{aligned} x &= r_0\cos(\omega t+\alpha)+V_0 t\cos(\omega t+\alpha) \\ y &= r_0\sin(\omega t+\alpha)+V_0 t\sin(\omega t+\alpha). \end{aligned} \tag{7}$$

Elimination of t from Equations (7) leads to the path of the particle P with respect to the fixed frame. The polar equation of the path is

$$r=r_0+\frac{V_0}{\omega}(\theta-\alpha), \tag{8}$$

which is a spiral with the pole at the point (r_0,α) and the initial line $\theta=\alpha$.

The following interesting results can also be obtained. If initially at $t=0, x=y=0$, i.e., if the particle starts from the origin, then $A=B=0$ and Equations (5) reduce to the form

$$\begin{aligned} x &= V_0 t\cos(\omega t+\alpha) \\ y &= V_0 t\sin(\omega t+\alpha). \end{aligned} \tag{9}$$

Hence its real path is a spiral having the pole at the origin and the initial line $\theta=\alpha$. Its polar equation is

$$r=\frac{V_0}{\omega}(\theta-\alpha). \tag{10}$$

If $\alpha=0$, i.e., if the particle moves along the X'-axis of the rotating frame with a uniform velocity V_0 starting from the origin, its real path is a spiral of Archimedes, defined by

$$r=\frac{V_0}{\omega}\theta. \tag{11}$$

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INEQUALITIES FOR INSCRIBED AND CIRCUMSCRIBED POLYGONS

M. S. KLAMKIN

In an interesting note, "A curious case of the use of mathematical induction in geometry," J. V. Uspensky (this MONTHLY, 34 (1927) 247–250), established the following three theorems:

THEOREM I. *If α and β denote the angles at the base of AB of a triangle ABC (Fig. 1) opposite to the sides CB and CA, then the following inequalities hold:*

$$\frac{CA}{BA} > \frac{\beta}{\alpha+\beta} \quad \textit{and} \quad \frac{CB}{AB} > \frac{\alpha}{\alpha+\beta}.$$

THEOREM II. *Two arcs AB and AC, neither exceeding a semicircumference, being taken on the same circle and the latter being greater of the two, the following inequality holds:*

$$\frac{AB}{AC} > \frac{\operatorname{arc} AB}{\operatorname{arc} AC}.$$

(See Fig. 2.)

THEOREM III. *Denote by P and P' the perimeters of two polygons inscribed in the same circle. If the greatest side of the second is less than the smallest side of the first, then $P' > P$.*

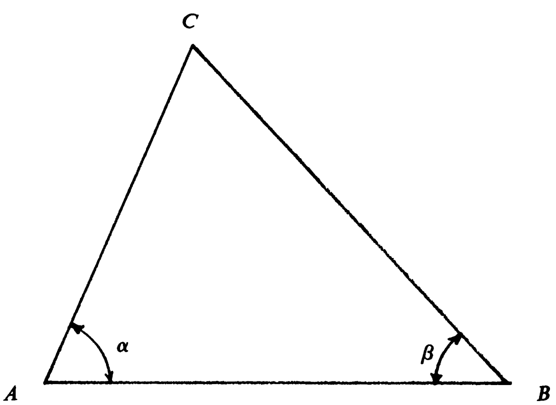

FIG. 1

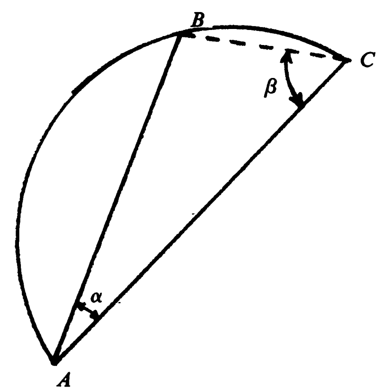

FIG. 2

In this note, we elaborate on Theorem III. Although it is not stated explicitly in the theorem, it is tacitly assumed from the context that the two polygons are convex and that the center of the circle lies in the interior of both polygons.

Uspensky's proof is a nice illustration of induction in geometry, which is not too prevalent (for other examples of geometric induction, see [1]). The inductive step appears in the proof of Theorem I where he first considers the case when angles α and β are commensurable so that $\alpha = p\delta$ and $\beta = q\delta$ where p and q are relatively prime integers. It is then supposed that the above inequalities are valid in every case when the sum $p+q$ is *less* than a given integer $N > 2$, from which it is shown to hold for $p+q=N$. Then the incommensurable case is reduced to the former case by a limiting process.

The proof here of Theorem III will be based on the convexity of $-\sin x$, which also leads to further related results.

Let the central angles subtended by the sides of $\mathcal{P}$ and $\mathcal{P}'$ from the center of the circle be denoted by $\theta_1, \theta_2, \ldots, \theta_m$ and $\theta_1', \theta_2', \ldots, \theta_n'$, respectively. Uspensky assumes that $0 < \theta_i, \theta_j' < \pi$, and

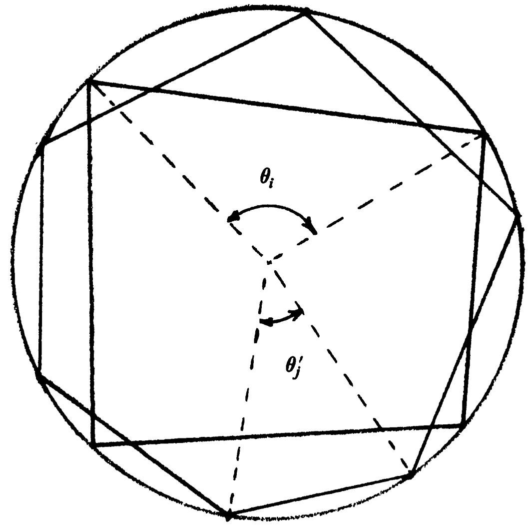

FIG. 3

$\max_j \theta_j' < \min_i \theta_i$. (See Fig. 3.) We weaken the latter condition slightly to

$$\max_j \theta_j' \leqslant \min_i \theta_i. \tag{1}$$

In order to apply the Majorization Theorem [2], [3], [4] for convex functions, it is also assumed without loss of generality that the angles are indexed to satisfy

$$\theta_1 \geqslant \theta_2 \geqslant \cdots \geqslant \theta_m \quad \text{and} \quad \theta_1' \geqslant \theta_2' \geqslant \cdots \geqslant \theta_n'.$$

It now follows from (1) that

$$\begin{aligned} &\theta_1 \geqslant \theta_1', \theta_1 + \theta_2 \geqslant \theta_1' + \theta_2', \ldots, \\ &2\pi = \theta_1 + \theta_2 + \cdots + \theta_m \geqslant \theta_1' + \theta_2' + \cdots + \theta_m', \\ &0 + \theta_1 + \theta_2 + \cdots + \theta_m \geqslant \theta_1' + \theta_2' + \cdots + \theta_{m+1}', \\ &\qquad \vdots \qquad\qquad\qquad \vdots \\ &0 + 0 + \cdots + 0 + \theta_1 + \theta_2 + \cdots + \theta_m = \theta_1' + \theta_2' + \cdots + \theta_n' = 2\pi. \end{aligned}$$

Then by the Majorization Theorem for any convex function $F(x)$,

$$\sum_{i=1}^{m} F(\theta_i) \geqslant \sum_{j=1}^{n} F(\theta_j'). \tag{2}$$

For strictly convex functions, there is equality if and only if $m = n, \theta_i = \theta_i' = \text{constant}$.

It is to be noted that in proving (2) under the rather restrictive conditions (1), one does not need the full power of the Majorization Theorem. (However, it will be needed subsequently.) For suppose that we have a set of points

$$a_i \leqslant A \leqslant b_j \leqslant B \leqslant c_k$$

and a strictly convex function $F(x)$. (See Fig. 4.) Then, for $A < x < B$, the curve of $F(x)$ is below the chord through the points $(A, F(A))$ and $(B, F(B))$. Hence, any weighted average of the points $(a_i, F(a_i))$ and $(c_k, F(c_k))$ is above this chord; any weighted average of the points $(b_j, F(b_j))$ is below the chord. Thus, if $p_i, q_j, r_k \geqslant 0$ and if

$$\sum p_i + \sum r_k = 1 = \sum q_j,$$

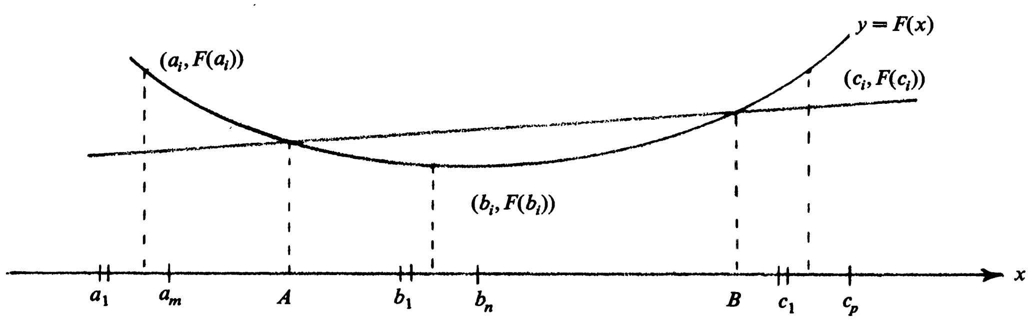

FIG. 4

then

$$\sum p_i(a_i, F(a_i)) + \sum r_k(c_k, F(c_k))$$

is above the chord while $\Sigma q_j(b_j, F(b_j))$ is below. Now if, in addition, $\Sigma p_i a_i + \Sigma r_k c_k = \Sigma q_j b_j$, then

$$\sum p_i F(a_i) + \sum r_k f(c_k) \geqslant \sum q_j F(b_j).$$

For the case in point, $0 \leqslant \theta_j' \leqslant \theta_i$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, and $\Sigma\theta_i = \Sigma\theta_j' = 2\pi$. It then follows easily that

$$\frac{n-m}{n} F(0) + \frac{1}{n} \sum F(\theta_i) \geqslant \frac{1}{n} \sum F(\theta_j').$$

Since $-\sin x$ is convex for $0 \leqslant x \leqslant \pi$ and since the perimeter P and area A of polygon $\mathcal{P}$ are given by

$$2R \sum \sin\theta_i/2 \text{ and } (R^2/2) \sum \sin\theta_i,$$

respectively, where R denotes the radius of the circle, and similar expressions for $\mathcal{P}'$, it follows from (2) that

$$P' \geqslant P \quad \text{and} \quad A' \geqslant A \tag{3}$$

with equality iff the polygons are congruent and regular.

It is geometrically intuitive that for the class of polygons $\mathcal{P}'$ satisfying (1), P' and A' will be a minimum for that polygon of $\mathcal{P}'$ whose angles are given by

$$\begin{aligned} \theta_j' &= \theta \equiv \min_i \theta_i, \qquad j = 1, 2, \ldots, k-1, \\ \theta_k' &= 2\pi - (k-1)\theta, \end{aligned} \tag{4}$$

where k is determined from

$$2\pi/\theta = k - 1 + f/\theta, \qquad 0 \leqslant f < \theta.$$

This means that the minimum polygon of class $\mathcal{P}'$ is as regular as possible with sides as long as possible. For example, if $\min\theta_i = 50°$, then the angles of the minimum polygon of class $\mathcal{P}'$ would have seven angles of 50° and one angle of 10°. A proof follows as before by comparing any polygon of class $\mathcal{P}'$ with the particular one given by (4). It is to be noted here that, although conditions (1) are not satisfied, the Majorization Theorem still applies.

It also follows in similar fashion that if we consider the family of n-gons inscribed in a given circle, the regular one has maximum area and maximum perimeter. Note that if the angles of one of these polygons are $\theta_1, \theta_2, \ldots, \theta_n$ where $\theta_1 \geqslant \theta_2 \geqslant \cdots \geqslant \theta_n$, then

$$\theta_1+\theta_2+\cdots+\theta_r \geqslant r\sum_{i=1}^{n}\theta_i/n$$

for $r=1,2,\ldots,n$.

We now give some analogous results for polygons circumscribed about a given circle. In view of the previous theorem, we might intuitively expect the

THEOREM. *If P, A and P', A', denote the perimeter and area, respectively, of two polygons $\mathcal{P}$ and $\mathcal{P}'$ circumscribed about a given circle such that the greatest side of $\mathcal{P}'$ is less than the smallest side of $\mathcal{P}$, then $P>P'$ and $A>A'$.* (5)

However, as stated, the theorem is not quite valid. First consider a circumscribed equilateral triangle and a circumscribed square. (In Fig. 5, we have left out the corresponding circumscribed square, which may detract from the subsequent argument.) The "theorem" does apply to these two figures. But now perturb the triangle slightly to form a circumscribed hexagon $A'E'C'D'B'F'A'$. The "theorem" does not apply to this hexagon and the previous square. It will, however, if we regard the square as an octagon by considering the four points of tangency of the sides with the circle as extra vertices. Consequently, for our theorem to apply to any circumscribed polygons, we will consider the points of tangency of the sides with the circle to be vertices also. This has the effect of doubling the number of vertices and decreasing the length of each side.

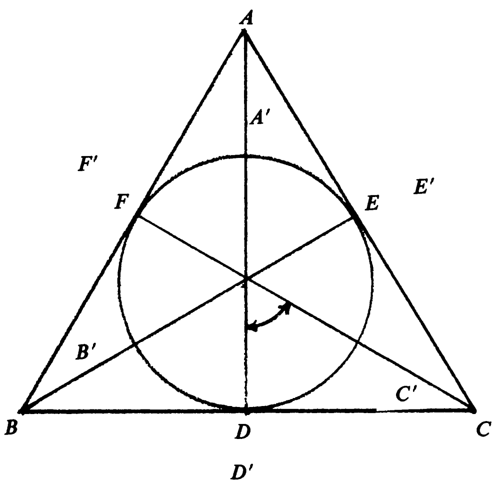

FIG. 5

To prove our theorem (5) as now interpreted, denote the angles subtended by the sides of $\mathcal{P}$ and $\mathcal{P}'$ with respect to the center by θ_i and θ_j', respectively. Then the perimeter and area of circumscribed polygon $\mathcal{P}$ are given by

$$R\sum \tan\theta_i \quad \text{and} \quad (R^2/2)\sum \tan\theta_i,$$

respectively, and by similar expressions for $\mathcal{P}'$. Since $\tan x$ is convex for $0 \leqslant x < \pi/2$, it follows by a similar argument as before that $P \geqslant P'$ and $A \geqslant A'$.

For the class of polygons $\mathcal{P}'$ satisfying the conditions in (5), P' and A' will be a maximum for that polygon of $\mathcal{P}'$ whose angles are again obtained as in (4).

The previous results can be extended to a wider class of admissible polygons by not requiring that the center of the circle lie in the interior of the polygons. In the case of inscribed polygons (see Fig. 6), we then require that the polygons $\mathcal{P}$ and $\mathcal{P}'$ have a common side V_1V_m and also that the angle inequalities pertain only to all the angles subtended by the sides with the

exception of the common side V_1V_m. In the case of circumscribed polygons (see Fig. 7), we then require that the polygons $\mathcal{P}$ and $\mathcal{P}'$ have a common side, V_1V_m, that is a chord of the circle and all the remaining sides are tangent to the circle. Again, the angle inequalities are to pertain only to all the angles subtended by the sides with the exception of the common side V_1V_m.

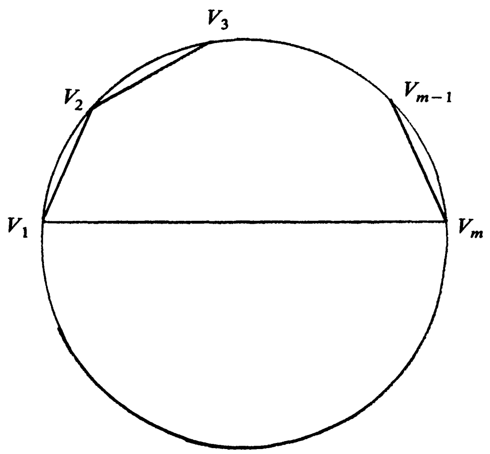

FIG. 6

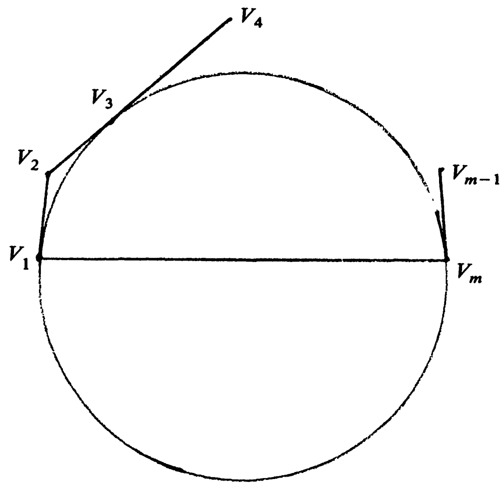

FIG. 7

In conclusion, we note that as a corollary to his Theorem II, Uspensky obtains, without calculus, that $(\sin x)/x$ is a decreasing function in $[0,\pi/2]$. Also, as corollaries of our results, without calculus,

$$n\sin(x/n) \quad \text{and} \quad -n\tan(x/n), \qquad (0\leqslant x/n\leqslant \pi/2)$$

are monotonically increasing functions of integral n.

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NONUNIQUE FACTORIZATION IN SUBSEMIGROUPS OF SEMIGROUPS WITH UNIQUE PRIME FACTORIZATION

GEBHARD GREITER

Factorial semigroups (that is, semigroups with unique factorization into primes) contain, as a rule, subsemigroups S that are *not* factorial; even more, if n and m are the numbers of irreducible factors in two factorizations of some $x\in S$, the quotient n/m may be arbitrarily large. That is shown by the following theorem:

THEOREM. *Let F be a factorial semigroup that is not a group. Then, for every $n\in\mathbb{N}$, $n\geqslant 2$, there is a subsemigroup S of F with $p_1p_2=p_3^n$ for some irreducible elements p_1,p_2,p_3 of S.*

Proof. Since F is not a group, F contains at least one prime element p. Given $n\in\mathbb{N}$, $n\geqslant 2$, let S be the semigroup of F generated by

$$p_1=p^{2n-1},\quad p_2=p^{2n+1}\quad\text{and}\quad p_3=p^4.$$

Then p_1,p_2,p_3 are irreducible elements of S with $p_1p_2=p_3^n$.

REMARK. For the factorial semigroup $\mathbb{Z}$ of rational integers, our theorem was first proved by B. Jacobson [1]. But in his proof he used a comparatively deep result of Dirichlet (stating the existence of primes in $a+b\mathbb{Z}$, if $\gcd(a,b)=1$).

Reference

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RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this department the MONTHLY *presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada T2N 1N4.*

DOES THERE EXIST A BANACH *-ALGEBRA WITHOUT IDENTITY WITH NO NONZERO POSITIVE FUNCTIONALS?

R. S. DORAN

A ***-algebra** is a complex associative linear algebra A with a mapping $x\to x^*$ of A into itself such that for $x,y\in A$ and complex λ: (a) $(x+y)^*=x^*+y^*$; (b) $(xy)^*=y^*x^*$; (c) $(\lambda x)^*=\bar{\lambda}x^*$($\bar{\lambda}$ is the complex conjugate of λ); and (d) $x^{**}=x$. The map $x\to x^*$ is called an **involution**; because of (d) it is clearly bijective. An algebra that is also a Banach space satisfying $\|xy\|\leqslant\|x\|\cdot\|y\|$ for all x,y is called a **Banach algebra**. A Banach algebra that is also a *-algebra is called a **Banach *-algebra**. A **C^*-algebra** is a Banach *-algebra satisfying $\|x^*x\|=\|x\|^2$ for all x.

Typical examples of Banach *-algebras are the complex numbers C with the usual multiplication, involution $\lambda^*=\bar{\lambda}$ (complex conjugation), and absolute value norm; the algebra $C(X)$ of bounded continuous complex-valued functions on a topological space X with pointwise multiplication $(fg)(t)=f(t)g(t)$, involution $f^*(t)=\overline{f(t)}$, and sup norm; the algebra $B(H)$ of bounded linear operators on a Hilbert space H with composition as multiplication, involution $T\to T^*$ (the adjoint of T), and operator norm; the group algebra $L^1(G)$ of a locally compact abelian group G with multiplication

$$(f*g)(t)=\int_G f(t-s)g(s)\,ds,$$

involution $f^*(t)=\overline{f(-t)}$, and L^1-norm; and the algebra $A(D)$ of continuous complex-valued functions on the closed unit disc D which are analytic on the interior of D with pointwise multiplication, involution $f^*(\lambda)=\overline{f(\bar{\lambda})}$, and sup norm. The first three of these examples are C^*-algebras.

For particular topological spaces X, variations of the involution in the second example can be

given. For instance, if $X=[0,1]$ with the usual topology, then $f^*(t)=\overline{f(1-t)}$ defines an involution in $C(X)$. As a second illustration, let $X=[0,1]\cup\{2,3\}$ with the usual topology of the reals and, for $f\in C(X)$, define $f^*(t)=\overline{f(t)}$ if $t\in[0,1]$, $f^*(2)=\overline{f(3)}$, and $f^*(3)=\overline{f(2)}$.

To see how extensive the class of Banach *-algebras are, we note that *every* Banach algebra A can be isometrically embedded as a closed two-sided ideal of a Banach *-algebra B. Indeed, let $B=A\times A$ and define

$$(x,y)+(w,z)=(x+w,y+z), \qquad (x,y)(w,z)=(xw,yz),$$
$$\lambda(x,y)=(\lambda x,\bar{\lambda}y), \qquad (x,y)^*=(y,x), \qquad \|(x,y)\|=\max\{\|x\|,\|y\|\}.$$

Then B is a Banach *-algebra and the map $x\to(x,0)$ is an isometric embedding of A in B. If e is an identity for A, then (e,e) is an identity for B.

A linear functional p defined on a *-algebra A is said to be **positive** if $p(x^*x)\geqslant 0$ for all $x\in A$. Given $t_0\in[0,1]$, the functional p defined on $C([0,1])$ by $p(f)=f(t_0)$ is clearly positive. Another example of a positive functional on $C([0,1])$ is $p(f)=\int_0^1 f(t)dt$. If ξ is a fixed vector in a Hilbert space H, then the functional p defined on $B(H)$ by $p(T)=(T\xi|\xi)$ is a positive functional.

Every positive functional p on a *-algebra A satisfies the following Schwarz inequality:

$$|p(y^*x)|^2\leqslant p(x^*x)p(y^*y) \tag{1}$$

for all $x,y\in A$ (see [**2**, *p.* 316]).

If A has an identity e, then it is easy to see that the Banach *-algebra $B=A\times A$ constructed above does not admit any nonzero positive functionals. Indeed, since $(e,-e)^*(e,-e)=-(e,e)$ and (e,e) is the identity for B, every positive functional on B must vanish at (e,e) and therefore by inequality (1) must be identically zero. Another example of a Banach *-algebra with identity admitting no nonzero positive functionals can be found in [**6**, p. 138].

PROBLEM. *Does there exist a Banach *-algebra without identity that admits no nonzero positive functionals?*

In the case of C^*-algebras (with or without identity) there are enough positive functionals to separate the points of the algebra (see [**1**, 2.6.1]). In view of the central role played by positive functionals in the representation theory of Banach *-algebras, an answer to the problem would be of considerable interest.

An excellent elementary introduction to Banach *-algebras (suitable for advanced undergraduates) can be found in [**5**], and much more extensive treatments are given in [**1**], [**3**], [**4**], and [**6**].

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THE SLATER CONDITION IN INFINITE-DIMENSIONAL VECTOR SPACES

JOCHEM ZOWE

Let Z be a real topological vector space, K a convex cone in Z, and b an element of K. Suppose the conical hull K_b of $K-\{b\}$ has non-empty topological interior. What can be said

about the interior of the smaller cone K? Does K have non-empty interior as well? Under the (merely technical) assumptions that Z is barreled and K is closed, the answer is yes! Show that this holds more generally for an arbitrary Z or give a counterexample.

The above question arises in optimization when minimizing a real functional f on a real normed space X subject to a constraint of the type $g(x) \in K$; here g maps X into the normed Z. If f restricted to $\{x \in X \mid g(x) \in K\}$ assumes a local minimum at x_0 and f, g are Fréchet-differentiable at x_0, then according to the famous *Kuhn-Tucker* condition there always exists a continuous linear functional l on Z such that

$$f'(x_0) + l \circ g'(x_0) = 0, \quad l(g(x_0)) = 0, \quad l(z) \geqslant 0 \quad \text{for all } z \in K,$$

provided the following regularity assumption on g is met (the so-called *Slater* condition):

(1) *there is some* $x \in X$ *such that* $g'(x_0)x + g(x_0) \in \operatorname{int} K$;

see the standard literature, e.g., Craven [1] or Kurcyusz [2]. It is easily seen that in the special case where $Z = \mathbb{R}^n$ (i.e., $g(x) = (g_1(x), \ldots, g_n(x))^T$) and $K = \{z \in \mathbb{R}^n \mid z_i \geqslant 0 \text{ for } i = 1, 2, \ldots, n\}$ the regularity condition (1) reduces to a condition on only that part of $g'(x_0)$ which is related to the components g_i of g which are "active" at x_0, i.e., for which $g_i(x_0) = 0$:

$$\textit{there is some } x \in X \textit{ such that } g_i'(x_0)x > 0 \textit{ for all } \\ i \in \{1, 2, \ldots, n\} \textit{ with } g_i(x_0) = 0.$$

Writing $K_{g(x_0)}$ for the conical hull of $K - \{g(x_0)\}$, this last condition can be rewritten as:

(2) *there is some* $x \in X$ *such that* $g'(x_0)x \in \operatorname{int} K_{g(x_0)}$.

The question arises if the well-known equivalence of (1) and (2) for finite-dimensional Z does also hold for infinite-dimensional Z. Note that the advantage of (2) compared with (1) is that, just as in the finite-dimensional case, (2) can be interpreted as a condition only on the "active" part of $g(x_0)$. To prove the equivalence of (1) and (2) one needs as a decisive step that

(3) $\operatorname{int} K_{g(x_0)} = \{\lambda g(x_0) \mid \lambda \in \mathbb{R}\} + \operatorname{int} K$,

and this remains an open question and does not seem to appear in the literature. Prove (or give a counterexample) that (3) holds for arbitrary normed Z and (closed) convex cone and show the equivalence of (1) and (2) in this general situation. Formula (3) can be proved for the special case where K is closed and Z is barreled, i.e., each radial (absorbing), convex circled and closed subset of Z is a neighborhood of the origin (see [3]; a Banach space, for example, is barreled). To do this, one shows that in the above situation the interior of K is non-empty if (and only if) the interior of $K_{g(x_0)}$ is non-empty (cf. [4]). To see this, suppose that int $K_{g(x_0)} \neq \varnothing$, i.e., $k - \lambda g(x_0) + U \subset K_{g(x_0)}$ with a suitable $k \in K$, $\lambda \in \mathbb{R}$ and a neighborhood U of the origin. It follows that $U \subset K_{g(x_0)} - k$, and thus X is equal to the conical hull of $K - (g(x_0) + k)$. By making use of the assumption that X is barreled, it is verified in a few further steps that $g(x_0) + k \in \operatorname{int} K$. Since the above assumptions on X and K play only a rather technical role in the proof, there is a real hope that (3) may hold without these assumptions, thus showing that (1) and (2) are equivalent in our general situation.

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CLASSROOM NOTES

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A NOTE ON RIEMANN-STIELTJES INTEGRALS

U. V. SATYANARAYANA

It is proved in most advanced-calculus courses that if a function f is non-negative and Riemann-integrable over $[a,b]$ then $\int_a^b f(x)dx \geqslant 0$, the proof of which is direct from the definition of integrability. However, not included in most advanced-calculus textbooks is the fact that if f is strictly positive on $[a,b]$ then $\int_a^b f(x)dx$ must also be positive. If the students have been introduced to measure theory, the usual proof relies on the fact that the discontinuities of a Riemann-integrable function form a set of measure zero.

In the following, I present an elementary proof of the above, and the only tool that the student needs is the fact that the intersection of a decreasing sequence of closed intervals is non-empty. In fact, more generally, we prove:

THEOREM. *Let $f(x) > 0$ and bounded on $[a,b]$, and let α be a non-constant monotonic non-decreasing function on $[a,b]$. Then the upper Riemann-Stieltjes (R-S) integral $\overline{\int}_a^b f d\alpha$ is positive.*

Taking $\alpha(x) = x$, the result for the Riemann integral follows.

For any partition $P: a = x_0 < x_1 < \cdots < x_n = b$ of $[a,b]$, let $M_i = \sup\{f(x): x_{i-1} \leqslant x \leqslant x_i\}$ and let $\delta_i = \alpha(x_i) - \alpha(x_{i-1})$, $1 \leqslant i \leqslant n$. Let us denote by $S(P,f,\alpha)$ the usual upper Riemann-Stieltjes sum $\sum_{i=1}^n M_i \delta_i$.

The proof of the theorem relies on a repeated use of the following:

LEMMA 1. *Let f be bounded on $[a,b]$ and let α be as in the theorem, with $\overline{\int}_a^b f d\alpha = 0$. Then, given $\varepsilon > 0$, there exists a sub-interval $[c,d]$ of $[a,b]$ (depending on ε) such that*

$$\sup_{c<x<d} f(x) < \varepsilon.$$

Proof. Since α is non-constant, $\alpha(b) - \alpha(a)$ must be positive. And since $0 = \overline{\int}_a^b f d\alpha = \inf\{S(P,f,\alpha): P \text{ is a partition of } [a,b]\}$, there must exist a partition P of $[a,b]$ such that

$$S(P,f,\alpha) < \varepsilon(\alpha(b) - \alpha(a)).$$

This clearly implies that there exists an i such that $M_i < \varepsilon$. Then the required sub-interval is $[x_{i-1}, x_i]$.

We now prove Lemma 2, which is an equivalent formulation of the Theorem.

LEMMA 2. *Let f be bounded and non-negative on $[a,b]$, and let α be as in the theorem, such that $\overline{\int}_a^b f d\alpha = 0$. Then f vanishes at least once in $[a,b]$.*

Proof. By the additivity of the upper integral, it follows that the upper integral is zero on every sub-interval of $[a,b]$, since $f(x) \geqslant 0$.

Taking $\varepsilon = 1$ in Lemma 1, we get a sub-interval, say $[a_1, b_1]$, of $[a,b]$ such that

$$\sup_{a_1<x<b_1} f(x) < 1.$$

Now since $\overline{\int}_{a_1}^{b_1} f d\alpha = 0$, we can use Lemma 1 again; taking, now, $\varepsilon = \frac{1}{2}$, we get a sub-interval $[a_2, b_2]$ of $[a_1, b_1]$ such that

$$\sup_{a_2 < x < b_2} f(x) < \frac{1}{2}.$$

Continuing this way, we obtain a decreasing sequence $\{[a_n, b_n]\}$ of closed intervals such that

$$\sup_{a_n < x < b_n} f(x) < \frac{1}{n}, \quad n = 1, 2, \ldots.$$

Since $\bigcap_{n=1}^{\infty}[a_n, b_n] \neq \varnothing$, there exists a number ξ such that $a_n \leqslant \xi \leqslant b_n$ for all natural numbers n. Hence,

$$f(\xi) \leqslant \sup_{a_n < x < b_n} f(x) < \frac{1}{n}, \text{ for all natural numbers } n,$$

showing that $f(\xi) \leqslant 0$. Since f is non-negative on $[a, b]$, it follows that $f(\xi) = 0$. This proves Lemma 2 and hence the Theorem.

The following corollary is immediate.

COROLLARY. *With f as in Lemma 2 and with $\alpha(x) = x$, the set of zeros of f is dense in $[a, b]$.*

REMARKS. Using measure theory, the corollary follows immediately from the fact that the Lebesgue integral of a non-negative function f on $[a, b]$ is zero if and only if f vanishes almost everywhere in $[a, b]$.

For a bounded Riemann integrable function f the result of this note may be established [**1**, p. 349] by showing that f has a point of continuity in $[a, b]$. The proof given above seems pedagogically preferable.

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THE SUM OF THE kTH POWERS OF THE FIRST n INTEGERS

HENRY J. SCHULTZ

In many calculus courses one must compute the area under the curve $y = x^k$ over the interval $[0, 1]$ by taking the limit of approximate areas determined by summing rectangles. For example, in Figure 1 we find that the area is

$$\lim_{n \to \infty} \sum_{m=1}^{n} \frac{1}{n}\left(\frac{m}{n}\right)^k = \lim_{n \to \infty} \frac{1}{n^{k+1}}\left(\sum_{m=1}^{n} m^k\right).$$

To complete the solution of this and many similar problems, it is then necessary to find (usually for small positive integral k) the sums

$$S_k(n) = \sum_{m=1}^{n} m^k. \tag{1}$$

The appropriate formulas are usually proved by induction or derived from simple geometric pictures. Some students are still not satisfied and want to know how they could discover formulas for arbitrary k. Unfortunately, the standard closed forms involve Bernoulli numbers or Stirling numbers of the second kind [**2**], which come from reasonably complicated recurrence relations. We shall derive a simple procedure for finding $S_k(n)$, k a positive integer, that is easy

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A PROOF OF THE FROBENIUS RANK EQUALITY

DONALD W. ROBINSON

Given the product $\alpha\beta\gamma$ of three matrices, the Frobenius inequality on rank states that

$$\operatorname{rank}\alpha\beta+\operatorname{rank}\beta\gamma\leqslant\operatorname{rank}\beta+\operatorname{rank}\alpha\beta\gamma.$$

Special cases of this inequality provide several familiar facts: for example, $\alpha=0$ implies rank $\beta\gamma\leqslant$ rank β; $\gamma=0$ implies rank $\alpha\beta\leqslant$ rank β; and if β is the n-by-n identity matrix, then rank $\alpha+$ rank $\gamma\leqslant n+$ rank $\alpha\gamma$. Thus, in particular, Sylvester's law is a consequence of the Frobenius inequality. (See for example [4, pp. 27–28] or [3, p. 11].)

Although it is not as well known, Frobenius [2] identified the "difference" between the right and left sides of the inequality as the rank of a product $\kappa\beta\nu$, where κ and ν are what he termed the "complete solutions" of the equations $\kappa(\beta\gamma)=0$ and $(\alpha\beta)\nu=0$, respectively. The purpose of this note is to provide a proof of this result for use in the classroom. In particular, the proof contains a simple demonstration of the Frobenius inequality and suggests some related results and extensions.

For convenience of presentation in an undergraduate class of linear algebra, this note is framed in the category of vector spaces over a given field. In particular, a vector-space morphism $\alpha:A\to B$ is said to be of finite rank provided its image space $\operatorname{Im}\alpha$ is of finite dimension; in this case, the rank of α is defined to be $r(\alpha)=\dim(\operatorname{Im}\alpha)$. The following fact about this category is used below: if $\alpha:A\to B$ is a morphism and A is of finite dimension, then both the image $\operatorname{Im}\alpha$ and the kernel $\operatorname{Ker}\alpha$ of α are of finite dimension and $\dim A=\dim(\operatorname{Ker}\alpha)+\dim(\operatorname{Im}\alpha)$. Also, in this paper the convention is used of reading morphism composition from left to right.

THEOREM. *Let*

$$A\xrightarrow{\alpha}B\xrightarrow{\beta}C\xrightarrow{\gamma}D$$

be a sequence of vector spaces and vector-space morphisms. Let $\kappa:\operatorname{Ker}\beta\gamma\to B$ be the insertion of the kernel of $\beta\gamma$ into B and $\nu:C\to C/\operatorname{Im}\alpha\beta$ be the natural projection of C onto its quotient space with respect to the image of $\alpha\beta$. If β is of finite rank, then so also are $\alpha\beta$, $\beta\gamma$, $\alpha\beta\gamma$, and $\kappa\beta\nu$, and

$$r(\alpha\beta)+r(\beta\gamma)+r(\kappa\beta\nu)=r(\beta)+r(\alpha\beta\gamma).$$

In particular,

$$r(\alpha\beta)+r(\beta\gamma)\leqslant r(\beta)+r(\alpha\beta\gamma).$$

Proof. First, since $\operatorname{Im}\alpha\beta\subseteq\operatorname{Im}\beta$, $\operatorname{Im}\alpha\beta\gamma\subseteq\operatorname{Im}\beta\gamma$, $(\operatorname{Im}\beta)\gamma\subseteq\operatorname{Im}\beta\gamma$, and $(\operatorname{Im}\alpha\beta)\gamma\subseteq\operatorname{Im}\alpha\beta\gamma$, then

$$\mathcal{C}:\frac{\operatorname{Im}\beta}{\operatorname{Im}\alpha\beta}\to\frac{\operatorname{Im}\beta\gamma}{\operatorname{Im}\alpha\beta\gamma};\quad x+\operatorname{Im}\alpha\beta\mapsto x\gamma+\operatorname{Im}\alpha\beta\gamma$$

is a vector-space morphism. Moreover, since $\operatorname{Im}\beta\gamma\subseteq(\operatorname{Im}\beta)\gamma$, $\mathcal{C}$ is surjective. Therefore, if $\operatorname{Im}\beta$ is of finite dimension, then so are $\operatorname{Im}\alpha\beta$, $\operatorname{Im}\beta\gamma$, and $\operatorname{Im}\alpha\beta\gamma$, and

$$\begin{aligned}r(\beta)-r(\alpha\beta)&=\dim(\operatorname{Im}\beta/\operatorname{Im}\alpha\beta)=\dim(\operatorname{Ker}\mathcal{C})+\dim(\operatorname{Im}\mathcal{C})\\&=\dim(\operatorname{Ker}\mathcal{C})+\dim(\operatorname{Im}\beta\gamma/\operatorname{Im}\alpha\beta\gamma)=\dim(\operatorname{Ker}\mathcal{C})+r(\beta\gamma)-r(\alpha\beta\gamma).\end{aligned}$$

That is,

$$r(\alpha\beta)+r(\beta\gamma)+\dim(\operatorname{Ker}\mathcal{C})=r(\beta)+r(\alpha\beta\gamma).$$

In particular, the Frobenius inequality is a consequence of this equality.

Second, the vector-space morphism $\kappa\beta\nu:\operatorname{Ker}\beta\gamma\to C/\operatorname{Im}\alpha\beta$ is a product of the morphism

$$\lambda:\operatorname{Ker}\beta\gamma\to\operatorname{Ker}\mathcal{C};\quad x\mapsto x\beta+\operatorname{Im}\alpha\beta$$

and the restriction

$$\mu:\operatorname{Ker}\mathcal{C}\to C/\operatorname{Im}\alpha\beta;\quad x+\operatorname{Im}\alpha\beta\mapsto x+\operatorname{Im}\alpha\beta$$

of the natural injection of $\operatorname{Im}\beta/\operatorname{Im}\alpha\beta$ into $C/\operatorname{Im}\alpha\beta$. Indeed, for every x in $\operatorname{Ker}\beta\gamma$,

$$x(\kappa\beta\nu)=((x\kappa)\beta)\nu=(x\beta)\nu=x\beta+\operatorname{Im}\alpha\beta=(x\beta+\operatorname{Im}\alpha\beta)\mu=(x\lambda)\mu=x(\lambda\mu).$$

Moreover, μ is clearly injective and λ is surjective: if $x\beta+\operatorname{Im}\alpha\beta\in\operatorname{Ker}\mathcal{C}$, then $x\beta\gamma\in\operatorname{Im}\alpha\beta\gamma, x\beta\gamma=y\alpha\beta\gamma$ for some $y\in A$, $(x-y\alpha)\beta\gamma=0$, and

$$(x-y\alpha)\lambda=(x-y\alpha)\beta+\operatorname{Im}\alpha\beta=x\beta-y\alpha\beta+\operatorname{Im}\alpha\beta=x\beta+\operatorname{Im}\alpha\beta.$$

Consequently,

$$\operatorname{Ker}\mathcal{C}\cong\operatorname{Ker}\mathcal{C}/\operatorname{Ker}\mu\cong\operatorname{Im}\mu=\operatorname{Im}\lambda\mu=\operatorname{Im}\kappa\beta\nu,$$

which together with the equality above completes the proof.

Analogous to the fact that $\mathcal{C}$ is an epimorphism is the fact that

$$\mathcal{A}:\frac{\operatorname{Ker}\alpha\beta\gamma}{\operatorname{Ker}\alpha\beta}\to\frac{\operatorname{Ker}\beta\gamma}{\operatorname{Ker}\beta};\quad x+\operatorname{Ker}\alpha\beta\mapsto x\alpha+\operatorname{Ker}\beta$$

is a monomorphism. A combination of these results suggest the following Corollary, the proof of which is left as an exercise for the reader.

COROLLARY. *Let the conditions and notation be as in the Theorem. Then*

$$0\to\frac{\operatorname{Ker}\alpha\beta\gamma}{\operatorname{Ker}\alpha\beta}\xrightarrow{\mathcal{A}}\frac{\operatorname{Ker}\beta\gamma}{\operatorname{Ker}\beta}\xrightarrow{\mathcal{B}}\frac{\operatorname{Im}\beta}{\operatorname{Im}\alpha\beta}\xrightarrow{\mathcal{C}}\frac{\operatorname{Im}\beta\gamma}{\operatorname{Im}\alpha\beta\gamma}\to 0,$$

where

$$\mathcal{A}:x+\operatorname{Ker}\alpha\beta\mapsto x\alpha+\operatorname{Ker}\beta$$

$$\mathcal{B}:x+\operatorname{Ker}\beta\mapsto x\beta+\operatorname{Im}\alpha\beta$$

$$\mathcal{C}:x+\operatorname{Im}\alpha\beta\mapsto x\gamma+\operatorname{Im}\alpha\beta\gamma,$$

is exact, and

$$r(\kappa\beta\nu)=\dim(\operatorname{Coker}\mathcal{A})=\dim(\operatorname{Im}\mathcal{B})=\dim(\operatorname{Ker}\mathcal{C}).$$

In conclusion, it is observed that the methods used in the proof above generalize to other categories. For example, the Theorem clearly extends to the category of R-modules, where the rank of an R-morphism whose image is of finite length is defined to be the length of its image. (For definitions see for example [**1**, pp. 212–214] or [**5**, pp. 158–163].)

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MATHEMATICAL EDUCATION

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THE FIRST WRITTEN EXAMINATIONS AT HARVARD COLLEGE

HUBERT KENNEDY

It is remarkable that the first written examinations ever demanded at the end of a year's course at Harvard College were in mathematics. It is even more remarkable that the two tutors who introduced them later held the highest offices at Harvard University: one became President and the other was the first Dean of the Graduate School. The two men were Charles W. Eliot (1834–1926) and James Mills Peirce (1834–1906).

Both Eliot and Peirce graduated from Harvard College with distinction in 1853. Eliot held second rank and Peirce ninth in a class of 88. Although both had, under the limited option then available, elected upper-level courses in mathematics, at the time of their graduation neither planned a career in teaching or mathematics. Eliot later wrote of himself:

> When I found myself a Bachelor of Arts I had no idea what profession I should follow; and after a vacation spent chiefly in travel, I returned to my father's house in Boston, and made serious efforts to supplement my college education. I joined a business college to learn bookkeeping, and took lessons in French and German, because neither at school nor at college had I been required to study these languages, or indeed been offered good opportunities to do so. [1, p. 97]

In the meantime, Peirce had entered the Law School. But when the opportunity of becoming a tutor (and retaining his rooms in the Harvard Yard) appeared, he did not continue his study of law, and indeed remained undecided about a final choice of career for several years. Eliot, however, had determined on a university career by the time they were appointed Tutors in Mathematics in 1854.

Peirce chose to teach the Freshman Class, the required subject for the first semester being Plane and Analytic Geometry, while Eliot taught the required Algebra to the Sophomores. (The following year they exchanged classes.) The two new tutors (who replaced the undoubtedly overworked Mr. Choate, who had just resigned) immediately instituted changes in the way of conducting classes and in the examinations. The change from the earlier recitation method was described by Peirce in his semiannual report to the Board of Overseers of Harvard University for the first term of the academic year 1854–55:

> In addition to the recitations, the class was required, three times in the term, to bring in written analyses of suitable portions of the text-book, the last being an analysis of the whole book. These exercises were designed to call attention to the connections of different parts of the subject and its method of treating it used in the text-book.
>
> Problems, solvable by means of theorems contained in the text-book, were occasionally proposed to exercise and test the skills and accuracy of those who chose to attempt them. Though these problems were voluntary and received no marks, solutions of them were offered by a very satisfactory number of students. The solutions were almost always correct and complete, and, in several instances, remarkable for neatness and originality. They were returned, after being examined, with the comments of the instructor. [3, vol 1, pp. 200–202]

During the second semester the Freshmen studied Plane and Spherical Trigonometry. Peirce again noted:

> The system of voluntary problems was continued; and to those who desired to attend a course of explanations was given of the methods of constructing maps. [3, vol. 1, p. 239]

The first written examinations were introduced by Peirce and Eliot at the end of their first year as tutors (not at the end of their second year, as stated by Henry James [5, vol. 1, p. 68]). Hugh Hawkins summed up their reasons:

> Offended by the dubious expertness and obvious absenteeism of the Overseers, the unequal difficulty of questions posed to different students, and the weight assigned to daily recitation marks, the young tutors obtained permission to substitute written examinations, which they graded themselves. [4, p. 15]

Although their permission had been "very reluctant," as Eliot later recalled, the Examining Committee was pleased with the result. Rev. Thomas Hill (later President of Harvard University, 1862–1868) reported for the Committee:

> The Sophomore and Freshman Classes have been exercised, once a fortnight, during the last term in a new mode of recitation; and the same mode was adopted for their examination. The class was brought together in Harvard Hall, and seated at tables in that spacious room, in such a manner as to prevent any assistance of one scholar by another. Here under the eye of tutors and proctors, each man labored for two hours; solving as many as he was able of a printed list of about twenty questions; recording his work and results in a blank book. These books were afterwards examined by the tutors and by the committee and marks were entered on the scale of merit in proportion to the number of examples performed, as well as to the correctness of results. Afterwards twelve of the best scholars in each class were orally examined, but the results of this oral examination were not entered upon the scale of merit. Both classes appeared remarkably well under this severe test, and reflected great credit upon the instruction of the tutors Eliot and Peirce. [3, vol. 1, pp. 276–277]

As a result of this experience, the Examining Committee recommended the following year that written examinations not only be held in mathematics but be considered for other subjects as well. There was some opposition to this, of course, but as Hawkins noted:

> The new arrangement had a strong appeal to faculty professionalism, and it spread to other departments. By increased reliance on written examinations in determining rank, the faculty freed class time from graded recitations. [4, p. 15]

Two years later Peirce again taught the Freshman Class, and this time he attached a copy of his Trigonometry examination to his report. There were eighteen questions, of which two were on the use of logarithms and three on navigation and surveying. Here are two examples:

> No. 3. How many parts of a plane triangle is it necessary to know, in order to solve it? How many parts of a spherical triangle?...
>
> No. 12. *Problem.* To solve a spherical right triangle, when one leg and the adjacent angle are given. [3, vol. 2, p. 71]

After four years as Tutor of Mathematics, Eliot advanced in his chosen academic career with an appointment as Assistant Professor of Mathematics and Chemistry. At the same time, Peirce resigned as Tutor to complete his studies in the Divinity School, which he had entered in 1857. His former classmate Edward Pearce was appointed Tutor in his place, and Peirce was given the position of Proctor, formerly held by Pearce, thus allowing him to retain his rooms in the Harvard Yard. (Indeed, this was the one constant in all of Peirce's changes; and he kept rooms in the Harvard Yard until 1880.)

After graduating from the Divinity School in 1859, Peirce preached in various churches in and around Boston, and briefly in Charleston, South Carolina. However, he was never "settled" as a Unitarian minister; and in 1861, when Eliot gave up the mathematical half of his assistant professorship to take charge of the chemical laboratories of the Lawrence Scientific School, Peirce accepted an appointment as Assistant Professor of Mathematics. He remained in Harvard's Mathematical Department the rest of his life; as tutor and professor, he taught mathematics there for nearly fifty years.

Of perhaps equal importance with his teaching was Peirce's activity as administrator. When Eliot was appointed President of Harvard University in 1869, he immediately promoted Peirce

to University Professor of Mathematics, and the two resumed the close collaboration that they had begun as tutors. This resulted in the further development of Harvard as a university. A Graduate Department was founded in 1872, and Peirce, as Secretary of the Academic Council, was its real leader. The first two Harvard Ph.D.'s were awarded the following year, one to Peirce's student, and later colleague, William Elwood Byerly (1849–1935). It was Byerly who later remarked that Peirce was "almost the father" of the Graduate School of Arts and Sciences, and Peirce was named its first Dean when it was founded in 1890. After five years in this position, he became Dean of the Faculty of Arts and Sciences for the next three years.

Eliot later recalled this period of their collaboration:

> He became in the College Faculty a steady advocate of every measure which enlarged the freedom of students and increased advanced instruction in Harvard College. His influence in the College Faculty was strong, partly because he was visibly disinterested, and partly because he was an ardent speaker and formidable antagonist in debate.... He had the vision of the new university, and was a strong member of the group that worked for it. [2, p. 9]

The story of Charles W. Eliot's long tenure as President of Harvard University, from 1869 until 1909, has been excellently told by Hawkins [4] and James [5] and need not be repeated here. Peirce's story is not so well known, since he has been overshadowed by his father, Benjamin Peirce (1809–1880), also professor of mathematics at Harvard and the most famous American mathematician of his day, and by James's brother, Charles S. Peirce (1839–1912), the mathematician-philosopher, who is often considered America's most original thinker. Though without the mathematical genius of his father, J. M. Peirce clearly surpassed him as a teacher; and although James was not as original as his irascible brother Charles, he was by far the more lovable.

While still a tutor, Peirce showed ability as a teacher, and he used his experience to write a textbook in Analytic Geometry [8]. This book was first used by him in the fall of 1857; it was successfully used at Harvard for many years. Apart from several small volumes of mathematical tables, this remained his only book.

For many years Peirce cherished the project of writing a treatise on quaternions, but he never accomplished it. His father had been an enthusiastic champion of this subject and, as early as 1848, offered a course that included "Hamilton's researches respecting quaternions." When Benjamin Peirce's health began to fail in the 1870's, James not only assumed administrative duties in the Mathematical Department but also taught advanced courses offered by his father. From 1878 he was regularly teaching a two-year course in quaternions.

Peirce's publications number about twenty, and only half of them deal with mathematics. (For an annotated list, see [7].) His most original mathematical contributions were in two articles on quaternions, in 1899 and 1904, but by then the popularity of quaternions had already passed and no one thought it worthwhile to follow up his discoveries.

Mathematics was not his only interest. When he died in 1906, the Boston *Herald* and *Transcript* noted that he was "considered the world's authority on quaternions" but added that Peirce, "though known as a student of higher mathematics to the world in general, was a patron of the arts, being a great lover of poetry and the theatre. He was an omnivorous reader of the poetry and literature of all races." Even less well known to the "world in general" was his interest in homosexuality. After long consideration, Peirce arrived at views that were very progressive for his time, and by 1891 he was forcefully expressing them in a correspondence with John Addington Symonds (1840–1893), historian of the Renaissance in Italy and a leading English advocate of reform of laws relating to homosexuals. (For Peirce's views on this subject, see [6].)

They were born within forty days of one another, but Eliot survived Peirce by twenty years. After leaving the Presidency in 1909, he continued to work for the benefit of higher education. Peirce died shortly before his planned retirement date, when he would have completed fifty

years of teaching mathematics. Although not in the public eye as Eliot was, Peirce's collaboration with him contributed much to the development of Harvard University, and this collaboration began in 1854 with their appointment as Tutors in Mathematics. According to Eliot:

> As young teachers of prescribed Mathematics in the Freshmen and Sophomore years we worked together with perfect accord and cooperation. Together we introduced certain improvements in the mode of conducting recitations in Mathematics. Together we obtained very reluctant permission from the College Faculty to conduct the final examinations for each year in writing—the first written examinations ever demanded at the end of a year's course in Harvard College. [2, pp. 8–9]

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PROBLEMS AND SOLUTIONS

EDITED BY VLADIMIR DROBOT

Send all proposed problems, in duplicate if possible, to Professor Vladimir Drobot, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053. Please include solutions, relevant references, etc.

An asterisk (*) indicates that neither the proposer nor the editors supplied a solution.

Solutions should be sent to the addresses given at the head of each problem set.

A publishable solution must, above all, be correct. Given correctness, elegance and conciseness are preferred. The answer to the problem should appear right at the beginning. If your method yields a more general result, so much the better. If you discover that a MONTHLY *problem has already been solved in the literature, you should of course tell the editors; include a copy of the solution if you can.*

PROBLEMS DEDICATED TO EMORY P. STARKE

Solutions of these Problems Dedicated to Emory P. Starke should be mailed to Prof. A. P. Hillman, Department of Mathematics & Statistics, University of New Mexico, Albuquerque, NM (USA) 87131, by October 31, 1980. To facilitate consideration, type with double spacing in the format used below. If acknowledgment is desired, include a self-addressed card, label, or envelope.

S 32. *Proposed by Walter Feit, Yale University.*

For integers m,n with $m>0$, let $N_m(n)$ denote the number of ways that a subset S of $\{0,1,\ldots,m-1\}$ and a sequence of $\pm$ signs can be chosen so that $n=\sum_{i\in S}\pm 2^i$. (The empty sum is 0 by definition.) For instance, $N_m(\pm[2^m-(-1)^m])=1+(-1)^m$; and so $\sum N_m(k)=2+(-1)^m$, this sum being taken over those values of k for which $[2^m-(-1)^m]|k$. Find $\sum N_m(k)$, if the sum is now taken over those values of k for which $\{[2^m-(-1)^m]/3\}|k$.

S 33. *Proposed by Simeon Reich, University of Southern California.*

Let $\{x_n\}, n=1,2,\ldots,$ be a sequence in L^p, $1<p<\infty$. Suppose that for some y in L^p, $\lim_{n\to\infty}\|x_n-ty\|$ exists for all $0\leqslant t\leqslant 1$. For which values of p does it follow that $\lim_{n\to\infty}\|x_n-ty\|$ exists for *all* $t\geqslant 0$?

What is the situation in other Banach spaces?

SOLUTIONS OF PROBLEMS DEDICATED TO EMORY P. STARKE

Weak Contraction Maps

S 8 [1979, 222]. *Proposed by R. Johnsonbaugh, Chicago State University.*

Call a function f from a metric space (M,d) into itself *a weak contraction map* if, whenever $x,y\in M$ with $x\neq y$, we have $d(f(x),f(y))<d(x,y)$.

(i) Give an example of a weak contraction map on a complete metric space with no fixed point.

(ii) Show that even on a compact metric space a weak contraction map need not be a contraction map; i.e., it need not satisfy $d(f(x),f(y))\leqslant cd(x,y)$ for $0<c<1$.

(iii) Prove that a weak contraction map on a compact metric space has a unique fixed point.

Solution by O. P. Lossers, Department of Mathematics, Eindhoven University of Technology, Eindhoven, the Netherlands. (i) Take $M=\mathbb{R}$ and f given by $f(x)=1+x-(x/(1+|x|))$. Clearly, $f(x)>x$ for all x, i.e., f has no fixed points and $0<f'(x)<1$ for all x so f provides a contradiction.

(ii) Take $M=[0,1]$ and $f(x)=\sin x$. On $(0,1]$ one has $0<f'<1$ so f is a weak contraction map. On the other hand, for all $0<c<1$ there exists an x such that $\sin x>cx$, i.e., f is not a contraction map.

(iii) Since $d(x,f(x))$ is a continuous function on M it attains a minimum, say at x_0. If $y_0=f(x_0)\neq x_0$ then $d(y_0,f(y_0))<d(x_0,f(x_0))$, which contradicts our assumption on x_0. So x_0 is a fixed point. Of course x_0 must be unique since any other fixed point x_1 would satisfy

$$d(x_0,x_1)=d(f(x_0),f(x_1))<d(x_0,x_1).$$

Also solved by K. F. Andersen (Canada), Anon, William H. Beckman, A. Brunnschweiler (Switzerland), F. S. Cater, Richard J. Driscoll, Thomas E. Elsner, Lee Erlebach & Zane C. Motteler, Peter Flusser, Joel Fowler & Sam Buss & Larry Brantley & Deborah Arangno, Gustaf Gripenberg (Finland), Ellen Hertz, G. A. Heuer & Karl W. Heuer, Roger Howe [(i) and (ii)], Michael Josephy (Costa Rica), Benjamin G. Klein, Man Kam Kwong, David Lantz, G. S. Lessels (Nigeria), John S. Lew, Mark Merriman, Albert A. Mullin, William Myers, Ivan Netuka & Jiří Veselý (Czechoslovakia), Stephen Noltie, Victor Pambuccian (Rumania), R. G. E. Pinch (England), Charles Riley, Ira Rosenholtz, T. Šalát (Czechoslovakia), Paul S. Schnare (Saudi Arabia), Stephen J. Shiffman, K. L. Singh, Richard D. Troxel, Gérard Vinel (France), Richard A. Vitale, L. E. Ward, Jr., J. H. Webb (South Africa), Bruce R. Wenner, and the proposer.

Editor's Notes. The references for some of these notes are given below.

Andersen stated that (iii) is somewhat generalized in **[2]** and is stated as Problem 3 on p. 110 of **[4]**. Cater proved the theorem: "Let f be a continuous function mapping a compact metric space M into itself such that $x\in M$, $x\neq fx$ implies that $d(fx,f^2x)<d(x,fx)$. Then f has a fixed point." Driscoll and also Erlebach & Motteler

added to part (iii): "For any $x_o \in M$ let $x_{n+1} = f(x_n)$ for $n = 0, 1, \ldots$; then x_n converges to the unique fixed point." Elsner noted that the argument for (iii) appears in [5]. Mullin added, "It appears that one can generalize (iii) to: Every weak contraction of a closed ball in a locally convex space has a fixed point." Pambuccian said that (iii) is known as Edelstein's Theorem and gave the references [1], [2], and [3]. Šalát cited [1] for (iii), Singh stated that (iii) follows as a corollary of Theorem 1 of [2], Vinel cited [2], and Webb cited [5].

References

1. D. G. Bennett and B. Fisher, On a fixed point theorem for compact metric spaces, Math. Mag., 47(1974) 40–41.

2. M. Edelstein, On fixed and periodic points under contractive mappings, J. London Math. Soc., 37 (1962) 74–79.

3. Gerald Jungck, An iff fixed point theorem, Math. Mag., 49 (1976) 32–34.

4. I. Kaplansky. Set Theory and Metric Spaces, Allyn & Bacon, Boston, 1972.

5. D. R. Smart, Fixed Point Theorems, Cambridge Tracts in Mathematics, no. 66, 1974.

Characterizing the Divisors of 24

S 9 [1979, 306]. *Proposed by M. S. Klamkin and A. Liu, University of Alberta.*

(a) Determine all positive integers n such that $\gcd(x,n)=1$ implies that $x^2 \equiv 1 \pmod{n}$.

(b) Determine all positive integers n such that $xy+1 \equiv 0 \pmod{n}$ implies that $x+y \equiv 0 \pmod{n}$.

Solution by Arnold Adelberg, Grinnell College, and Jeffrey M. Cohen, graduate student, University of Pittsburgh (independently). (a) We show that n satisfies the condition iff $n|24$. First, if n has a prime divisor $p>3$ and q is the product of all the prime divisors of n different from p, then the Chinese Remainder Theorem implies the existence of an x with $x \equiv 1 \pmod{q}$ and $x \equiv 2 \pmod{p}$. [Let $q=1$ if n is a power of p.] Then $\gcd(x,n)=1$ and $x^2 \equiv 4 \not\equiv 1 \pmod{p}$ so that $x^2 \not\equiv 1 \pmod{n}$. Next, if $n=2^r3^s$, then $\gcd(5,n)=1$ and $5^2 \equiv 1 \pmod{n}$ iff $n|24$.

Conversely, if $n|24$ then $\gcd(x,2)=1$ implies $x^2 \equiv 1 \pmod{8}$ and $\gcd(x,3)=1$ implies $x^2 \equiv 1 \pmod{3}$ so that n satisfies the condition.

(b) Let A and B be the sets of integers n satisfying the conditions in (a) and (b), respectively. We show that $A=B$.

Let $n \in A$. Then $xy+1 \equiv 0 \pmod{n}$ implies that $\gcd(x,n)=1, x^2 \equiv 1 \pmod{n}$, and

$$x+y \equiv x+x^2y \equiv x(1+xy) \equiv x \cdot 0 \equiv 0 \pmod{n}.$$

Thus $A \subseteq B$.

Now let $n \in B$. Then $\gcd(x,n)=1$ implies that there is an integer y with $xy \equiv -1 \pmod{n}$ which implies $x+y \equiv 0 \pmod{n}$ and so $x^2 \equiv x(-y) \equiv 1 \pmod{n}$, i.e., $B \subseteq A$. Hence $A=B$.

Editor's Note. M. J. DeLeon established several generalizations of S 9 dealing with the property of a pair (m,n) of positive integers such that $\gcd(a,m)=1$ implies $a^n \equiv 1 \pmod{m}$. He also referred to problem B-1 of the December 6, 1969, William Lowell Putnam Mathematical Competition, which asked for a proof that $n \equiv -1 \pmod{24}$ implies $24|\sigma(n)$.

Also solved by Robert Breusch, Stephen D. Bronn, Randall J. Covill, Thomas C. Craven, Michael Doob (Canada), Hugh M. Edgar, Milton Eisner, Thomas E. Elsner, Lorraine L. Foster, Robert Gilmer, Gustaf Gripenberg (Finland), David Hammer, V. Kannen & D. V. R. Krishnaiah (India), Joseph B. Klerlein, L. Kuipers (Switzerland), Man Kam Kwong, O. P. Lossers (Netherlands), Nicholas A. Martin (Canada), Tim McMillan, Roberto Mena, Mark Merriman, William Myers, Victor Pambuccian (Rumania), Robert Patenaude, George Shulman, Joseph Silverman, Sahib Singh, M. K. Siu (Hong Kong), Kenneth W. Spackman, E. P. Starke, F. B. Strauss, Ernst Trost (Switzerland), University of South Alabama Problem Group, John T. Ward, and the proposers.

ELEMENTARY PROBLEMS

Solutions of these Elementary Problems should be mailed in duplicate to Dr. J. L. Brenner, 10 Phillips Road, Palo Alto, CA (USA) 94303, by October 31, 1980. Please place the solver's name and mailing address on each (double-spaced) sheet. Include a self-addressed card or label (for acknowledgment).

E 2835. *Proposed by Michael Golomb, Purdue University.*

Let $-1<a_0<1$, and define recursively $a_n=[\frac{1}{2}(1+a_{n-1})]^{\frac{1}{2}}$, $n>0$. Find the limits $(n\to\infty)$ of $A_n=4^n(1-a_n)$, $B_n=a_1a_2\cdots a_n$, $C_n=4^n(B-a_1a_2\cdots a_n)$.

E 2836. *Proposed by Joseph E. Valentine, University of Texas at San Antonio.*

Show that an absolute geometry (no parallel postulate) is Euclidean (or Riemannian) if some triangle has the property that a median and the segment joining the midpoints of the other two sides bisect each other.

E 2837*. *Proposed by C. W. Scherr, University of Texas at Austin*

Let a_{ij} be the side of a triangle that connects vertices i and j. Let m_i be the median from vertex i. Elementary application of the law of cosines yields the relation

$$a_{12}^{\alpha}+a_{23}^{\alpha}+a_{31}^{\alpha}=\lambda^{\alpha/2}(m_1^{\alpha}+m_2^{\alpha}+m_3^{\alpha}),$$

valid for all triangles when $\alpha=2$ or $\alpha=4$ and $\lambda=4/3$. Find an expression for λ in the limit as α goes to zero. Find the class of triangles for which the relation is valid for a fixed λ and arbitrary α.

E 2838. *Proposed by U. Abel and F. Boukal, University of Bielefeld, West Germany.*

Let X be a measure space with measure m. For any family $\mathcal{A}$ of measurable subsets of X define $m(\mathcal{A})=m(\cup_{A\in\mathcal{A}}A)$. Let $\mathcal{C}$ be a covering of X by measurable subsets and let k be the minimal cardinality of a subcollection of $\mathcal{C}$ that covers X. Prove that for any $c>0$ there exists a polynomial algorithm for finding a subcollection $\mathcal{B}$ of $\mathcal{C}$ of cardinality k so that $m(\mathcal{B})\geqslant(1-c^{-1})m(X)$.

E 2839. *Proposed by J. W. P. Hirschfeld, University of Sussex, England.*

The equation $x^2+x+d=0$ has two solutions or none over $GF(2^h)$ according as the trace, $T(d)$, of d is zero or one, where $T(d)=\sum_{i=0}^{h-1}d^{2^i}$. Show that the number of non-zero elements in $GF(2^h)$ for which $T(d)=T(d^{-1})=0$ is $\frac{1}{4}\{2^h-3-2(-\sqrt{2})^h\cos(h\cos^{-1}1/\sqrt{8})\}$.

E 2840. *Proposed by Felix T. Smith, SRI International.*

A real symmetric orthogonal matrix is to be constructed when one of the rows, say $C_{1j}=\alpha_j$, with $\Sigma\alpha_j^2=1$, is specified. For $n>2$, there are 2 and only 2 solutions; give a general expression for the (real) C_{ij} in terms of the α_j. What happens when the matrix is symmetric and unitary—when the C_{ij} are complex and $C_{1j}=\alpha_j+i\beta_j$?

SOLUTIONS OF ELEMENTARY PROBLEMS

Equivalent Sets of Axioms

E 2289 [1971, 405]. *Proposed by John Corcoran, State University of New York at Buffalo.*

Let A be a finite system of independent axioms in a first order logic. Let $\&A$ be the conjunction of the members of A. Then singleton $\&A$ is at least as small as any independent set equivalent to A. Under what conditions is there an independent set equivalent to A and at least as large as any such independent set?

Solution by M. J. Pelling, University of Malaya, Kuala Lumpur, Malaysia. Call a first-order theory K n-complete if there is a finite set of n closed formulae $C_1,\ldots,C_n$ such that $\vdash_K C_1\vee C_2$

$\vee \cdots \vee C_n$, each theory $K \cup \{C_i\}$ is complete, $1 \leqslant i \leqslant n$, and n is least with these properties. Then 1-completeness is completeness in the usual sense and the significance of n-completeness is that there are exactly n possible assignments of truth values to the closed formulae in a model for K.

Given A, let K be the first-order theory whose sole axiom is $\neg \& A$: then there exists an independent set equivalent to A and at least as large as any such independent set if and only if K is n-complete for some n. Furthermore, if K is n-complete, then any independent set equivalent to A contains at most n formulae, and this is attained.

To see this, suppose first that K is not n-complete for any n. Then, given any $n \geqslant 1$, K will admit n models $M_1, \ldots, M_n$ and closed formulae $C_1, \ldots, C_n$ such that C_i is true in M_i but false in all M_j for $j \neq i$; we may also suppose $\vdash C_i \rightarrow \neg \& A$ for each i. Define $B_i = \& A \vee C_1^* \vee \cdots \vee C_{i-1}^* \vee C_{i+1}^* \vee \cdots \vee C_n^*$ for $1 \leqslant i \leqslant n$ where $C_i^* = C_i \& \neg (C_1 \vee \cdots \vee C_{i-1} \vee C_{i+1} \vee \cdots \vee C_n)$. Then the axiom set $B_1, B_2, \ldots, B_n$ is obviously equivalent to A, and it is also an independent set because, in the model M_i for K, B_i is false but all B_j for $j \neq i$ are true. It follows that there is no independent set equivalent to A and at least as large as any other such indepedent set.

Now suppose that K is n-complete and let $C_1, \ldots, C_n$ be a set of closed formulae as in the definition above of n-completeness. Because n is minimal, $\vdash_K C_i \rightarrow \neg (C_1 \vee \cdots \vee C_{i-1} \vee C_{i+1} \vee \cdots \vee C_n)$ so that the argument of the preceding paragraph can be applied to the C_i with M_i any model of $K \cup \{C_i\}$ to deduce that there is an independent axiom set of n formulae equivalent to A.

However, if $B_1, \ldots, B_k$ is any other independent axiom set equivalent to A, then by the independence K must have a model in which B_i is false but all $B_j, j \neq i, 1 \leqslant j \leqslant k$ are true; and this will be so for each of the k possible values of i. Since K admits only n possible assignments of truth values to its closed formulae in a model it follows $k \leqslant n$.

A Determinant Involving Derivatives

E 2767 [1979, 307]. *Proposed by James W. Burgmeier, University of Vermont.*

Let f be a function with sufficiently many derivatives and let D_n be the determinant

$$D_n = \begin{vmatrix} f' & f & 0 & 0 & \cdots & 0 & 0 \\ \frac{f''}{2!} & f' & f & 0 & \cdots & 0 & 0 \\ \frac{f'''}{3!} & \frac{f''}{2!} & f' & f & \cdots & 0 & 0 \\ \vdots & & & & & & \\ \frac{f^{(n)}}{n!} & \frac{f^{(n-1)}}{(n-1)!} & & & \cdots & & f' \end{vmatrix}.$$

Show that

$$D_{n+1} = f' D_n - \frac{1}{n+1} f D_n'.$$

Solution by A. A. Jagers, Technische Hogeschool Twente, Enschede, Netherlands, and D. Z. Djoković, University of Waterloo (independently). The assertion is clearly true if $f \equiv 0$. If $f \neq 0$, let D_m denote the matrix, and let det D_m denote its determinant (a slight change in notation). Let $A_m(f)$ be the $m+1 \times m+1$ matrix obtained from D_m by appending first row $(f, 0, \ldots, 0)$ and last column $[0, \ldots, 0, f]$. Then $A_m(f) \cdot A_m(1/f) = I$. Thus det $D_m = (-1)^m f^{m+1} (1/f)^{(m)}/m!$. The assertion follows.

Also solved by Leon Gerber, O. P. Lossers (Netherlands), Otto G. Ruehr, and the proposer.

Gerber refers to V. F. Ivanoff, this MONTHLY, 55 (1948) 491. Ruehr cites Muir's *Determinants*, Dover, 1960, p. 703.

A Formula Involving $S_k = \Sigma m^k$

E 2770 [1979, 307]. *Proposed by Warren Page, New York City Community College.*

Let n and N be fixed positive integers and let

$$S_k = \sum_{m=1}^{n} m^k \quad \text{for } k = 1, 2, \ldots, N.$$

Prove

(a)

$$\sum_{h=1}^{N} \sum_{k=1}^{h} \binom{h+1}{k} S_k = \frac{n+1}{n}\left[(n+1)^{N+1} - (N+1)n - 1\right],$$

and

(b)

$$\sum_{h=1}^{N} \sum_{k=1}^{h} (-1)^h \binom{h+1}{k} S_k = \begin{cases} \dfrac{n+1}{n+2}\left[1 - (n+1)^{N+1}\right] & \text{for odd } N, \\ \dfrac{(n+1)^2}{n+2}\left[(n+1)^N - 1\right] & \text{for even } N \end{cases}$$

Solution by Joseph Wiener, Pan American University. Define $S_k(a,d) = \Sigma_{m=1}^{n}(a+(m-1)d)^k, k = 1, \ldots, N$. Then $S_k = S_k(1,1)$. It will be proved that

$$\text{(A)} \quad \sum_{h=1}^{N} \sum_{k=1}^{h} \binom{h+1}{k} d^{h+1-k} S_k(a,d) = [(a+nd)^N - 1](a+nd)^2/(a+nd-1)$$
$$- (a^N - 1)a^2/(a-1) - (d^N - 1)nd^2/(d-1),$$

and

$$\text{(B)} \quad \sum_{h=1}^{N} \sum_{k=1}^{h} (-1)^h \binom{h+1}{k} d^{h+1-k} S_k(a,d) = [(-a-nd)^N - 1](a+nd)^2/(a+nd+1)$$
$$- [(-a)^N - 1]a^2/(a+1) - [(-d)^N - 1]nd^2/(d+1).$$

The proof depends on the identity

$$(e^{dx} - 1)T(x) = e^{(a+nd)x} - e^{ax}, \qquad (*)$$

where $T(x) = \Sigma_{m=1}^{n} \exp[a+(m-1)d]x$. Differentiation of (*) $h+1$ times gives (with $x = 0$)

$$\sum_{k=1}^{h} \binom{h+1}{k} d^{h+1-k} S_k(a,d) = (a+nd)^{h+1} - a^{h+1} - nd^{h+1}.$$

From this, (A) and (B) follow directly.

Editor's Note. L. Carlitz defined $F_N(\lambda) = \Sigma_{n=1}^{N} \lambda^h \Sigma_{k=1}^{h} \binom{h+1}{k} S_k$, and computed another generalization:

$$F_n(\lambda) = (n+1)\{\lambda^{N+1}(n+1)^{N+1} - \lambda(n+1)\}/[\lambda(n+1) - 1] - (n+1)\{\lambda^{N+1} - \lambda\}/[\lambda - 1].$$

Also solved by K. F. Andersen (Canada), P. Bracken (Canada), C. A. De Carlucci, L. L. Foster, N. Franceschine, F. Gerrish (England), F. S. Gillespie, S. H. Greene, R. E. D. Henderiks (Netherlands), A. A. Jagers (Netherlands), L. Kuipers (Switzerland), J. Lee, D. F. Lockhart, M. Merriman, R. Morelli, V. N. Murty, R. G. Nath, B. E. Rhoades, O. G. Ruehr, N. M. Ruggles, J. Silverman, V. Upatisringa, U. of South Alabama Problem Group, M. Vowe (Switzerland), A. Venetoulias, G. Vinel (France), J. T. Wand, P. Zwier, and the proposer.

The Polynomial Congruences $x^k \equiv x$, $\Pi(x-a_i) \equiv 0$

E 2773 [1979, 393]. *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

Problem E 2704 [1978, 198] asked for the number of solutions to $x^2 = x$ in the ring $\mathbb{Z}_n$ of integers modulo n. Question: What is the number of solutions in $\mathbb{Z}_n$ of $x^3 = x$? (See 1979, 397.)

I. *Solution by Stuart S.-S. Wang, Oakland University, Rochester, Michigan, and Robert Gilmer, Florida State University (independently).* More generally, consider the congruence $x^k \equiv x \bmod n$, $k > 2$, $n > 1$. Denote by $f_k(n)$ the number of solutions of this congruence in $\mathbb{Z}_n$. If $n = P_1 P_2 \cdots P_s$ is the factorization of n into powers of distinct primes, then clearly $f_k(n) = f_k(P_1) \cdots f_k(P_s)$. Hence, we consider first the case of a prime power $m = p^e$. In this case, the set of units and the set of nilpotent elements form a partition of the ring $\mathbb{Z}_m$. It is evident that 0 is the only nilpotent element such that $x^k = x$. Thus, $f_k(m) = 1 + g_k(m)$, where $g_k(m)$ is the number of units x of $\mathbb{Z}_m$ such that $x^{k-1} = 1$. For p odd or for $m = p^e = 2$ or 4, the multiplicative group of units of $\mathbb{Z}_m$ is cyclic of order $\phi(m)$, and hence $g_k(m) = (k-1, \phi(m))$ in these cases. On the other hand, if $p = 2$ and $e \geqslant 3$, the group of units of $\mathbb{Z}_m$ is the direct product of a cyclic group of order 2 and a cyclic group of order 2^{e-2}. Hence, if $k-1$ is odd, then $g_k(m) = 1$; and if $k-1$ is even, $g_k(m) = 2(k-1, 2^{e-2})$. In summary, $f_k(n) = \Pi_{i=1}^s [1 + (k-1, \phi(P_i))]$ if n is not divisible by 8; whereas, if 8 divides n and if the labeling is such that P_1 is a power of 2, then $f_k(n) = 2\Pi_{i=2}^s [1 + (k-1, \phi(P_i))]$ for k even, while $f_k(n) = [1 + 2(k-1, P_1/4)]\Pi_{i=2}^s [1 + (k-1, \phi(P_i))]$ for k odd. Applying these formulas to the case $k = 3$ yields $f_3(n) = 5 \cdot 3^{s-1}$ if $8|n$, $f_3(n) = 2 \cdot 3^{s-1}$ if $2|n$ but $4 \nmid n$, and $f_3(n) = 3^s$ otherwise.

II. *Solution by Robert Gilmer.* As a second generalization, we determine, for $n \geqslant 2$ and for each monic polynomial $f(X) \in \mathbb{Z}[X]$ that is a product of (not necessarily distinct) linear factors over $\mathbb{Z}$, the number $N(f, n)$ of distinct roots of $f(X)$ in $\mathbb{Z}_n$. As in Solution I, it is enough to consider the special case where $n = p^e$ is a prime power, but the details are different.

If $e = 1$, the $\mathbb{Z}_p$ is a field, and the number of solutions of $f(X)$ in $\mathbb{Z}_p$ is clear. Assume that $N(f, p^i)$ has been determined for all f and for $i < e$. Consider $N(f, p^e)$, where $f = (X - a_1) \cdots (X - a_t)$. Write $f = f_0 f_1 \cdots f_{p-1}$, where f_i is the product over all $X - a_j$ with $a_j \equiv i \bmod p$. The integer a is a root of $f \bmod p^e$ if and only if $f_s(a) \equiv 0 \bmod p^e$, where $a \equiv s \bmod p^e$; this is true since $f_j(a)$ is relatively prime to p^e for $j \neq s$. Thus, $N(f, p^e) = \Sigma_{i=0}^{p-1} N(f_i, p^e)$. Determining $N(f_0, p^e), \ldots, N(f_{p-1}, p^e)$ is basically the same problem (p times), for $f_i(X)$ is mapped onto a polynomial of the type $f_0(X)$ under the $\mathbb{Z}$-automorphism of $\mathbb{Z}[X]$ determined by mapping X to $X + i$. Therefore, we consider the problem of determining $N(f_0, p^e)$.

If we reduce $f_0 \bmod p^e$, we obtain a polynomial $f_0^* = (X - [b_1 p])^{e(1)} \cdots (X - [b_s p])^{e(s)} \in \mathbb{Z}_{p^e}[X]$, where $0 \leqslant b_1 < \cdots < b_s \leqslant (p^e - 1)$ and where $[b_i p]$ denotes the class of $b_i p \bmod p^e$. By definition, $N(f_0, p^e)$ is the number of distinct roots of f_0^* in $\mathbb{Z}_{p^e}$. Since it is clear that any root $[y]$ of f_0^* in $\mathbb{Z}_{p^e}$ must be such that $p|y$, we have the problem of determining the number of integers a between 0 and $p^{e-1} - 1$ such that $[0] = ([ap] - [b_1 p])^{e(1)} \cdots ([ap] - [b_s p])^{e(s)} = [p]^g([a] - [b_1])^{e(1)} \cdots ([a] - [b_s])^{e(s)}$, where $g = e(1) + e(2) + \cdots + e(s)$. If $g \geqslant e$, then $f_0^*([ap]) = [0]$ for each such a—that is, $N(f_0, p^e) = p^e$ if $g \geqslant e$. If $g < e$, then $N(f_0, p^e)$ is the number of integers a between 0 and $p^{e-1} - 1$ such that $(a - b_1)^{e(1)} \cdots (a - b_s)^{e(s)}$ is divisible by p^{e-g}. The integers between 0 and $p^{e-1} - 1$ partition into p^{e-g} classes $\bmod p^{e-g}$, each class containing $p^{e-1}/p^{e-g} = p^{g-1}$ elements. It follows that $N(f_0, p^e) = p^{g-1} N(g_0, p^{e-g})$, where $g_0 = (X - b_1)^{e(1)} \cdots (X - b_s)^{e(s)}$; $N(g_0, p^{e-g})$ is known since the exponent $e - g$ is less than e, and this completes the solution of the problem.

Also solved by H. L. Abbott, Stephen D. Brown, Lee Erlebach, Lorraine L. Foster, Zachary Franco, Gloria G. Gagola, Michael Josephy (Costa Rica), Robert Lumbert, Nicholas A. Martin (Canada), Robert A. Melter, William Myers, Robert Patenaude, Problem Solving Group of Rand Afrikaans University (South Africa), K. Rama (India), K. N. Rajeswari, D. J. Samuelson, Jeffrey Shallit, Nan-Shan Shou (Hong Kong), Sahib Singh, Lawrence Somer,

Rony Teitler (U.K.), Stuart S.-S. Wang (two other solutions), Gregory P. Wene, Robert C. Williams, Edward T. Wong, and the proposer.

Editor's Note: Melter refers to Problem 104 (Canadian Math. Bull., 8 (1965), 669) and to Ian G. Connell, *The Ring Scheme* F_q (Canadian Math. Bull., 15 (1972), 79–85).

$n^e \equiv n \pmod b$

E 2776 [1979, 393]. *Proposed by Alan Wayne, Holiday, Florida.*

(a) In the decimal system, find all twelve-digit positive integers n such that n^{102} ends at the right in the digits of n.

(b)* Is there a corresponding solution to the problem in numeration systems other than base ten?

Solution (part (a)) by H. L. Abbott, University of Alberta. The conditions of the problem require $n^{102} \equiv n \pmod{10^{12}}$, $10 \nmid n$, $(n,10) \neq 1$. To see that $(n,10)\neq 1$, suppose $(n,10)=1$. Then $n^{101} \equiv 1 \pmod{10^{12}}$, so that $n=1$, since $\phi(10^{12})=4\cdot 10^{11}$.

Clearly, both relations $n(n^{101}-1)\equiv 0 \mod 2^{12}$, $n(n^{101}-1)\equiv 0 \mod 5^{12}$ must hold. Further, either

$$n\equiv 0 \mod 2^{12}, \text{ and } n^{101}-1\equiv 0 \mod 5^{12}, \tag{*}$$

or else

$$n\equiv 0 \mod 5^{12}, \text{ and } n^{101}-1\equiv 0 \mod 2^{12}. \tag{**}$$

The only solution of (*) less than 10^{13} is 81, 787, 109, 376. The answer required (from **) is $N=918, 212, 890, 625$.

Editor's Note. L. L. Foster noted that mod 10^{12}, $N=5^e, e=2^{11}$.

Solution (part (b)) by David Witte, undergraduate, University of Wisconsin at Madison. The solutions are sorted out by the following theorem.

THEOREM. *Let $b,t,k>1$ be positive integers. Then, in the enumeration system to the base b, there is a t-digit number n such that n^k ends in the digits of n if, and only if, either:*
(i) *b is not a prime power; or*
(ii) *$k-1$ is not relatively prime to $\phi(b^t)$.*

Proof. ($\Rightarrow$) Suppose $n^k=n \pmod{b^t}$, and $b^{t-1}\leqslant n<b^t$. We may assume b is a prime power, say $b=p^c$. Since $p^{ct}|(n^k-n)$, either $p^{ct}|n$ or $p^{ct}|n^{k-1}-1$. The former implies $n=0$, so the latter must be the case.

Let m be the order of n in the multiplicative group of integers mod p^{ct}. Then $m|k-1$, Lagrange's Theorem implies that $m|\phi(p^{ct})$, and because $n\neq 1$, we have $m>1$. Hence $k-1$ and $\phi(p^{ct})$ have a common factor.

($\Leftarrow$) (i) Suppose b is not a prime power. Then, from the solution to problem E 2704, we know that there is an integer $x\neq 0,1 \pmod{b^t}$ satisfyimg $x^2=x \pmod{b^t}$. Note that this implies $x^k=x \pmod{b^t}$. We also have $(1-x)^2=(1-x) \pmod{b^t}$. Either x or $1-x$ has a t-digit residue mod b^t.

(ii) Suppose that $b=p^c$ is a prime power, and there is a common prime divisor q of $k-1$ and $\phi(p^{ct})=(p-1)p^{ct-1}$. If $q=p$, then $n=1+p^{ct-1}$ is the desired t-digit solution.

Now assume $q\neq p$. Let $1,a,a^2,\ldots,a^{q-1}$ be the cyclic group of order q in the multiplicative group of the integers mod p^{ct}. We have $p^{ct}|a^q-1=(a-1)(a^{q-1}+\cdots+1)$. Since $a\neq 1 \pmod{p^{ct}}$, it follows that $p|a^{q-1}+\cdots+1$, and hence $a\neq 1 \pmod p$ since $q\neq p$. Therefore $p^{ct}|a^{q-1}+\cdots+1$.

Then some a^n has residue at least $(p^{ct}-1)/(q-1) \pmod{p^{ct}}$. Since $q<p$, we have $(p^{ct}-1)/(q-1)>(p^{ct}-1)/(p-1)>p^{ct-1}\geqslant p^{c(t-1)}$. Hence the residue of a^n is a t-digit number. It is the

desired solution.

Also solved by D. M. Bloom, L. L. Foster, Jinku Lee, Robert Lumbert, George Shulman, and the proposer.

Integers Relatively Prime to b in $[nb/a]$

E 2777 [1979,393]. *Proposed by I. Borosh, H. Diamond, M. Gbur, & D. Hensley, University of Illinois, Urbana-Champaign.*

Let b/a be a reduced fraction greater than one. Let $r=r(a,b)$ denote the number of integers relatively prime to b in the sequence

$$[b/a],[2b/a],\dots,[(a-1)b/a].$$

State and prove a rule for determining r as a function of a and b.

Here $[x]$ denotes the greatest integer in x. (Application: The number of primes $p \leqslant x$ lying in the sequence $\{[jb/a]\}_{j=1}^{\infty}$ is asymptotic to $rx/(\phi(b)\log x)$, where ϕ denotes Euler's function. This application is problem 7.16 of *Les nombres premiers*, W. J. Ellison, in collaboration with M. Mendès-France, Hermann, Paris, 1975.)

I. *Solution by Lorraine L. Foster, California State University, Northridge.* (It is implied that $b>a\geqslant 2$.) We shall prove that

$$r(a,b)=\sum_{d|b}\mu(d)[(a-1)/d]. \qquad (*)$$

For $k=1,2,\dots,a-1$, write $kb=q_ka+r_k, 0<r_k<a$. Since $(a,b)=1$, it is clear that $(b,q_k)=1\Leftrightarrow (b,r_k)=1$. Also, the r_k are $\{1,2,\dots,a-1\}$ in some order. Next, $q_k=[kb/a]$. Thus $r(a,b)$ is the number of integers k such that $(k,b)=1$, $1\leqslant k\leqslant a-1$. The formula (*) follows by use of the usual inclusion-exclusion argument.

II. *Solution by Ernst W. Trost, Zurich, Switzerland.* (Continuation of Solution I.) To find the order of magnitude of $r(a,b)$, we use the fact (Hardy and Wright, Introduction to the Theory of Numbers, 3rd ed., p. 260) that the number of integral divisors of n is $O(n^{\delta})$ for all positive δ. Since Euler's totient $\phi(n)$ is given by $\phi(n)=n\Sigma_{d|n}\mu(d)/d$, we find $r(a,b)=(a-1)\phi(b)/b+O_{\delta}(\min[b^{\delta},a])$.

Also solved by H. Diamond.

ADVANCED PROBLEMS

Solutions of these Advanced Problems should be mailed in duplicate to Prof. R. C. Lyndon, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109 (USA), by October 31, 1980. The solver's full post-office address should be on each sheet.

6300. *Proposed by Jack Garfunkel, Flushing, New York.*

Chebycheff showed that, if two numbers are chosen at random from the set of natural numbers, the probability that they will be relatively prime is $6/\pi^2$. Show that if two numbers are chosen at random from the Fibonacci sequence $1,1,2,3,5,\dots$, the probability P that they will be relatively prime satisfies the inequalities $8/\pi^2>P>7/\pi^2$.

6301*. *Proposed by Clark Kimberling, University of Evansville.*

Suppose that $\{s_n\}$ is a sequence of positive integers such that

$$0\leqslant s_{n+m}-s_n-s_m\leqslant K, \qquad m,n=1,2,3,\dots,$$

for some positive integer K. Let N be a positive integer. Must there exist real numbers $a_1,a_2,\dots,a_K$ such that

$$s_n=\sum_{k=1}^{K}[a_in] \quad \text{for } n=1,2,\dots,N?$$

(Here $[x]$ denotes the greatest integer less than or equal to x.)

6302. *Proposed by P. Q. Perlmutter, The Colorado College.*

Let $\{F_\alpha\}_{\alpha\in A}$ be a collection of closed subsets of $\mathbb{R}$. Show that if this collection is a chain, then $F=\cup_{\alpha\in A}F_\alpha$ is an $\mathbb{F}_\sigma$ set. [By a chain, we mean $F_\alpha\subset F_\beta$ or $F_\beta\subset F_\alpha$ for all $\alpha,\beta\in A$.] Can this property be generalized to other classes of Borel sets?

6303*. *Proposed by the editors.*

Under what conditions on A can the leading principal $t\times t$ minor of the real orthogonal matrix A be replaced by a symmetric $t\times t$ minor so that the property of orthogonality is preserved? (See E 2840, p. 489 in this issue.)

SOLUTIONS OF ADVANCED PROBLEMS

Convex Bodies in n-space

6089* [83, 293]. *Proposed by E. Ehrhart, Université de Strasbourg, France.*

Let K be a convex body in R_n of Jordan content $V(K)>(n+1)^n/n!$ and with centroid at the origin. Does $K\cup(-K)$ contain a convex body C, symmetric in the origin, for which $V(C)>2^n$?

Comments by the proposer. (1) The case $n=2$ is proved by the proposer in *Comptes Rendus Acad. Sci. Paris*, 258 (1964) 4885–4887. (2) He believes the assertion is true for $n=3$ and has proved it in this case provided K is a tetrahedron. (3) He believes it is false for $n=4$, and probably for all $n\geqslant 4$.

Best Rank-k Approximation to a Matrix

6125 [1976, 818]. *Proposed by Simeon Reich, Tel Aviv University, Tel Aviv, Israel*

For a given $n\times n$ complex matrix A of rank r, and an integer k, $1\leqslant k\leqslant r$, a best rank-k approximation of A is a matrix $A_{(k)}$ of rank k satisfying $\|A-A_{(k)}\|=\inf\{\|A-X\|:X$ is an $n\times n$ matrix of rank $k\}$ where $\|A\|=(\text{trace }A^*A)^{\frac{1}{2}}$.

Show that if A is normal, then $A^j_{(k)}$ is a best rank-k approximation of A^j for all $j\geqslant 1$, but that this is no longer true for arbitrary A. (Cf. Ben-Israel and Greville, *Generalized Inverses*, Wiley, New York, 1974, p. 250.)

Solution by the proposer. Since $\|\cdot\|$ is unitarily invariant, it follows that if A is normal, then $A=UDU^*$ and $A_{(k)}=UD_{(k)}U^*$ where U is unitary and $D,D_{(k)}$ are diagonal. It is clear that $D^j_{(k)}$ is a best rank-k approximation of D^j for all $j\geqslant 1$. Consequently, $A^j_{(k)}=UD^j_{(k)}U^*$ is a best rank-k approximation of $A^j=UD^jU^*$ for all $j\geqslant 1$.

Now let $A=\begin{pmatrix}0&2\\1&0\end{pmatrix}$. Then $A_{(1)}=\begin{pmatrix}0&2\\0&0\end{pmatrix}$, $\|A^2-A^2_{(1)}\|=2\sqrt{2}$, but $\|A^2-B\|=2$ for $B=\begin{pmatrix}2&0\\0&0\end{pmatrix}$.

The Differences of the Partition Function

6137 [1977, 141; 1978, 830]. *Proposed by I. J. Good, Virginia Polytechnic Institute and State University.*

Let $p(n)$ denote the number of partitions of n $(n=1,2,\ldots)$, and let k denote an integer greater than 3. Prove that $\Delta^k p(n)$ $(n=1,2,\ldots)$ is a sequence of alternating terms.

III. *Comment by P. T. Bateman, University of Illinois, Urbana.* Bateman and Erdős [*Mathe-*

matika, 3 (1956) 1–14] proved that for any fixed positive integer k the sequence $\triangle^k p(n)$ ($n=1,,\ldots$) is positive for all sufficiently large n. In fact, we proved a much more general result.

Bound on Zeros of a Polynomial

6237 [1978, 770]. *Proposed by Emeric Deutsch, Polytechnic Institute of New York.*

Show that every zero z of the complex polynomial

$$f(z)=z^n+a_1z^{n-1}+a_2z^{n-2}+\cdots+a_{n-1}z+a_n$$

satisfies $-\beta\leqslant \operatorname{Re} z\leqslant\alpha$, where α and β are the unique positive roots of the equations

$$x^n+(\operatorname{Re}a_1)x^{n-1}-|a_2|x^{n-2}-|a_3|x^{n-3}-\cdots-|a_{n-1}|x-|a_n|=0,$$
$$x^n-(\operatorname{Re}a_1)x^{n-1}-|a_2|x^{n-2}-|a_3|x^{n-3}-\cdots-|a_{n-1}|x-|a_n|=0,$$

respectively.

Solution by the proposer. The equations in x can be written in the form

$$x\pm\operatorname{Re}a_1=|a_2|/x+|a_3|/x^2+\cdots+|a_n|/x^{n-1}\quad(x\neq 0).$$

Now it is clear that these equations have a unique positive root since, as x varies from 0 to $+\infty$, the left-hand side increases from $\pm\operatorname{Re}a_1$ to $+\infty$ and the right-hand side decreases from $+\infty$ to 0.

If z were a zero of $f(z)$ satisfying $\operatorname{Re}z>\alpha$, then we would have

$$\begin{aligned}\alpha+\operatorname{Re}a_1<\operatorname{Re}(z+a_1)&\leqslant|z+a_1|\\&=|a_2/z+a_3/z^2+\cdots+a_n/z^{n-1}|\\&\leqslant|a_2|/|z|+\ldots+|a_n|/|z|^{n-1}\\&<|a_2|/\alpha+|a_3|/\alpha^2+\cdots+|a_n|/\alpha^{n-1}\end{aligned}$$

(since $|z|\geqslant\operatorname{Re}z>\alpha$). This strict inequality contradicts the definition of α. Consequently, $\operatorname{Re}z\leqslant\alpha$. The lower bound for $\operatorname{Re}z$ is obtained by applying the upper bound to the polynomial $(-1)^nf(-z)$.

Editor's remark. The statement in the problem is similar to Cauchy's result regarding the absolute values of the zeros of a polynomial. (See, for example: A. S. Householder, *The Numerical Treatment of a Single Nonlinear Equation*, McGraw-Hill, New York, 1970, p. 70.)

Also solved by Tom M. Apostol, L. E. Clarke (England), A. J. Koeping (England), Morris Marden, Q. G. Mohammad (India), and Robert Vermes.

Evaluations of Trigonometric Series

6241 [1978, 828]. *Proposed by Robert Baillie, Computer-Based Education Research Laboratory, University of Illinois.*

Prove

$$\text{(A)}\qquad \sum_{n=1}^{\infty}\frac{\sin(n)}{n}=\sum_{n=1}^{\infty}\left(\frac{\sin(n)}{n}\right)^2=\frac{\pi-1}{2},$$

$$\text{(B)}\qquad \sum_{n=1}^{\infty}\frac{\sin^2(n)}{n^4}=\frac{(\pi-1)^2}{6}.$$

I. *Solution by the proposer* (*revised by the editors*). It is well known and easily checked that

$$\sum_{n=1}^{\infty}\frac{1}{n}\sin nx=g(x)=-\frac{1}{2}(\pi-x),\ -\pi\leqslant x\leqslant 0;\ \frac{1}{2}(\pi-x),\ 0<x\leqslant\pi.$$

Let $f(x)=(\pi-1)x/2$, $|x|\leqslant 1$; $(\pi-x)/2$, $1\leqslant x\leqslant\pi$. By direct computation of the Fourier coefficients we have

$$f(x)=\sum_{n=1}^{\infty} n^{-2}\sin(n)\sin(nx).$$

Since f and g agree, in particular, at $x=1$, we get the first formula in (A) from $g(1)$ and the second from $f(1)$. Then (B) follows from Parseval's formula applied to f.

II. *Solution of part (A) by P. Henrici (Eidgenössische Technische Hochschule, Zurich, Switzerland).* The functions

$$G_k(\omega):=\begin{cases}1, & \omega=0,\\ \left(\dfrac{\sin\omega}{\omega}\right)^k, & \omega\neq 0,\end{cases}$$

$k=1,2$, are the Fourier transforms of

$$F_1(\tau):=\begin{cases}\dfrac{1}{2}, & |\tau|<1,\\ 0, & |\tau|\geqslant 1\end{cases},\qquad F_2(\tau):=\begin{cases}\dfrac{1}{2}-\dfrac{1}{4}|\tau|, & |\tau|<2,\\ 0, & |\tau|\geqslant 2,\end{cases}$$

respectively, so that

$$G_k(\omega)=\int_{-\infty}^{\infty}F_k(\tau)e^{-i\omega\tau}d\tau,$$

as may be verified directly. The relations (A) now follow from the Poisson summation formula

$$\sum_{k=-\infty}^{\infty}F(k\eta+\tau)=\frac{1}{\eta}\text{P.V.}\sum_{n=-\infty}^{\infty}G\left(\frac{2\pi n}{\eta}\right)\exp\left(\frac{2\pi in\tau}{\eta}\right)$$

by letting $\tau=0$ and $\eta=2\pi$. [Applying the Poisson formula to Fourier transforms of functions with small support frequently yields startling results; see this MONTHLY, 80 (1973) 18–25.]

Comment by the editors: Part (B) can also be established by this method.

III. *Solution of part (B) by Richard Johnsonbaugh, Chicago State University.* We derive a general formula for $\sum_1^\infty(\sin^2 n)/n^{2k}$. Our result depends on the formula

$$\alpha(\theta)=\sum_1^\infty(\cos 2n\pi\theta)/(2n^{2k})=\frac{\pi^{2k}2^{2k-2}(-1)^{k-1}}{(2k)!}\left[\phi_{2k}(\theta)+(-1)^{k-1}B_k\right].$$

(See Bromwich, *Introduction to the Theory of Infinite Series*, 1926, p. 370.) The $\{\phi_k\}$ are multiples of the Bernoulli polynomials and the $\{B_k\}$ are the Bernoulli numbers. Using the half-angle formula we find

$$\sum_1^\infty(\sin^2 n)/(n^{2k})=\sum_1^\infty(1-\cos 2n)/(2n^{2k})=\alpha(0)-\alpha(1/\pi)$$
$$=\left[\pi^{2k}2^{2k-2}(-1)^k\phi_{2k}(1/\pi)\right]/((2k)!).$$

Taking $k=1, k=2$ for which $\phi_2(x)=x^2-x$, $\phi_4(x)=x^4-2x^3+x^2$, we obtain

$$\sum_1^\infty(\sin^2 n)/n^2=(\pi-1)/2\quad\text{and}\quad\sum_1^\infty(\sin^2 n)/n^4=(\pi-1)^2/6.$$

Solutions were received from 67 readers.

Editor's comment. Most solvers either combined various sums obtained from reference books to get (A) and (B), or started from the Fourier series for $(\pi-x)/2$, integrated it, and used trigonometric identities. E. T. H. Wang (Wilfrid Laurier University) noticed (as did the proposer) that (A) is a striking analog of the well-known equation

$\int_0^\infty x^{-1}\sin x\,dx = \int_0^\infty x^{-2}\sin^2 x\,dx = \pi/2$. A. J. van der Poorten (Australia) derived both (A) and (B) from the theory of polylogarithms; R. de Buda (Toronto) used Shannon's sampling theorem (perhaps better known to mathematicians as the cardinal interpolation series); Daniel Weisser (Berkeley) used the general theory of zeta functions.

The Classical Series $\Sigma(-1)^n n^{-1}\log n$

6243 [1978, 828]. *Proposed by Emil Grosswald, Temple University.*

Find in closed form the sum S of the conditionally convergent series

$$\sum_{n=2}^{\infty}(-1)^n n^{-1}\log n.$$

Editor's Note. M. S. Klamkin points out that the same problem, except for a change of sign, was proposed by him as Problem 4592 [1954, 350; 1955, 588] and that generalizations appeared in Problem 4762 [1958, 635] and in Briggs and Chowla, The power series coefficients of $\zeta(s)$, this MONTHLY, 62 (1955) 323. John R. Hatcher and Eli L. Isaacson also draw attention to the paper of Briggs and Chowla. Paul F. Byrd refers to solutions in E. Hansen, A Table of Series and Products (44.1.8), 288, and in A. D. Wheelon, A Short Table of Summable Series (9.106). Geoffrey B. Campbell (Australia) notes that a solution appears in G. H. Hardy, Collected Works, Oxford, 1969, vol. 4, p. 476. John Todd remarks that there is quite a literature on this problem and its extensions, as an example for the application of Euler's method for accelerating the convergence of an alternating series, and refers to J. J. Y. Liang and John Todd, The Stieltjes constants, J. Research Nat. Bur. Standards, 76B (1972) 161–178, and to John Todd, Basic numerical mathematics, I, Numerical Analysis, Birkhäuser 1979, Problem 6.5. D. J. Newman remarks that the problem is old.

The sum of the series is $\gamma\log 2-\frac{1}{2}(\log 2)^2$, where γ is Euler's constant.

Also solved by W. A. Al-Salam & A. Meir, Tom M. Apostol, K. F. Andersen, Anon. & Tomas Schonbek, J. M. Ash, Günter Bach (West Germany), David Borwein, Robert Breusch, Geoffrey B. Campbell (Australia; two solutions), R. C. Carson, Paul Chauveheid (Belgium), L. E. Clarke (England), Charles A. De Carlucci, Emeric Deutsch, J. S. Frame, Nick Franceschine III, Leon Gerber, G. Gonnet, Sidney Heller, P. Henrici (Switzerland), Eli L. Isaacson, A. A. Jagers (Netherlands), Richard Johnsonbaugh, William J. Knight, O. P. Lossers (Netherlands), L. E. Mattics, J. G. Mauldon, Robert M. McLeod, A. McD. Mercer, Armal Mercier, V. N. Murty, D. J. Newman, Johannes C. C. Nitsche, Hermann Schmidt (Germany), Edwin Shapiro, Joseph Silverman, Allen Stenger, Lajos Takács, L. Van Hamme (Belgium), G. C. Wake (New Zealand), David Weisser, Ken Yocum, and the proposer.

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR., AND LYNN A. STEEN
with the assistance of the mathematics departments of St. Olaf, Carleton, and Macalester Colleges
COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

Introductory Combinatorics. By Richard A. Brualdi. North-Holland, Amsterdam, The Netherlands, 1977. First reprint, 1978. x+374 pp. $19.95. (Telegraphic Review, June-July 1978.)

The enrollments in many advanced undergraduate mathematics courses have rapidly diminished in recent years, but not in courses in combinatorics. The reason is that combinatorics

is an increasingly useful subject with applications in many fields, including computing, electrical engineering, and economics. Accordingly, several textbooks have been written to meet the increased demand. One of the better ones is the book under review.

It begins with a chapter of several interesting and enticing examples to illustrate the nature of the subject. Unfortunately, the second chapter, on the pigeonhole principle, is perhaps one of the most difficult in the text and might have been placed later in the development. Although the pigeonhole principle is easily grasped, its applications are frequently nontrivial. Ramsey's theorem, which generalizes the pigeonhole principle, might best be omitted by many instructors. The next five chapters are very readable, are written in a leisurely style, and contain numerous examples. Chapter 3 treats the familiar topics of combinations and permutations of both sets and multisets. Binomial coefficients, binomial coefficient identities, the binomial theorem, and the multinomial theorem are discussed in Chapter 4. Chapter 5 contains a clear explanation of the inclusion-exclusion principle with applications to combinations with repetitions, to derangements, and to other forbidden-position problems. Recurrence relations are studied in Chapter 6, where, unfortunately, the author's convention for the Fibonacci numbers is not the usual one. The material on solving linear recurrence relations is presented nicely. There is also a section on differences, a subject which is covered in much greater detail in the older, classical texts on combinatorics. Chapter 7 is concerned with generating functions and their applications.

Chapters 1–7 should constitute the core of a course in combinatorics. The author remarks in the preface that in a one-semester course a class should cover these chapters and perhaps two of the next three chapters, which are independent of each other. I found this advice fairly accurate. I chose Chapter 8, on the marriage theorem and bipartite graphs, and Chapter 10, an introduction to graph theory. We didn't quite complete these chapters. However, I had supplemented Chapter 7 with about two and a half lectures on Bernoulli numbers. The introduction to graph theory in Chapter 10 is especially clear. Graph theory is such an important subject, however, that some of my students, especially those in computer science, wished to devote more time to it. Chapter 11 is a continuation of the study of graphs, while Chapter 12 concludes the text with a consideration of optimization problems. Generally, the content of the last five chapters demands a little bit more sophistication than the previous chapters. Pólya's theory of counting is not discussed in the text.

An unusually large number of exercises is one of the good features of this book. Most of the exercises are fairly straightforward, but almost every chapter contains some very challenging exercises. In a few exercises, the intent is rather hazy. The book contains surprisingly few misprints and errors. Perhaps the most glaring misprint is the mislabeling of the graph for the Königsberg bridge problem on page 231. The printing is quite attractive and uncrowded. The text is laudably readable and detailed in explanations and examples. The author has gauged very well the style of writing with which upper-level undergraduates in mathematics, computer science, and engineering will feel comfortable. Thus Brualdi's text can be highly recommended to all instructors in search of a readable textbook for an undergraduate combinatorics course.

BRUCE C. BERNDT, University of Illinois at Urbana–Champaign

The Theory of Information and Coding: A Framework for Communication (volume 3 of *Encyclopedia of Mathematics and Its Applications*, edited by Gian-Carlo Rota). By Robert J. McEliece. Addison-Wesley, Reading, Massachusetts, 1977 (2nd printing, with revisions, 1979). xvi + 302 pp. $24.50. (Telegraphic Review, April 1978.)

I have been hooked on algebraic coding theory since reading Norman Levinson's paper "Coding Theory: A Counterexample to G. H. Hardy's Conception of Applied Mathematics" in this MONTHLY (March 1970). In recent years, my courses in linear algebra, abstract algebra, and

discrete mathematics have reflected my addiction to coding and a parallel interest in information theory. Last year a group of students requested that I teach a course on information theory and coding theory. I was ecstatic, but in trouble: As far as I knew there was no text that gave extensive treatment to Shannon's theory of information and algebraic coding theory at a level appropriate for advanced undergraduates. Fortunately, as the students deposited their petition requesting the course on my chairman's desk, the mailman deposited an Addison-Wesley description of McEliece's book on my desk—a solution to my problem.

The book is divided into three parts: Introduction; Part I, Information Theory (which is devoted to Shannon's theory and includes channel and source coding theorems); Part II, Coding Theory (which includes block and convolutional codes and source codes). I taught a one-quarter course (ten weeks, four meetings per week) from the book to a class consisting of twelve junior- and senior-level students (two computer-science majors, four mathematics majors and six electrical-engineering majors), emphasizing channel coding. Each student had taken a course in abstract algebra or linear algebra or probability, beyond their five quarters of calculus and differential equations, prior to registering for the course. Less preparation would not be sufficient to enable a student to profit from McEliece's book.

We spent a week on the Introduction. The author introduces elementary linear codes, demonstrates that the use of such codes can reduce bit error probabilities, and discusses just how small bit error probabilities can be made in view of Shannon's theorem. I thought the introduction was excellent and the students agreed—according to one of them "the introduction was worth the price of the book."

We spent three tough weeks on Part I. This time included periodic visits to Chapter 1 (Entropy and Mutual Information), which the class found intimidating even though we regarded it as a reference section and heeded the author's warning that "it is a mistake to read it first." Still, the class felt that the effort required to get the channel coding theorem was well spent. There were quibbles over the author's introduction of the capacity cost function. The subtlety of introducing it to "underscore" the duality between channel coding and source coding was lost on some of the more practical-minded students, especially since the author admitted that the most important capacity function is identically zero.

The remainder of the quarter was devoted to Part II. We discussed most of the material in Chapters 7 (Linear Codes), 8 (BCH, Goppa and Related Codes), and 9 (Convolutional Codes) in detail. The exposition in Part II is excellent: one never feels inundated by technicalities as is the case in many introductions to coding theory. Theorems flow from examples and are supported by numerous excellent problems.

Overall the only deficiencies which I can point out are generality and some redundancy. However, these characteristics are inherent to the encylopedic nature of the book and do not detract from its value as a text. I plan to use McEliece's book the next time I teach a course on information and coding.

GARY J. SHERMAN, Rose-Hulman Institute of Technology

MISCELLANEA

39. Mathematicians are like foreigners: if you talk to them, they translate it into their own language, and then it immediately becomes something quite different.

—Goethe, *Maximen und Reflexionen*, no. 1279.

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, P, L. *Proceedings of the International Congress of Mathematicians, Helsinki 1978*. Ed: Olli Lehto. U. of Helsinki, Hallituskatu 15, Helsinki, 1979. *V. 1*, 506 pp; *V. 2*, 511 pp. [ISBN: 951-41-0352-1] Texts of opening and closing ceremonies, of Fields medal award presentations, of plenary and invited addresses. LAS

BASIC, T(13: 1). *Mathematics, An Everyday Experience, Second Edition*. Charles D. Miller, Vern E. Heeren. Scott F, 1980, 561 pp, $15.95. [ISBN: 0-673-15279-0] First edition has been revised (TR, April 1976) and expanded by addition of material on basic arithmetic, basic algebra, consumer mathematics, the metric system, logic and sets. Both exercises and text tend to be somewhat livelier and more imaginative than in many books at this level. JS

PRECALCULUS, T(13: 1). *College Algebra: A Skills Approach, Lecture Version*. J. Louis Nanney, John L. Cable. Allyn, 1980, x + 307 pp, $14.95 [ISBN: 0-205-06914-2]; *Trigonometry: A Skills Approach, Lecture Version*, 1979, ix + 246 pp, $13.95. [ISBN: 0-205-06920-7] Poor use of radian measure. Not enough emphasis on thinking; problem-solving skills must be applied. LLK

FOUNDATIONS, P??? *Structure Theory: Languages, Science & Aesthetics*. Eugene H. Hussey. Demecon Pub, 1978, xii + 375 pp, $14. This privately published tract contains a lot of vague generalities. For example, to quote from the preface: "[T]his book concerns the structure of scientific demonstration. [So far, so good.] This structure can be said to start by sending the two components of regularity into a label by coordination. ...Iteration of structure extends scientific demonstration along one's environment." Caveat Emptor! GHM

FOUNDATIONS, S(14-16), L. *Well-Ordered Sequences*. B.J. Ball. J. Undergrad. Math. (Dept. of Math., Guilford Coll., Greensboro, NC 27410), 10 pp, $3 (P). This short monograph in undergraduate mathematics gives an unsophisticated treatment of some of the properties of well-ordered sets. Presented in the heuristic style of R.L. Moore, it should whet the appetite of the curious student for further study. The flavor is spiced by examples from point-set topology without worrying over indigestible chunks of logic. Many assertions are left as exercises, making this an attractive starting resource for independent study projects. GHM

FOUNDATIONS, P, L. *Foundational Studies, Selected Works*. Andrzej Mostowski. North-Holland, 1979. *V. I*, Stud. in Logic and Found. of Math., V. 93, xlvi + 635 pp [ISBN: 0-444-85102-X]; *V. II*, viii + 605 pp, $134.25 set. [ISBN: 0-444-85103-8] Approximately half of Mostowski's most influential works, including *Thirty Years of Foundational Studies* in its entirety. Papers not originally in English are translated here for the first time. Opens with a very brief biography, a complete bibliography, and five brief surveys of Mostowski's contributions to decidability and recursion, to the foundations of set theory, to model theory, to logical calculi, and to the foundation of second order arithmetic. LAS

NUMBER THEORY, T(18: 1), S, P, L. *Metric Theory of Diophantine Approximations*. Vladimir G. Sprindžuk. Trans: Richard A. Silverman. V.H. Winston (Distr: Wiley), 1979, xiii + 156 pp, $19.95. [ISBN: 0-470-26706-2] The subject of diophantine approximation has undergone enormous progress in the last few years and this book presents in a unified manner these recent accomplishments. The first half focuses on the theory of approximation of independent quantities and the second is devoted to the theory of approximation of dependent quantities. Includes a substantial bibliography. CEC

NUMBER THEORY, S(18), P. *Transcendence in Fields of Positive Characteristic*. J.M. Geijsel. Math. Centre Tracts No. 91. Math Centrum, 1979, x + 115 pp, (P). [ISBN: 90-6196-157-2] A study of transcendence properties of fields with a non-archimedean valuation and positive characteristic. Includes a survey of known results and the proof of a theorem which generalizes the Wade analogue of the Gelfond-Schneider theorem. CEC

NUMBER THEORY, T(17: 2), S, P, L. *Probabilistic Number Theory*. P.D.T.A. Elliott. Springer-Verlag. *I: Mean-Value Theorems*, Grund. math. Wissenschaften, B. 239, 1979, xxii + 392 pp, $35 [ISBN: 0-387-90437-9]; *II: Central Limit Theorems*, Grund. math. Wissenschaften, B. 240, 1980, xviii + 341 pp, $35. [ISBN: 0-387-90438-7] An encyclopedic two-volume work which broadly encompasses the area of probabilistic number theory. The author also imparts a rich sense of the history of the subject. Introductory chapters include necessary concepts from measure theory and probability. A superb list of references is included. No exercises. CEC

ALGEBRA, S(16-18), P, L. *Special Functions and Linear Representations of Lie Groups*. Jean Dieudonné. CBMS Reg. Conf. in Math., No. 42. AMS, 1980, iii + 59 pp, $8.40 (P). [ISBN: 0-8218-1692-6] This brief presentation serves its apparent purpose well, giving an enticing and lucid introduction to some of the flavor and theory of representations of Lie groups and some of the connections to classical special functions. A wide range is covered, including such topics as SU(2), SL(2,R), spherical functions, Fourier and Plancherel transforms, automorphic, and Bessel functions. No proofs, but references are given. No index. JS

ALGEBRA, S(17-18), P, L. *Representation Theory of Lie Groups.* M.F. Atiyah, *et al.* London Math. Soc. Lect. Notes Ser., No. 34. Cambridge U Pr, 1979, v + 341 pp, $24.50 (P). [ISBN: 0-521-22636-8] Ten papers from the 1977 Oxford Symposium on Lie groups. Part I (Mackey, Bott, McDonald, Simms, Atiyah) is introductory and expository; Part II (Schmid, Helgason, Kostant, Kazhdan, Lusztig) is more technical but still quite general. References and index. JS

CALCULUS, T(14: 2). *Intermediate Calculus, Multivariable Functions and Differential Equations with Applications.* James F. Hurley. Saunders, 1980, xv + 765 pp. [ISBN: 0-03-056783-1] Clear presentation, good illustrations and examples, ample exercises. Choice of topics and level of writing excellent. LLK

CALCULUS, T(13-14: 1, 2). *Calculus for the Social, Managerial, and Life Sciences, Second Edition.* Laurence D. Hoffmann. McGraw, 1980, ix + 429 pp, $16.95. [ISBN: 0-07-029317-1] First edition (1975) was *Practical Calculus for the Social and Managerial Sciences* (see TR, October 1975). Informal and intuitive. Clear and readable. Techniques favored over theory. Applications-oriented. Functions through Lagrange multipliers. Changes from first edition include more drill problems, fewer proofs, chapter tests with answers, algebra review, material on differentials, trigonometry and mathematics of finance. For one semester, two quarters, or two semesters. Worth a look. JK

REAL ANALYSIS, T**(14-15: 1, 2), S, L. *Elementary Analysis: The Theory of Calculus.* Kenneth A. Ross. Springer-Verlag, 1980, viii + 264 pp, $18.80. [ISBN: 0-387-90459-X] In many ways, just the right book for a course bridging the gap between elementary calculus and an in-depth study of real analysis. Non-encyclopedic and not overpowering. Readable throughout. Proofs carefully detailed at first but less so in later chapters. Numerous attractive exercises with solutions for most odd-numbered ones. Optional sections on metric spaces to broaden readers' horizons. Choice treatment. JK

DIFFERENTIAL EQUATIONS, P. *Hypoelliptic Boundary-Value Problems.* J. Barros-Neto, Ralph A. Artino. Lect. Notes in Pure and Appl. Math., V. 53. Dekker, 1980, v + 90 pp, $19.75 (P). [ISBN: 0-8247-6886-8] A survey of hypoelliptic boundary-value problems in the constant coefficients case. Constructs reproducing kernels for these problems and examines the regularity properties of these kernels. TRS

DIFFERENTIAL EQUATIONS, P. *Hyperbolic Differential Polynomials and their Singular Perturbations.* Jacques Chaillou. Trans: J.W. Nienhuys. Math. and its Appl., V. 3. Reidel, 1979, xv + 168 pp, $31.50. [ISBN: 90-277-1032-5] An examination of how various families of linear hyperbolic differential operators with constant coefficients behave when the order of the operators or the multiplicities of their characteristics change for some parameter value. For scientists with an understanding of distribution theory. TRS

DIFFERENTIAL EQUATIONS, P. *Quantitative Analysis in Sobolev Imbedding Theorems and Applications to Spectral Theory.* M.S. Birman, M.Z. Solomjak. Amer. Math. Soc. Transl., Ser. 2, V. 114. AMS, 1980, viii + 132 pp, $26. [ISBN: 0-8218-3064-3] Lectures delivered at the 10th summer school on boundary value problems held at Nalchik in July 1972. LAS

NUMERICAL ANALYSIS, P. *The Numerical Solution of Nonlinear Operator Equations by Imbedding Methods.* C. den Heijer. Math. Centre Tracts, No. 107. Math Centrum, 1979, 164 pp, Dfl. 20 (P). [ISBN: 90-6196-175-0] Numerical methods for solving F(x) = 0, where F is a nonlinear mapping on Hilbert space, are developed by imbedding the equation in an initial value problem and by solving that problem by integration procedures of Runge-Kutta type. TRS

NUMERICAL ANALYSIS, T(15: 1). *A Practical Guide to Computer Methods for Engineers.* Terry E. Shoup. P-H, 1979, xii + 255 pp, $16.95. [ISBN: 0-13-690651-6] Written by an engineer for engineers, this text presents the basic numerical algorithms for solving algebraic equations, eigenvalue problems, ordinary and partial differential equations, optimization problems. Algorithms analyzed with a view to their virtue in particular engineering situations. Exercises. TRS

FUNCTIONAL ANALYSIS, T(16-18: 1). *Analyse Hilbertienne.* Laurent Schwartz. Hermann, 1979, 297 pp, 68F (P). [ISBN: 2-7056-5897-1] An excellent treatment of introductory functional analysis on Banach and Hilbert spaces. Examples drawn mostly from differential and integral equations. The austerity of the presentation is occasionally relieved by an "ouf! ouf!". Exercises and solutions. TRS

FUNCTIONAL ANALYSIS, T*(18), P. *Theory of Operator Algebras I.* Masamichi Takesaki. Springer-Verlag, 1979, vii + 415 pp, $39.50. [ISBN: 0-387-90391-7] A beautiful presentation of self-adjoint algebras of bounded operators on Hilbert space. Starting with the fundamentals of Banach algebras and C*-algebras, the main text concerns the duality theory of operator algebras. Chapters in text start with general introductions and end with historical notes. Exercises. References. TRS

FUNCTIONAL ANALYSIS, P. *Lecture Notes in Mathematics-731: Duality for Crossed Products of von Neumann Algebras.* Yoshiomi Nakagami, Masamichi Takesaki. Springer-Verlag, 1979, ix + 139 pp, $9 (P). [ISBN: 0-387-09522-5] The authors present the dualized version of the Arveson-Connes spectral analysis, the integrability of an action, dominant actions, the comparison theory of 1-cocycles. TRS

OPTIMIZATION, T*(14), S, L. *Elementary Linear Programming with Applications, International Edition.* Bernard Kolman, Robert E. Beck. Comp. Sci. and Appl. Math. Acad Pr, 1980, xiii + 399 pp, $17.95. [ISBN: 0-12-417865-0] A clear, honest treatment of linear programming methods: simplex, duality, sensitivity analysis, degeneracy. Contains sections on integer programming and special linear programming topics, e.g., transportation problems, critical path methods. Many exercises and examples--a very nice book. TAV

OPTIMIZATION, P. *Discrete Optimization.* Ed: P.L. Hammer, E.L. Johnson, B.H. Korte. North-Holland, 1979. *I,* Annals of Discrete Math., No. 4, xii + 299 pp, $58.50 [ISBN: 0-444-853227]; *II,* Annals of Discrete Math., No. 5, vi + 453 pp, $78. [ISBN: 0-444-853235] Proceedings of two back-to-

back conferences held in British Columbia in August 1977: the Advanced Research Institute on Discrete Optimization and Systems Applications and the Discrete Optimization Symposium. JAS

ANALYSIS, P, *The First Nonlinear System of Differential and Integral Calculus.* Michael Grossman. Mathco (Box 240, Rockport, MA 01966), 1979, x + 85 pp, $15 (P). Detailed account (without the easily supplied proofs) of the exponential calculus (formerly called the geometric calculus), one of the non-Newtonian calculi constructed by Grossman and Katz in 1967 and presented in *Non-Newtonian Calculus* in 1972 (see TR, May 1973). JK

ANALYSIS, T(14-16: 1), S, L. *Generalised Functions.* R.F. Hoskins. Ellis Horwood, 1979, 192 pp, $39.95. [ISBN: 0-85312-105-2] An introduction to generalized functions which is both elementary and honest, presuming only elementary calculus. Topics: calculus review, two chapters on the delta function, time-invariant linear systems, transforms, distributions. Exercises. A good text, except for the price. TRS

ANALYSIS, P, *Product Integration with Applications to Differential Equations.* John D. Dollard, Charles N. Friedman. Ency. Math. and its Appl., V. 10. A-W, 1979, xxii + 253 pp, $24.50. [ISBN: 0-201-13509-4] Introduced by Volterra in 1887 as a tool for solving systems of linear differential equations, the product integral is to the product as the ordinary integral is to the sum. This book is an attractive, elementary introduction to the product integral and includes a general survey of the subject by P.R. Masani. TRS

ALGEBRAIC GEOMETRY, P, *Vector Bundles on Complex Projective Spaces.* Christian Okonek, Michael Schneider, Heinz Spindler. Progress in Math., No. 3. Birkhäuser Boston, 1980, vii + 389 pp, $18 (P). [ISBN: 3-7643-3000-7] An extended and updated version of a course presented in Göttingen in the winter semester 1978/79. However, this is an introduction designed for readers with a basic knowledge of "analytic (or) algebraic geometry." Each section includes a paragraph with historical remarks and references to further results. JAS

DIFFERENTIAL GEOMETRY, P, *Geometry of the Laplace Operator.* Ed: Robert Osserman, Alan Weinstein. Proc. of Symposia in Pure Math., V. 36. AMS, 1980, vii + 323 pp, $18. [ISBN: 0-8218-1439-7] Expanded versions of most of the invited papers and a few contributed papers from the AMS symposium held at the University of Hawaii, March 27-30, 1979. JAS

GEOMETRY, S***(13-16), P, L***, *Circles, A Mathematical View.* D. Pedoe. Dover, 1979, viii + 87 pp, $2.75 (P). [ISBN: 0-486-63698-4] A most welcome, enlarged republication. Aside from a few minor corrections, the text proper is unchanged. Additions include exercises (with solutions) on the contents of the first three of the book's four chapters and a brief list of books for further reading. N.A. Court warmly reviewed the first edition of this delightful gem (ER, January 1959). Buy the book, and savor it. JK

GEOMETRY, T(17-18), S, P, *Projective Geometries Over Finite Fields.* J.W.P. Hirschfeld. Clarendon Pr, 1979, xii + 474 pp, $45. [ISBN: 0-19-853526-0] An account of finite geometry that truly starts from "scratch" and begins (another volume is to follow) a complete comprehensive account of the subject. While treating virtually all aspects of the subject and presenting general and esoteric results, the pace is sufficiently relaxed to work through the details for many particular examples of projective planes. Fine bibliography. SS

PROBABILITY, P*, *Controlled Markov Processes.* E.B. Dynkin, A.A. Yushkevich. Grund. der math. Wissenschaften, B. 235. Springer-Verlag, 1979, xvii + 289 pp, $39.80. [ISBN: 0-387-90387-9] A systematic presentation of the current theory of multistage Markovian decision processes. Beginning with discrete models with a finite time horizon, the text proceeds to more complex models with increasing abstraction. A large number of applications are developed. Contains several appendices and a useful bibliography. TAV

STATISTICS, S*, P*, *BMDP-79, Biomedical Computer Programs, P-Series.* Ed: W.J. Dixon, M.B. Brown. U of Calif Pr, 1979, xiii + 880 pp, $13.50 (P). [ISBN: 520-03569-0] Revision of the 1977 BMDP Manual of statistical programs. Content is identical to the 1977 Manual, except for a revision of the transformation section and the addition of sections describing three new programs: Stepwise Logistic Regression, General Mixed Model ANOVA--Equal Cells, and K-Means Clustering. RSK

STATISTICS, P, *Lecture Notes in Control and Information Science-21: Optimal Experiment Design for Dynamic System Identification.* Martin B. Zarrop. Springer-Verlag, 1979, x + 197 pp, $14.50 (P). [ISBN: 0-387-09841-0] This is a publication of the author's thesis work in which it is assumed that a model of some process is known, and that the experimenter has only to determine the form of the single linear input and/or the single linear output of a dynamic system in order to maximize information from an experiment. The same problem for static systems has been treated in statistics literature; this book reports on a new research approach to the dynamic problem. AWR

COMPUTER PROGRAMMING, S, L, *Problems for Computer Solution, Second Edition.* Donald D. Spencer. Hayden, 1979, 128 pp, $5.95 (P). Topically arranged, partially graded collection of language-independent problems intended as a supplement to any programming textbook. Exercises (in arithmetic, algebra, trigonometry, probability, statistics, number theory, business, chemistry, physics and on games and puzzles) range in difficulty from trivial to somewhat challenging, in appeal from zero upwards. Few hints. No answers. Source book for the teacher unwilling or unable to make up routine exercises. JK

COMPUTER PROGRAMMING, T(14-15: 1), *Programming Assembler Language.* Peter Abel. Reston Pub, 1979, xiii + 400 pp, $14.95. [ISBN: 0-8359-5658-X] Intended as an introductory text and primary reference on IBM 360 and 370 assembly language. Assembly and execution; characters, decimal and

binary data; arithmetic, bit manipulation and use of registers; I/O and external storage; macros and subprograms; OS and DOS. Some previous high-level experience would be helpful. RWN

COMPUTER PROGRAMMING, T(13-15: 1), S, *Introduction to Pascal.* Jim Welsh, John Elder. P-H, 1979, xiv + 282 pp, $14.95. [ISBN: 0-13-491522-4] A comprehensive introduction to Pascal based on courses taught at Queen's University of Belfast. Suitable even for a reader learning to program for the first time. Includes extensive "case-study" programs, and an appendix containing the Pascal syntax diagrams. LAS

COMPUTER SCIENCE, T?(14: 1), S, *Discrete Structures of Computer Science.* Leon S. Levy. Wiley, 1980, 310 pp, $17.95. [ISBN: 0-471-03208-5] Descriptive introduction to a large number of topics, most of which are not sufficiently explored and developed in the text and exercises to promote thorough understanding. Chapter titles: Sets, Functions, Relations (including matrices and Boolean algebras); Directed Graphs; Algebraic Systems (groupoids, groups, homomorphisms); Formal Systems; Trees; Programming Applications. LCL

COMPUTER SCIENCE, T(17-18: 1), S, P, L, *Transductions and Context-Free Languages.* Jean Berstel. Teubner Stuttgart, 1979, 278 pp, (P). [ISBN: 3-519-02340-7] A theory of formal languages is presented with an emphasis on rational transductions and their use for the classification of context-free languages. A graduate level text which assumes a prerequisite of a semester course in formal languages and automata theory. Includes an adequate supply of exercises and a bibliography. CEC

COMPUTER SCIENCE, T*(16-18: 1, 2), L, *Algorithms: Theory, Complexity and Efficiency.* Lydia I. Kronsjö. Wiley, 1979, xv + 361 pp, $41.25. [ISBN: 0-471-99752-8] A readable presentation of the "new" theory of algorithms engineered by the development of computers. The emphasis is on the algorithms, their derivation, and their efficiency; not on programming. Each of the nine chapters ends with a dozen or so problems. There is a substantial bibliography and a serviceable index. JAS

COMPUTER SCIENCE, T*(13-18: 1, 2), S, *Computer Organization: Hardware/Software.* G.W. Gorsline. P-H, 1980, x + 309 pp, $19.95. [ISBN: 0-13-165290-7] Emphasis on structure and organization common to many stored program computers. Conceptual level is chiefly that of the block diagram with little attention to the architectural and register transfer levels. Begins with the PMS system, data types, addressing structures, and registers. Includes chapters on instructions and modalities, control units, memories, data paths and interrupts, central processor, effects of miniaturization, networks. Chapter questions. Bibliography. Index. RJA

COMPUTER SCIENCE, S(14-16), P, *TRS-80 Interfacing, Book 1.* Jonathan A. Titus. Howard W. Sams, 1979, 190 pp, $8.95 (P). [ISBN: 0-672-21633-7] An introduction to the signals available within Radio Shack's TRS-80 computer that can be used to control external devices under control of Basic programs. LAS

COMPUTER SCIENCE, S(14-16), P, *The Z-80 Microcomputer Handbook, Third Printing.* William Barden, Jr. Howard W. Sams, 1979, 304 pp, $8.95 (P). [ISBN: 0-672-21500-4] A detailed guide to the hardware and software of the Z-80 microprocessor (the heart of Radio Shack's TRS-80), together with a survey of various microcomputers that use it. LAS

COMPUTER SCIENCE, T(14-16: 1), L, *Text Processing: Algorithms, Languages, and Applications.* Allen B. Tucker, Jr. Comp. Sci. and Appl. Math. Acad Pr, 1979, xii + 171 pp, $16.50. [ISBN: 0-12-702550-2] Two chapters on the use of PL/I and SNOBOL for text processing; a survey of packages (KWIC for concordances, SCRIPT for formatting, CMS for interactive editing, FAMULUS for bibliographic retrieval); literature review (with extensive references). Suitable as a text or as a reference. LAS

COMPUTER SCIENCE, T(16-18: 2), L, *Computer Image Processing and Recognition.* Ernest L. Hall. Comp. Sci. and Appl. Math. Acad Pr, 1979, xvii + 584 pp, $29.50. [ISBN: 0-12-318850-4] A substantial text for the senior or graduate computer scientist or electrical engineer. Includes computer experiments and problems for the student and a great deal of bibliographic material. Covers the gamut from physical optics to artificial intelligence. Emphasizes five major areas: image enhancement and restoration, three dimensional reconstruction, digital television and image compression, segmentation and description, scene matching and recognition. JAS

COMPUTER SCIENCE, S(13-18), *How to Build Your Own Working 16-Bit Microcomputer.* Ken Tracton. TAB Books, 1979, 95 pp, $3.95 (P). [ISBN: 0-8306-9813-2] A technical description of the architecture of the Texas Instruments TMS9900 16-bit processor and its family of support chips leading to discussion of system design and the use of peripherals. JAS

COMPUTER SCIENCE, T(14-18: 1, 2), S, P, L, *Structured Programming: Theory and Practice.* Richard C. Linger, Harlan D. Mills, Bernard I. Witt. A-W, 1979, xiv + 402 pp, $18.95. [ISBN: 0-201-14461-1] This text is based on the conviction that complex software development must be based on mathematical principles embodied in the dictates of structured programming. In particular, topics include communication in the software development process, mathematical properties of programs, and program reading, verification and writing. Beautifully formatted text. Exercises. Index. RJA

COMPUTER SCIENCE, T(16-18: 1, 2), L, *Interactive Computer Graphics: Data Structures, Algorithms, Languages.* Wolfgang K. Giloi. P-H, 1978, xiii + 354 pp, $20.95. [ISBN: 0-13-469189-X] The first half of this text presents picture structures (data structures whose data is the graphic data of a picture) and the theory of representing and manipulating them. The remainder of the book starts with a brief description of interactive hardware followed by a study of languages and display programming systems. Based on a two-quarter sequence taught at the University of Minnesota. Extensive bibliography, reasonable index. Several typographical errors, some disturbing. JAS

COMPUTER SCIENCE, P, L*. *Computer Arithmetic.* Ed: Earl E. Swartzlander, Jr. Benchmark Papers in Elec. Eng. and Comp. Sci., V. 21. Dowden, Hutchinson & Ross, 1980, xiii + 378 pp, $45. [ISBN: 0-87933-350-2] Reprinting of 42 key papers, several from obscure sources, on algorithms for arithmetic and evaluation of elementary functions. Contains numerous references and an extensive bibliography. LAS

COMPUTER SCIENCE, P. *Advances in Computers, Volume 18.* Ed: Marshall C. Yovits. Acad Pr, 1979, xiii + 308 pp, $31. [ISBN: 0-12-012118-2] Survey papers on image processing, computer chess, software science, computer assisted instruction, and Soviet software. LAS

COMPUTER SCIENCE, T(17-18: 1), S, L. *Methods and Applications of Interval Analysis.* Ramon E. Moore. SIAM, 1979, xi + 190 pp, $16. [ISBN: 0-89871-161-4] Rounding errors or errors caused by uncertain initial data are of concern in machine computing. Interval analysis, a branch of mathematics developed as a tool to cope with such problems, is explained in this book to scientists and engineers as well as to applied mathematicians. AWR

COMPUTER SCIENCE, S(16-18), P. *A Computational Logic.* Robert S. Boyer, J. Strother Moore. Acad Pr, 1979, xiv + 397 pp, $29.50. [ISBN: 0-12-122950-5] An implementation of and elaboration on the original ideas of John McCarthy concerning the use of recursive functions and induction (instead of quantification) to understand computer programs, and the use of computers to aid the generation of proofs. Includes a description of the formal theory and the many techniques for proving theorems in it (special emphasis on induction). Several major examples. LCL

COMPUTER SCIENCE, S(17-18), P. *Operating Systems, An Advanced Course.* Ed: R. Bayer, R.M. Graham, G. Seegmüller. Springer-Verlag, 1979, x + 593 pp, $19.80 (P). [ISBN: 0-387-09812-7] Contains the lecture notes of an advanced course held at the Technical University of Munich in 1977 and 1978. Includes articles by various authors on the object model, naming and binding of objects, kernel design, protection, synchronization in layered systems, reliability, data base operating systems, networks, system specification, decentralized systems with largely autonomous nodes. RJA

COMPUTER SCIENCE, T(16: 1), S, L. *Introduction to Automata Theory, Languages, and Computation.* John E. Hopcroft, Jeffrey D. Ullman. A-W, 1979, x + 418 pp, $20.95. [ISBN: 0-201-02988-X] A textbook which includes the study of automata, context-free grammars, Turing machines, undecidability, computational complexity and intractable problems. Includes numerous examples and exercises along with an extensive bibliography. CEC

COMPUTER SCIENCE, P. *Combinatorial Complexes: A Mathematical Theory of Algorithms.* Peter H. Sellers. Reidel, 1979, xv + 184 pp, $18.95. Introduces a mathematical theory of algorithms aimed toward finding new algorithms, contrasting with the usual mathematical approach which is to analyze known algorithms. The first chapter is devoted to formalization of desired programs and to some associated theory. Subsequent chapters give applications. AWR

COMPUTER SCIENCE, T, S(13-18), L. *Microcomputer Experimentation with the Intel SDK-85.* Lance A. Leventhal, Colin Walsh. P-H, 1980, xi + 340 pp, $12.95. [ISBN: 0-13-580860-X] An "introductory laboratory" for readers who want to get into the nuts and bolts of machine language programming based on the readily available single board SDK-85 computer using the Intel 8085. Each section (laboratory) has a vocabulary list, "do it" instructions and problems, and a "key point summary." This book does not substitute for the standard 8085 manuals but can be used without them. Covers input and output, arrays, arithmetic, timing and more. JAS

COMPUTER SCIENCE, T(14-15: 1), L. *Minicomputer Systems, Organization, Programming, and Applications (PDP-11), Second Edition.* Richard H. Eckhouse, Jr., L. Robert Morris. P-H, 1979, xix + 491 pp, $20.50. [ISBN: 0-13-583914-9] A re-organization and extension by 100 pages of the first edition (TR, February 1977). Clearer development, better use of programming examples, and more nearly accurate and thorough description of the PDP/11 and Macro-11. Several real-time applications are used to demonstrate the relationship between hardware and software. Overall, appears to be much improved both as a text and as a reference. RWN

COMPUTER SCIENCE, T(16-18). *Digital Image Processing.* Kenneth R. Castleman. P-H, 1979, xvi + 429 pp, $25. [ISBN: 0-13-212365-7] A mathematically sophisticated (although the "rigor is reasonably relaxed") treatment of image processing as a tool for solving practical problems with an emphasis on applications. Based on a course given at Caltech with examples drawn in part from on-going work at the Jet Propulsion Laboratory. Includes a broad range of topics from geometry to hardware using physics and Fourier transforms as needed. Substantial bibliography and index. JAS

SYSTEMS THEORY, T(17: 1), P. *Lecture Notes in Biomathematics-28: Mathematical Aspects of Reacting and Diffusing Systems.* Paul C. Fife. Springer-Verlag, 1979, iv + 185 pp, $9.80 (P). [ISBN: 0-387-09117-3] Consideration of modelling chemical and biological systems which react and diffuse; techniques for analysis of scalar and systems models. RBK

SYSTEMS THEORY, P. *Methodology in Systems Modelling and Simulation.* Ed: Bernard P. Zeigler, *et al.* North-Holland, 1979, xv + 537 pp, $65.75. [ISBN: 0-444-853405] Refereed papers from an August 1978 meeting held at the Weizmann Institute in Rehovot, Israel. Contains an index of major subjects treated in the volume, notes from a panel discussion on the future of modelling and simulation, and a brief annotated bibliography of related references. LAS

SYSTEMS THEORY, P. *A Technique for Software and Systems Design.* R.J. Lano. North-Holland, 1979, xii + 119 pp, $29.25. [ISBN: 0-444-84354-5] Presents the N^2 chart as an implementation tool and methodology for the tabulation, definition, analysis, and description of functional interactions and interfaces in large systems. RJA

APPLICATIONS (BIOLOGY), P. *Lecture Notes in Biomathematics-32: Systems Theory in Immunology.* Ed: Carlo Bruni, *et al.* Springer-Verlag, 1979, xi + 273 pp, $16 (P). [ISBN: 0-387-09728-7] Proceedings of a "Working Conference" held in Rome in May 1978. JAS

APPLICATIONS (BIOLOGY), P. *Lecture Notes in Biomathematics-31: The Geometry of Population Genetics.* Ethan Akin. Springer-Verlag, 1979, iv + 205 pp, $14 (P). [ISBN: 0-387-09711-2] Two fundamental results of the quantitative theory of natural selection state that along the solution curves of the selection differential equation mean fitness is constantly increasing, and that the direction of motion is the direction of greatest increase. Akin explains these results, and others, by using techniques of Riemannian geometry to develop a suitable metric (and hence an appropriate gradient) to interpret selection, recombination and mutation geometrically. LAS

APPLICATIONS (BIOLOGY), S(16-18), P, L. *Lecture Notes in Biomathematics-30: Mathematical Models in Cell Biology and Cancer Chemotherapy.* Martin Eisen. Springer-Verlag, 1979, ix + 431 pp, $22 (P).[ISBN: 0-387-09709-0] "Biology has become a quantitative science...mathematical models are absolutely essential for research in modern cell kinetics." An introduction to the use of optimal control theory in chemotherapy, to balance the lethal effects of drugs against the different growth kinetics of normal and cancerous cells. LAS

APPLICATIONS (CONTROL THEORY), P. *Lecture Notes in Control and Information Sciences-20: Stability of Adaptive Controllers.* Bo Egardt. Springer-Verlag, 1979, v + 158 pp, $9.80 (P). [ISBN: 0-387-09646-9] A study, based on a Ph.D. thesis at Lund Institute of Technology, of a general algorithm for design of adaptive systems and self-tuning regulators of plants whose operation parameters are not known. The stability properties of this algorithm are analysed and sufficient conditions for boundedness of closed-loop signals are given. JAS

APPLICATIONS (ELECTRONICS), S. *Problems and Solutions in Logic Design, Second Edition.* D. Zissos. Oxford U Pr, 1979, 223 pp, $26. [ISBN: 0-19-859362-7] This *Second Edition* (*First Edition*, TR, May 1976) has a new chapter on pulse-driven circuits and new problems have been added. LLK

APPLICATIONS (ENGINEERING), S(13-16). *Mathematical Formulae for Engineering and Science Students, Third Edition.* S. Barnett, T.M. Cronin. Bradford U Pr, 1979, viii + 76 pp, £3.65 (P). [ISBN: 0-901945-35-8] Only change from previous edition is a new index. (*Second Edition*, TR, November 1977.) TRS

APPLICATIONS (ENGINEERING), S(13-18). *Computerist's Handy Databook/Dictionary.* Clayton L. Hallmark. TAB Books, 1979, 96 pp, $3.95 (P). [ISBN: 0-8306-1069-3] Presents the common factual information about computers and computing to the layman in the form of charts, lists, tables, formulas, symbols. Sections on number systems and codes, logic symbols and engineering units, number related codes, machine related codes, and microprocessors. Second half of the text is devoted to a dictionary of computing terms and the meaning of microcomputer abbreviations and acronyms. Index. RJA

APPLICATIONS (INFORMATION RETRIEVAL), P. *Systems Analysis for Information Retrieval.* Helen M. Townley. Westview Pr, 1978, xiv + 121 pp, $13.50. [ISBN: 0-233-96920-9] Intended as a guide to mechanizing information services. Useful to novices and to systems staff and information officers. Presentation is practically oriented throughout. Commences with descriptions of the components of information storage systems. Continues with the requirements of the initial system study, types of storage, strategies for searching, and the implementation of the final design. Three appendices. Reading list. Index. RJA

APPLICATIONS (MANAGEMENT SCIENCE), T(15-17: 1), S. *Management Science: Cases and Applications.* Raj Aggarwal, Inder Khera. Holden-Day, 1979, xiii + 226 pp, $12.95 (P). [ISBN: 0-8162-0096-3]; *Solutions Manual*, 151 pp, (P). [ISBN: 0-8162-0097-1] Includes supplementary text with short structured case problems. Introduction to case analysis and report writing. Data included enables students to proceed with solution. Illustrates applications of specific techniques. Suitable for management science or operations research courses which stress applications. Solutions provided in manual. WC

APPLICATIONS (PHYSICS), T(18: 1), P, L. *Similarity, Self-Similarity, and Intermediate Asymptotics.* G.I. Barenblatt. Trans: Norman Stein, Milton VanDyke. Consultants Bureau (Distr: Plenum), 1979, xvii + 218 pp, $35. [ISBN: 0-306-10956-5] Explores the applications of dimensional analysis to a wide range of problems in mathematical physics, chemistry, biology, and engineering. Specific, detailed applications to hydrodynamics, elasticity, and turbulence are discussed, demonstrating the power of the theory of self-similar asymptotics. A careful translation, of interest to theorists and experimentalists alike. TRS

APPLICATIONS (PHYSICS), P. *Lecture Notes in Mathematics-760: Canonical Gibbs Measures.* H.O. Georgii. Springer-Verlag, 1979, viii + 190 pp, $11.80 (P). [ISBN: 0-387-09712-0] A study of some extensions of de Finetti's representation theorem for interacting particle systems, for both the discrete and the continuous case. LCL

APPLICATIONS (SYNERGETICS), P. *Pattern Formation by Dynamic Systems and Pattern Recognition.* Ed: H. Haken. Springer-Verlag, 1979, viii + 305 pp, $34. [ISBN: 0-387-09770-8] Collection of papers presented at the International Symposium on Synergetics held at Elmau, Bavaria, Germany, from April 30 to May 5, 1979. Index of contributors. RJA

Reviewers Whose Initials Appear Above

Richard J. Allen, St. Olaf; William Carlson, St. Olaf; Clifton E. Corzatt, St. Olaf; Lorraine L. Keller, St. Olaf; Roger B. Kirchner, Carleton; Richard S. Kleber, St. Olaf; Joseph Konhauser, Macalester; Loren C. Larson, St. Olaf; George H. Mills, St. Olaf; R.W. Nau, Carleton; A. Wayne Roberts, Macalester; Thomas R. Savage, St. Olaf; John Schue, Macalester; Seymour Schuster, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; T.A. Vessey, St. Olaf.

NEWS AND NOTICES

EDITED BY PAUL A. HAEDER, University of Nebraska at Omaha

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036

SUGGESTION BOX

Members of the MAA are encouraged to send in suggestions, questions, etc., about the operations of the Association. Communications will be referred to the appropriate officer of the Association for answering; from time to time, those of general interest may also be answered in one or both of the official journals. Communications should be addressed to: Suggestion Box, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036

PERSONAL ITEMS

University of Wyoming: Associate Professor *B. G. Roth* has been promoted to Professor. Assistant Professor *M. Olson* has been promoted to Associate Professor.

University of Texas, Austin: Associate Professor *John D. Dollard* has been promoted to Professor. New Professors on the staff are *Kenneth Kunen*, University of Wisconsin, and *Haskell P. Rosenthal*, University of Illinois. *Arnold W. Miller*, University of Wisconsin, is a new Assistant Professor. Professor *John Heisler*, St. Edwards University, Austin, Texas, and *Landon A. Colquitt*, Texas Christian University, are Visiting Professors.

D.W. Post College, Long Island University: Assistant Professor *Geoffrey Berresford* has been promoted to Associate Professor. Associate Professor *Susan Andima* has been granted tenure.

Middle Tennessee State University: Professor *Paul Hutcheson* has returned to full time teaching from Director of the Computer Center. Associate Professor *Earl E. Keese* has been promoted to Professor. Dr. *J. C. Hankins* has joined the faculty as Assistant Professor.

Chicago State University: Assistant Professor, *Gary Webb* has been promoted to Associate Professor. Dr. *Tim Carroll* has been appointed Assistant Professor.

Duquesne University: Associate Professor *Charles A. Loch* has been appointed Chairman. Assistant Professor *Rosaline Lee* has been promoted to Associate Professor. Dr. *John Baker* has been appointed Assistant Professor.

Kearney State College, Kearney, Nebraska: Professor *Theodora Nelson* has retired with the title of Professor Emeritus. Professor *L. M. Larsen* has returned to full time teaching from Department Chairman.

University of Massachusetts, Amherst: Associate Professor *M. K. Bennett* has been promoted to Professor. Professor *Doris S. Stockton* was awarded two citations for her books *Essential College Algebra* and *Essential Trigonometry* from the New Jersey Institute of Technology at the Twelfth Annual New Jersey Writers' Conference.

California State University, Sacramento: Associate Professor *Wallace Etterbeck* has been promoted to Professor. Professor *Charles Hagopian* has been guest speaker at a number of out-of-state conferences and professional meetings.

Florida State University: Assistant Professor *Steven F. Bellenot* has been promoted to Associate Professor. Professor *O. G. Harrold, Jr.* has retired with the title of Professor Emeritus.

Oakland University, Rochester, Michigan: *Darrell Schmidt*, Marshall University, and *Nancy Shoemaker*, SUNY-Albany, have been appointed Assistant Professors. Visiting Assistant Professor *Stuart Wang* has been appointed Assistant Professor.

University of California, Berkeley: Professor *Shiing-shen Chern* has been recalled to active service. A symposium on Differential Geometry was held in his honor in June, 1979. Professor *Stephen Smale* is a Miller Research Professor.

University of Saskatchewan, Saskatoon: *Remeshwar D. Gupta* is a new Assistant Professor on the staff. *James E. Totten* has accepted a position as Assistant Professor at Caribou College, Kamloops, British Columbia.

Stephen F. Austin State University, Nacogdoches, Texas: Dr. *Thomas A. Atchison*, Mississippi State, has been appointed Professor and Chairman, Department of Mathematics and Statistics. Assistant Professors *R. W. Yeagy* and *C. W. Proctor* have been promoted to Associate Professors. Dr. *W. I. Layton*, Professor and Chairman, retired August 31, 1979, after 29 years of service. He is now associated with Newberry College in Newberry, South Carolina.

University of New Haven, Connecticut: *W. Thurman Whitley*, Marshall University, has been appointed Associate Professor and Chairman of the Department of Mathematics. *James H. Fife*, Ph.D. candidate at Yale University, has been appointed Instructor and *Baldev K. Sachdeva*, University of Wisconsin at Milwaukee, has been appointed Visiting Assistant Professor.

Hofstra University, Hempstead, New York: *Douglas Bauer*, Stephens Institute, and *Rochelle Leibowitz*, University of South Carolina, have been appointed Assistant Professors.

Dr. *Josephine Ingle*, Blue Valley Middle School, Stanley, Kansas, and Dr. *Walter Reid*, University of Colorado, have been appointed Adjunct Assistant Professors at the University of Wisconsin, Eau Claire.

Dr. *Donald Quarles*, Stephens Institute of Technology, has joined the staff at the IBM Research Center, Yorktown Heights, New York.

Associate Professor *Michael Olinick*, Middlebury College, Vermont, has been appointed Chairman. He was Visiting Exchange Professor at San Diego State University for Spring, 1979.

Associate Professor *George M. Rosenstein, Jr.*, Franklin and Marshall College, Lancaster, Pennsylvania, has been promoted to Professor.

Professor *John Selden, Jr.*, Head of the Department of Mathematics at Bayero University, Kano, Nigeria, has been elected to the Council of the recently established research-oriented Nigerian Mathematical Society.

Associate Professor *Leonard A. Asimow*, University of Wyoming, is Visiting Associate Professor at Syracuse University.

Dr. *Walter Michaelis*, Idaho State University, has been appointed Visiting Assistant Professor at the University of Montana.

Dr. *Alan Stickney*, previously with the Freshman Honors Program at the University of Delaware, has been appointed Assistant Professor at Wittenberg University, Springfield, Ohio.

Janet A. McLeavey is a Visiting Professor at the University of Rhode Island.

Professor *Bernard L. Madison*, formerly Vice-Chairman at Louisiana State, has been named Chairman and Professor of Mathematics at the University of Arkansas.

Chairman *Louis P. Pushkarsky*, Mathematics Department, Trenton Junior College, Trenton, Missouri, has been elected President of the Missouri Association of Community and Junior Colleges.

Assistant Professor *Jane Malbrock*, Kean College of New Jersey, has been promoted to Associate Professor.

Associate Professor *Thomas Arbutiski*, Community College of Allegheny County, Pennsylvania, has been promoted to Professor.

Associate Professor *Stephen I. Gendler*, Clarion State College, Clarion, Pennsylvania, has been promoted to Professor.

Dr. *Patricia Henry*, Weber State College, Ogden, Utah, has been named Chief Reader of Mathematics, Educational Testing Service.

Associate Professor *Dennis Zill*, Loyola Marymount University, Los Angeles, has been promoted to Professor.

J. F. Traub of Carnegie-Mellon University has been appointed as Edwin Howard Armstrong Professor of Computer Science, Professor of Mathematics, and Chairman of the Computer Science Department at Columbia University.

Francis Hannick, Ph.D., University of Montana, has been appointed Assistant Professor at Mankato State University, Mankato, Minnesota.

Dr. *Steven M. Shew*, formerly at Olivet College, has been appointed Associate Professor at Grand Canyon College, Phoenix, Arizona.

Assistant Professor *Dennis Wildfogel*, Stockton State College, Pamona, New Jersey, has been promoted to Associate Professor.

Visiting Assistant Professor *Catherine Gates*, Franklin and Marshall College, Lancaster, Pennsylvania, has accepted a position with the Honeywell Corporation in Massachusetts.

Associate Professors *Richard D. Byrd* and *Siemion Fajtlowicz*, University of Houston, have been promoted to Professors.

Dr. *Edward W. Brande*, S. J., Fordham University, has been appointed Professor and Academic Vice-President at St. Peter's College, Jersey City, New Jersey.

Professor *Robert J. Serfling*, Florida State University, has been appointed Professor at The Johns Hopkins University.

Associate Professor *Jeanne Wright*, University of Wisconsin-Oshkosh, has accepted a faculty position at St. Leo's College, St. Leo, Florida.

Professor Emeritus *H. J. Ettlinger*, University of Texas, celebrated his 90th birthday on September 1, 1979, which was also the 61st wedding anniversary of Professor and Mrs. Ettlinger.

Professor *H. E. Lacey* of the University of Texas at Austin has been appointed Head of the Department of Mathematics of Texas A & M University, effective August 1, 1980.

Professor *Karl Goldberg*, a retired research mathematician with the National Bureau of Standards, died at his home in Rockville, Maryland, on June 2, 1979, at the age of 50.

TOPOLOGY CONFERENCE HONORING A SIXTY-FIFTH BIRTHDATE ANNIVERSARY

With National Science Foundation support, the Department of Mathematics of the University of Texas, Austin, sponsored a Topology Conference on October 20-23, 1979 in honor of *R. H. Bing*'s 65th birthday. The topic of special concern at the conference was the *Topology of Manifolds*. Approximately 140 mathematicians attended.

SLOAN RESEARCH FELLOWSHIPS

The Alfred P. Sloan Foundation is expanding its 25 year old program of fellowships in mathematics to include two fellowships specifically designated for applied mathematicians. The deadline for nominations is September 15, 1980. Direct applications are not accepted.

Sloan Research Fellowships are intended to provide flexible research support to especially promising young faculty members at an early stage of their careers.

For information and nomination forms write: Sloan Research Fellowships, Alfred P. Sloan Foundation, 630 Fifth Avenue, New York, New York 10020.

ENVIRONMETRICS '81

Environmetrics '81, a unique conference stressing innovative applications of statistics and mathematics to problems in environmental quality, will be held in Washington, D.C., on April 6-8, 1981. It is being sponsored by the Society for Industrial and Applied Mathematics (SIAM), the

SIAM Institute for Mathematics and Society (SIMS), and the U. S. Environmental Protection Agency. SIMS is the principal sponsor and will conduct the conference.

The conference will include 10 topic areas: *Environmental media:* air, water, noise, toxic substances, radiation, thermal pollution, solid wastes, pesticides, water supply; *Health and ecological effects:* toxicology, epidemiology, chronic effects, acute effects, biological effects, risk assessment; *Source monitoring:* effluent limitations, emission inventory estimation, control strategy assessment; *Ambient monitoring:* design, exposure (human exposure), quality assurance, data storage and summarization; *Measurement systems:* network design, instrument calibration, scaling, trace levels, data shortage and summarization, linking observed levels to individual exposure; *Standards:* compliance, risk, exposure, controls, indices; *Models of pollutant transport and fate:* initialization, solution techniques, evaluation; *Interface problems:* acid rain; *Prediction and trend analysis:* time series models, forecasting techniques; and *Relevant computer "packages":* data base management, statistical software packages.

To submit a contributed paper for presentation at Environmetrics '81, an author must provide the Program Committee with a one-page typewritten summary by October 15, 1980. Summaries, and all other communications, should be mailed to: Environmetrics '81, SIMS, 33 South 17th Street, Philadelphia, PA 19103.

REPORT ON PROBLEM SOLVING COURSES PLANNED

A subcommittee of the Mathematical Association of America (MAA) Committee on the Teaching of Undergraduate Mathematics plans a survey of problem solving courses in mathematics at the secondary and undergraduate levels. Its chair, Alan Schoenfeld, says that the job of the subcommittee is to prepare a report which

1. Describes the "state of the art" in problem solving courses,
2. lists available resources for teaching problem solving (and possibly creates some such resources), and
3. makes recommendations regarding a) the place of problem solving in the curriculum, b) ways to teach it.

The subcommittee plans to distribute a questionnaire to persons teaching problem solving courses. If you are teaching such a course, or know of someone who is, please let them know; if you have ideas as to what should be on the questionnaire, about useful resources, or about possible contributions the subcommittee might make, please get in touch. Contact Alan H. Schoenfeld, Mathematics Department, Hamilton College, Clinton, N.Y., 13323.

ARTICLES COMING SOON IN MATHEMATICAL MONTHLY

Some historical remarks concerning degree theory, Dr. Hans-Willi Siegberg
Poly's Counting Theorem via Tensors, Professor Russell Merris
Irrational Suma and Twin Primes, Solomon W. Golomb
The first proof of the quadratic reciprocity law, revisited, Professor Ezra Brown
Robbins' theorem for mixed-multigraphs, Frank Boesch and R. Tindell
Elementary characterization of classical optima, John L. Troutman and W. Hrusa
Increasing the participation of college women in mathematics-based fields, Lenore Blum and Steven Givant
Simple analytic proof of the prime number theorem, Donald J. Newman
A less strange version of Milnor's proof of Brouser's fixed point theorem, C. A. Rogers
Teaching mathematical problem solving skills, Alan H. Schoenfeld
The Jordan theory is simple! A. Galperin and Z. Waksman

CONFERENCE ON NUMERICAL ANALYSIS OF SEMICONDUCTOR DEVICES AND INTEGRATED CIRCUITS

NASECODE II Conference: The second international conference on the Numerical Analysis of Semiconductor Devices and Integrated Circuits will be held in Dublin, Ireland, from 17th to 19th June 1981 under the auspices of the Numerical Analysis Group, Dublin. The conference is sponsored by IEEE (Electron Devices Society), IEE (Irish Branch), Royal Irish Academy and Irish Mathematical Society.

Contributed papers are solicited on any topic relevant to the numerical simulation, optimization and computer aided design of semiconductor devices or integrated circuits. The preliminary version of such a paper should be submitted not later than FRIDAY, 20th MARCH 1981. It must be accompanied by a separate one-page abstract. The final version of an accepted contribution should be delivered to the registration desk on the first day of the conference. It must be typed according to our instructions on special paper supplied by us and it should be at most five pages long.

The proceedings of the conference will be published in book form in August 1981. They will contain the full texts of both invited lectures and the contributed papers. Registered participants will receive one free copy.

The registration fee is £90 and all communications should be addressed to NASECODE II Conference, 39 Trinity College, Dublin 2, Ireland, telephone No. (01) 772941 ext. 1889 or 1949; telex No. 5442 or 31166; telegraphic address TRINITY DUBLIN.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

MARCH MEETING OF THE FLORIDA SECTION DEDICATED TO THE MEMORY OF JAKE GOLIGHTLY

The Thirteenth Annual Spring Meeting of the Florida Section of the MAA was held on March 7 and 8, 1980 at Jacksonville University. There were 165 registrants attending the meeting.

Six invited addresses were presented as follows:

The Three Crises in Mathematics: Logicism, Intuitionism and Formalism, Professor Ernst Snapper of Dartmouth College
Bernoulli's String Problem for Rubber Bands, Professor Stuart Antman of the University of Maryland
Mathematical Models and Existence Theorems, Professor Dorothy L. Bernstein, Brown University
Learning to Teach College Mathematics, Professor Alan Wayne, Pasco Hernando Community College
Introducing Mathematical Models to Secondary School Teachers, Professor Bill Caldwell, Univ. of No.FL
Mathematics in Biology, Professor James Keesling, University of Florida.

In conjuction with the meeting there was a State Articulation Conference. The following talks and panel discussions were presented:

The Crisis in Secondary School Mathematics, Panel: Elroy J. Bolduc, University of Florida, Gene Nichols, Florida State University, Don Lichtenberg, University of South Florida, and Maurice E. Nott, Chairman, St. Petersburg Junior College
T'day's Mathematics, Charles (Chuck) Miller, American River Community College, Sacramento, California
The Secondary Teacher's Point of View, Panel: Michael Caballero, Teacher, Leto High School, Tampa, Morita Eng, Florida's Teacher of the Year, Sandalwood Senior-Junior High School, Jacksonville, Daryle May, Director of Teacher Education Programs, Jacksonville University, and Chairman, Bill Jordan, Seminole Junior College
Articulation Update, Panel: Members of Committee for Articulation in Mathematics, Chairman, Bill Rice, St. Petersburg Junior College.

A Saturday morning session was sponsored by Pi Mu Epsilon, the Mathematics Honorary Fraternity. The following talks were presented: Donna Corson, Florida Eta Chapter, University of North Florida, *A Cure for Instant Insanity Using Multi-graphs*
A Discussion on the Multinomial Theorem, Katherine F. Rowell, Georgia Epsilon Chapter, Valdosta State College
Frames: A Knowledge Representation and Organization System for A Computer, Tina Patterson, Florida Epsilon Chapter, University of South Florida.

The following papers were presented to the section:

Polynomial Equations Satisfied Over a Commutative Ring, Robert Gilmer, Florida State University
An Application of Quaternions to Flight Simulation, Edward P. Shaughnessy, Reflectone, Inc.
A Counter-Example to the Bounded Orbit Conjecture, Stephanie M. Boyles, University of Florida
An Integer Valued Hausdorff Like Metric, Carolyn R. Johnson, University of Florida
A Remarkable Simple Closed Curve: Revisited, O. G. Harrold, Florida State University
A Pointwise Periodic Map on a Chainable Continuum is Periodic, Edwin Duda, University of Miami
Span and Semi-span of Metric Continua, J. Kell, University of Miami
Finite to one Open Mappings on Circularly Chainable Continua, P. Bartick, University of Miami
Deducing the Properties of Singularities of Functions From Their Taylor Series Coefficients, B. Guerrieri and C. Hunter, Florida State University
Radicals of Semi-group Rings, Eleanor G. Turman, University of Florida
Primes in Arithmetical Sequences, Herman Simon, University of Miami
Hereditary Paracompactness of the Structure Spaces of Certain Rings, J. D. McKnight, University of Miami
A Technical Introduction to Microcomputers, Gareth Williams, Stetson University
Strong Q-Groups, Dennis Kletzing, Statson University
Weak Continuity of Nonlinear Mappings, Leonard J. Lipkin, University of North Florida
The Use of History in a Statistics Class, Edwin G. Landauer, Naval Nuclear Power School
Some Observations About the 1977-78 National Assessment in Mathematics, Eugene D. Nichols, Florida State University
A Computer Program for Determining the Euler's Path, Gary Word and Sidney H. Kung, Jacksonville U
Project Oasis: The Search for Extraterrestrial Intelligence, Gerhard Ritter, University of Florida.

The Association For Women in Mathematics held a Panel Discussion on *Women Mathematicians, Their Work and Their Special Problems.* Members of the Panel were Bettye Ann Case, Tallahassee Community College; Betty Lichtenberg, University of South Florida; moderator Elizabeth Magarian, Stetson University.

The Florida Section has divided the State into seven areas and in the fall of 1979 Mini-Sectional Meetings were held in five of the seven areas. These local meetings were organized so that everyone in the State could attend a meeting without extensive traveling. Teachers from Junior High Schools, High Schools and Colleges were invited to attend. The programs dealt primarily with teaching and articulation. Reports of these meetings are available from the Secretary.

The luncheon-business meeting was held Saturday, March 8, 1980. Chairman Frederick Hoffman presided at the meeting. Committee reports were presented. Professor Robert Gilmer of Florida State University was elected chairman-elect; Professors Raymond Roth of Rollins College and Ernest Ross of St. Petersburg Junior College were elected as Vice-Chairmen.

FRANK L. CLEAVER, *Secretary*

SPRING MEETING OF THE N.J. SECTION

The spring meeting of the N.J. Section of the MAA was held on Saturday, March 15, at Cherry Hill, New Jersey in conjunction with NCTM. About 40 members of the section were present for the meeting.

The meeting was called to order at 9:00 A.M. by the section chairperson, Jean Lane. Our first speaker was John M. Cozzolino of the Wharton School of the University of Pennsylvania, whose talk was entitled *The Mathematics of Risk Analysis*. This was followed by a brief business meeting at which revisions to the section's by-laws were adopted. Our second and final speaker was Kenneth C. Wolff of Montclair State College who discussed *Weaving Applications into the Content of Traditional Courses*. He pointed out the advantages of UMAP modules and the NCCCD modules (put out by SUNY-Stoney Brook) to instructors who want to inject applications into their Math courses.

The meeting was adjourned at 12 Noon.

JAMES MAGLIANO, *Secretary*

OKLAHOMA-ARKANSAS MAA SECTION MEETING

This meeting was held at Westark Community College on March 28-29, in Fort Smith, Arkansas. The attendance was approximately one hundred and twenty five with forty papers given by faculty and students in the section. The highlights of the meeting were the N. A. Court Lecture given by Dr. Walter Rudin, University of Wisconsin-Madison, and the Invited Address given by Dr. R. D. Anderson, Louisiana State University. A unique feature of this meeting was that there were fifteen undergraduate student papers presented in three undergraduate student sessions. Each session was presided over by an undergraduate student.

The following talks completed the program:

A Note on Powers of Sums of Powers of Consecutive Positive Integers, Thomas F. Peter, University of Arkansas at Little Rock

A Characterization of Cut Techniques in Integer Programming, Richard Greenhaw, Oklahoma Christian College

Does the Quadratic Equation Have Greek Roots? David Row, University of Oklahoma

A Peculiar Trigonometric Identity, $\tan\frac{\alpha+\beta}{2}=\frac{\cos\alpha-\cos\beta}{\sin\beta-\sin\alpha}$, Tetsundo Sekiguchi, University of Arkansas at Fayetteville

The Property of Being a Baire Space is Semi-Topological, T. R. Hamlett, Arkansas Tech University

SQ-Closed Spaces, Travis Thompson, University of Arkansas at Pine Bluff

Rarely Continuous Functions II, Paul E. Long, University of Arkansas at Fayetteville

Comparing Infinities, Milan Aiken, University of Oklahoma

A Missile's Destructive Effect on Adjacent Targets, Dorothy Haynes, Oklahoma State University

A Characteristic of Continuity for Real Valued Functions, Martha Ellen Waggoner, Arkansas Tech University

Determining the Hours of Daylight at a Fixed Latitude, Nancy Stevens, Oklahoma State University

A Tool for Use in Livestock Management, David Race, Arkansas Tech University

Properties Preserved by Continuous Functions, Anna Belk, Arkansas Tech University

The Linear Algebra of Post-Tensioned Tendons in a Concrete Slab Roof, Page Maxon, Oklahoma State U

Measures of Convexity of $\mathbb{R}$, John Merrill, Hendrix College

Applications of the Runge-Kutta Numerical Method in the Development of a Kinetic Model of Air Pollution, Kyle R. Anderson, Oklahoma State University

Singular Functions, Sandra Cousins, Hendrix College

The Winding Number of Curves on Surfaces, Morris Marx, University of Oklahoma

A Generalized Convolution for Three Functions, J. R. Choike and I. I. Kotlarski, Oklahoma State U

On Positive Solutions of Second Order Differential Equations, Stan Eliason, University of Oklahoma

On the Norm Sequences of Operators, Naoki Kimura, University of Arkansas at Fayetteville

The MAA Placement Testing Program, Bernard L. Madison, University of Arkansas at Fayetteville

Simple Iteration in Calculus, William Murray, Hendrix College

Use of Real World Problems in Teaching General Mathematics, Jeanne Agnew, Oklahoma State U

Greek Influence on Mathematics Education, Verbal Snook, Oral Roberts University

Incidence Rings and the Isomorphism Problem, Joel K. Haack, Oklahoma State University

Linear and Geometric Algebra Over Rings With Many Units, Bernard R. McDonald, University of Oklahoma

A Characterization of Groups, M. K. Sen, University of Arkansas at Fayetteville

Determination of Blind Flight Paths Which Occur During Helicopter Instrument Approaches, James Yates, Central State University

Mathematical Models and Computation in the Law, Ray Keown, University of Arkansas at Fayetteville

Solutions of Sparsely Perturbed System of Linear Algebraic Equations, David Hsu, University of Arkansas at Little Rock

What is the Four-Five Hole Problem? Philip C. Almes, Phillips University

Approximate Solutions of Two-Point Boundary Value Problems by Numerical Inversion of Laplace Transform, Alan M. Johnson, University of Arkansas at Little Rock

Compact Hankel Operators, Jeffrey Butz, University of Oklahoma

Some Oscillatory Properties of Third and Fourth Linear Differential Equations, Abdelali Benharbit, Oklahoma State University

The Dirac Delta Function as a Generalized Function, Kevin R. Wadleigh, Oral Roberts University

Spectral Theory of Linear Operators on Complex Banach Spaces, Martin L. Traviolia, Oral Roberts U

Mathematics in 19th Century Russia, David Sutherland, Hendrix College

Women in Mathematics, Cassandra Scrimshire, Hendrix College.

ROCKY MOUNTAIN SECTION MEETING

The 63rd Annual Meeting of the Rocky Mountain Section of MAA was held March 28-29, 1980, on the campus of the University of Colorado in Boulder with 122 members of MAA in attendance. Professor Dorothy Bernstein, President of the Mathematical Association of America, gave the annual banquet address, *A Differential Equation of Literary Criticism.*

The program included two panel discussions:

Transferability of College Credits, moderated by Professor Allan Skillman of Casper College and including as panelists, Professor Ardel Boes of Colorado School of Mines; Professor Corrine Brase of Arapahoe Community College; Professor Burnett Meyer of the University of Colorado at Boulder; and Professor Duane Porter of the University of Wyoming.

Nontraditional Uses of Computers, moderated by Professor Laurel Rodgers and including as panelists Professor Dorothy Bernstein, Brown University; Dr. Bertram Herzog, University of Colorado at Boulder; and Professor Austin Brown, Jr., Colorado School of Mines.

There were fourteen 20-minute papers contributed.

On the Teaching of Projective Geometry, Professor Arne Magnus, CSU
Second-chance Examinations: an Evaluation, Professor David Ballew, SDSM&T
Computer Graphics in the Calculus Classroom, Professor Austin B. Brown, Jr. and Professor Robert S. Fisk, CSM
Some Parallels Between Digital Computers and the Human Mind, Dr. Ira Becker, Ball Brothers
Optimal Channel Assignment and Chromatic Graph Theory, William K. Hale, National Telecommunications and Information Administration, U.S. Department of Commerce
A Conjecture on Consecutive Composites, II, Professor C. Albert Grimm, SDSM&T
Uses of Woodbury's Formula, Professor Dale Rognlie, SDSM&T
An Effective Interactive Introduction to Mathematics for Elementary Teachers (and Others), Professor Earl Hasz, MSC
Cooperative Education in Mathematics, Professor W. E. Brumley, CSU
The Hungarian Magic Cube — A Search for a Solution, Professor Les Shader, UW
Modules in the Mathematics Classroom, Professor Joan R. Hundhausen, CSM
Which is the Most Beautiful Polygon? Professor John H. Hodges, UCB
Four Mutually Tangent Circles — the Second Circle Varying, Professor Hung C. Li, USC
A Locus Determined by the Three Real Roots of the Reduced Cubic Equation, Professor F. Max Stein, CSU

Professor Laurel Rodgers of the University of Colorado at Colorado Springs, Chairman of the Section, presided at the Annual Business Meeting. The new officers for 1980-81 are

Chairperson: Professor *William Ramaley,* Ft. Lewis College
Chairperson-Elect: Professor *John Gill,* University of Southern Colorado
Vice-Chairperson: Professor *Aubrey Owen,* Community College of Denver
Program Chairperson: Professor *Stephen Shiffman,* Colorado College

Professor *Rebekka Struik* of the University of Colorado reported on the Section's High School Lectureship Program, and Professor *Bernstein* discussed such programs at the national level. Professor *Bernstein* also encouraged MAA membership promotion for those students who plan on college careers. Professor *Duane Porter*'s Governor's Report was read.

DAVID BALLEW, *Secretary*

MARCH MEETING OF THE WISCONSIN SECTION

The Spring Meeting of the Wisconsin Section of the MAA was held at the University of Wisconsin at Milwaukee, March 28-29, 1980, with 83 MAA members and 17 non-members registering. Chairman Norbert J. Kuenzi (US-Oshkosh) presided. The program had been drawn up by Chairman-Elect Anthony E. Barkauskas (UW-LaCrosse) and local arrangements were handled by Robert Hall (UW-Milwaukee).

The invited addresses were:

Modeling Within Mathematics, Peter Hilton (Battelle Research Institute)
Hadamard, Hotelling, Harwit: A New Application of Some Old Mathematics, Neal J. A. Sloane (Bell Telephone Laboratories). Unfortunately, Professor Sloane was too ill to attend and present his address.

The following additional talks were given:

Velocity Changes and Weak Mixing in Ergodic Flows, David Yurchak (UW-Milwaukee)
The Missionaries and Cannibals Learn Graph Theory, Timothy V. Fossum (UW-Parkside)
The Programmable Calculator and the Bored High School Student, Florence N. Greville
Developing Number Concepts in Young Children, Kathleen L. Briggs (UW-Milwaukee)
Some Results in Reducibility in Ordinary Differential Equations, M. M. Subramaniam (UW Center-West Bend)
Auto-homeomorphisms of a 2-manifold Induced by 'Twists', Norman Frisch (UW-Oshkosh)
Two Variable Power Series--A Topic for Undergraduates, Jonathan Kane (UW-Madison)
Mathematics in the Social and Biological Sciences: Applications from Project UMAP, Philip D. Straffin (Beloit College)
Mathematical Models of Energy Economics, William Holahan (UW-Milwaukee Economics Department)
Folds, Pleats, and Halos—A Slide Presentation, Walt Tape (UW-Eau Claire)
The Inverse Prediction Problem in Regression, Paul J. Campbell (Beloit College)
A Description of MATH TIPS as Used in a Precalculus Course, Frederic Tufte (UW-Platteville)
Torsion-Theoretic Generalization of Semisimple Modules, William Lau (UW-Milwaukee)
Individualizing the Intermediate Algebra Course—A Report, J. D. Wine (UW-LaCrosse)
The Logarithmic Spiral—A Slide Presentation, Eli Maor (UW-Eau Claire)
Conservative Graphs, David W. Bange (UW-LaCrosse)

A Genuine Application of Descartes' Rule of Signs, D. D. Freund (UW Center-Waukesha)
Counting Non-isomorphic Patterns, or There's More Than One Way to Hook a Rug, Rick Poss (St. Norbert College)
Public Key Cryptosystems and College Algebra, Harvey Fox (UW Center-Waukesha)

A special swap session was held for Apple microcomputer users.

Anthony E. Barkauskas (UW-LaCrosse), Chairman-Elect, succeeded to the office of Section Chairman (one-year term). *Karl Beres* (Ripon College) was elected Chairman-Elect (one-year term). *Paul J. Campbell* (Beloit College) continues as Secretary-Treasurer (second year of three-year term) and *Gary D. Klatt* (UW-Whitewater) as Section Governor (last year of three-year term).

PAUL J. CAMPBELL, *Secretary-Treasurer*

APRIL MEETING OF THE SOUTHEASTERN SECTION

The fifty-ninth annual meeting of the Southeastern Section was held on April 11-12, 1980 at Appalachian State University in Boone, North Carolina. A total of 238 persons attended the meeting, including 41 students and 197 members of the Association. The local arrangements were handled by Professor *Theresa Early.*

Three invited addresses were given: Professor *R. P. Boas* (Editor of *The American Mathematical Monthly)* of Northwestern University on *Serious Mathematics from Unpromising Material;* Professor *Bruce C. Berndt* of the University of Illinois on *Ramanujan's Notebooks;* and Professor *Billy F. Bryant* (Section Lecturer) of Vanderbilt University on *Isaac Newton: The Man and His Papers.*

Officers elected for 1980-81 are: Chairman, *Lida K. Barrett,* University of Tennessee at Knoxville; Chairman-elect, *James R. Garrett,* Lenoir Rhyne College; Vice-Chairman, *Cathrine C. Aust,* Clayton Junior College; Section Lecturer, *Daniel D. Warner,* Clemson University, all for a one year term.

At the business meeting, it was announced that the winner of the Section prize for the best performance on the Putnam examination was *Edward J. Rak* of the University of North Carolina at Chapel Hill.

The Section voted to hold its 1981 meeting at Emory University, Atlanta, Georgia.

The following papers were presented:

Braids: A hands-on experience in abstract algebra, L. R. King, Davidson College
Alternative Rings with $x^{n+1}p(x) = x^n$, Tae-il Suh, East Tennessee State University
A special class of Bell polynomial, F. T. Howard, Wake Forest University
Super associativity, Mary F. Neff, Emory University
Some topological implications of the Gauss-Bonnet Theorem, Thomas C. Spangler, Davidson College
Computation of Mersenne primes, Joy Coles, Emory University
The design of an instructional programming language, Margaret Francel, Emory University
Simulating a large-scale computer without going fishing, James L. Benjamin, Emory University
Elementary group theory and round robin tournaments, Benjamin G. Klein, Davidson College
Optical Illusions: An application for third-semester calculus, David A. Smith, Duke University
An alternate method for teaching remedial algebra, Sandra C. McLaurin, UNC at Wilmington
MAA placement testing program, Thomas A. Carnevale, Clemson University (Visiting)
Solving certain nonhomogeneous equations, Garland Jackson, Auburn University, at Montgomery
Card dealing and Pascal's triangle, John Neff, Georgia Institute of Technology
On powers of a matrix, S. B. Khleif, Tennessee Technological University
On Pascal's pyramid and its applications, S. C. Saxena, Coastal Carolina College, USC
An educational assistance program for educationally disadvantaged students, H. Tom Mathews, The University of Tennessee-Knoxville
A modified mastery format used in a college algebra course for students in the educational assistance program, Margaret H. Myers, The University of Tennessee-Knoxville
Increasing the accessibility of college mathematics to visually handicapped students, Virginia V. Jory, Georgia Institute of Technology
A note on integration by parts, W. R. Spickerman, J. W. Daniels, East Carolina University
Student evaluation of instruction in two trigonometry classes, W. R. Spickerman, Robert N. Joyner, East Carolina University
Making plane domains into domains of attraction, Kenneth R. Gurganus, UNC-Wilmington
Suprema of Fourier partial sums, Lyndell M. Kerley, East Tennessee State University
Cauchy integral of tempered distributions, Richard D. Carmichael, Wake Forest University
Term by term dyadic differentiation, Charles H. Powell, The University of Tennessee-Knoxville
Spherical harmonics: The old and new, Kenneth I. Gross, UNC-Chapel Hill
A graphical study of 'The Robber Bridegroom' by Eudora Welty, John S. Bayne, Dale C. Hill, Clemson U
How to be free (if you're a semigroup), Lee Ann Taylor, Emory University
A decision theory model for water well drilling, Rhonda Aull, Clemson University
Enumeration of 3-uniform hypergraphs, Joel C. Fowler, Emory University
Predicting storm erosion of beaches, Richard H. Burkhart, UNC-Wilmington
Permutable primes, Jean Bevis, Jan List Boal, Georgia State University
Finite rank modifications and generalized inverses of Fredholm operators, Sylvia T. Bozeman, Spelman College; Luis Kramarz, Emory University
Autocorrelation vs. power spectral density, Leland L. Long, Tennessee Technological University
Some residual intersection properties of rational normal curves, C. Ray Wylie, Furnam University
Some symmetric octahedra, Donald Aplin, Winthrop College
Real valued functions defined on a topological space X, Ernest P. Lane, Appalachian State University
Preservation of the cell separation by refinable maps, Paul R. Patten, Brewton-Parker College
Bernet's formula for a recursive sequence of order k, W. R. Spickerman, East Carolina University.

INEZ C. GENTRY, *Secretary*

THE MATHEMATICAL ASSOCIATION OF AMERICA
THE SIXTY-THIRD ANNUAL MEETING OF THE ASSOCIATION

The Sixty-third Annual Meeting of the Association was held at the Convention Center in San Antonio, Texas, from Saturday to Monday, January 5-7, 1980, in conjunction with Meetings of the American Mathematical Society, the Society for Women in Mathematics, the Mathematicians Action Group, and the National Council of Teachers of Mathematics. There were 2758 registrants including 1581 members of the Association. Sessions of the MAA were held on Saturday and Sunday afternoons at 1:00 P.M. and on Monday morning at 8:30 A.M.

FIRST SESSION OF THE ASSOCIATION

Joint Session with SIAM:

Panel Discussion: " A Report on Upper Level Mathematical Modeling Courses And Seminars."

A panel discussion with Professor T. Gilmer Proctor, Clemson University, Moderator: Professor E. Earl Burch, Clemson University, Professor Donald A. Drew, Rensselaer Polytechnical Institute, Professor Robert E. O'Malley, Jr., University of Arizona.

Panel Discussion: " Mexican-Americans and Mathematics" or "Where Have All of the Chicanos Gone?"

A panel discussion with Professor Manuel P. Berriozabal, the University of Texas at San Antonio, Moderator; Professor Richard J. Griego, Professor David A. Sanchez, University of New Mexico, Professor Richard A. Tapia, Rice University, Professor William Y. Velez, University of Arizona.

According to a 1974 National Science Foundation Report on Women and Minorities in Science and Engineering, the total number of mathematicians in the United States was 45,200, less than 100 of which were Mexican-American. Thus, even though Mexican-Americans comprise more than 3% of the population nationally, less than 1/4 of 1% of the professional mathematicians are Mexican-American. Similar underrepresentations exist in other scientific and engineering areas. Less than 1/2 of 1% (15 out of 3274) of the Mathematics doctorates between 1973 and 1976 were awarded to Mexican-Americans. Some causes of this professional underrepresentation can be found in the historical background of the Mexican-American in the United States, the inferior educational preparation of the Mexican-American particularly at the precollege level, and the poor track record of affirmative action programs.

"Mathematics and Artificial Intelligence" by Professor Douglas Hofstadter, Indiana University.

Artificial intelligence research in mathematics has traditionally depended heavily on the automation of deduction. It was stressed that deduction is only a part of mathematics and that intuition and imagery are as important, if not more so. Intelligent mathematics certainly requires imagery. Some examples of imagery were presented: (1) The "Magic Cube" which affords a graphic understanding of a large finite group; (2) A proof via imagery of van der Waerden's Theorem on arithmetic sequences in partitions of the integers; (3) the imagery at the symbol-level is the proof of Euler's identity (used in the Prime Number Theorem). These examples were proffered as a challenge for workers in Artificial Intelligence to consider.

RETIRING PRESIDENTIAL ADDRESS: "Drama in Mathematics: A Demonstration with Partitions" by Professor Henry L. Alder, University of California, Davis.

The teaching of mathematics can be greatly enhanced if the beauty, excitement, surprises, in short, the drama in mathematics, is emphasized. The speaker demonstrated this by using as an example the fascinating history of the discovery of the existence and nonexistence of certain types of partition identities over the last 80 years, comparing, in particular, their proofs by combinatorial methods and by use of generating functions. He included some very recent surprising developments and mentioned some open problems.

SECOND SESSION OF THE ASSOCIATION

"Growing Crystals Mathematically" by Dr. Lynn O. Wilson, Bell Telephone Laboratories.

It is often desirable to grow semiconductor crystals whose dopant content is controlled and uniform. Integrated circuit chips, for example are fabricated from such crystals. This talk described how an industrial mathematician helped provide crystal growers with some insight into the dopant incorporation process. In addition to describing the crystal growth problem itself, the talk explained the process of solving such a problem and exhibited some of the rewards and frustrations inherent in this type of work.

"The Microcomputer as Teacher" by Professor Everett Hafner, Hampshire College.

A demonstration and discussion of examples in support of the notion that a small computer possesses the gift for teaching things that many of us find difficult or impossible to learn elsewhere. It exhibits rates of convergence; it produces models of physical systems; it stimulates us to form and to examine mathematical conjectures; it produces vivid graphs of the behavior of functions; it makes sense of long lists, such as those derived from measurements in physics; and it urges us to explore the heart of the computer itself. An Apple-II Microcomputer, driving an array of video monitors, was used to display programs and graphs.

"Pegboard Solitaire" by Professor Hugh L. Montgomery, University of Michigan.

Pegboard Solitaire is played by jumping one peg over another, and removing the peg jumped over. In the classical problem one starts with a board in the shape of a Greek cross, and the object is to leave only one peg on the board. After a review of empirical observations, the talk proceeded to a mathematical analysis of the problem. One aspect of the puzzle is completely described in terms of a group of order 16, while another aspect has been settled very elegantly by J. H. Conway. Taken together, these observations make the original problems very easy, and bring much more difficult problems within research.

Session on Mathematics for Two-Year College Faculty: Professor Roland H. Lamberson, Des Moines Area Community Colleges, Presider.

"Mathematical Proofs, Goofs, and Spoofs" by Professor Warren Page, New York City Community College.

"How Many Mathematicians...?" by Professor Larry Curnutt, Bellevue Community College, Bellevue, Washington.

"How To Get (At Least) a Fair Share of the Cake" by Professor Kenneth Rebman, California State University at Hayward.

Business Meeting of the Association: Announcement of the recipients of the Nineteenth Award for Distinguished Service to Mathematics and the Chauvenet Prize for 1980. (The recipients of these awards are announced in the February, 1980, issue of the MONTHLY.)

THIRD SESSION OF THE ASSOCIATION

"Arms Races, Prison Guard Schedules, and Mercator's World Map: Applications from Project UMAP" by Dr. Ross L. Finney, Educational Development Center.

To help educational institutions meet the increasing demands for education in professional uses of mathematics, the National Science Foundation-supported Undergraduate Mathematics Applications Project (UMAP) is producing self-contained, lesson-length, instructional units from which undergraduate students can learn professional applications of mathematics in such fields as biomedical sciences, economics, American politics, harvesting, international relations, numerical methods, computer science, earth science, and navigation. One hundred ten units have been published, one hundred fifty more are in preparation, and many additional manuscripts are expected. UMAP also produces expository monographs, of paperback book length, that treat mathematics and its applications in greater depth than do the lesson-length modules.

"Seeing and Believing: The Geometry of Binocular Space Perception is Hyperbolic" by Professor Albert A. Blank, Carnegie-Mellon University.

The binocular visual is not, as once thought, a simple range finding system for localizing objects in three-dimensional space. The relation between perception and physical stimulus is multivalent. Moreover, the intrinsic metric of perception is not related to the euclidean metric of the physical frame in any simple way. The curvature of the intrinsic metric is empirically found to be negative and nearly constant. Thus the intrinsic geometry of binocular vision is essentially Lobachevskian. Remarkable unexpected phenomena, found in the course of experimental and analytic investigation of the metric theory, were discussed.

"Topological Games" by Professor David Gale, University of California at Berkeley.

The title refers to certain games played on graphs in which players alternately choose either varieties or edges with the object of forming certain "winning" paths. The most familiar example is the game of Hex. The interest in these games arise from their relation to several areas of "serious" (as opposed to recreational) mathematics, notably elementary algebraic topology, combinatorics (matroid theory), and most recently the theory of computational complexity. An informal semi-historical survey of the subject was given in which these connnections were illustrated.

"Big Matrices and Little Kernels: The Examples that Come Before the Theorems" by Professor Paul R. Halmos, Indiana University.

A leisurely non-technical stroll through a gallery of infinite matrices and integral kernels. Purpose: to exhibit the concrete examples that the abstract theory of integral operators comes from.

SPECIAL SESSIONS OF THE ASSOCIATION

CONCERT AND LECTURE: On Saturday evening there was a most enjoyable program consisting of the popular lecture, "Mathematical Problems in ESP Research," by Professor Persi W. Diaconis, Stanford University, and a concert for violoncello and piano featuring Professors Louis Rowen, Bar Ilan University, and Leonard Gillman, University of Texas. The concert program follows:

Sonata No. 3 in G. minor (BWV 1029)	BACH
Sonata in A major	FRANCK
Intermission	
Sonata No.4 in C major (op. 102, No. 1)	BEETHOVEN
Sonata in G. minor (op. 19)	RACHMANINOFF

MAA Mini-Course: "Precalculus and Calculus with Hand-Held Calculators"

A mini-course on hand-held calculators in the elementary calculus sequence was held on Friday and Sunday evenings. Participants were provided with TI-58 calculators for use during the workshop. The Friday session was designed for those participants without previous experience with programmable calculators. The Sunday session was open to those who had attended the earlier session as well as to persons experienced with programmable calculators. The mini-course was conducted by Professors Harry P. Allen and John O. Riedl, Jr. of Ohio State University.

MAA Swap Session: "Section-Sponsored Short Courses"

Professors John D. Faires, Youngstown State University, Sanford L. Segal, University of Rochester, and Donald B. Small, Colby College conducted this session on Friday evening.

MAA Swap Session: "Student Participation at Section Meetings"

Professors Milton D. Cox, Miami University, Robert Eslinger, Hendrix College, and Fredric J. Zerla, University of Southern Florida conducted this session on Friday evening.

"Africa and Mathematics" by Professor Donald M. Hill, Florida A&M University

This Sunday evening talk emphasized the need for cooperation at all levels between individuals, institutions, and nations. Special emphasis was given to the Third World. Through slides and artifacts a look at Zairian society provided insight into the opportunities and problems of teaching mathematics there. The general educational system was described and the mathematics curriculum was discussed.

CUPM Panel Discussion: "General Mathematical Sciences Program" by Professor Alan C. Tucker, SUNY at Stony Brook, Moderator.

The CUPM Panel on a General Mathematical Sciences Program (MSP) is developing recommendations for a model Mathematical Sciences major. A preliminary set of recommendations and their underlying philosophy will be discussed with the audience. The recommendations cover the calculus sequence, upper-level core mathematics, computer science, operations research/modeling, and statistics.

Film Showings: Film showings were held on Sunday from 7:00-9:20 P.M.:

7:00-7:25	Quaternions: A Herald of Modern Algebra--a B.B.C. broadcast as part of the Open University's History of Mathematics Course
7:30-7:47	Maurits Escher: Painter of Fantasies
7:50-7:59	Powers of Ten
8:00-8:07	Circle Circus
8:10-8:22	The Hypercube: Projections and Slicings
8:22-9:20	Challenge in the Classroom (R. L. Moore)

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors met on Friday, January 4, 1980, at 9:00 A.M. in the Corte Real Room of the Hilton Palacio del Rio Hotel with 44 members present. Among the items of business transacted were the following:

The Board elected Professor G. Baley Price, University of Kansas, as a member of the Finance Committee for the four-year term 1980-83.

Professor Vladimir Drobot, University of Santa Clara, was elected Associate Editor of the MONTHLY in charge of the Problems Section.

The Board elected Professor Susan J. Devlin, Bell Telephone Laboratories, and Richard K. Guy, University of Calgary, as Governors-at-Large for the three-year term, 1980-82.

The Board voted to continue MAA sponsorship of JPCM for the five-year term extending through the conclusion of 1984.

The Board elected Professor George E. Andrews, Pennsylvania State University, as the Hedrick Lecturer for 1980. The Lectures will be delivered in August at the meeting in Ann Arbor, Michigan.

The Board voted to accept the following grants:

1. A grant of $1000 from the Polaroid Foundation for support of "Women and Mathematics" (WAM).
2. A grant of $500 from the John Hancock Life Insurance Corporation for support of WAM.

Dr. Alfred B. Willcox, MAA Executive Director, reported that the Association's membership is presently 9 more than one year earlier.

It was reported that the net cost of the new headquarters during 1976 was $16,000. If MAA had continued to rent its headquarters space, it was said that the rental would have been approximately $35,000 to $40,000.

BUSINESS MEETING OF THE ASSOCIATION

President Dorothy L. Bernstein presented the Award for Distinguished Service to Mathematics to Professor Henry L. Alder, University of California, Davis. She called upon the Secretary to read the citation he had prepared to accompany this Award. This citation appears in the February, 1980, issue of the MONTHLY. Professor Alder was present to accept the Award and he thanked all of the members of the Association for having been given the opportunity to serve them as MAA Secretary and as MAA President. A bound copy of the citation was presented to Mrs. Alder in honor of her fine support for the many activities of Professor Alder.

President Bernstein next presented the Chauvenet Prize for 1980 to Professor Heinz Bauer of the University of Erlangen-Nurnberg. This Prize was presented for the paper, "Approximation and Abstract Boundaries," which appeared in this MONTHLY, 85(1978), 632-47. Professor Bauer was present to accept the Prize and said that he was very grateful. Biographical material and a description of the paper for which the award was made appears in the February, 1980, issue of the MONTHLY.

The Secretary presented his report on the actions taken by the Board of Governors at their meeting on Friday. It was later announced that Professor Lynn A. Steen, St. Olaf College had been elected First Vice-President of the Association for the two-year term 1980-81.

David P. Roselle, Secretary

ACADEMIC MEMBERS ELECTED BY THE ASSOCIATION

At its meeting on January 4, 1980, the Board of Governors elected as institutional members:

The Royal Thai Army Academy
Saint Augustine's College

David P. Roselle, Secretary

CALENDAR OF FUTURE MEETINGS

Sixtieth Summer Meeting, University of Michigan, Ann Arbor, August 18–20, 1980.

Sixty-fourth Annual Meeting, San Francisco, California, January 9–11, 1981.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, last weekend in April or first weekend in May. Deadline for papers six weeks before meeting.

EASTERN PENNSYLVANIA AND DELAWARE, Saturday before Thanksgiving.

FLORIDA, early March. Deadline for paper titles two weeks before meeting.

ILLINOIS, first Friday/Saturday in May.

INDIANA

INTERMOUNTAIN

IOWA, third weekend in April. Deadline for papers February 1.

KANSAS, March or April. Deadline for papers January 1.

KENTUCKY, early April. Deadline for papers six weeks before meeting.

LOUISIANA–MISSISSIPPI, Mississippi State University, Mississippi State, February 13–14, 1981.

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Goucher College, Towson, Maryland, November 14–15, 1980.

METROPOLITAN NEW YORK, spring. Deadline for papers two weeks before meeting.

MICHIGAN, first Friday and Saturday in May. Deadline for papers six weeks before meeting.

MISSOURI, late March/early April. Deadline for papers January 31.

NEBRASKA, April.

NEW JERSEY, Union College, Cranford, October 25, 1980.

NORTH CENTRAL, end of October and April. Deadline for papers October 1 and April 1.

NORTHEASTERN, Saturday after Thanksgiving, and third week in June in odd-numbered years.

NORTHERN CALIFORNIA, first or second Saturday in February.

OHIO, John Carroll University, University Heights, October 17–18, 1980.

OKLAHOMA–ARKANSAS, (approx.) Friday and Saturday of first weekend in April. Deadline for papers three weeks before meeting.

PACIFIC NORTHWEST, Central Washington University, Ellensburg, Washington, June 20–21, 1980.

ROCKY MOUNTAIN, last weekend in April or first in May. Deadline for papers eight weeks before meeting.

SEAWAY, first Saturday in November and Saturday in late April. Deadline for papers six weeks before meeting.

SOUTHEASTERN

SOUTHERN CALIFORNIA, first or second Saturday in March.

SOUTHWESTERN, usually in April. Deadline for papers two weeks before meeting.

TEXAS, Friday and Saturday in early April. Deadline for papers March 1.

WISCONSIN, Friday and Saturday between mid-April and first week in May. Deadline for papers six weeks before meeting.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Toronto, Canada, January 3–8, 1981.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES, Sheraton National Hotel, Arlington, Virginia, October 9–13, 1980.

AMERICAN MATHEMATICAL SOCIETY, University of Michigan, Ann Arbor, August 19–22, 1980.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Amherst, Massachusetts, June 23–26, 1980.

ASSOCIATION FOR COMPUTING MACHINERY, Nashville, Tennessee, October 27–29, 1980.

ASSOCIATION FOR SYMBOLIC LOGIC, San Francisco, California, January 9–10, 1981.

ASSOCIATION FOR WOMEN IN MATHEMATICS, University of Michigan, Ann Arbor, August 18–22, 1980.

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHEMATIQUES, Montreal, June 3–5, 1980.

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS, University of Michigan, Ann Arbor, August 18–21, 1980.

MU ALPHA THETA, Georgia Tech., Atlanta, August 3–6, 1980.

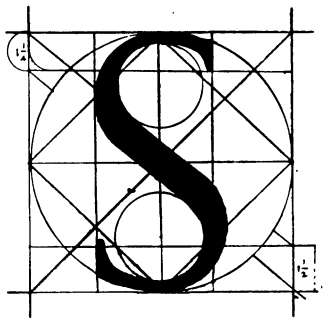
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, St. Louis, Missouri, April 22–25, 1981.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Four Seasons Sheraton, Toronto, Canada, May 4–6, 1981.

PI MU EPSILON, University of Michigan, Ann Arbor, August 18–20, 1980.

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS



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Donald E. Knuth

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AUGUST-SEPT

THE AMERICAN MATHEMATICAL MONTHLY

Volume 87, Number 7

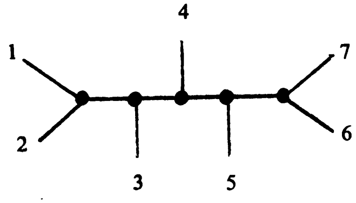
The Heart of Mathematics
by P. R. Halmos

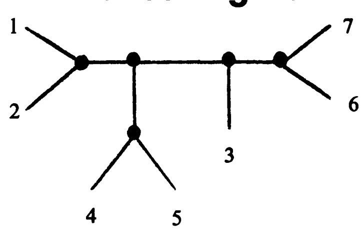
Brouwer's Fixed Point Theorem Again

History of Optimization

Polynomials, Logarithmic Means, Linear Codes, Holomorphic Functions

Comparing Trees, Covering Reals, Measuring Lemniscates





Detailed contents on cover 4

1980

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THE MATHEMATICAL ASSOCIATION OF AMERICA

VOLUME 87 NUMBER 7

(ISSN 0002-9890)

NOTICE TO AUTHORS

Research papers per se are unsuitable; see statement of policy (Vol. 85, p. 1).

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Estimated median time from submission to publication: Main Articles 15 months, Progress Reports 6 months, Math. Notes 24 months, Research Problems 6 months, Classroom Notes 14 months, Math. Education 10 months.

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THE HEART OF MATHEMATICS

P. R. HALMOS

Introduction. What does mathematics *really* consist of? Axioms (such as the parallel postulate)? Theorems (such as the fundamental theorem of algebra)? Proofs (such as Gödel's proof of undecidability)? Concepts (such as sets and classes)? Definitions (such as the Menger definition of dimension)? Theories (such as category theory)? Formulas (such as Cauchy's integral formula)? Methods (such as the method of successive approximations)?

Mathematics could surely not exist without these ingredients; they are all essential. It is nevertheless a tenable point of view that none of them is at the heart of the subject, that the mathematician's main reason for existence is to solve problems, and that, therefore, what mathematics *really* consists of is problems and solutions.

"Theorem" is a respected word in the vocabulary of most mathematicians, but "problem" is not always so. "Problems," as the professionals sometimes use the word, are lowly exercises that are assigned to students who will later learn how to prove theorems. These emotional overtones are, however, not always the right ones.

The commutativity of addition for natural numbers and the solvability of polynomial equations over the complex field are both theorems, but one of them is regarded as trivial (near the basic definitions, easy to understand, easy to prove), and the other as deep (the statement is not obvious, the proof comes via seemingly distant concepts, the result has many surprising applications). To find an unbeatable strategy for tic-tac-toe and to locate all the zeroes of the Riemann zeta function are both problems, but one of them is trivial (anybody who can understand the definitions can find the answer quickly, with almost no intellectual effort and no feeling of accomplishment, and the answer has no consequences of interest), and the other is deep (no one has found the answer although many have sought it, the known partial solutions require great effort and provide great insight, and an affirmative answer would imply many non-trivial corollaries). Moral: theorems can be trivial and problems can be profound. Those who believe that the heart of mathematics consists of problems are not necessarily wrong.

Problem Books. If you wanted to make a contribution to mathematics by writing an article or a book on mathematical problems, how should you go about it? Should the problems be elementary (pre-calculus), should they be at the level of undergraduates or graduate students, or should they be research problems to which no one knows the answer? If the solutions are known, should your work contain them or not? Should the problems be arranged in some systematic order (in which case the very location of the problem is some hint to its solution), or should they be arranged in some "random" way? What should you expect the reader to get from your work: fun, techniques, or facts (or some of each)?

All possible answers to these questions have already been given. Mathematical problems have quite an extensive literature, which is still growing and flowering. A visit to the part of the stacks labeled QA43 (Library of Congress classification) can be an exciting and memorable revelation, and there are rich sources of problems scattered through other parts of the stacks too. What follows is a quick review of some not quite randomly selected but probably typical problem collections that even a casual library search could uncover.

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His mathematical interests are in measure and ergodic theory, algebraic logic, and operators on Hilbert space, with excursions to probability, statistics, topological groups, and Boolean algebras.—*Editors*

Hilbert's Problems. The most risky and possibly least rewarding kind of problem collection to offer to the mathematical public is the one that consists of research problems. Your problems could become solved in a few weeks, or months, or years, and your work would, therefore, be out of date much more quickly than most mathematical exposition. If you are not of the stature of Hilbert, you can never be sure that your problems won't turn out to be trivial, or impossible, or, perhaps worse yet, just orthogonal to the truth that we all seek—wrongly phrased, leading nowhere, and having no lasting value.

A list of research problems that has had a great effect on the mathematical research of the twentieth century was offered by Hilbert in the last year of the nineteenth century at the International Congress of Mathematicians in Paris **[3]**. The first of Hilbert's 23 problems is the continuum hypothesis: is every uncountable subset of the set $\mathbb{R}$ of real numbers in one-to-one correspondence with $\mathbb{R}$? Even in 1900 the question was no longer new, and although great progress has been made since then and some think that the problem is solved, there are others who feel that the facts are far from fully known yet.

Hilbert's problems are of varying depths and touch many parts of mathematics. Some are geometric (if two tetrahedra have the same volume, can they always be partitioned into the same finite number of smaller tetrahedra so that corresponding pieces are congruent?—the answer is no), and some are number-theoretic (is $2^{\sqrt{2}}$ transcendental?—the answer is yes). Several of the problems are still unsolved. Much of the information accumulated up to 1974 was brought up to date and collected in one volume in 1976 **[5]**, but the mathematical community's curiosity did not stop there—a considerable number of both expository and substantive contributions has been made since then.

Pólya-Szegő. Perhaps the most famous and still richest problem book is that of Pólya and Szegő **[6]**, which first appeared in 1925 and was republished (in English translation) in 1972 and 1976. In its over half a century of vigorous life (so far) it has been the mainstay of uncountably many seminars, a standard reference book, and an almost inexhaustible source of examination questions that are both inspiring and doable. Its level stretches from high school to the frontiers of research. The first problem asks about the number of ways to make change for a dollar, the denominations of the available coins being 1, 5, 10, 25, and 50, of course; in the original edition the question was about Swiss francs, and the denominations were 1, 2, 5, 10, 20, and 50. From this innocent beginning the problems proceed, in gentle but challenging steps, to the Hadamard three circles theorem, Tchebychev polynomials, lattice points, determinants, and Eisenstein's theorem about power series with rational coefficients.

Dörrie. "The triumph of mathematics" is the original title (in German) of Dörrie's book **[1]**. This is a book that deserves to be much better known than it seems to be. It is eclectic, it is spread over 2000 years of history, and it ranges in difficulty from elementary arithmetic to material that is frequently the subject of graduate courses.

It contains, for instance, the following curiosity attributed to Newton (Arithmetica Universalis, 1707). If "a cows graze b fields bare in c days, a' cows graze b' fields bare in c' days, a'' cows graze b'' fields bare in c'' days, what relation exists between the nine magnitudes a to c''? It is assumed that all fields provide the same amount of grass, that the daily growth of the fields remains constant, and that all the cows eat the same amount each day." Answer:

$$\det\begin{pmatrix} b & bc & ac \\ b' & b'c' & a'c' \\ b'' & b''c'' & a''c'' \end{pmatrix}=0.$$

This is Problem 3, out of a hundred.

The problems lean more toward geometry than anything else, but they include also Catalan's question about the number of ways of forming a product of n prescribed factors in a multiplicative system that is totally non-commutative and non-associative ("how many different

ways can a product of n different factors be calculated by pairs?," Problem 7), and the Fermat-Gauss impossibility theorem ("the sum of two cubic numbers cannot be a cubic number," Problem 21).

Two more examples should give a fair idea of the flavor of the collection as a whole: "every quadrilateral can be considered as a perspective image of a square" (Problem 72), and "at what point of the earth's surface does a perpendicularly suspended rod appear the longest?"(Problem 94). The style and the attitude are old-fashioned, but many of the problems are of the eternally interesting kind; this is an excellent book to browse in.

Steinhaus. My next mini-review is of a Polish contribution, Steinhaus [7], which (like Dörrie's) has exactly 100 problems, and they are genuinely elementary and good solid fun. When someone says "problem book" most people think of something like this one, and, indeed, it is an outstanding exemplar of the species. The problems are, however, not equally interesting or equally difficult. They illustrate, moreover, another aspect of problem solving: it is sometimes almost impossible to guess how difficult a problem is, or, for that matter, how interesting it is, till after the solution is known.

Consider three examples. (1) Does there exist a sequence $\{x_1, x_2, \ldots, x_{10}\}$ of ten numbers such that (a) x_1 is contained in the closed interval $[0,1]$, (b) x_1 and x_2 are contained in different halves of $[0,1]$, (c) each of x_1, x_2, and x_3 is contained in a different third of the interval, and so on up through $x_1, x_2, \ldots, x_{10}$? (2) If 3000 points in the plane are such that no three lie on a straight line, do there exist 1000 triangles (meaning interior and boundary) with these points as vertices such that no two of the triangles have any points in common? (3) Does there exist a disc in the plane (meaning interior and boundary of a circle) that contains exactly 71 lattice points (points both of whose coordinates are integers)?

Of course judgments of difficulty and interests are subjective, so all I can do is record my own evaluations. (1) is difficult and uninteresting, (2) is astonishingly easy and mildly interesting, and (3) is a little harder than it looks and even prima facie quite interesting. In defense of these opinions, I mention one criterion that I used: if the numbers $(10, 1000, 71)$ cannot be replaced by arbitrary positive integers, I am inclined to conclude that the corresponding problem is special enough to be dull. It turns out that the answer to (1) is yes, and Steinhaus proves it by exhibiting a solution (quite concretely: $x_1 = .95$, $x_2 = .05$, $x_3 = .34$, $x_4 = .74$, etc.). He proves (the same way) that the answer is yes for 14 instead of 10, and, by three pages of unpleasant looking calculation, that the answer is no for 75. He mentions that, in fact, the answer is yes for 17 and no for every integer greater than 17. I say that's dull. For (2) and (3) the answers are yes (for all n in place of 1000, or in place of 71).

Glazman-Ljubič. The book of Glazman and Ljubič [2] is an unusual one (I don't know of any others of its kind), and, despite some faults, it is a beautiful and exciting contribution to the problem literature. The book is, in effect, a new kind of textbook of (finite-dimensional) linear algebra and linear analysis. It begins with the definitions of (complex) vector spaces and the concepts of linear dependence and indepedence; the first problem in the book is to prove that a set consisting of just one vector x is linearly independent if and only if $x \neq 0$. The chapters follow one another in logical dependence, just as they do in textbooks of the conventional kind: Linear operators, Bilinear functionals, Normed spaces, etc.

The book is not expository prose, however; perhaps it could be called expository poetry. It gives definitions and related explanatory background material with some care. The main body of the book consists of problems; they are all formulated as assertions, and the problem is to prove them. The proofs are not in the book. There are references, but the reader is told that he will not need to consult them.

The really new idea in the book is its sharp focus: this is really a book on functional analysis, written for an audience who is initially not even assumed to know what a matrix is. The ingenious idea of the authors is to present to a beginning student the easy case, the transparent

case, the motivating case, the finite-dimensional case, the purely algebraic case of some of the deepest analytic facts that functional analysts have discovered. The subjects discussed include spectral theory, the Toeplitz-Hausdorff theorem, the Hahn-Banach theorem, partially ordered vector spaces, moment problems, dissipative operators, and many other such analytic sounding results. A beautiful course could be given from this book (I would love to give it), and a student brought up in such a course could become an infant prodigy functional analyst in no time.

(A regrettable feature of the book, at least in its English version, is the willfully unorthodox terminology. Example: the (canonical) projection from a vector space to a quotient space is called a "contraction", and what most people call a contraction is called a "compression". Fortunately the concept whose standard technical name is compression is not discussed.)

Klambauer. The last addition to the problem literature to be reviewed here is Klambauer's **[4]**. Its subject is real analysis, and, although it does have some elementary problems, its level is relatively advanced. It is an excellent and exciting book. It does have some faults, of course, including some misprints and some pointless repetitions, and the absence of an index is an exasperating feature that makes the book much harder to use than it ought to be. It is, however, a great source of stimulating questions, of well known and not so well known examples and counterexamples, and of standard and not so standard proofs. It should be on the bookshelf of every problem lover, of every teacher of analysis (from calculus on up), and, for that matter, of every serious student of the subject.

The table of contents reveals that the book is divided into four chapters: Arithmetic and combinatorics, Inequalities, Sequences and series, and Real functions. Here are some examples from each that should serve to illustrate the range of the work, perhaps to communicate its flavor, and, I hope, stimulate the appetite for more.

The combinatorics chapter asks for a proof of the "rule for casting out nines" (is that expression for testing the divisibility of an integer by 9 via the sum of its decimal digits too old-fashioned to be recognized?), it asks how many zeroes there are at the end of the decimal expansion of 1000!, and it asks for the coefficient of x^k in $(1+x+x^2+\cdots+x^{n-1})^2$. Along with such problems there are also unmotivated formulas that probably only their father could love, and there are a few curiosities (such as the problem that suggests the use of the well ordering principle to prove the irrationality of $\sqrt{2}$). A simple but striking oddity is this statement: if m and n are distinct positive integers, then

$$m^{n^m} \neq n^{m^n}.$$

The chapter on inequalities contains many of the famous ones (Hölder, Minkowski, Jensen), and many others that are analytically valuable but somewhat more specialized and therefore somewhat less famous. A curiosity the answer to which very few people are likely to guess is this one: for each positive integer n, which is bigger

$$\sqrt{n}^{\sqrt{n+1}} \quad \text{or} \quad \sqrt{n+1}^{\sqrt{n}}?$$

The chapter on sequences has the only detailed and complete discussion that I have ever seen of the fascinating (and non-trivial) problem about the convergence of the infinite process indicated by the symbol

$$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$$

Students might be interested to learn that the result is due to Euler; the reference given is to the article *De formulis exponentialibus replicatis*, Acta Academica Scientiarum Imperialis Petropolitanae, 1777. One more teaser: what is the closure of the set of all real numbers of the form $\sqrt{n}-\sqrt{m}$ (where n and m are positive integers)?

The chapter on real functions is rich too. It includes the transcendentality of e, some of the basic properties of the Cantor set, Lebesgue's example of a continuous but nowhere differentiable function, and F. Riesz's proof (via the "rising sun lemma") that every continuous monotone

function is differentiable almost everywhere. There is a discussion of that vestigial curiosity called Osgood's theorem, which is the Lebesgue bounded convergence theorem for continuous functions on a closed bounded interval. The Weierstrass polynomial approximation theorem is here (intelligently broken down into bite-size lemmas), and so is one of Gauss's proofs of the fundamental theorem of algebra. For a final example I mention a question that should be asked much more often that it probably is: is there an example of a series of functions, continuous on a closed bounded interval, that converges absolutely and uniformly, but for which the Weierstrass M-test fails?

Problem Courses. How can we, the teachers of today, use the problem literature? Our assigned task is to pass on the torch of mathematical knowledge to the technicians, engineers, scientists, humanists, teachers, and, not least, research mathematicians of tomorrow: do problems help?

Yes, they do. The major part of every meaningful life is the solution of problems; a considerable part of the professional life of technicians, engineers, scientists, etc., is the solution of mathematical problems. It is the duty of all teachers, and of teachers of mathematics in particular, to expose their students to problems much more than to facts. It is, perhaps, more satisfying to stride into a classroom and give a polished lecture on the Weierstrass M-test than to conduct a fumble-and-blunder session that *ends* in the question: "Is the boundedness assumption of the test necessary for its conclusion?" I maintain, however, that such a fumble session, intended to motivate the student to search for a counterexample, is infinitely more valuable.

I have taught courses whose entire content was problems solved by students (and then presented to the class). The number of theorems that the students in such a course were exposed to was approximately half the number that they could have been exposed to in a series of lectures. In a problem course, however, exposure means the acquiring of an intelligent questioning attitude and of some technique for plugging the leaks that proofs are likely to spring; in a lecture course, exposure sometimes means not much more than learning the name of a theorem, being intimidated by its complicated proof, and worrying about whether it would appear on the examination.

Covering Material. Many teachers are concerned about the amount of material they must cover in a course. One cynic suggested a formula: since, he said, students on the average remember only about 40% of what you tell them, the thing to do is to cram into each course 250% of what you hope will stick. Glib as that is, it probably would not work.

Problem courses do work. Students who have taken my problem courses were often complimented by their subsequent teachers. The compliments were on their alert attitude, on their ability to get to the heart of the matter quickly, and on their intelligently searching questions that showed that they understood what was happening in class. All this happened on more than one level, in calculus, in linear algebra, in set theory, and, of course, in graduate courses on measure theory and functional analysis.

Why must we cover everything that we hope students will ultimately learn? Even if (to stay with an example already mentioned) we think that the Weierstrass M-test is supremely important, and that every mathematics student must know that it exists and must understand how to apply it—even then a course on the pertinent branch of analysis might be better for omitting it. Suppose that there are 40 such important topics that a student *must* be exposed to in a term. Does it follow that we must give 40 complete lectures and hope that they will all sink in? Might it not be better to give 20 of the topics just a ten-minute mention (the name, the statement, and an indication of one of the directions in which it can be applied), and to treat the other 20 in depth, by student-solved problems, student-constructed counterexamples, and student-discovered applications? I firmly believe that the latter method teaches more and teaches better. Some of the material doesn't get *covered* but a lot of it gets *discovered* (a telling

old pun that deserves to be kept alive), and the method thereby opens doors whose very existence might never have been suspected behind a solidly built structure of settled facts. As for the Weierstrass M-test, or whatever else was given short shrift in class—well, books and journals do exist, and students have been known to read them in a pinch.

Problem Seminars. While a problem course might be devoted to a sharply focused subject, it also might not—it might just be devoted to fostering the questioning attitude and improving technique by discussing problems widely scattered over several fields. Such technique courses, sometimes called Problem Seminars, can exist at all levels (for beginners, for Ph.D. candidates, or for any intermediate group).

The best way to conduct a problem seminar is, of course, to present problems, but it is just as bad for an omniscient teacher to do all the asking in a problem seminar as it is for an omniscient teacher to do all the talking in a lecture course. I strongly recommend that students in a problem seminar be encouraged to discover problems on their own (at first perhaps by slightly modifying problems that they have learned from others), and that they should be given public praise (and grade credit) for such discoveries. Just as you should not tell your students all the answers, you should also not ask them all the questions. One of the hardest parts of problem solving is to ask the right question, and the only way to learn to do so is to practice. On the research level, especially, if I pose a definite thesis problem to a candidate, I am not doing my job of teaching him to do research. How will he find his next problem, when I am no longer supervising him?

There is no easy way to teach someone to ask good questions, just as there is no easy way to teach someone to swim or to play the cello, but that's no excuse to give up. You cannot swim for someone else; the best you can do is to supervise with sympathy and reinforce the right kind of fumble by approval. You can give advice that sometimes helps to make good questions out of bad ones, but there is no substitute for repeated trial and practice.

An obvious suggestion is: generalize; a slightly less obvious one is: specialize; a moderately sophisticated one is: look for a non-trivial specialization of a generalization. Another well-known piece of advice is due to Pólya: make it easier. (Pólya's dictum deserves to be propagated over and over again. In slightly greater detail it says: if you cannot solve a problem, then there is an easier problem that you cannot solve, and your first job is to find it!) The advice I am fondest of is: make it sharp. By that I mean: do not insist immediately on asking the natural question ("what is ...?", "when is ...?", "how much is ...?"), but focus first on an easy (but nontrivial) yes-or-no question ("is it ...?").

Epilogue. I do believe that problems are the heart of mathematics, and I hope that as teachers, in the classroom, in seminars, and in the books and articles we write, we will emphasize them more and more, and that we will train our students to be better problem-posers and problem-solvers than we are.

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A LESS STRANGE VERSION OF MILNOR'S PROOF OF BROUWER'S FIXED-POINT THEOREM

C. A. ROGERS

In a recent note [1], John Milnor gives a proof of the "hairy dog theorem" and deduces the Brouwer fixed-point theorem as a consequence. Milnor describes his proof as strange but elementary. In this note we give a less strange and more direct proof of the following retraction theorem, which is well known to be equivalent, in a completely elementary and fairly simple way, to the Brouwer fixed-point theorem.

THEOREM 1. *It is not possible for a continuous function* $\mathbf{f}$ *to map the unit ball* $B^n=\{\mathbf{x}\,|\,\|\mathbf{x}\|\leqslant 1\}$ *of n-dimensional Euclidean space onto the unit sphere* $S^{n-1}=\{\mathbf{x}\,|\,\|\mathbf{x}\|=1\}$ *and to satisfy*

$$\mathbf{f}(\mathbf{x})=\mathbf{x}$$

for all $\mathbf{x}$ *on* S^{n-1}.

The theorem is an immediate consequence of two lemmas.

LEMMA 1. *If there were a continuous map of* B^n *onto* S^{n-1} *leaving each point of* S^{n-1} *fixed, then there would be a continuously differentiable map with these properties.*

LEMMA 2. *It is not possible for a continuously differentiable function to map* B^n *onto* S^{n-1} *and to leave each point of* S^{n-1} *fixed.*

The proof of the first lemma uses standard ideas; the proof of the second lemma uses the ideas of Milnor.

Proof of Lemma 1. Let $\mathbf{f}$ map B^n continuously onto S^{n-1} and suppose that $\mathbf{f}(\mathbf{x})=\mathbf{x}$ for all $\mathbf{x}$ on S^{n-1}. Then $\mathbf{f}(\mathbf{x})-\mathbf{x}$ is continuous on B^n, vanishes on S^{n-1}, and satisfies

$$\|\mathbf{f}(\mathbf{x})-\mathbf{x}\|\leqslant 2 \tag{1}$$

on B^n. So we can choose θ with $\frac{3}{4}<\theta<1$ so that

$$\|\mathbf{f}(\mathbf{x})-\mathbf{x}\|<\tfrac{1}{4}, \quad \text{for } \theta\leqslant\|\mathbf{x}\|\leqslant 1. \tag{2}$$

Let $\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n$ be the unit vectors along the coordinate axes. By the Weierstrass approximation theorem we can choose polynomials $P_i(x_1,x_2,\ldots,x_n)$, $1\leqslant i\leqslant n$, so that

$$\Big\|\sum_{i=1}^{n} P_i(x_1,x_2,\ldots,x_n)\mathbf{e}_i-(\mathbf{f}(\mathbf{x})-\mathbf{x})\Big\|<\tfrac{1}{4}, \tag{3}$$

for all $\mathbf{x}$ with $\|\mathbf{x}\|\leqslant 1$. Write

$$\mathbf{P}(\mathbf{x})=\sum_{i=1}^{n} P_i(x_1,x_2,\ldots,x_n)\mathbf{e}_i,$$

for convenience. Again, using the Weierstrass approximation theorem, we can choose a polynomial Q satisfying

$$\tfrac{3}{4}\leqslant Q(r^2)\leqslant 1, \qquad 0\leqslant r\leqslant\theta; \qquad |Q(r^2)|\leqslant 1, \qquad \theta\leqslant r\leqslant 1; \qquad Q(1)=0.$$

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Write

$$\mathbf{g}(\mathbf{x})=\mathbf{x}+Q(\|\mathbf{x}\|^2)\mathbf{P}(\mathbf{x}).$$

For $0\leqslant\|\mathbf{x}\|\leqslant\theta$, we have

$$\begin{aligned}\|\mathbf{g}(\mathbf{x})\|&=\|\mathbf{x}+Q(\|\mathbf{x}\|^2)\mathbf{P}(\mathbf{x})\|\\&=\|\mathbf{f}(\mathbf{x})+Q(\|\mathbf{x}\|^2)\{\mathbf{P}(\mathbf{x})-\mathbf{f}(\mathbf{x})+\mathbf{x}\}+\{Q(\|\mathbf{x}\|^2)-1\}\{\mathbf{f}(\mathbf{x})-\mathbf{x}\}\|\\&\geqslant\|\mathbf{f}(\mathbf{x})\|-|Q(\|x\|^2)|\cdot\|\mathbf{P}(\mathbf{x})-\mathbf{f}(\mathbf{x})+\mathbf{x}\|-|1-Q(\|\mathbf{x}\|^2)|\cdot\|\mathbf{f}(\mathbf{x})-\mathbf{x}\|\\&\geqslant1-1\cdot\tfrac14-\tfrac14\cdot2=\tfrac14.\end{aligned}$$

Similarly, for $\theta\leqslant\|\mathbf{x}\|\leqslant1$, we have

$$\begin{aligned}\|\mathbf{g}(\mathbf{x})\|&=\|\mathbf{x}+Q(\|\mathbf{x}\|^2)\{\mathbf{P}(\mathbf{x})-\mathbf{f}(\mathbf{x})+\mathbf{x}\}+Q(\|\mathbf{x}\|^2)\{\mathbf{f}(\mathbf{x})-\mathbf{x}\}\|\\&\geqslant\|\mathbf{x}\|-|Q(\|\mathbf{x}\|^2)|\big[\,\|\mathbf{P}(\mathbf{x})-\mathbf{f}(\mathbf{x})+\mathbf{x}\|+\|\mathbf{f}(\mathbf{x})-\mathbf{x}\|\,\big]\\&\geqslant\theta-1\cdot\big[\tfrac14+\tfrac14\big]\geqslant\tfrac14.\end{aligned}$$

So

$$\|\mathbf{g}(\mathbf{x})\|\geqslant\tfrac14\quad\text{for } \|\mathbf{x}\|\leqslant1. \tag{4}$$

For $\|\mathbf{x}\|=1$, we have

$$\mathbf{g}(\mathbf{x})=\mathbf{x}.$$

Now each component of $\mathbf{g}$ is a polynomial in $x_1,x_2,\dots,x_n$. So $\mathbf{g}$ is continuously differentiable and so is $\mathbf{h}$, defined by

$$\mathbf{h}(\mathbf{x})=\mathbf{g}(\mathbf{x})/\|\mathbf{g}(\mathbf{x})\|$$

for $\|\mathbf{x}\|\leqslant1$. Clearly, $\mathbf{h}$ is a continuously differentiable map of B^n onto S^{n-1}, leaving each point of S^{n-1} fixed.

Proof of Lemma 2. Suppose that $\mathbf{f}$ is a continuously differentiable map of B^n onto S^{n-1}, leaving each point of S^{n-1} fixed. Write

$$\begin{aligned}\mathbf{g}(\mathbf{x})&=\mathbf{f}(\mathbf{x})-\mathbf{x},\\\mathbf{f}_t(\mathbf{x})&=\mathbf{x}+t\mathbf{g}(\mathbf{x})=(1-t)\mathbf{x}+t\mathbf{f}(\mathbf{x}),\end{aligned}$$

for $\|\mathbf{x}\|\leqslant1$ and $0\leqslant t\leqslant1$.

As $\mathbf{f}$ is continuously differentiable, so is $\mathbf{g}$, and there is a constant C such that

$$\|\mathbf{g}(\mathbf{y})-\mathbf{g}(\mathbf{x})\|\leqslant C\|\mathbf{y}-\mathbf{x}\|$$

for all $\mathbf{x},\mathbf{y}$ in B^n. If $0\leqslant t<1/C$, and $\mathbf{f}_t(\mathbf{x})=\mathbf{f}_t(\mathbf{y})$, then

$$\begin{aligned}\|\mathbf{x}-\mathbf{y}\|&=\|t\mathbf{g}(\mathbf{y})-t\mathbf{g}(\mathbf{x})\|\\&\leqslant tC\|\mathbf{y}-\mathbf{x}\|,\end{aligned}$$

and $\mathbf{x}=\mathbf{y}$ as $tC<1$. Thus, the map $\mathbf{f}_t$ from B^n to B^n is injective when $0\leqslant t<1/C$.

As the partial differential coefficients of $\mathbf{g}$ with respect to $x_1,x_2,\dots,x_n$ are uniformly bounded, the Jacobian matrix

$$\left(\frac{\partial\mathbf{f}_t}{\partial x_1},\frac{\partial\mathbf{f}_t}{\partial x_2},\dots,\frac{\partial\mathbf{f}_t}{\partial x_n}\right)=I_n+t\left(\frac{\partial\mathbf{g}}{\partial x_1},\frac{\partial\mathbf{g}}{\partial x_2},\dots,\frac{\partial\mathbf{g}}{\partial x_n}\right) \tag{5}$$

is dominated by its diagonal and so is nonsingular provided $0\leqslant t\leqslant t_0$, with t_0 a sufficiently small positive number. Now, for $0\leqslant t\leqslant t_0$, the inverse function theorem tells us that $\mathbf{f}_t$ maps the interior of B^n into an open set, G_t say, contained in B^n. Consider any point $\mathbf{e}$ in B^n that is not in G_t for some t with $0\leqslant t\leqslant t_0$. Join $\mathbf{e}$ to any point $\mathbf{g}$ of G_t and choose a point $\mathbf{b}$ on the line segment

$\mathbf{e},\mathbf{g}$ on the boundary of G_t. As the image of B^n under $\mathbf{f}_t$ is compact, $\mathbf{b}=\mathbf{f}_t(\mathbf{x})$ for some $\mathbf{x}$ in B^n. As $\mathbf{b}$ is not in G_t, $\mathbf{x}$ is not in the interior of B^n and so has $\|\mathbf{x}\|=1$. Hence $\mathbf{b}=\mathbf{x}$, and $\mathbf{e}$ as well as $\mathbf{b}$ lies on the boundary of B^n. As $\mathbf{f}_t$ maps S^{n-1} onto itself, we see that, when $0\leqslant t\leqslant t_0$, $\mathbf{f}_t$ maps B^n bijectively to itself.

Now consider the integral

$$I(t)=\int_{B^n}\frac{\partial \mathbf{f}_t}{\partial \mathbf{x}}\,d\mathbf{x}=\int\cdots\int_{B^n}\det\left(\frac{\partial \mathbf{f}_t}{\partial x_1},\frac{\partial \mathbf{f}_t}{\partial x_2},\dots,\frac{\partial \mathbf{f}_t}{\partial x_n}\right)dx_1\,dx_2\cdots dx_n,$$

for $0\leqslant t\leqslant 1$. When $0\leqslant t\leqslant t_0$, we have a formula for the volume V_n of the unit ball B_n. Thus $I(t)$ has the constant value V_n for $0\leqslant t\leqslant t_0$. But it is clear from (5) that $I(t)$ is a polynomial in t. Hence $I(t)$ is constant and has the positive value V_n for all t. But, we have

$$\mathbf{f}_1\cdot\mathbf{f}_1=1$$

identically, so that

$$\frac{\partial \mathbf{f}_1}{\partial x_i}\cdot\mathbf{f}_1=0,\qquad 1\leqslant i\leqslant n,$$

and

$$\det\left(\frac{\partial \mathbf{f}_1}{\partial x_1},\frac{\partial \mathbf{f}_1}{\partial x_2},\dots,\frac{\partial \mathbf{f}_1}{\partial x_n}\right)=0,$$

for all $\mathbf{x}$ in B^n. Thus $I(1)=0$, and we have the required contradiction.

Note added December 1979. Dr. Roger Smart has explained to me that Brouwer's fixed-point theorem for continuously differentiable maps can be obtained directly from Lemma 2, without use of Lemma 1. The general case of Brouwer's theorem then follows by a simpler application of the Weierstrass approximation theorem. Theorem 1 then follows easily from the general Brouwer theorem.

Reference

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ON THE DEVELOPMENT OF OPTIMIZATION THEORY

ANDRÁS PRÉKOPA

1. Introduction. Farkas's famous paper of 1901 [23] became a principal reference for linear inequalities after the publication of the paper of Kuhn and Tucker, "Nonlinear Programming,"

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in 1951 [53]. In that paper Farkas's fundamental theorem on linear inequalities was used to derive necessary conditions for optimality for the nonlinear programming problem. The results obtained led to a rapid development of nonlinear optimization theory. The work of John [50] containing similar but weaker results for optimality, published in 1948, has been generally known, but it was not until a few years ago that Karush's work [51] of 1939 became widely known, although essentially the same result was obtained by Kuhn and Tucker [53] in 1951.

In this paper we call attention to some important work done in the last century and before. We show that fundamental ideas about the necessary optimality conditions for nonlinear optimization subject to inequality constraints can be found in papers by Fourier, Cournot, and Farkas, as well as by Gauss, Ostrogradsky, and Hamel.

To start to describe the early development of optimization theory it is very helpful to look at the first two sentences in Farkas's paper [23]:

> The natural and systematic treatment of analytical mechanics has to have as its background the inequality principle of virtual displacements first formulated by Fourier and later by Gauss. The possibility of such a treatment requires, however, some knowledge of homogeneous linear inequalities that may be said to have been entirely missing up to now.

We see that Farkas had a definite reason for developing the theory of linear inequalities. He was Professor of Theoretical Physics at the University of Kolozsvár. We can certainly assume that he himself had already applied his inequality theorem to the problem of mechanical equilibrium. In fact, he reported first "on the applications of the mechanical principle of Fourier" at the session of the Hungarian Academy held on December 17, 1894. This was published later in Hungarian [13] and in German [14]. We shall analyze this paper in more detail in further sections. We present here, however, a brief summary of the problem and the principal result.

The method of Lagrange for finding extrema of functions subject to equality constraints was published in 1788 in his famous book *Mécanique Analytique* as a tool for finding the stable equilibrium state of a mechanical system. In the case when a potential exists, the Lagrangian problem can be formulated in the following manner:

$$\begin{array}{ll} \text{minimize} & g(x) \\ \text{subject to} & g_i(x)=0, \quad i=1,\dots,m, \end{array}$$

where the objective function is the potential. The case of inequality constraints was first investigated in 1798 by Fourier [34]. If a potential exists, then Fourier's problem is

$$\begin{array}{ll} \text{minimize} & g(x) \\ \text{subject to} & g_i(x)\geqslant 0, \quad i=1,\dots,m. \end{array}$$

Assuming that the constraining and the objective functions are differentiable and some regularity conditions hold, the Lagrangian necessary condition for the equilibrium states that at the minimizing point x^* the gradient $\nabla g(x^*)$ can be expressed as a suitable linear combination of the gradients $\nabla g_i(x^*)$, $i=1,\dots,m$.

For Fourier's problem the necessary condition for equilibrium was proved by Farkas in 1894 [13] and 1898 [17]. This condition, formulated without proof for special cases by Cournot in 1827 [4] and for the general case by Ostrogradsky in 1834 [67], is that at the minimizing point x^* the gradient $\nabla g(x^*)$ can be expressed as a linear combination with *nonnegative* coefficients of those gradients $\nabla g_i(x^*)$ whose corresponding constraints hold with equality at x^*.

Figure 1 illustrates the situation. The constraints restricting a mass point are five half-spaces. Choosing the coordinate system so that the x_1, x_2 plane coincides with the plane of the ground, the potential corresponding to the gravitational force is given by the well-known formula mgx_3, provided x_3 is small as compared to the radius of the earth. At the equilibrium point x^*, four constraints hold with equality and the cone generated by their negative gradients contains the vector of the gravitational force.

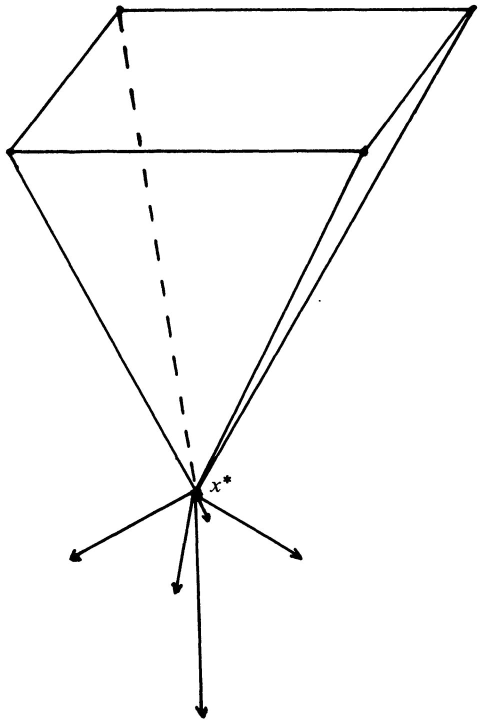

FIG. 1. The mass point is in stable equilibrium at x^*; the vector of the gravitational force is in the cone generated by the negatives of the gradients of the constraining functions at x^*.

Now we need to say a few words about those mechanical principles which played an important role in the development of optimization theory.

2. On the Principles of Mechanical Equilibrium. The principle of virtual work was enunciated by Johann Bernoulli in 1717. It appeared first in a book by Varignon in the same year. Let us quote Lagrange [55, I, pp. 21, 22]:

> The principle of virtual velocity can be given the following general form.
>
> If a system of any number of bodies or points on each of which any forces act is in equilibrium, and one gives the system some small displacement on the basis of which each point moves an infinitesimal distance that will express its virtual velocity, the sum of the forces, each multiplied by the distance moved by the corresponding point in the direction of the force, will always be equal to zero, if infinitesimal displacements in the direction of the forces are taken as positive, and those in the opposite direction, negative.
>
> John Bernoulli is the first as far as I know, who observed this great general form of the principle of virtual velocity and its usefulness for the solution of the problems of Statics. One can see this in one of his Letters to Varignon, dated 1717, which Varignon placed at the beginning of the ninth Section of his New Mechanics, a Section which is entirely devoted to showing by means of different applications the truth and the use of the principle in question.

This principle was considered by Lagrange as an axiom of Mechanics. On page 27 of the same work we read:

> In this law lies what is generally called the principle of virtual velocity, which has been recognized for a long time as the fundamental principle of equilibrium, as we have described it in the preceding Section, and we can consequently consider it as a kind of an axiom of Mechanics.

For the case of a conservative system of forces, i.e., when the forces are given as negative partial

derivatives of a scalar function, the principle, first enunciated by Courtivron, applies. Concerning this, Lagrange writes the following [55, I., p. 70]:

> ...which yields a further principle of Statics, namely that out of all configurations that are successively assumed by the system, the one in which it has the greatest or the smallest kinetic energy is also that in which it would have to be placed in order for it to remain in equilibrium. [See Courtivron, Mémoires de l'Académie des Sciences of 1748 and 1749.]

Lagrange gave sufficient conditions that the potential takes its minimum. As Bertrand remarked, his proof was incomplete and Dirichlet later gave a correct proof [10], [11].

The mechanical principle of Fourier was first published in 1798 in his paper "Mémoire sur la Statique." This concerns the case of inequality constraints. Farkas remarked [13, p. 458] that the declaration of the inequality principle is not the main contribution of the paper. It has the subtitle: "Contenant la démonstration du principe des vitesses virtuelles." This very general "proof," however, has not been accepted (see, e.g., [75]); thus the main merit of the paper is still, contrary to Farkas's remark, the statement of the inequality principle. We quote from page 488 of Fourier's paper [34]:

> As it frequently happens that the points of the system are only constrained by fixed obstacles, without being attached to them, it is evident that there are possible displacements that do not satisfy the equations of constraint: one also sees that for these displacements the moment of the resulting forces is necessarily positive, since the direction of these forces has to be orthogonal to the resisting surfaces. Thus the sum of the moments of the applied forces is positive for all displacements of this kind; but it is impossible to displace a solid body that is in equilibrium so that the total moment of the applied forces will be negative. Finally, if one considers the resistances as forces, which provide us, as we know, with the means of estimating these resistances, the body can be considered free and the sum of the moments is zero for all possible displacements.

The moment of the forces $P, Q, R, \ldots$ acting on a mechanical system is defined as the sum of scalar products

$$P\delta p + Q\delta q + R\delta r + \cdots \tag{2.1}$$

where $\delta p, \delta q, \delta r, \ldots$ are variations of the displacements. The Bernoulli principle declares (2.1) to be equal to zero, while the Fourier principle declares (2.1) to be less than or equal to zero in case of equilibrium. Fourier was using the "fluxion," as it turns out from other parts of his paper, instead of the displacement, these two being negatives of each other. This explains why Fourier required (2.1) to be nonnegative.

If a potential V exists, i.e., if we have

$$P = -\frac{\partial V}{\partial p}, \; Q = -\frac{\partial V}{\partial q}, \; R = -\frac{\partial V}{\partial r}, \; \ldots,$$

where the derivatives on the right denote vectors, then the requirement that (2.1) is less than or equal to zero takes the form:

$$\frac{\partial V}{\partial p}\delta p + \frac{\partial V}{\partial q}\delta q + \frac{\partial V}{\partial r}\delta r + \cdots \geqslant 0. \tag{2.2}$$

On the left side we have a total differential. A correct mathematical proof—according to our present standards—cannot be found in Fourier's work. We know, in particular, that (2.2) cannot be derived as a necessary condition that V takes its minimum, without some constraint qualification. Fourier remarked that in the case where his principle reduces to (2.2), we can find the equilibrium state by minimizing (!) the function V.

In 1829 Gauss [38] again enunciated the inequality principle without mentioning Fourier. He added a footnote explaining his ideas in more detail.

> According to the principle of virtual velocities this equilibrium requires that the sum of products of all three factors, namely, each of the masses m, m', m'', etc., the line segments $cb, c'b', c''b''$, etc., and the

projections of the possible motions of the corresponding points in accordance with the constraints, should be equal to zero, as it is ordinarily stated, or more exactly that the above sum should never be positive.

Footnote:

The ordinary expression tacitly assumes such constraints as allow the opposite for every possible motion, as, e.g., that a point must remain on a certain surface, that the distance between two points should be fixed, and so on. But this is an unnecessary restriction and does not always correspond to Nature. The surface of an impenetrable body does not compel a material point to remain on it but merely prevents it from going to the other side; an inextensible but flexible cord connecting two points prevents only the increase but not the decrease of the distance, etc. Why should we not prefer to express the law of virtual velocities from the very beginning so that it covers all cases?

In 1834 Ostrogradsky also enunciated the inequality principle at a session of the Académie Impériale des Sciences de Saint-Pétersbourg. Voss remarks [75, p. 74] that therefore in Russia it is also called Ostrogradsky's principle. It is interesting to read what Ostrogradsky [68] wrote about the inequality principle:

It is very surprising to see that in the new edition of the Mécanique Analytique, published at a time when the full extent of the principle of virtual velocity was already known, Lagrange not only did not make any use of the observation that in the equilibrium of forces the total moment may acquire a negative value, but in a way he ruled it out when it turned up naturally in the proof he gave for the principle of virtual velocity; meanwhile, failing to take account of it, this great mathematician incompletely enumerated the possible displacements in most of the questions of the first part of the Mécanique Analytique and it is easy to recognize that the displacements he neglected to consider are not forbidden by any condition, so that even when all the equations that he established for the equilibrium are satisfied, equilibrium may not occur.

In this memoir we propose to expound the analysis of the use of the principle of virtual velocity considered in its utmost generality and to complete the solution of various questions handled in the first part of the Mécanique Analytique.

Lagrange's method of multipliers was published in the first volume of *Mécanique Analytique*, pp. 77–79, as a tool for finding the equilibrium of a mechanical system using Bernoulli's principle. The validity of the method was proved by Lagrange entirely on an algebraic basis. (This ingenious method was unable to attract mathematics students and teachers for a long time. The method of presentation used nowadays by many instructors is to prove the necessary condition first in the case of inequality constraints, give a geometric meaning to this, and then refer to the case of equality constraints. This can make the students more enthusiastic about this theory.) We shall see in Sections 4–7 what happened to Fourier's principle. Now we devote a few pages to Farkas's theorem.

3. The Theorem of Farkas on Linear Inequalities. Gyula Farkas (1847–1930), a former member of the Hungarian Academy of Sciences, contributed primarily to theoretical physics. In this respect his results in Mechanics and Thermodynamics are the best known. For further information concerning his life and work see [32], [33], [65], [66], [73].

Farkas's 1901 paper [23] is generally cited for the fundamental theorem on homogeneous linear inequalities. In what follows we shall write vectors in column form, and the transpose will be denoted by a prime.

Let $g_1, \dots g_M, g$ denote m-component vectors and form the following homogeneous linear inequalities:

$$g_i'x \geqslant 0, \quad i = 1, \dots, M, \tag{3.1}$$

$$g'x \geqslant 0. \tag{3.2}$$

If (3.2) holds for every x for which all inequalities in (3.1) hold, then we say that the inequality (3.2) is a consequence of the system of inequalities (3.1). Farkas's theorem is the following.

THEOREM 3.1. *The inequality* (3.2) *is a consequence of the inequalities* (3.1) *if and only if there*

exist nonnegative numbers $\lambda_1,\ldots,\lambda_M$ *such that*

$$g=\lambda_1 g_1+\cdots+\lambda_M g_M. \tag{3.3}$$

Farkas published this theorem first in 1894 and 1895 **[13]**,**[14]**. However, the proof contains a gap. The second proof that he gave in 1896 **[15]** in Hungarian is also incomplete. Essentially the same paper appeared in German in 1899 **[16]**. The first complete proof was published in Hungarian in 1898 **[17]** and in German in 1899 **[18]**. This proof is included in his best-known paper **[23]**. It is instructive to see what kind of error he made in the first proof. We shall quote from **[14]**.

First he shows that we may suppose that there are as many linearly independent inequalities in the system, of which one consequence is considered, as the number of variables. Then he represents the coefficient vector of the consequence inequality as a linear combination of the other coefficient vectors. Now if, for the sake of simplicity, the first n inequalities are linearly independent, then we can express the multipliers belonging to these in terms of the others in the following manner

$$\begin{aligned}\lambda_1&=I_0+I_1\lambda_{n+1}+I_2\lambda_{n+2}+\cdots\\ \lambda_2&=K_0+K_1\lambda_{n+1}+K_2\lambda_{n+2}+\cdots\\ &\;\vdots\\ \lambda_n&=\cdots.\end{aligned} \tag{3.4}$$

What Farkas then writes can be summarized as follows: a necessary condition that it is impossible for all λ-values simultaneously to assume nonnegative values is that, in (3.4), or in one of its equivalent forms that can be obtained using subsequent eliminations, at least one right-hand side has the two properties that its first term is negative and the coefficients of the remaining terms are either negative or zero.

Farkas seems to have had in mind a procedure similar to what is used in the dual (simplex) method. We know, however, that cycling is possible there **[2]**. We can correct the proof by applying the lexicographic dual method **[57]**; a short presentation is given in **[70]**. Let us summarize the whole corrected proof briefly.

Proof of Farkas's theorem. First we remark that if we apply a linear transformation $y=Bx$, where B is a nonsingular square matrix, to the inequalities (3.1), (3.2), then the new (3.2) inequality will be a consequence of the new (3.1) system. Furthermore, proving Farkas's theorem for the new inequalities, we see that (3.3) is satisfied with the same nonnegative multipliers $\lambda_1,\ldots,\lambda_M$.

Let us assume first that the system (3.1) contains as many linearly independent relations as the number of variables. Consider the linear programming problem:

$$\begin{aligned}&\text{minimize } 0\cdot\lambda_1+\cdots+0\cdot\lambda_M\\ &\text{subject to } \lambda_1 g_1+\cdots+\lambda_M g_M=g,\quad \lambda_1\geqslant 0,\ldots,\lambda_M\geqslant 0.\end{aligned} \tag{3.5}$$

Starting from any basis and applying the lexicographic dual method, at the end we reach a system of the form (3.4) where either $I_0\geqslant 0$, $K_0\geqslant 0,\ldots,$ or there exists a row in which the first term on the right-hand side is negative and all variables there have positive or zero coefficients. The second case cannot occur. In fact the lexicographic dual method guarantees that the column vectors of (3.4) are obtained from the vectors $g_1,\ldots,g_M,g$ by a nonsingular linear transformation; thus the system of linear inequalities (in the variables $y_1,\ldots,y_n$):

$$\begin{array}{llll} y_1 & \geqslant 0 & -I_1y_1-K_1y_2-\cdots & \geqslant 0\\ \quad y_2 & \geqslant 0 & -I_2y_1-K_2y_2-\cdots & \geqslant 0\\ \quad\ddots & \;\vdots & & \\ \qquad y_n & \geqslant 0 & & \end{array} \tag{3.6}$$

has the consequence

$$I_0 y_1 + K_0 y_2 + \cdots \geqslant 0. \tag{3.7}$$

Now, if, e.g., we had $I_0 < 0$, $I_1 \leqslant 0$, $I_2 \leqslant 0, \ldots$, then the vector of components $y_1 = 1, y_2 = \cdots = y_n = 0$ would satisfy (3.6) and would not satisfy (3.7).

If in (3.1) we have $h(<M)$ linearly independent relations and for the sake of simplicity $g_1, \ldots, g_h$ are linearly independent, then we choose n-component vectors $d_1, \ldots, d_{n-h}$ so that $B = (g_1, \ldots, g_h, d_1, \ldots, d_{n-h})$ is a nonsingular matrix and apply the transformation $y = Bx$ for the inequalities in (3.1) and (3.2). Then (3.1) will depend only on $y_1, \ldots, y_h$ and, since the inequality in (3.2) is a consequence of the former ones, $y_{h+1}, \ldots, y_n$ will have zero coefficients there also. Now we can forget about $y_{h+1}, \ldots, y_n$ and apply the above reasoning to the system of variables $y_1, \ldots, y_h$. At the end we can reestablish $y_{h+1}, \ldots, y_n$ everywhere with zero coefficients and reach the original inequalities by setting $y = Bx$.

Thus we have proved Farkas's theorem. We have also proved

THEOREM 3.2. *If* (3.2) *is not a consequence of the system* (3.1), *then there is a linear transformation $y = Bx$ with nonsingular square matrix B such that for at least one i, the variable y_i has negative coefficients in the new inequality* (3.2) *and has nonnegative coefficients in the new system* (3.1)

4. Necessary Condition for Equilibrium. We consider a mechanical system the state of which is described by the vector $x \in R^n$, which is subject to the following constraints:

$$g_i(x) \geqslant 0, \qquad i = 1, \ldots, m, \tag{4.1}$$

where the functions $g_i : R^n \to R$, $i = 1, \ldots, m$ are differentiable.

Denote by $X_1, \ldots, X_n$ the components of the forces acting on the system and suppose that equilibrium is reached at x^*. Then by Fourier's principle the inequality

$$X_1 \delta x_1 + \cdots + X_n \delta x_n \leqslant 0 \tag{4.2}$$

is satisfied, where $\delta x_1, \ldots, \delta x_n$ are variations of the coordinates $x_1, \ldots, x_n$, i.e., small quantities with the property that the vector of components

$$x_1^* + \delta x_1, \ldots, x_n^* + \delta x_n \tag{4.3}$$

satisfies (4.1). In what follows we shall use general terms, but our statements can be made exact under not very restrictive mathematical conditions.

The inactive constraints in (4.1), i.e., those for which we have strict inequality at the point x^*, do not restrict small changes in the coordinates. Thus to obtain the conditions on small changes we only have to consider the constraints active at x^*. Let us assume that these are the first M constraints. Then, writing dx_1 instead of δx_1, the Fourier principle requires that

$$X_1 dx_1 + \cdots + X_n dx_n \leqslant 0 \tag{4.4}$$

and the increments are subject to the inequalities:

$$\frac{\partial g_i(x^*)}{\partial x_1} dx_1 + \cdots + \frac{\partial g_i(x^*)}{\partial x_n} dx_n \geqslant 0, \qquad i = 1, \ldots, M. \tag{4.5}$$

Now we can forget about the order of magnitude of the quantities $dx_1, \ldots, dx_n$ satisfying the above inequalities. In fact if $dx_1, \ldots, dx_n$ satisfy a homogeneous linear inequality, then the same holds for $tdx_1, \ldots, tdx_n$ where t is any nonnegative constant.

Let X denote the vector with components $X_1, \ldots, X_n$. We shall write it in row form and also use the convention that gradients are row vectors.

If x^* is an equilibrium point, then, using Fourier's principle, we find that the linear inequality (4.4) is a consequence of the system of linear inequalities (4.5); hence, by Farkas's theorem, there exist nonnegative numbers $\lambda_1, \ldots, \lambda_M$ such that

$$X + \lambda_1 \nabla g_1(x^*) + \cdots + \lambda_M \nabla g_M(x^*) = 0 \tag{4.6}$$

If the system of forces is conservative, i.e., there exists a potential $V(x)$ so that at every point of the state space we have

$$X_i = -\frac{\partial V}{\partial x_i}, \qquad i=1,\ldots,n, \tag{4.7}$$

then (4.4) becomes

$$\frac{\partial V}{\partial x_1} dx_1 + \cdots + \frac{\partial V}{\partial x_n} dx_n \geqslant 0. \tag{4.8}$$

Here on the left-hand side we have a total differential. Having a potential, we can start from (4.8), by applying Courtivron's principle, which states that if a mechanical system is in stable equilibrium then the potential has a local minimum, and observing that (except for pathological cases) the total differential is nonnegative at the minimum point of the function.

We can ask: Who—if anybody—deduced equation (4.6) from the inequality principle for mechanical equilibrium? To answer this question we can start by analyzing the papers cited by Farkas in **[23]**, and we should also look at the volumes of *Enzyklopädie der Mathematischen Wissenschaften* published at the beginning of this century. In the four volumes on mechanics there are two papers (Voss **[75]** and Stäckel **[71]**) that mention theories of statics and dynamics under inequality constraints. These authors and Farkas cite about thirty books and papers; some of them are unfortunately not available to the author, but Farkas, Voss, and Stäckel together very likely give a good picture of the history of the subject. Voss writes **[75**, p. 74]:

> This case, not considered by Lagrange, was first considered, independently of Fourier, by Gauss and by Ostrogradsky . . . therefore in Russia the Fourier principle is also called Ostrogradsky's principle. In France the Fourier principle has been less often neglected; A. A. Cournot developed Ostrogradsky's equations as early as 1827.

One paper by Ostrogradsky **[67]** is also cited by Farkas; Stäckel mentions **[68]** also.

Voss does not refer to Farkas; but Stäckel cites the papers **[14]**, **[16]**. We can forgive Stäckel for not recognizing the importance of Farkas's work, because the subject of his paper is "Elementare Dynamik."

Dynamical problems subject to inequality constraints were considered by Gibbs in 1879 **[39]**. Nonnegativity of the multipliers in some special cases are proved by Mayer **[58]**, **[59]** and Zermelo **[76]**. Farkas gave a more general form of the theory in 1906 **[24]**.

Fourier, Cournot, Ostrogradsky, and Farkas seem to be the principal contributors to the present form of the necessary condition of static equilibrium. Cournot and later Ostrogradsky presented the equations (4.6) in the form of a conjecture, as we may say now. Farkas proved (4.6) by relying on Fourier's work concerning the first part of the theorem, considering a conservative system of forces, where—as it is obvious today but did not seem necessary for those in the last century dealing with mechanical problems—a constraint qualification is needed. We have to add to these that the work of Cournot was based to a great extent on the work of Poinsot **[69]**.

Fourier and Farkas both contributed further important ideas to optimization theory. Let us summarize briefly their principal results in this respect.

Fourier—anticipated the formulation of the linear programming problem in 1824 [**36**, II., p. 325–328], **[40]**;
—formulated the inequality principle for the mechanical equilibrium in 1798 **[34]**;
—initiated the parametric solution of homogeneous linear inequalities in 1826 **[35]**.

Farkas —proved the basic theorem concerning homogeneous linear inequalities, first mentioned in 1894 **[13]** and first complete proof in 1898 **[17]**;
—gave a rigorous proof for the "dual form" of Fourier's mechanical inequality principle, first in 1894–1895 **[13]**, **[14]**;
—gave an elegant parametric representation for the solutions of homogeneous linear inequalities, first in 1898 **[19]**.

5. The Work of Cournot on the Problem of Equilibrium. Antoine-Augustin Cournot (1801–1877), a famous polyhistor, was one of the greatest mathematical economists. The work of his that we analyze here was done as a contribution to mechanics. It is interesting to remark that, though optimization theory is an important tool of mathematical economics, in his famous book *Recherches sur les Principes Mathématiques de la Théorie des Richesses*, no application can be found of his previous investigation of the problem of mechanical equilibrium. Since the first publication of this book in 1838 several editions have appeared, e.g., [7] with the critiques of Walras, Bertrand, and Pareto, and introduction and biographical notes by Lutfalla, and [8] with the notes of Irving Fisher.

Cournot became *docteur ès sciences* in 1829 in Paris. His thesis [5],[6] contributed to dynamics, where he applied his earlier result published in 1827 [4]. One year earlier he published an elementary paper [3] on inequalities; there is no anticipation in that paper of any form of Farkas's theorem.

In the paper [4] that is at present most important for us, Cournot does not refer to the work of Fourier. He seems to have been unaware of the enunciation of the inequality principle by Fourier in 1798. Cournot rediscovered the principle but also derived the necessary conditions for equilibrium. We quote from his short paper [4, p. 166]:

> . . . It often happens that the constraints of the system cannot be expressed by equations; from these constraints there result conditions of equilibrium that have always been considered as not being derivable from the principle of virtual velocity and being outside of this principle (Mécanique de M. Poisson, tom. I, p. 241). Our purpose here is to show that when the constraints of the system can be expressed algebraically by inequality signs, the principle of virtual velocity, suitably modified, still applies and provides the conditions of equilibrium by a uniform method.
>
> Let us assume that a system is subject to a certain number of similar constraints expressed by the inequalities
>
> $$I>0, \quad I'>0, \quad \text{etc.} \qquad J<0, \quad J'<0, \quad \text{etc.},$$
>
> the signs $>$ and $<$ always being supposed not to exclude the case of the equality.
>
> One cannot, in general, differentiate an inequality like an equation, or derive a relation between the increments of the variables; but when equilibrium conditions are sought, only two cases can appear. Either the nature of the system is such that one could suppress the constraints without changing its state, as happens if cords that connect some of its points are not taut or if the points to which certain surfaces form impenetrable obstacles do not touch these surfaces: in this case equilibrium must occur independently of the constraints and it is unnecessary to take them into account; or the system is so situated that the constraints produce their effect and can decrease the number of conditions necessary for equilibrium: then one has the present coordinate values $I=0$, $J=0$, etc., and if one varies these values, their increments will not be entirely arbitrary; on the basis of the primitive conditions, they must satisfy the relations:
>
> $$\text{(a)}\ \delta I>0, \quad \delta I'>0, \quad \text{etc.} \qquad \delta J<0, \quad \delta J'<0, \quad \text{etc.}$$
>
> In addition, when the system is subject to constraints of the kind that we are considering here, it is clear that one could suppress them, provided that forces were applied to produce the same effect; thus the resistance of a surface can be replaced by the application of a suitably oriented force normal to the surface; the constraint of a taut cord, by the application of a force oriented in the direction of the tension in the cord; etc. Let $F, F', \ldots,$ be the forces applied directly to the system in the directions $f, f', \ldots,$; and P, P' the auxiliary forces that take the place of obstacles and are directed along lines $p, p', \cdots$; on the basis of the principle of virtual velocity we will have the fundamental equation
>
> $$F\delta f + F'\delta f' + \text{etc.} + P\delta p + P'\delta p' + \text{etc.} = 0.$$
>
> Now it is easy to see that all virtual movements in which P and δp, P' and $\delta p'$, etc., would have opposite signs, would tend to overcome the obstacles that the surfaces, cords, etc., oppose to the different points of the system, and are consequently incompatible with the constraints of the system, expressed by the inequalities (a). The only movements compatible with the constraints are those for which P and p, $\delta P'$ and $\delta p'$ have the same sign and for which, as a consequence of this, the quantities $P\delta p, P'\delta p'$, etc., are essentially positive.
>
> Thus, disregarding the auxiliary forces P, P', etc., and the terms containing them in the equation of the virtual velocities, for the constraints that have to be satisfied by the system we have
>
> $$\text{(b)}\ F\delta f + F'\delta f' + \text{etc.} < 0.$$

The above "proof" given by Cournot concerns the validity of the Fourier principle. In the paper **[4]** Cournot then derives the nonnegativity of the Lagrangian multipliers for special cases similar to some of those investigated by Lagrange in the first part of *Mécanique Analytique* in case of equality constraints. When dealing with the case of a system of points lying on planes parallel to the xy-plane, Cournot refers to the work of Poinsot **[69]**, first published in 1803. In fact for this special case Poinsot already obtained the conditions of equilibrium.

The papers **[3]**, **[4]** of Cournot seem to be less well known than his other works. They are not listed in the bibliography of his works published in **[7]**.

6. The Work of Ostrogradsky on the Problem of Equilibrium. Mikhail Vasilevich Ostrogradsky (1801–1862) contributed important results to mechanics (among other subjects). He was a student in Paris and attended the courses of Fourier, Poisson, Cauchy, and other well-known French mathematicians. He returned to Russia (St. Petersburg) in 1828. In 1830 he became an extraordinary member, and in 1832 an ordinary member, of the Academy of St. Petersburg.

We should mention two papers of Ostrogradsky concerning the problem of mechanical equilibrium. Farkas refers only to **[67]**, presented in 1834 before the Academy, but the other **[68]** is an improved version of his theory. In the earlier paper four applications are mentioned: (a) a point that is allowed to move in one part of the space subdivided by a surface; (b) the pivoted polygon; (c) the flexible cord; (d) the incompressible liquid. Further problems are mentioned concerning dynamics.

In **[68]** Ostrogradsky refers to (what we now call) Farkas's theorem as an obvious algebraic fact and derives the equation of equilibrium (4.6). We reproduce here part of pages 589 and 590 of **[68]**:

> Assume that the quantities $\delta s, \delta s', \delta s'', \delta s''', \ldots,$ belong not only to those displacements of the system, of which the reaction forces are capable, but also to all other displacements, whether possible or not; or rather consider $\delta s, \delta s', \delta s'', \delta s''', \ldots,$ to be entirely arbitrary. We have to express that the reaction forces $R, R', R'', R''', \ldots,$ are not capable of producing any displacement for the system that satisfies the conditions
>
> $$\delta L > 0, \quad \delta L_1 > 0, \quad \delta L_2 > 0, \quad \delta L_3 > 0, \quad \ldots \tag{15}$$
>
> the sign $>$ does not exclude equality.
>
> Now we know that a system of forces is capable of every displacement that contributes a positive value to the total moment and of none of those that correspond to negative or zero values of the total moment. Thus, for the reaction forces to be incapable of producing any displacements satisfying the conditions (15), it is necessary that their moment is negative or zero for these displacements; in other words it is necessary that the function
>
> $$R\delta s \cos\psi + R'\delta s' \cos\psi' + R''\delta s'' \cos\psi'' + R'''\delta s''' \cos\psi''' + \cdots$$
>
> in which $\psi, \psi', \psi'', \psi''', \ldots,$ designate the angles $\widehat{R\delta s}, \widehat{R'\delta s'}, \widehat{R''\delta s''}, \widehat{R'''\delta s'''}, \ldots,$ respectively, and which consequently represents the moment of the forces $R, R', R'', R''', \ldots,$ is negative or zero whenever $\delta s, \delta s', \delta s'' \delta s''', \ldots,$ satisfy (15).
>
> The solution of the question of making the function
>
> $$R\delta s \cos\psi + R'\delta s' \cos\psi' + R''\delta s'' \cos\psi'' + R'''\delta s''' \cos\psi''' + \cdots$$
>
> negative or zero whenever the functions of the same kind, $\delta L, \delta L_1, \delta L_2, \delta L_3, \ldots,$ are positive or zero, belongs to the most elementary algebra. It is necessary and sufficient that $R\delta s \cos\psi + R'\delta s' \cos\psi' + R''\delta s'' \cos\psi'' + R'''\delta s''' \cos\psi''' + \cdots$ can be reduced to a linear function of $\delta L, \delta L_1, \delta L_2, \delta L_3, \ldots,$ with negative coefficients. Thus we only have to make, for arbitrary $\delta s, \delta s', \delta s'', \delta s''', \ldots,$
>
> $$R\delta s \cos\psi + R'\delta s' \cos\psi' + R''\delta s'' \cos\psi'' + R'''\delta s''' \cos\psi''' + \cdots = \lambda\delta L + \lambda_1\delta L_1 + \lambda_2\delta L_2 + \lambda_3\delta L_3 + \cdots$$
>
> and add the condition that all λ's be negative. Or, if one wants to avoid considering negative λ's, one can make
>
> $$R\delta s \cos\psi + R'\delta s' \cos\psi' + R''\delta s'' \cos\psi'' + R'''\delta s''' \cos\psi''' + \cdots = -(\lambda\delta L + \lambda_1\delta L_1 + \lambda_2\delta L_2 + \lambda_3\delta L_3 + \ldots)$$

then all λ's will be positive. By the last equation and the preceding one it is evident that the moment $R\delta s\cos\psi + R'\delta s'\cos\psi' + R''\delta s''\cos\psi'' + R'''\delta s'''\cos\psi''' + \ldots$ will be negative or zero whenever the functions $\delta L, \delta L_1, \delta L_2, \delta L_3, \ldots,$ will be positive or zero.

If we transpose all terms to the same side, the equation of equilibrium of the reaction forces becomes

$$R\delta s\cos\psi + R'\delta s'\cos\psi' + R''\delta s''\cos\psi'' + R'''\delta s'''\cos\psi''' + \cdots + \lambda\delta L + \lambda_1\delta L_1 + \lambda_2\delta L_2 + \lambda_3\delta L_3 + \cdots = 0. \quad (16)$$

It has to hold for all $\delta s, \delta s', \delta s'', \delta s''', \ldots,$ of arbitrary magnitude and direction. But one should not forget to add to the equation (16) the inequalities

$$\lambda > 0, \quad \lambda_1 > 0, \quad \lambda_2 > 0, \quad \lambda_3 > 0, \quad \ldots. \quad (17)$$

Ostrogradsky made two errors. One consists of asserting Farkas's theorem without proof. Ostrogradsky's authority was so strong that many authors took the theorem for granted **[58]**, **[59]**, **[64]**, **[75]**. The second error was discovered by Study, as communicated by Mayer **[58**, p. 225]. On page 583 of **[68]** Ostrogradsky infers that some of his inequalities should reduce to equations of equilibrium. This would have made it possible to determine the multipliers in the equation at equilibrium. He thought that those multipliers should be equal to zero which would turn out to be negative if we did not have the inequality constraints.

7. The Work of Farkas on the Problem of Equilibrium. Since the two papers **[13]**, **[14]** are essentially the same, we quote from the German version **[14]**. In the introductory part of the paper we read the following:

The purpose of this paper is to show that, with a suitable modification, the method of multipliers of Lagrange can be carried over to the Fourier principle also.

In the first section of the paper Farkas deals with his inequality theorem. His proof is incomplete, however, as we pointed out. In the second section the necessary condition of equilibrium is derived:

...the constraint expressions go over to the following

$$\begin{aligned} &\sum F\delta q = 0, \quad \sum G\delta q = 0, \quad \ldots, \\ &\sum S\delta q \geqslant 0, \quad \sum T\delta q \geqslant 0, \quad \ldots, \end{aligned} \quad (11)$$

and the fundamental inequality is

$$\sum Q\delta q < 0 \quad \text{or} \quad -\sum Q\delta q \geqslant 0. \quad (12)$$

The constraint equations have to be expressed in the form of inequalities so that the system of constraint conditions appears as follows:

$$\begin{aligned} \sum F\delta q &\geqslant 0, & \sum G\delta q &\geqslant 0, \quad \ldots, \\ -\sum F\delta q &\geqslant 0, & -\sum G\delta q &\geqslant 0, \quad \ldots, \\ \sum S\delta q &\geqslant 0, & \sum T\delta q &\geqslant 0, \quad \ldots. \end{aligned} \quad (13)$$

The Fourier principle requires that the inequality (12) is satisfied by all systems of values δq that satisfy (13). It will be shown that this happens only when there are positive multipliers such that the coefficients Q can be represented as homogeneous linear functions of the coefficients $F, G, \ldots, -F, -G, \ldots, S, T, \ldots$. Let $\phi', \psi', \ldots, \phi'', \psi'', \ldots, \lambda, \mu, \ldots,$ denote these positive multipliers. We must have

$$-Q = (\phi' - \phi'')F + (\psi' - \psi'')G + \cdots + \lambda S + \mu T + \cdots.$$

The differences $\phi' - \phi'', \psi' - \psi'', \ldots,$ may also take negative values, consequently the Fourier principle will be satisfied by those Q values that can be determined from equations such as

$$Q + \phi F + \psi G + \cdots + \lambda S + \mu T + \cdots = 0, \quad (14)$$

where $\phi, \psi, \ldots, \lambda, \mu, \ldots,$ have the same values in the expressions for each Q and moreover $\phi, \psi', \ldots,$ are

completely arbitrary, but $\lambda, \mu, \ldots,$ are arbitrary nonnegative quantities. Conversely, the fundamental inequality (12) follows from the system (11) by using (14) and simple procedures.

In a further section of the paper "the two main types of application" are presented: (a) the equilibrium equation for tangential solid bodies; (b) the equilibrium equation for nonsolid bodies.

8. On the Constraint Qualification. The "constraint qualification" is of fundamental importance, not only from the point of view of nonlinear optimization, but also from the point of view of mechanics. Soon we shall see why.

At the beginning of this century the axiomatic foundation of the mathematical and physical sciences was an important activity. In his famous paper **[49]** Hilbert urges mathematicians to axiomatize two "physical disciplines": probability theory and mechanics. The paper of Hamel **[46]** of 1909 is an attempt in the direction of the axiomatic foundation of classical mechanics. The Fourier inequality principle was unfortunately not included. An improved version of his axiomatics is contained in his paper **[47]**, which appeared in 1927. There the inequality principle of Fourier is already mentioned as one of the axioms: Axiom II$2k$ on page 17. On page 33 concerning "Das Energieprinzip," Axiom II$5c\beta$ declares, denoting the potential by U, that $\delta U \geqslant 0$ is a necessary condition for equilibrium. Hamel had to establish consistency between the two axioms and this could only be done by a "constraint qualification"; in fact we find it in Axiom II$5c\gamma$, on page 33. It requires that to every mass particle there correspond a scalar function u such that the following equality holds:

$$\delta U = \sum dm \nabla u \delta r, \tag{6.1}$$

where dm denotes the mass and r the state of a particle. If instead of (6.1) we require

$$\delta U = \nabla U \delta r, \tag{6.2}$$

where r denotes the state of the whole system of particles, then we obtain essentially the Karush-Kuhn-Tucker constraint qualification. Equation (6.2) implies (6.1) because we can generate the u functions for the purpose of satisfying (6.1) in such a way that in U we subsequently fix all variables except for those belonging to one particle.

Unfortunately Hamel was unaware of the existence of Farkas's theorem. Even in his *Theoretische Mechanik*, first published in 1949, this theorem is not referred to, though part of the pages 69–70, 517–518, are devoted to the Fourier inequality principle. There we also see that his "constraint qualification" is not derived rigorously and his reasoning concerning the Fourier principle is more heuristic.

9. Miscellaneous Remarks Concerning Linear Inequalities. Farkas's most important results concerning the Fourier principle and the theory of linear inequalities are summarized in his papers **[13]**, **[14]**, **[23]**.

The parametric representation of the solutions of linear inequalities was initiated by Fourier **[35]**. Minkowski gave the representation of all solutions using extremal rays **[60]**. In the parametric representation given by Farkas **[19]**, **[23]**, it is particularly simple to generate the rays whose convex combinations constitute all solutions of the linear inequalities. Writing $z_1, \ldots, z_n$ instead of $dx_1, \ldots, dx_n$ in (4.5) and (4.6), we can represent all solutions of (4.5) in a parametric form, insert it into (4.4), and, if the number of variables and the active constraints in (4.5) is not large, we can solve the equilibrium problem in some cases. For this we have to know which are the active constraints in (4.5). This method was recommended by Farkas **[19]**. The application of the Farkas parametric representation technique for linear programming is perhaps best done as in the paper by Uzawa **[74]**. He completes one inequality to equality (if necessary), eliminates the nonzero constants on the right-hand side from the other constraints, inserts the parametric form of the solutions of these into the remaining equality and the objective function, and finds the optimal solution.

The papers **[21]**, **[22]** in Hungarian and German, respectively, have the same content. Farkas gave a further mechanical application of his theorem on linear inequalities. This concerns the decomposition of the forces of reaction in a mechanical system into shocks and others satisfying the negatives of the constraints given for the displacements (Voss mentions in **[75**, p. 25] that this kind of distinction between the forces is attributed to Painlevé). The mathematical results of the paper are reproduced in **[23]**. The book **[20]**, in Hungarian, does not contain new results as compared to his earlier papers and to the paper **[23]**.

Between 1902 and 1917 Farkas did not publish on linear inequalities. In 1917, after Haar generalized Farkas's theorem for the inhomogeneous case, Farkas returned to the area and published further papers **[28]**, **[29]**, **[31]** on systems of linear inequalities. The papers **[25]**, **[26]**, **[27]**, **[30]** are also partially relevant; **[30]** is the German version of **[25]**.

Haar wrote three papers on linear inequalities **[42]**, **[43]**, **[44]**. The first two are essentially Hungarian and German versions of the same paper. He gave the following generalization for Farkas's theorem (we shall use the notation of Section 3).

THEOREM 6.1. *If the linear inequality*

$$g'x \geqslant b \tag{6.1}$$

is a consequence of the linear inequalities

$$g_i'x \geqslant b_i, \quad i=1,\dots,M, \tag{6.2}$$

i.e., if every x satisfying (6.2) *also satisfies* (6.1), *then there exist nonnegative constants* $\lambda, \lambda_1, \dots, \lambda_M$ *such that for every $x \in R^n$ we have*

$$g'x - b = \sum_{i=1}^{M} \lambda_i (g_i'x - b_i) + \lambda. \tag{6.3}$$

This is the same statement referred to by Kuhn and Tucker in a footnote supplied to von Neumann's manuscript **[61]**. This concerns the duality theorem of linear programming proved by Gale, Kuhn, and Tucker **[37]**.

Haar remarked **[42]**, **[44]**, that the theory of linear inequalities was developed by Farkas and Minkowski. The famous book of Minkowski **[60]** also contains results on linear inequalities (pages 39–45 in both editions). He considered a finite system of homogeneous linear inequalities from two points of view: to represent them in a parametric form and to discover those which are superfluous, i.e., can be omitted without changing the set of solutions. This latter problem led him to prove a theorem closely related to Farkas's theorem mentioned in Section 3. For the sake of completeness, we present here the exact statement of Minkowski's theorem.

We assume that in the system (3.1) the number of linearly independent relations equals n, the number of components of x. This causes no loss of generality, as Minkowski remarked. We also assume that there exists at least one x for which (3.1) holds with all strict inequalities. Such an x is called essential. An essential x is said to be an extremal ray of the cone (3.1) if x is not the sum of two nonzero vectors which satisfy (3.1) and neither of them is a constant multiple of the other.

THEOREM 6.2. *Among the linear forms $g_i'x \geqslant 0$, $i=1,\dots,M$ those which are* 0 *for $m-1$ linearly independent extremal solutions have the property that each of them is essential and all other forms can be expressed as linear combinations of them with nonnegative weights.*

For further references to early papers on linear inequalities see **[9]**.

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Remarks. In the list given below the name of Farkas appears with the initials J. or Gy. Up to the beginning of this century it was customary to translate the given name into the language of the paper. "J." refers to the German name Julius; "Gy." refers to the Hungarian name Gyula, and these two names were supposed to be the same.

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MISCELLANEA

40.

GIVE ME AN EPSILON AND I WILL TREAT IT WELL

RAY BOBO

$x=\varnothing$	Found alone and wanting,
	I now languish unbegun;
$\varnothing\to\{\varnothing\}$	Oh, to have your arms about me,
	Be transformed into the one.
$\delta=\{\varepsilon\}$	Closeness for our angels
	Is returned with an embrace;
$x+y\to\infty$	So that by incorporation,
	We can soar to loving space.
$\partial\int f=f$	In meaning full,
$\int\partial f=f+C$	In asking all, Amen.

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A poem involving Mathematics usually expresses either a mathematical idea (for example, Soddy's "The Kiss Precise," *Nature*, 137 (1936) 1021) or the poet's feelings about some aspect of Mathematics (for example, Millay's well-known sonnet "Euclid alone has looked on Beauty bare"). The poem printed above is one of a very small number in which poetical ideas are expressed in the language of Mathematics. We may think of it as being an application of Mathematics to Poetry. Readers desiring further exegesis should consult Professor Bobo.—R. P. B.

EDITED BY DEBORAH TEPPER HAIMO AND FRANKLIN TEPPER HAIMO

Material for this department should be sent to Professor Deborah Tepper Haimo, Department of Mathematical Sciences, University of Missouri, St. Louis, MO 63121.

ROOTS OF POLYNOMIALS IN ALGEBRAIC EXTENSIONS OF FIELDS

I. M. ISAACS

The following result is known. (For instance, see [1] or page 88 of [3].)

THEOREM 1. *Let $F \subseteq E$ be an algebraic field extension. Then E is algebraically closed iff every nonconstant polynomial in $F[X]$ has a root in E.*

This theorem is not quite the triviality it may appear to be at first glance. If one knows that all polynomials in $F[X]$ *split* over E, then it is an easy exercise to show that E is algebraically closed. Under the weaker hypothesis of Theorem 1, however, this conclusion is considerably more difficult to prove. (It is more difficult to find in the literature, too. A search of about a dozen books that deal with field extensions was able to uncover only one proof of this result and two cases where at least a part of Theorem 1 appears as a problem.)

The purpose of this note is to prove the following generalization of Theorem 1.

THEOREM 2 . *Let E_1 and E_2 be algebraic extensions of F. Then E_1 and E_2 are F-isomorphic iff the two sets of polynomials*

$$\mathcal{S}_i = \{ f \in F[X] \| f \text{ has a root in } E_i \}$$

are equal.

In other words, algebraic extensions of F are characterized by the sets of polynomials over F that have at least one root in the extension. Equivalently, the algebraic extension $E \supseteq F$ is determined up to F-isomorphism by the set of minimal polynomials $m_F(\alpha)$ over F of the elements $\alpha \in E$. Note that Theorem 1 follows immediately from Theorem 2 by taking $E_1 = E$ and $E_2 = \bar{F}$, the algebraic closure of F.

Most of the work for Theorem 2 is contained in the proof of the following.

THEOREM 3. *Let $E \supseteq F$ be an algebraic extension and let $K \supseteq F$. Suppose, for every field L with $F \subseteq L \subseteq E$ and $|L:F| < \infty$, that there exists an F-isomorphism of L into K. Then there exists an F-isomorphism of E into K.*

Proof. Suppose $F \subseteq L \subseteq E$ and $\sigma : L \to K$ is an F-isomorphism. We shall say that (L, σ) is an *extendible pair* if, for every field L_1 with $L \subseteq L_1 \subseteq E$ and $|L_1 : L| < \infty$, the map σ extends to an isomorphism of L_1 into K. By hypothesis, then, (F, identity) is an extendible pair.

Let $\mathcal{E}$ be the set of extendible pairs and partially order $\mathcal{E}$ by $(L, \sigma) \leqslant (M, \tau)$ if $L \subseteq M$ and the restriction $\tau | L = \sigma$ holds. We claim that $\mathcal{E}$ satisfies the hypotheses of Zorn's lemma.

Let $\mathfrak{L} \subseteq \mathcal{E}$ be nonempty and linearly ordered and let (L, σ) be the union of the fields and maps in $\mathfrak{L}$. We must show that (L, σ) is an extendible pair. Clearly σ is an F-isomorphism. Let $L \subseteq L_1 \subseteq E$ with $|L_1 : L| < \infty$. We must extend σ to L_1. To do this, choose a finite set $S \subseteq L_1$ such that $L_1 = L[S]$ and let Ω be the set of all F-isomorphisms $F[S] \to K$. Since $|F[S] : F| < \infty$, we have $|\Omega| < \infty$.

Suppose, for each $\omega \in \Omega$, that there exists $(M_\omega, \pi_\omega) \in \mathfrak{L}$ such that no extension of π_ω to $M_\omega[S]$ is also an extension of ω. Since $|\Omega| < \infty$ and $\mathfrak{L}$ is linearly ordered, we can find $(M, \pi) \in \mathfrak{L}$ such that $(M, \pi) \geqslant (M_\omega, \pi_\omega)$ for all $\omega \in \Omega$. Since (M, π) is an extendible pair, π can be extended to

$\theta: M[S]\to K$ and $\theta|F[S]\in\Omega$. Say $\theta|F[S]=\omega$. Then $\theta|M_\omega$ is a simultaneous extension of π_ω and ω, and this is a contradiction.

We have established that there exists $\omega\in\Omega$ such that, for every $(M,\pi)\in\mathfrak{L}$, there is a common extension θ of ω and π to $M[S]$. These maps θ are uniquely determined and it follows that, as (M,π) runs over $\mathfrak{L}$, the union of the various θ's is a well-defined map $L[S]\to K$. Thus $(L,\sigma)\in\mathfrak{S}$ as desired.

We may now apply Zorn's lemma and choose a maximal element $(U,\phi)\in\mathfrak{S}$. It suffices to show that $U=E$. If not, choose $\alpha\in E-U$ and let Λ be the set of extensions of ϕ to $U[\alpha]$. By the maximality of (U,ϕ), no pair $(U[\alpha],\lambda)$ is extendible for $\lambda\in\Lambda$. For each such λ, then, there exists a field L_λ with $U[\alpha]\subseteq L_\lambda\subseteq E$ and $|L_\lambda : U[\alpha]|<\infty$ such that λ cannot be extended to L_λ. Let L be the compositum of the fields L_λ for $\lambda\in\Lambda$.

Since $|U[\alpha]:U|<\infty$, we have $|\Lambda|<\infty$ and thus $|L:U|<\infty$. Since (U,ϕ) is extendible, ϕ has an extension $\theta: L\to K$ and $\theta|U[\alpha]\in\Lambda$. Say $\theta|U[\alpha]=\lambda$. Then $\theta|L_\lambda$ is an extension of λ to L_λ, and this is a contradiction. The proof is complete. ∎

We need the following lemma. (This is Problem 21 on page 177 of [2]. It follows from Lemma 1.2 of [4], which is in fact much stronger.)

LEMMA 4. *A vector space over an infinite field is not the union of any finite collection of proper subspaces.*

The next result is a slight variation of Theorem 2, which follows easily from it.

THEOREM 5. *Let $E\supseteq F$ be an algebraic extension and let $\mathfrak{S}=\{m_F(\alpha)|\alpha\in E\}$. Let $K\supseteq F$ and assume that every $f\in\mathfrak{S}$ has a root in K. Then there exists an F-isomorphism of E into K.*

Proof. By Theorem 3, it suffices to show that an F-isomorphism $L\to K$ exists for every L with $F\subseteq L\subseteq E$ and $|L:F|<\infty$. Fix such an L and let $\bar{K}$ be an algebraically closed field containing K. Let Ω be the set of F-isomorphisms $L\to\bar{K}$. Note that $|\Omega|<\infty$, since $|L:F|<\infty$.

For $\omega\in\Omega$, let M_ω be the inverse image of K under ω, so that $F\subseteq M_\omega\subseteq L$. We claim that L is the union of the M_ω for $\omega\in\Omega$. To see this, let $\alpha\in L$ and choose a root $\beta\in K$ for $m_F(\alpha)$. It follows that there exists an F-isomorphism $\phi: F[\alpha]\to F[\beta]\subseteq K$, and ϕ may be extended to an F-isomorphism $\omega: L\to\bar{K}$. Thus $\omega(\alpha)=\phi(\alpha)=\beta\in K$, and $\alpha\in M_\omega$ as desired.

We conclude now that $L=M_\omega$ for some $\omega\in\Omega$. If $|F|=\infty$, this is immediate from Lemma 4. If $|F|<\infty$, then $|L|<\infty$ and we have $L=F[\alpha]$ for some α. Now $\alpha\in M_\omega$ for some ω and again we have $L=M_\omega$. It follows that ω maps L into K, and the proof is complete. ∎

Proof of Theorem 2. Assume $\mathfrak{S}_1=\mathfrak{S}_2$. By Theorem 5, there exist (not necessarily surjective) F-isomorphisms $\phi: E_1\to E_2$ and $\theta: E_2\to E_1$. Let $L=\phi(\theta(E_2))\subseteq E_2$. Let $\alpha\in E_2$ and put $f=m_F(\alpha)$. Since L and E_2 are F-isomorphic, f has equal numbers of roots in L and in E_2 and it follows that $\alpha\in L$. Thus $L=E_2$ and ϕ is an F-isomorphism of E_1 onto E_2 as desired. The converse is trivial. ∎

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THE POWER AND GENERALIZED LOGARITHMIC MEANS

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1. Introduction. For nonnegative real numbers x and y and $\lambda \neq 0$, let

$$A(\lambda)=A(x,y;\lambda)=\left(\frac{x^\lambda+y^\lambda}{2}\right)^{1/\lambda} \tag{1.1}$$

be the λth power mean of x and y, and define

$$A(0)=\sqrt{xy}\ . \tag{1.2}$$

It is well known that $A(\lambda)$ is continuous and nondecreasing in λ, so, in particular,

$$A(0)\leqslant A\left(\frac{1}{3}\right)\leqslant A\left(\frac{2}{3}\right)\leqslant A(1). \tag{1.3}$$

Tung-Po Lin has proved [7, Theorem 1] that

$$L(x,y)=\frac{x-y}{\ln x-\ln y}\leqslant A\left(\frac{1}{3}\right) \tag{1.4}$$

with equality if and only if $x=y$. Theorem 1 of this paper is an inequality dual to (1.4). It turns out that Lin's Theorem 1 and the present Theorem 1 are two "singular cases" of a general family (see Theorem 2 below) of inequalities between power means and generalized logarithmic means. We show that Lin's method of proof already suffices to establish these general inequalities. As adumbrated in [6], stronger and even more general inequalities have been obtained by Leach and Sholander. It is hoped that this note will draw attention both to their interesting results and to the unsolved problems in [8].

Define

$$U(x,y)=e^{-1}(y^y/x^x)^{1/(y-x)}. \tag{1.5}$$

THEOREM 1. *We have*

$$A\left(\frac{2}{3}\right)\leqslant U(x,y) \tag{1.6}$$

with equality if and only if $x=y$.

The generalized logarithmic mean $u(x,y;\alpha)$ is defined by

$$u=u(\alpha)=u(x,y;\alpha)=\left[\frac{x^\alpha-y^\alpha}{\alpha(x-y)}\right]^{1/(\alpha-1)} \tag{1.7}$$

for $\alpha\neq 0$, 1. If we let $\alpha\to 0$ in (1.7) the average u tends to the logarithmic mean $L(x,y)$ that occurs on the left-hand side of (1.4); if we let $\alpha\to 1$ we get the average $U(x,y)$ that occurs on the right-hand side of (1.6). We remark that $u(-1)=A(0)$, $u(\frac{1}{2})=A(\frac{1}{2})$, $u(2)=A(1)$, and $u(\alpha)$ is nondecreasing in α. For a discussion of these means, see [8].

THEOREM 2. *If* $-1<\alpha<\frac{1}{2}$ *or* $2<\alpha$, *then*

$$u(\alpha)\leqslant A\left(\frac{\alpha+1}{3}\right). \tag{1.8}$$

If $\alpha<-1$ or $\frac{1}{2}<\alpha<2$, then (1.8) holds with the inequality *reversed*. Equality holds when $\alpha=-1$, $\frac{1}{2}$, or 2; otherwise equality can hold only when $x=y$. Moreover, (1.8) is not always true if $(\alpha+1)/3$ is replaced by any smaller number when α lies in the first two ranges, or by any larger number for α in the latter two ranges.

2. Proof of Theorem 1. We begin by giving rational bounds for ln t that are surprisingly good for $1 \leqslant t \leqslant 2$.

LEMMA 2.1. *For $t \geqslant 1$ we have*

$$\frac{8(t^3-1)}{3(t+1)^3} \leqslant \ln t \leqslant \frac{(t+1)(t^3-1)}{3t(t^2+1)}. \tag{2.1}$$

Proof. For the right-hand inequality, note that both sides are equal when $t=1$, so it suffices to show that the right side has a larger derivative for $t>1$. But this reduces to verification of the trivial inequality

$$0 \leqslant (t^3-1)(t-1)^2. \tag{2.2}$$

We omit the proof of the left-hand inequality since it is proved similarly and is not needed below.

LEMMA 2.2. *For $t \geqslant 1$ we have*

$$1 \leqslant \frac{\ln t}{t-1} + (3/2)\ln\left[\frac{2}{1+t^{-2/3}}\right]. \tag{2.3}$$

Proof. The right-hand side becomes continuous if we define it to have the value 1 at $t=1$. Hence it suffices to show that it has a nonnegative derivative for $t \geqslant 1$, in other words,

$$0 \leqslant -\frac{\ln t}{(t-1)^2} + \frac{1}{t(t-1)} + \frac{1}{t(1+t^{2/3})}. \tag{2.4}$$

In fact, it suffices to show this with t replaced by t^3, and this reduces to Lemma 1.

To prove the Theorem, we may assume $y>x$ and set $t=y/x$ in Lemma 2.2. We find that

$$(3/2)\ln\left[\frac{x^{2/3}+y^{2/3}}{2}\right] \leqslant -1+\ln y - \frac{x(\ln x - \ln y)}{y-x}, \tag{2.5}$$

and the result follows upon exponentiation. Note that strict inequality holds everywhere above, unless $t=1$.

It is easy to see that if $\lambda > 2/3$, there are values of x and y whose ratio lies arbitrarily close to 1 such that

$$U(x,y) < A(\lambda). \tag{2.6}$$

Expand the logarithm of each side of (2.6) as a power series in the parameter δ where $y/x = 1+\delta$. The terms of order zero and one on both sides are the same, while the right-hand side has a larger term of order δ^2.

In [7], Lin shows that (1.4) is similarly false if $1/3$ is replaced by a smaller number.

3. Proof of Theorem 2. We call the numbers -1, 0, .5, 1, 2 *nodes*. Theorem 2 for the nodes $\alpha=0$ and $\alpha=1$ are Theorem 1 of Lin's paper and Theorem 1 of this paper, respectively. Equality is immediate at the other nodes. Hence we can assume that α is not a node.

Let β be a real number different from -1, and set $t=y/x$. Without loss of generality, $t>1$. Set

$$\lambda = (\beta+1)/3, \quad z=t^\lambda, \quad \text{and } \mu=\lambda^{-1}. \tag{3.1}$$

Then

$$r = \frac{u(\alpha)}{A(\lambda)} = \alpha^{1/(1-\alpha)} 2^\mu \frac{(z^\mu - 1)^{1/(1-\alpha)}}{(z+1)^\mu (z^{\alpha\mu}-1)^{1/(1-\alpha)}}. \tag{3.2}$$

Now set $z=(1+w)/(1-w)$, where $-1<w<1$. We obtain

$$r^{1-\alpha}=\alpha\frac{(1+w)^{\mu}-(1-w)^{\mu}}{(1+w)^{\mu\alpha}-(1-w)^{\mu\alpha}}. \tag{3.3}$$

By the binomial expansion, the right-hand side of (3.3) is the ratio $g(\mu)/g(\mu\alpha)$, where

$$g(\mu)=1+(\mu-1)(\mu-2)w^2/3!+(\mu-1)(\mu-2)(\mu-3)(\mu-4)w^4/5!+\cdots. \tag{3.4}$$

Now

$$(\mu-1)(\mu-2)<(\mu\alpha-1)(\mu\alpha-2) \tag{3.5}$$

is equivalent to

$$(1-\alpha)(\alpha-\beta)<0. \tag{3.6}$$

This shows that, by varying β in a neighborhood of α, we can force the ratio $g(\mu)/g(\mu\alpha)$ to be either >1 or <1, provided w is sufficiently small. But $w\to 0$ as $t\to 1$, so the last assertion of the Theorem follows.

Next, let $\alpha=\beta$. Then the coefficients of w^2 are the same in $g(\mu)$ and $g(\mu\alpha)$, namely,

$$(2-\alpha)(1-2\alpha)/6(\alpha+1)^2. \tag{3.7}$$

Hence the general term of $[g(\mu)-g(\mu\alpha)]$ has the form

$$\frac{(2-\alpha)(1-2\alpha)}{6(\alpha+1)^2}[(\mu-3)(\mu-4)\cdots(\mu-2k+1)(\mu-2k)$$
$$-(\mu\alpha-3)(\mu\alpha-4)\cdots(\mu\alpha-2k+1)(\mu\alpha-2k)]\frac{w^{2k}}{(2k)!}. \tag{3.8}$$

For $q\geqslant 2$ the inequality

$$(\mu-2q+1)(\mu-2q)\geqslant(\mu\alpha-2q+1)(\mu\alpha-2q) \tag{3.9}$$

is equivalent to

$$(\alpha-1)/(\alpha+1)\geqslant 0, \tag{3.10}$$

a condition independent of q. Thus by grouping the factors on the right of (3.8) into pairs, we find that $g(\mu)-g(\mu\alpha)$ has the same sign as

$$(2-\alpha)(1-2\alpha)(\alpha-1)/(\alpha+1). \tag{3.11}$$

Since the exponent on the left of (3.3) is $1-\alpha$, the theorem follows.

4. Remarks. Lin proves in [7] that for $\lambda>0$ and any $\epsilon>0$, there are $x,y>0$ such that $L(x,y)/A(\lambda)<\epsilon$. For any generalized logarithmic mean $u(\alpha)$ with $2>\alpha>0$ no such phenomenon can occur, since for $t=y/x\geqslant 1$ we have

$$1\geqslant\frac{u(\alpha)}{A(1)}=2\alpha^{1/(1-\alpha)}(t-1)^{1/(1-\alpha)}/(t+1)(t^{\alpha}-1)^{1/(1-\alpha)}\to 2\alpha^{1/(1-\alpha)}>0 \tag{4.1}$$

as $t\to\infty$.

For $c>0$ let

$$P_c(u)=[u(1-u)]^{(c-2)/2}, \tag{4.2}$$

and for $t\neq 0$ define

$$M(t,c)=M(t,c;x,y) \tag{4.3}$$

by

$$[M(t,c)]^t\int_0^1 P_c(u)\,du=\int_0^1[ux+(1-u)y]^t P_c(u)\,du. \tag{4.4}$$

The averages $M(t,c)$ are studied in [1], [2], [5], [9] and are also mentioned in [4, p. 133 and p. 174]. The referee points out that the averages L, U, $u(\alpha)$, and $A(\alpha)$ studied here correspond (in the obvious limiting sense) to $M(-1,2)$, $M(0,2)$, $M(\alpha-1,2)$, and $M(\alpha,0)$. The referee poses the question of determining for fixed x and y whether, in its domain of definition, $M(t,c)$ varies monotonically along rays emanating from $(1,-1)$ in the (t,c) plane.

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ON THE WEIGHT DISTRIBUTIONS OF COSETS OF A LINEAR CODE

JAMES R. SCHATZ

In memory of William H. Reynolds

1. Introduction. In the study of the error-correction and error-detection capabilities of a linear code C, it is desirable to determine the weight distribution of each coset of C. For most codes this task is far too difficult to accomplish. In this note we consider a method of obtaining some partial information on the distribution of vectors in the cosets of C. All codes considered are linear and binary.

Let C be a code of length n over $F=GF(2)$. Let S denote the set of cosets of C. Note that any partition of S into disjoint subsets induces a partition of F^n. Our approach to obtaining partial information on the distribution of vectors in the cosets of C is to partition the set S into disjoint subsets and then to consider the distribution of vectors with respect to the induced partition of F^n. We first recall the necessary basic definitions.

C is called an (n,k,d)-code if C is a k-dimensional subspace of F^n and the minimum (Hamming) weight of a code word in C is d. The number $t=[(d-1)/2]$ is called the packing radius of C. The (Hamming) weight (that is, the number of nonzero components) of a vector $v\in F^n$ is denoted by $|v|$. A vector w is called a coset leader if $|w|\leqslant|v|$ for all $v\in w+C$. The weight of a coset $w+C$ is defined to be the weight of any coset leader of $w+C$.

2. The Parameters $N(m,s)$. We now consider the partition of the set S of cosets of C into subsets consisting of all cosets of a given weight. To study the distribution of vectors with respect to the induced partition of F^n, we introduce some parameters.

DEFINITION. Let C be an (n,k,d)-code. For $0\leqslant s\leqslant m\leqslant n$, $N(m,s)$ denotes the number of weight m vectors of F^n that belong to weight s cosets of C.

Two immediate consequences of this definition are: for $0\leqslant m\leqslant n$,

$$N(m,0)+N(m,1)+\cdots+N(m,m)=\binom{n}{m}, \tag{1}$$

and

$$N(m,s)=0 \qquad \text{if } m+s<d \quad\text{and}\quad m\neq s. \tag{2}$$

We obtain (1) by noting that every vector of weight m belongs to a coset of weight at most m, while (2) follows from the fact that if two vectors belong to the same coset of C their difference belongs to C.

The following theorem expresses $N(m,s)$, for $s \leqslant t$, in terms of the weight distribution of C. Some applications of this result are presented in the next section.

THEOREM. *Let C be an (n,k,d)-code with packing radius t. For $1 \leqslant s \leqslant t$ and $s \leqslant m \leqslant n$,*

$$N(m,s)=\sum_{r=0}^{s}\binom{m+s-2r}{m-r}\binom{n-(m+s-2r)}{r}A_{m+s-2r}, \tag{3}$$

where A_i denotes the number of weight i code words of C.

Proof. Fix m and s such that $1 \leqslant s \leqslant t$ and $s \leqslant m \leqslant n$. For $c \in C$ define

$$M(c)=\{v\in F^n: |v|=m \quad \text{and} \quad v=w+c \quad \text{for some} \quad w\in F^n \quad \text{with} \quad |w|=s\}.$$

Observe first that if $c_1, c_2 \in C$, $c_1 \neq c_2$, then $M(c_1)\cap M(c_2)=\varnothing$. For, if $w_1+c_1=w_2+c_2$, $|w_1|=|w_2|=s$, then $w_1+w_2=c_1+c_2\in C$, but $|w_1+w_2| \leqslant 2s \leqslant 2t < d$, which is a contradiction. A similar argument shows that, since $s \leqslant t$, each vector of weight s is the unique leader in its coset. Thus, if $v \in M(c)$ for some c, then v is a weight m vector in a weight s coset. On the other hand, if v is a weight m vector belonging to a coset with a weight s leader w, then $v+w\in C$ and $v\in M(v+w)$. Hence,

$$N(m,s)=\left|\bigcup_{c\in C} M(c)\right|=\sum_{c\in C}|M(c)|. \tag{4}$$

Now fix $c \in C$. If $v,w\in F^n$, $|v|=m$, $|w|=s$, $|v\cap w|=r$, and $c=v+w$, then $|c|=|v|+|w|-2|v\cap w|=m+s-2r$. Hence a necessary condition for $M(c)$ to be nonempty is that $|c|=m+s-2r$ for some r, $0\leqslant r\leqslant s$. Fix r, $0 \leqslant r \leqslant s$, and let $|c|=m+s-2r$. We claim that:

$$|M(c)|=\binom{m+s-2r}{m-r}\binom{n-(m+s-2r)}{r}. \tag{5}$$

To see this we need only note that a choice of $m-r$ coordinate places from the support of c together with a choice of r coordinate places from the complement of the support of c determines exactly one pair of vectors $v,w\in F^n$ with $|v|=m$, $|w|=s$, and $v+w=c$. (Of course, for some m, s, and r, the value of the second binomial coefficient in (5) will be 0.) Combining (4) and (5) we obtain (3).

3. Applications. A number of basic results of coding theory can be proved by considering the parameters $N(m,s)$. Some examples will now be given.

As noted in the proof of the theorem above, if C is an (n,k,d)-code with packing radius t, then for $s \leqslant t$ each vector of weight s is the unique leader in its coset. Since there are 2^{n-k} cosets of C, it follows that

$$\sum_{i=0}^{t}\binom{n}{i}\leqslant 2^{n-k}. \tag{6}$$

This is known as the Hamming (or sphere-packing) bound. A code C for which equality holds in (6) is called perfect. (All perfect linear codes are known. See [4].) We will first show how (3) together with (1) and (2) can be used to determine the weight distribution of a perfect code. It is remarkable that the complete weight distribution of a perfect code is determined by the three parameters n, k, and d.

(1) *The weight distribution of the $(n=2^r-1,\ k=n-r,\ d=3)$-Hamming code satisfies the recurrence relation $A_0=1$, $A_1=0$, and, for $m \geqslant 1$,*

$$(m+1)A_{m+1}+A_m+(n-m+1)A_{m-1}=\binom{n}{m}. \tag{7}$$

Proof. A Hamming code is perfect, with packing radius 1. Thus, a vector not in the code belongs to a coset of weight 1. That is, $N(m,1)=\binom{n}{m}-A_m$, for all m. However, by (3), $N(m,1)=(m+1)A_{m+1}+(n-m+1)A_{m-1}$, and (7) follows.

(2) *The* (23,11,7)-*Golay code has weight distribution* $A_0=A_{23}=1$, $A_7=A_{16}=253$, $A_8=A_{15}=506$, $A_{11}=A_{12}=1288$.

Proof. The binary Golay code is perfect, with packing radius 3. Hence for all m,

$$N(m,1)+N(m,2)+N(m,3)=\binom{23}{m}-A_m. \tag{8}$$

Since $d=7$, $A_1=A_2=\cdots=A_6=0$. Now by (2), (3), and (8):

$$\binom{23}{4}=N(4,3)=\binom{7}{4}A_7,$$

and so $A_7=253$. Similarly,

$$\binom{23}{5}=N(5,2)+N(5,3)=\binom{7}{5}A_7+\binom{8}{5}A_8,$$

from which $A_8=506$ follows. One continues in this manner using (3) and (8) to determine the complete weight distribution. We note, however, that in most definitions of the Golay code it is clear that the all-1 vector belongs to the code and so we deduce that $A_{23}=1$ and $A_m=A_{23-m}$ for all m. Knowing this shortens the computations.

(3) A generalization of (3) and (8) to codes over $GF(q)$ allows one to determine the weight distributions of the nonbinary Hamming codes as well as the ternary Golay code.

(4) Of course using (1), (2) and (3) as in the examples above we can obtain some of the familiar necessary conditions on a perfect code. For example, if an $(n,k,d=2t+1)$-code is perfect, we can argue that

$$\binom{n}{t+1}=N(t+1,t)=\binom{d}{t+1}A_d,$$

and so $\binom{d}{t+1}$ must divide $\binom{n}{t+1}$.

When s exceeds the packing radius of a code, $N(m,s)$ cannot be calculated in general. However, some simple estimates lead to the following refinement of the Hamming bound.

(5) The Specialized Johnson Bound [2]. *If C is an $(n,k,d=2t+1)$-code then*

$$\sum_{i=0}^{t}\binom{n}{i}+\left(\binom{n}{t+1}-\binom{n}{t}[(n-t)/(t+1)]\right)/[n/(t+1)]\leqslant 2^{n-k}. \tag{9}$$

Proof. Let a denote the number of cosets of weight $t+1$. Since the sum of two vectors in the same coset of C belongs to C, two vectors of weight $t+1$ in the same coset must have disjoint support. Moreover a weight $t+1$ vector in a weight t coset must be disjoint from the leader. Hence $N(t+1,t)\leqslant\binom{n}{t}[(n-t)/(t+1)]$ and $N(t+1,t+1)\leqslant a[n/(t+1)]$. But, by (1) and (2), $N(t+1,t)+N(t+1,t+1)=\binom{n}{t+1}$, so

$$\left(\binom{n}{t+1}-\binom{n}{t}[(n-t)/(t+1)]\right)/[n/(t+1)]\leqslant a.$$

Counting cosets we obtain (9).

The Johnson bound shows, for example, that there does not exist a (22,14,5)-code. It also rules out the existence of a perfect (90,78,5)-code. As an exercise the reader might consider the following problem: Does there exist a (15,8,5)-code?

Further applications of the results above are possible. Perhaps the most important feature of the $N(m,s)$ parameters is that they provide a unified approach to some of the elementary results of coding theory. For a more extensive introduction to coding theory the reader should see **[1]**, **[3]**, and **[5]**.

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COWLING'S THEOREM ON A DIRICHLET FINITE HOLOMORPHIC FUNCTION IN THE DISK

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Let $\mathfrak{D}$ be the family of functions f holomorphic in $U=\{|z|<1\}$ such that

$$S(f)\equiv \iint_U |f'(z)|^2 dx\,dy < +\infty \qquad (z=x+iy).$$

Set

$$\overline{M}(r,f)=\sum_{n=0}^{\infty}|a_n|r^n \qquad (0<r<1)$$

for $f(z)=\sum_{n=0}^{\infty}a_n z^n\in\mathfrak{D}$. V. F. Cowling [1, Theorem] proved that for each member $f\in\mathfrak{D}$,

$$\lim_{r\to 1}\left(\log\frac{1}{1-r}\right)^{-\frac{1}{2}}\overline{M}(r,f)=0. \tag{1}$$

Setting for $f\in\mathfrak{D}$,

$$M(r,f)=\max_{|z|=r}|f(z)|, \qquad 0<r<1,$$

and observing that $M(r,f)\leqslant\overline{M}(r,f)$, $0<r<1$, one can assert that $-\frac{1}{2}$ in (1) is best possible from the following:

THEOREM. *For each constant p, $0<p<\frac{1}{2}$, there exists $f\in\mathfrak{D}$ such that*

$$\liminf_{r\to 1}\left(\log\frac{1}{1-r}\right)^{-p}M(r,f)\geqslant 1. \tag{2}$$

Proof. Set

$$f(z)=\left(\frac{1}{z}\log\frac{1}{1-z}\right)^p=\sum_{n=0}^{\infty}a_n z^n \qquad (f(0)=1)$$

in U. It then follows from the estimate due to J. E. Littlewood [2, (2), p. 93] that there exist a constant $A>0$ and a natural number k such that

$$|a_n|\leqslant An^{-1}(\log n)^{p-1} \quad \text{for } n\geqslant k+1.$$

Therefore,

$$\begin{aligned}\pi^{-1}S(f)&=\sum_{n=0}^{\infty}n|a_n|^2\\ &\leqslant\sum_{n=0}^{k}n|a_n|^2+A^2\sum_{n=k+1}^{\infty}n^{-1}(\log n)^{2p-2}<+\infty\end{aligned}$$

because $2p-2<-1$. The inequality (2) now follows from

$$M(r,f) \geqslant f(r) = r^{-p}\left(\log\frac{1}{1-r}\right)^{p}, \qquad 0<r<1.$$

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UNSOLVED PROBLEMS

EDITED BY RICHARD GUY

In this department the MONTHLY *presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.*

HOW ALIKE ARE TWO TREES?

TEMPLE F. SMITH AND MICHAEL S. WATERMAN

Let τ_n be the set of binary trees with the same n labeled terminal vertices. (Such trees are connected graphs which have no cycles and with each vertex of valence one or three.) Let e be an interior edge joining vertices u and v. Deletion of e, u, and v produces four subtrees T^1, T^2, T^3, and T^4, where both T^1 and T^2 are adjacent to u and both T^3 and T^4 are adjacent to v. A **nearest neighbor interchange about** e consists in making T^1 adjacent to v and making one of T^3 or T^4 adjacent to u. See Figure 1.

Define the distance between tree $T\in\tau_n$ and $S\in\tau_n$ to be the minimum number of nearest neighbor interchanges to change tree T into tree S. An example of two trees in τ_7 with a distance of two is shown in Figure 2. Questions of interest include (i) an efficient algorithm to compute the distance and (ii) characterizing the pairs of trees in τ_n which maximize the distance. These problems have received some attention [2], but only partial results are known. Even the maximum distance attainable for two elements of τ_n is unknown.

There are many applications of such trees; the one motivating the problems here is that of representing possible evolutionary relationships of contemporary organisms. Many schemes exist for reconstructing evolutionary relationships, and they often give distinct binary trees. It is of interest, then, to compare these trees. The distance in this paper implies that we weigh speciation equally whether it occurred early or relatively recently.

Dobson [1] has given a survey of techniques to compare trees. None of the measures she gives seems to us to be satisfactory from a biological point of view, and therefore we devised the new metric on τ_n [2]. While we are satisfied with our metric, it lacks an efficient computation algorithm. For $n=6$, the metric has been analyzed by Fitch and Siegel [personal communication].

For a given interior edge e, there is an associated *partition* $\pi_e=\{A,B\}$ of the terminal vertices

into two sets. It is routine to show that the collection of all such partitions represents the tree [2].

We did devise a technique which was feasible for computation, but we could not prove that it always calculates the required distance. It searches for the least number of nearest neighbor interchanges to achieve a partition in T identical with that in S. Then the algorithm considers each "side" of the partition independently. Some results [2] given below suggested this technique. An optimal path is any sequence of nearest neighbor interchanges changing tree T into tree S which achieves the distance between T and S.

(i) If π^T is a partition in T and $\pi^S = \pi^T$ is a partition in S, then π^T will not change in any optimal path.

(ii) If an optimal path has partition π_e associated with edge e which is equal to a partition in S, then e is crossed at most once by any end vertex.

(iii) If T and S have no equal partitions but some nearest neighbor interchange in T yields an equal partition, then that nearest neighbor interchange is on some optimal path.

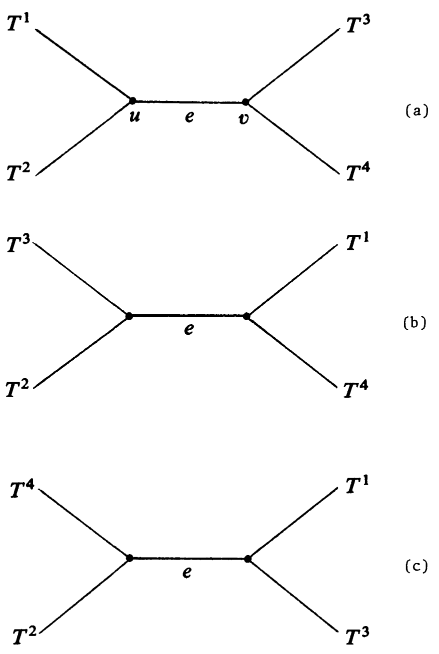

FIG. 1. Binary tree with subtrees T^1, T^2, T^3, and T^4 (a) and the two resulting trees from nearest neighbor interchanges about edge e.

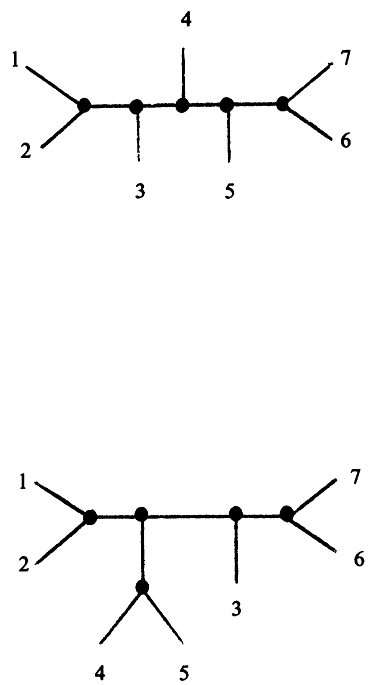

FIG. 2. Two binary trees a distance of two apart.

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COVERING SETS OF REALS

R. J. GARDNER

Suppose E is a set of real numbers and $\{r_n\}$ is a sequence of positive real numbers with $r_n \geqslant r_{n+1}$ for each n. Then E is said to have a **fine cover of order** $\{r_n\}$ if there exist open intervals I_n with $E \subseteq \cap_{m=1}^{\infty} \cup_{n=m}^{\infty} I_n$ and $|I_n| \leqslant r_n$. (Here $|I_n|$ denotes the length of I_n.)

The study of fine covers was motivated by a result of A. Hyllengren [4], which uses them to give necessary and sufficient conditions that a set of complex numbers be contained in the set of Valiron deficient values of a meromorphic function of finite order. (By replacing intervals by spheres and length by diameter we can study fine covers in any metric space. Hyllengren does so in the plane; here, however, we consider only sets of reals.) Previous attempts to estimate the size of such sets, using Hausdorff measures or capacity, were only partially successful. Very recently a further application has been found, by D. S. Lubinsky [6], in the theory of Padé approximation. Here the problem is to estimate the size of the exceptional set for convergence of Padé approximants. The solution, in the case of nondiagonal approximation, is again in terms of fine covers rather than Hausdorff measures or capacity; so the concept of a fine cover seems to be a useful one.

PROBLEM 1. If E has a fine cover of order $\{r_n\}$, and $0<k<1$, does E have a fine cover of order $\{kr_n\}$?

We conjecture that the answer is, in general, no. It is clearly true in special cases, for example if $r_{n+1} \leqslant kr_n$ for large n, or if $\Sigma r_n = \infty$, but we feel that a counterexample, even with E compact, exists. The construction of such a counterexample might be much harder if $\{r_n\}$ were prescribed, for example, $r_n = 1/n^2$.

Connections between the theory of fine covers and Hausdorff measure or capacity have been established in [1], [3], and [5], and have prompted some questions on fine covers for which the corresponding answers in measure theory are at least partially known. For example:

PROBLEM 2. If E is a Borel set with no fine cover of order $\{r_n\}$, is there a compact set $K \subseteq E$ which also has no fine cover of order $\{r_n\}$?

Again we conjecture the answer to be negative in general. To state a related problem, we need a definition. A set E has a **layering** of order $\{r_n\}$ if for each $m = 1,2,\ldots,$ there exist open intervals $I_n(m)$, $n = m, m+1, \ldots,$ with $E \subseteq \cup_{n=m}^{\infty} I_n(m)$ and $|I_n(m)| \leqslant r_n$, $n = m, m+1, \ldots.$

PROBLEM 3. If a set E has a layering of order $\{r_n\}$, does E have a fine cover of order $\{r_n\}$?

This can easily be proved for *compact E*; it follows that a counterexample that is a Borel set would also be a counterexample for Problem 2.

These problems are also stated in [2], where some related results may be found.

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THE LENGTH OF A LEMNISCATE

GEORGE PIRANIAN

The original question. In 1962, D. J. Newman [3] asked whether for each polynomial

$$P(z)=\prod_{m=1}^{n}(z-a_m)$$

the length $L(P,t)$ of the level set

$$E(P,t)=\{z:|P(z)|=t\}$$

is an increasing function of t.

The crude answer. If $n=1,2,\ldots$ and $Q_n(z)=z^n-1$, then the lemniscate $E(Q_n,1)$ is a rosette of n leaves, each of diameter greater than 1; therefore $L(Q_n,1)>2n$. On the other hand, the lemniscate $E(Q_n,2)$ consists of a simple closed curve that touches alternately the two circles

$$|z|=3^{1/n}\sim 1+(\log 3)/n \quad \text{and} \quad |z|=1.$$

It is fairly easy to see that the sequence $\{L(Q_n,2)\}$ is bounded. We deduce that there exist only finitely many indices n for which $L(Q_n,t)$ is an increasing function of t on the entire half-line $0\leqslant t<\infty$.

Some computations. Because the equation $Q_n(z)=1$ has the solution $z=2^{1/n}$, each leaf in the rosette $E(Q_n,1)$ has diameter $2^{1/n}$, and therefore

$$L(Q_n,1)>2n\cdot 2^{1/n}.$$

If $t>1$, the curve $E(Q_n,t)$ oscillates between the circles

$$|z|=(t+1)^{1/n} \quad \text{and} \quad |z|=(t-1)^{1/n},$$

touching each of them n times. Its length is clearly less than the length of the outer circle plus $2n$ times the distance between the two circles. In other words,

$$L(Q_n,t)<2\pi(t+1)^{1/n}+2n\left[(t+1)^{1/n}-(t-1)^{1/n}\right];$$

in particular,

$$L(Q_n,2)<2n\left[(1+\pi/n)3^{1/n}-1\right],$$

and therefore

$$\frac{L(Q_n,2)}{L(Q_n,1)}<\frac{(1+\pi/n)3^{1/n}-1}{2^{1/n}}$$

The numerator is a decreasing function of n, and its value is less than 1 when $n=6$. Also, its value is less than $2^{1/5}$ when $n=5$. Therefore $L(Q_n,2)<L(Q_n,1)$ when $n=5,6,\ldots$.

For the case $n=4$, we refine our method by estimating $L(Q_4,1)$ more carefully. In $E(Q_4,1)$ we inscribe a polygonal curve having vertices not only at the origin and on the circle $|z|=2^{1/4}$, but also on the circle $|z|=1$. Because $|Q_4(e^{i\theta})|=1$ when $4\theta=\pi/3$, the distance between the point $2^{1/4}$ and either of the two nearest points of $E(Q_4,1)$ on the unit circle is

$$\left[(2^{1/4}-\cos\pi/12)^2+(\sin\pi/12)^2\right]^{1/2}\sim 0.595.$$

This gives a lower bound 8(1.595) for $L(Q_4,1)$. Our upper bound for $L(Q_4,2)$ is approximately 8(1.34), and therefore $L(Q_4,2)<L(Q_4,1)$.

Residual questions and new problems. It appears highly plausible that $L(Q_3,t)<L(Q_3,1)$, for some t greater than 1. The case of Q_1 is trivial, because $L(Q_1,t)=2\pi t$. The determination whether $L(Q_2,t)$ is an increasing function may require extraordinarily delicate computations,

especially if the answer is affirmative. Conceivably, one can prove that the derivative of $L(Q_2,t)$ has a zero at $t=1$ and that it has no other zeros.

As long as a component $E_m(P,t)$ of $E(P,t)$ bounds only one finite domain and this domain contains no zeros of P', the length of $E_m(P,t)$ is an increasing function of t. To see this, we observe that some analytic function element

$$f(w)=\sum_0^\infty c_n w^n$$

provides an inverse mapping of the disk $|w|<t$ onto the interior of $E_m(P,t)$. By a theorem of G. H. Hardy (see [**1**, p. 270] or [**2**, Section 23]), the mean value of $|f'|$ on the circle $|w|=\tau$ is an increasing function of τ, for $0<\tau<t$, and therefore the length of $E_m(P,\tau)$ is also an increasing function of τ. If E_m is convex, we can obtain this result directly from elementary geometric considerations; but even if P is a cubic, some components of $E(P,t)$ may bound nonconvex finite domains containing exactly one zero of P (see [**4**, p. 1276]), and therefore the proof of the general assertion requires Hardy's theorem.

We have seen that for each index n, the function $L(Q_n,t)$ increases in the interval $0<t<1$. Do the left- and right-hand derivatives of $L(Q_n,t)$ exist at $t=1$, and are they equal? Is it true that for $n=3,4,\ldots$, the function $L(Q_n,t)$ decreases in some interval $1<t<T_n$ and increases everywhere in (T_n,∞)? What is the asymptotic behavior of T_n, as $n\to\infty$?

The problem is manageable in principle, for we can write the equation $|Q_n(re^{i\theta})|=t$ in the form

$$r^{2n}-2r^n\cos n\theta+1-t^2=0.$$

Solving this equation for r and using elementary calculus, we can express $L(Q_n,t)$ as a definite integral. Unfortunately, the integrand is so complicated that the determination of $\operatorname{sgn} dQ_n/dt$ is extremely difficult. Preliminary explorations with a computer may help.

Added in proof. For other computations concerning $L(Q_n,t)$, see I. N. Baker, Length of a graph, this MONTHLY, 71 (1964) 217-218.

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MISCELLANEA

41. The feeling which tempts persons to this problem [circle-squaring] is that which, in romance, made it impossible for a knight to pass a castle which belonged to a giant or an enchanter. I once gave a lecture on the subject: a gentleman who was introduced to it by what I said remarked, loud enough to be heard by all around, "Only prove to me that it is impossible, and I will set about it this very evening."

—A. De Morgan, A Budget of Paradoxes, vol. 2, p. 210 (1915; Dover reprint, 1954).

DIVISIBILITY PROPERTIES OF SOME CYCLOTOMIC SEQUENCES

J. C. LAGARIAS AND A. M. ODLYZKO

This note shows how some basic techniques of algebraic number theory can be used to answer completely some questions recently raised in the Problems section of this MONTHLY [6]. We consider the sequence

$$P_N = P_N(f) = \prod_{j=1}^{N} f(\zeta_N^j) \tag{1}$$

where $f(x)$ is a fixed polynomial with integer coefficients and $\zeta_N = \exp(2\pi i/N)$. Since P_N is an algebraic integer in $Q(\zeta_N)$ and invariant under all the automorphisms $\zeta_N \to \zeta_N^k$, $(k,N)=1$ of that field, P_N is actually a rational integer. We are interested in the divisibility properties of the sequence P_N. This problem was first examined by Pierce [3] and later treated exhaustively by D. H. Lehmer [2]. Using Lehmer's methods, we describe a finite computation which determines for a given prime power p^k the set of N for which $p^k | P_N$. We apply these methods to the special case $f(x) = x^4 + x + 1$ to prove generalized and corrected forms of conjectures made by P. Bruckman in [6].

We first give an alternative expression for P_N. Letting

$$f(x) = \sum_{i=0}^{d} a_i x^i = a_d \prod_{i=1}^{d} (x - \theta_i),$$

we find

$$P_N = a_d^N \prod_{j=1}^{N} \prod_{i=1}^{d} (\zeta_N^j - \theta_i) = a_d^N (-1)^{Nd} \prod_{i=1}^{d} (\theta_i^N - 1). \tag{2}$$

An immediate consequence of (2) is [2, p. 472]:

LEMMA 1. *$P_N(f)$ satisfies a linear recurrence of degree at most 2^d.*

Proof. Let $g_0(x) = x - a_d$ and $g_k(x) = \prod (x - a_d \theta_{i_1} \cdots \theta_{i_k})$ where the product runs over the $\binom{d}{k}$ distinct k-tuples of roots of $f(x)$. Note each $g_k(x) \in \mathbb{Z}[x]$ and set

$$g(x) = \text{l.c.m.}\{ g_k(x) \mid 0 \leqslant k \leqslant d \} = \sum_{l=0}^{L} c_l x^l,$$

a monic polynomial in $\mathbb{Z}[x]$ of degree at most 2^d. Then $u(N) = P_N$ satisfies the recurrence

$$\sum_{l=0}^{L} c_{L-l} (-1)^{ld} u(N-l) = 0 \tag{3}$$

since, by (2), P_N is a sum of elements of the form $(-1)^{Nd} a_d^N (\theta_{i_1} \cdots \theta_{i_k})^N$, each of which separately satisfies (3). ∎

For $f(x) = x^4 + x + 1$, which was considered in [6], $g_0(x) = g_4(x) = x - 1$; so $g(x)$ has degree $\leqslant 15$, showing that the maximum 2^d is not always attained. In fact, in this case $g(x) = (x-1)(x^4+x+1)(x^6-x^4-x^3-x^2+1)(x^4+x^3+1)$ has degree exactly 15 [2, p. 473].

In analyzing divisibility properties of $P_N(f)$, since $P_N(f \cdot g) = P_N(f) \cdot P_N(g)$, it suffices to consider the case that $f(x)$ is irreducible. At this point we designate the primes $p | a_0 a_d$ as *exceptional* and exclude them in what follows. They are exactly the primes $p | c_0 c_L$ for which the recurrence (3) is not invertible (mod p). For all nonexceptional primes p, the linear recurrence (3) by itself guarantees that $P_N(f)$ is periodic (mod p^k) and, hence, that the solutions to $p^k | P_N$ comprise a finite set of arithmetic progressions.

Our goal is to show that these arithmetic progressions can be described in terms of prime ideals in the field $K=Q(\theta)$ formed by adjoining a fixed root $\theta=\theta_1$ of $f(x)$. By definition of the norm $N_{K/Q}$ we can rewrite (2) as:

$$P_N=a_d^N(-1)^{Nd}N_{K/Q}(\theta^N-1). \tag{4}$$

Next we write the ideal factorization in K of a rational prime p as $(p)=\mathfrak{p}_1\cdots\mathfrak{p}_g$ where $\mathfrak{p}_i$ is of degree f_i, i.e., $N_{K/Q}\mathfrak{p}_i=p^{f_i}$. We can analyze divisibility of P_N by rational prime powers p^k by analyzing divisibility of the principal ideal $(1-\theta^N)$ by the prime ideals $\mathfrak{p}_i$; for if $\mathfrak{p}_1^{a_1}\cdots\mathfrak{p}_g^{a_g}\|1-\theta^N$ then from (4) we obtain

$$p^{a_1f_1+\cdots+a_gf_g}\|P_N. \tag{5}$$

(As usual, $p^\alpha\|n$ means $p^\alpha|n$ and $p^{\alpha+1}\nmid n$.) In what follows we will not consider rational primes p that do not factor into distinct prime ideals in K. Such p are called ramified primes. It is known that all the ramified primes divide Disc($f(x)$), the discriminant of $f(x)$, and hence there are only a finite number of them. The methods presented below could be extended to handle ramified nonexceptional primes.

We now consider divisibility of $(1-\theta^N)$ by a prime ideal $\mathfrak{p}$ lying over an unramified nonexceptional rational prime p. Note θ is invertible $(\operatorname{mod}\mathfrak{p})$ for such a prime ideal $\mathfrak{p}$, and we let $\operatorname{ord}_\mathfrak{p}\theta$ denote the least positive r for which $\theta^r\equiv 1\ (\operatorname{mod}\mathfrak{p})$.

LEMMA 2. *Let $\mathfrak{p}$ be a prime ideal in K lying over a rational prime p unramified in K, and suppose $\mathfrak{p}$ is relatively prime to θ. If $\mathfrak{p}\nmid(2)$ let $\mathfrak{p}^a\|(1-\theta^{\operatorname{ord}_\mathfrak{p}\theta})$. Then for all $r\geq 0$,*

$$\mathfrak{p}^{a+r}\|(1-\theta^N)\Leftrightarrow \operatorname{ord}_\mathfrak{p}\theta\,|\,N \quad\textit{and}\quad p^r\|N. \tag{6}$$

If $\mathfrak{p}|(2)$ let $\mathfrak{p}^a\|(1-\theta^{2\operatorname{ord}_\mathfrak{p}\theta})$. Then for all $r\geq 0$,

$$\mathfrak{p}^{a+r}\|(1-\theta^N)\Leftrightarrow \operatorname{ord}_\mathfrak{p}\theta\,|\,N \quad\textit{and}\quad 2^{r+1}\|N. \tag{7}$$

Proof. Suppose $\mathfrak{p}\nmid(2)$. We first prove $\mathfrak{p}^{a+r}\|(1-\theta^{p^r\operatorname{ord}_\mathfrak{p}\theta})$ by induction on r. It is true for $r=0$, and suppose

$$\theta^{p^r\operatorname{ord}_\mathfrak{p}\theta}=1+\beta \quad\text{with}\quad \mathfrak{p}^{a+r}\|(\beta).$$

Using the binomial theorem and the fact that $p\geq 3$,

$$\begin{aligned}\theta^{p^{r+1}\operatorname{ord}_\mathfrak{p}\theta}&=(1+\beta)^p\\&=1+p\beta+\binom{p}{2}\beta^2+\beta^3M\\&\equiv 1+p\beta\quad(\operatorname{mod}\mathfrak{p}^{a+r+2}).\end{aligned} \tag{8}$$

Since p is unramified, $\mathfrak{p}\|(p)$, and $\mathfrak{p}^{a+r+1}\|(\theta^{p^{r+1}\operatorname{ord}_\mathfrak{p}\theta}-1)$ completes the induction step. Next note that

$$\begin{aligned}\theta^{n\cdot p^r\operatorname{ord}_\mathfrak{p}\theta}&=(1+\beta)^n\\&\equiv 1+n\beta\ (\operatorname{mod}\mathfrak{p}^{a+r+1}).\end{aligned} \tag{9}$$

If $(n,p)=1$ then $\mathfrak{p}\nmid n$ and $\mathfrak{p}^{a+r}\|(\theta^{n\cdot p^r\operatorname{ord}_\mathfrak{p}\theta}-1)$, completing the proof in this case. When $p=2$, the proof is similar, except (8) holds only for $a+r\geq 2$. The hypothesis $\mathfrak{p}^a\|(1-\theta^{2\operatorname{ord}_\mathfrak{p}\theta})$ guarantees $a\geq 2$. ∎

In order to use Lemma 2, we must obtain the prime ideals $\mathfrak{p}$. The prime factorization of (p) in K can be obtained for almost all p by factoring $f(x)$ $(\operatorname{mod}p)$ [**1**, p. 27].

PROPOSITION. *Let $K=Q(\theta)$, $f(x)$ the minimal polynomial for θ, p a rational prime not dividing* Disc($f(x)$). *If*

$$f(x) \equiv f_1(x) \cdots f_g(x) \pmod{p} \tag{10}$$

where the $f_i(x)$ *are irreducible* mod$p$ *then the*

$$\mathfrak{p}_i = (p, f_i(\theta)) \tag{11}$$

are distinct prime ideals in K *lying over* (p),

$$(p) = \mathfrak{p}_1 \cdots \mathfrak{p}_g$$

and

$$N_{K/Q}\mathfrak{p}_i = p^{\deg f_i(x)}. \quad \blacksquare \tag{12}$$

Using (11), we can determine $\mathrm{ord}_{\mathfrak{p}_i}\theta$ directly as the smallest positive l with $x^l = 1$ in the finite field $\mathbb{Z}/p\mathbb{Z}[x]/(f_i(x))$. We can also calculate the value of a for which $\mathfrak{p}_i^a \| (1-\theta^l)$ by a more complicated version of this approach.

There are some restrictions on the possible values of $\mathrm{ord}_{\mathfrak{p}}\theta$ that can be inferred directly from knowledge of (12). Let 0_K denote the ring of integers of K.

LEMMA 3. *Let* $\mathfrak{p}$ *be a prime ideal in the field* $K = \mathbb{Q}(\theta)$ *lying over a rational prime* p, *and suppose* $\mathfrak{p}$ *is relatively prime to* θ. *Set* $N_{K/Q}\mathfrak{p} = p^f$, *noting* $f \leqslant d$. *Then*

(i) $\mathrm{ord}_{\mathfrak{p}}\theta \mid p^f - 1$,

(ii) $\mathrm{ord}_{\mathfrak{p}}\theta \nmid p^g - 1$ *for any* g *with* $0 < g < f$.

Proof. The residue class field $0_K/\mathfrak{p} = K_{\mathfrak{p}}$ is a finite field with p^f elements, which is generated over GF(p) by θ if $\mathfrak{p} \nmid (\theta)$. The multiplicative group of $K_{\mathfrak{p}}$ is cyclic, so $\mathrm{ord}_{\mathfrak{p}}\theta \mid p^f - 1$. Since θ generates $K_{\mathfrak{p}}$, it cannot be in any field GF(p^g) with $g < f$; hence $\mathrm{ord}_{\mathfrak{p}}\theta \nmid p^g - 1$. ∎

We can sometimes calculate $\mathrm{ord}_{\mathfrak{p}}\theta$ for all prime ideals $\mathfrak{p}$ dividing (p) in $\mathbb{Q}(\theta)$ directly from knowledge of the exact divisibility of the $P_l(f)$ by powers of p for small l and the factorization (10), by using Lemma 3 and (5). The following application illustrates this.

We add a word on calculating $P_N(f)$. For small n and d we can calculate $P_N(f)$ using (1) by approximating $f_N = \cos(2\pi/N) + i\sin(2\pi/N)$ numerically and using the fact that P_N is an integer. This method is extremely inefficient but easy to program. The most efficient method available expresses θ^N as a rational linear combination of $1, \theta, \ldots, \theta^{N-1}$:

$$\theta^N = a_{0N} + a_{1N}\theta + \cdots + a_{d-1,N}\theta^{d-1} \tag{13}$$

and uses the fact that each sequence $u(N) = a_{iN}$ satisfies the linear recurrence

$$\sum_{l=0}^{d} a_{d-l}u(N-l) = 0$$

of degree d. Then (4) is used to compute P_N. For details, see Lehmer [**2**, p. 472].

Application. Consider the polynomial $f(x) = x^4 + x + 1$ of Problem 6044. Then $\mathrm{Disc}(f(x)) = \mathrm{Disc}(K) = 229$ is a prime. Since $f(x)$ is irreducible (mod 2), the ideal (2) is prime in $Q(\theta)$ of norm $2^4 = 16$. By Lemma 3, $\mathrm{ord}_{(2)}\theta \mid 15$. In fact $2 \nmid P_3$ and $2 \nmid P_5$ while $2^4 \| P_{15}$. Hence $(2) \| 1 - \theta^{15}$. By computation $2^8 \| P_{30}$ so by (5), (7), and Lemma 2,

$$2^{4k} \| P_N \Leftrightarrow 15 | N \quad \text{and} \quad 2^{k-1} \| N,$$

which proves Bruckman's conjectures (A) and (B). (Note that our definition of P_N differs from Bruckman's by a factor of 3.) Next $f(x) \equiv (x^3 + x^2 + x - 1)(x-1) \pmod 3$ so $(3) = \mathfrak{p}_1\mathfrak{p}_2$ with $N_{K/Q}\mathfrak{p}_1 = 3$, $N_{K/Q}\mathfrak{p}_2 = 3^3$. In fact $3 \| P_1$, so $(1-\theta)$ can be divisible only by a first degree prime, i.e., $\mathfrak{p}_1 \| (1-\theta)$. Also $3^4 \| P_{13}$ while $\mathfrak{p}_1 \| (P_{13})$ by Lemma 2, so $\mathfrak{p}_2 \| (1-\theta^{13})$. (Note $13 | (3^3 - 1)$ as required.) Hence by Lemma 2 and (5),

$$3^k \| P_N \Leftrightarrow \text{either} \quad (1)\ 3^{k-1} \| N,\ 13 \nmid N$$

$$\text{or} \quad (2)\ 4 | k,\ 3^{k/4-1} \| N \text{ and } 13 | N,$$

which is a corrected form of Bruckman's conjectures (C) and (D). Next $(5)=\mathfrak{p}_1\mathfrak{p}_2$ with $N_{K/Q}\mathfrak{p}_1=5$, $N_{K/Q}\mathfrak{p}_2=5^3$. In fact $\mathrm{ord}_{\mathfrak{p}_1}\theta=4$, $\mathrm{ord}_{\mathfrak{p}_2}\theta=124$, $\mathfrak{p}_1\|(1-\theta^{124})$, so that

$$5|P_N \Leftrightarrow 4|N,$$

$$5^2|P_N \Leftrightarrow 20|N \text{ or } 124|N,$$

which shows conjecture (E) is correct, but (F) is incorrect. Next (7) is prime in K. Hence $7|P_{2400}$ by Lemma 3; so conjecture (G) is false. Also $7\nmid P_{48}$ and always occurs as a power of 7^4 when it divides any P_N. Finally $(11)=\mathfrak{p}_1\mathfrak{p}_2$ with $N_{K/Q}\mathfrak{p}_1=11$, $N_{K/Q}\mathfrak{p}_2=11^3$, $\mathrm{ord}_{\mathfrak{p}_1}\theta=10$, $\mathrm{ord}_{\mathfrak{p}_2}\theta=1330$, with associated $a(\mathfrak{p}_i)=1$, so that by Lemma 2:

$$11^k|P_N \Leftrightarrow 10|N \text{ and either}$$

$$\text{(i) } 11^{k-1}|N,\ 133\nmid N, \text{ or}$$

$$\text{(ii) } 4|k,\ 11^{k/4-1}\|N,\ 133|N,$$

which includes conjecture (H).

REMARKS. (i) Relatively little can be said about the exponent a for which $\mathfrak{p}^a\|(1-\theta^{\mathrm{ord}_{\mathfrak{p}}\theta})$. It is not always 1. For example, in the rationals $\mathrm{ord}_{1093}2=364$ but $1093^2|(1-2^{364})$. (ii) Using cyclotomic polynomials we may factor θ^N-1. Let $\Phi_d(x)$ be the dth cyclotomic polynomial. Then

$$\theta^N-1=\prod_{d|N}\Phi_d(\theta)$$

with $\Phi_d(\theta)\in K$. Postnikova and Schinzel [4] and Schinzel [5] have shown that for all $N\geqslant N_0(\theta)$, each $\Phi_N(\theta)$ contains a prime ideal divisor $\mathfrak{p}$ such that $\mathfrak{p}\nmid\Phi_k(\theta)$ for $1\leqslant k\leqslant N-1$. The analogous result for $N_{K/Q}\Phi_N(\theta)$ seems to be an open problem.

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MIKUSIŃSKI OPERATORS WITHOUT THE TITCHMARSH THEOREM

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1. Introduction. In 1950 J. Mikusiński published his fundamental papers on operational calculus [4], [5]. Although the theory has developed in many directions since then [1], [5], a reading of his book [5] indicates that his initial motivation was in part pedagogical; that is, how could one rigorously present Laplace-transform theory to engineers (and others) without the complex-variable techniques necessary to prove the inversion theorem? His elementary proof of the Titchmarsh theorem enabled him to obtain a field of operators in which most of the basic

operations of analysis could be replaced by straightforward algebraic manipulations. The resultant theory is elegant and enables one to handle many complicated analytic objects, such as delta functions, in a relatively elementary way. One would have hoped that this technique would have become standard in the teaching of operational methods. Apparently this has not been the case. No new texts have appeared in almost twenty years, and the method is rarely mentioned in either elementary or advanced treatises on differential equations or systems analysis.

From our teaching experience we feel that it is certainly possible to incorporate much of this material into a junior-level differential-equations course, as the algebra involved is no more complicated than that covered in a beginning linear-algebra course. As a result we would advocate using Mikusiński operators as a substitute for the more traditional Heaviside operators and/or the Laplace transform. The main obstacle, in our opinion, is the necessity of proving the Titchmarsh theorem, a nontrivial task. Although the proof given by Mikusiński in **[6]** is elementary, in that it uses no more than standard calculus theorems, it is long and generally above the level of many students in such courses. We propose, therefore, to develop an approach that entirely obviates the need for the Titchmarsh theorem. (For yet another approach to Mikusiński operators, see the article by Struble [7] and references therein.)

The argument we use rests on the ability to construct from a given ring a *ring* of quotients. The construction is standard **[8]** and was used by Mikusiński himself in another context. The result, as is shown in Section 2, enables one to obtain quickly much of the apparatus of operational calculus without the use of either complex-variable theory or the Titchmarsh theorem. Perhaps the recognition of the relatively elementary algebra needed to obtain a powerful analytic tool will spur the use of more modern techniques by both engineers and mathematicians.

2. Mikusiński Operators. Let $C(R_+)$ be the space of complex-valued continuous functions on $[0,\infty)$. Under pointwise addition and scalar multiplication, $C(R_+)$ is a complex vector space. If multiplication of f and g is defined by

$$(f*g)(t)=\int_0^t f(\tau)g(t-\tau)\,d\tau \tag{1}$$

then $C(R_+)$ becomes a commutative, associative algebra. (As usual, $f*g$ is called the convolution of f and g.) By virtue of the Titchmarsh theorem, $C(R_+)$ is an integral domain with respect to $*$, and $\mathfrak{M}$ will denote its field of quotients. $\mathfrak{M}$ is called the Mikusiński field and it is easily shown to be an algebra over the complex numbers **[2]**, **[6]**. It is well known that $\mathfrak{M}$ contains many of the objects necessary to do analysis on $[0,\infty)$. In particular, if we let

$$h(t)=1, \quad t\geqslant 0, \tag{2}$$

denote the Heaviside unit function, then $h^{-1}=s\in\mathfrak{M}$ is Mikusiński's generalized operator of differentiation **[6]**. It has the property that for $f\in C^1(R_+)$

$$sf=f'+\{f(0)\}\delta \tag{3}$$

where δ is the identity element in $\mathfrak{M}$ and $\{f(0)\}$ is a scalar element **[6]**. Since many of the applications of $\mathfrak{M}$ require only the existence of elements that are generated by s, the full field structure is not always needed. Because of this we examine a somewhat different construction of s from that given above. To do this requires the introduction of some algebra well known to algebraists but possibly not as familiar to analysts **[8]**. For completeness we include a brief summary.

DEFINITION. *Let $\mathfrak{R}$ be a ring. ($\mathfrak{R}$ does not necessarily have an identity.) A subset $\mathbb{S}$ of $\mathfrak{R}$ is said to be a multiplicative system if $a\in\mathbb{S}$ and $b\in\mathbb{S}\Rightarrow ab\in\mathbb{S}$.*

THEOREM. *Let $\mathfrak{R}$ be a ring and $\mathfrak{D}$ the set of elements that are not divisors of zero; then $\mathfrak{D}$ is a multiplicative system.*

Proof. Let $a \in \mathfrak{D}$ and $b \in \mathfrak{D}$. Suppose $ab \notin \mathfrak{D}$. Then there exists $c \neq 0$ such that $abc = 0$. Thus $bc = 0$, giving $c = 0$ since $b \in \mathfrak{D}$. Thus we arrive at a contradiction, showing that $\mathfrak{D}$ is closed under multiplication. ■

Now let $V = \mathfrak{R} \times \mathfrak{D}$. V can be made into a ring by defining addition and multiplication in the usual way:

$$(a,b)\times(c,d) = (ac,bd), \tag{4}$$

and

$$(a,b) + (c,d) = (ad + bc, bd).$$

$\times$ and $+$ are well defined since $\mathfrak{D}$ is a multiplicative system. We define a relation R on V by $(a,b)\mathrm{R}(c,d)$ iff $ad - bc = 0$. It can be shown that R is an equivalence relation and $+$ and $\times$ are preserved under R. Let $Q(V)$ be the set of equivalence classes of V under R. Since $+$ and $\times$ pass to the quotient, $Q(V)$ is a ring under the induced operations. Elements of $Q(V)$ are denoted by $[a,b], a \in \mathfrak{R}, b \in \mathfrak{D}$. $Q(V)$ has an identity δ given by $[a,a], a \in \mathfrak{D}$. In general, though, $Q(V)$ is not a field. However, it is easily shown that $[a,b]$ has an inverse in $Q(V)$ iff $a \in \mathfrak{D}$, and then $[a,b]^{-1} = [b,a]$. In addition, if $\mathfrak{R}$ is a vector space over a field K then $Q(V)$ has the structure of an algebra **[2]**.

The above construction is now applied to $C(R_+)$. Although it is known that $C(R_+)$ is a domain, we do not make that assumption. First, observe that $\mathfrak{D}$ is nonempty, since $h \in \mathfrak{D}$. In fact if $h * f = 0$ then $\int_0^t f(\tau)\,d\tau = 0$. Since f is continuous, differentiation gives $f(t) = 0, t \geqslant 0$. Thus $h * f = 0 \Rightarrow f = 0$ so that h is not a zero divisor. Consequently $Q(V), V = C(R_+) \times \mathfrak{D}$, is well defined.

It is now immediate that $s \in Q(V)$, since h can be represented in $Q(V)$ by $[h^2, h] = \mathcal{H}$, and $\mathcal{H}^{-1} = [h, h^2] = s$ exists. One may now use $Q(V)$ to solve ordinary differential equations with constant coefficients in the usual way. We illustrate this with an example.

Example. Consider the initial value problem

$$\begin{cases} u'' + u = f(t), \\ u(0) = u_0, \quad u'(0) = v_0, \quad f(t) \in C(R_+). \end{cases} \tag{5}$$

Since $u'' \in C^2(R_+)$ we get, using (3), that

$$u'' = s^2 u - \{u_0\} s - \{v_0\}\,\delta.$$

Thus

$$s^2 u + u = f(t) + \{u_0\}\,\delta + \{v_0\}\,\delta. \tag{6}$$

Now

$$s^2 u + u = (s^2 + \delta) u = (s + i\delta)(s - i\delta) u.$$

If we observe that $(s - \lambda\delta) e^{\lambda t} = \delta$, then $(s - \lambda\delta)^{-1}$ exists and (6) can be solved as

$$u = u_0 (s^2 + \delta)^{-1} s + (s^2 + \delta)^{-1} v_0 + (s^2 + \delta)^{-1} f.$$

But

$$\begin{aligned} (s^2 + \delta)^{-1} &= (s - i\delta)^{-1}(s + i\delta)^{-1} \\ &= e^{-it} * e^{it} = \sin t \end{aligned}$$

and

$$(s^2 + \delta)^{-1} s = s \sin t = \cos t,$$

giving

$$u(t) = u_0 \cos t + v_0 \sin t + \int_0^t \sin(t - \tau) f(\tau)\,d\tau.$$

Further application of the theory requires the introduction of an appropriate notion of sequential convergence. In [6] Mikusiński gave the following definition.

DEFINITION. *Let* $\{a_n\}$ *be a sequence in* $\mathfrak{M}$. *We say that* $\{a_n\}$ *converges to a iff there exists a function* $p \in C(R_+)$ *such that* pa_n *and* $pa \in C(R_+)$ *for all n, and* pa_n *converges uniformly on compact subsets of* R_+ *to* pa.

Using the fact that $\mathfrak{M}$ is a field, it was shown that limits are unique and that many properties of convergence in linear topological spaces hold. (Note: Convergence in $\mathfrak{M}$ is not topological [7].) Since we do not assume that $Q(C(R_+)\times\mathfrak{M})$ is a field, a slight modification of Mikusiński's definiton of convergence is required.

DEFINITION. *Let* $\{a_n\}$ *be a sequence in* $Q(C(R_+)\times\mathfrak{D})$. *We say that* $\{a_n\}$ *converges to* $a \in Q$ *iff there exists* $p \in \mathfrak{D}$ *such that pa and* $pa_n \in C(R_+)$ *for all n, and* pa_n *converges uniformly on compact subsets of* R_+ *to* pa.

THEOREM 2. *If* $\{a_n\}$ *converges to a then a is unique.*

Proof. Assume that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} a_n = b$ and $b \neq a$. Then there exist $p \in \mathfrak{D}$ and $q \in \mathfrak{D}$ such that $pa_n \to pa$ and $qa_n \to qb$. This gives $\lim_{n\to\infty} qpa_n = \lim_{n\to\infty} pqa_n = pqa$ and $\lim_{n\to\infty} pqa_n = pqb$. By uniqueness of limits in $C(R_+)$ we get $pq(a-b)=0$. Since $pq \in \mathfrak{D}, a=b$, giving a contradiction. ■

As in [6] the expected properties of convergence follow. Using these, it is possible to introduce series of operators and to develop the theory of partial differential equations.

Finally we point out that other operational calculi may be treated in a similar way [1]. The crux of our approach is that one need find only algebraic systems in which some, but not necessarily *all*, elements have inverses. For practical purposes this removes the necessity of establishing theorems of the Titchmarsh type and should expand the scope of application of modern operational methods.

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A METRIC MODEL OF PLANE EUCLIDEAN GEOMETRY

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1. Introduction. The purpose of this note is to present a model of Euclidean geometry that satisfies all the axioms of Birkhoff's metric approach. Besides giving a proof of the relative consistency of Euclidean geometry, the model will provide the student in a typical sophomore- or junior-level college course in geometry with an interesting example of the interplay of various fields in mathematics. Our techniques will involve those from geometry, linear algebra, calculus, and differential equations. The model is simple enough that many of the details can be left to the student.

This model grew out of the realization by the author that unlike many college courses (e.g., linear algebra, abstract algebra, topology, differential geometry, probability, analysis) elementary geometry suffers from a lack of examples. It is the nature of the subject to develop in an axiomatic manner. Typically, only at the end of a course is the consistency of the axioms proved by giving a model. Further, we know that there are very few models available: any two models of Euclidean plane geometry are isometric, and so are any two models of hyperbolic geometry once the unit of length is normalized.

It is the author's feeling that a course in elementary geometry from an advanced standpoint, such as given by Moise [2], is improved by the presence of a model for which the various axioms can be tested as presented. Hence we developed the model that follows. (Similar considerations led to Millman's development of the upper-half-plane model of hyperbolic geometry in [1].) It should be noted that the model presented by Moise [2, p. 373] does not contain the development of angle measurement but settles for the angular congruence axioms. Furthermore, his verification depends rather heavily on the use of isometries and thus must appear rather late in the course.

The general flavor of the presentation is linear algebraic, although we assume no knowledge of linear algebra other than addition, scalar multiplication, and the inner product of ordered pairs. Our motivation comes from linear algebra and a desire to exploit the standard inner product. The development of angular measure, which will use an integral definition of the inverse cosine function, could easily be expanded to a full treatment of trigonometry from an analytic viewpoint. Our notation and terminology will be as in [2].

2. The Incidence, Distance, and Plane Separation Axioms. The underlying set for our model will be $\mathbb{S}=\mathbb{R}^2$. If $P\neq Q$ both belong to $\mathbb{S}$, we set $L_{PQ}=\{X=P+t(Q-P)|t\in\mathbb{R}\}$ and let our set of lines be $\mathcal{L}=\{L_{PQ}|P\neq Q\}$.

THEOREM 1. $\{\mathbb{S},\mathcal{L}\}$ *is a plane incidence geometry; i.e.,*
(a) *Given* $P\neq Q$ *in* $\mathbb{S}$, *there is a unique line* $L\in\mathcal{L}$ *with* $P, Q\in L$.
(b) *Every line has at least two points.*
(c) *There exist three noncollinear points.*

Proof. The only part students find difficult is the uniqueness in (a). If $P, Q\in L_{RS}$ it is easy to show that $L_{PQ}\subset L_{RS}$ since $P=R+u(S-R)$ and $Q=R+v(S-R)$. These equations may be inverted to $R=P-(u/(v-u))(Q-P)$ and $S=P+((1-u)/(v-u))(Q-P)$ to imply $L_{RS}\subset L_{PQ}$, so that $L_{PQ}=L_{RS}$. ∎

We shall denote the inner product of $X=(x_1,x_2)$ and $Y=(y_1,y_2)$ by $\langle X,Y\rangle=x_1y_1+x_2y_2$. As usual, the length of X is $\|X\|=\sqrt{\langle X,X\rangle}$.

In the development of angular measure, the Cauchy-Schwarz Inequality will be needed. Since it can also be used to prove the triangle inequality for our distance function, it is reasonable to introduce it now.

LEMMA 2 (Cauchy-Schwarz Inequality). $|\langle X,Y\rangle|\leqslant\|X\|\,\|Y\|$ *with equality if and only if either* $Y=(0,0)$ *or* $X=tY$ *for some* $t\in\mathbb{R}$.

Proof. Consider the discriminant of the quadratic $f(t)=\|X-tY\|^2$ and note that f cannot have distinct real roots. ∎

The distance function will be familiar Euclidean distance expressed in vector form: $d(P,Q)=\|P-Q\|$.

THEOREM 3. *The function* $d\colon\mathbb{S}\times\mathbb{S}\to\mathbb{R}$ *given by* $d(P,Q)=\|P-Q\|$ *is a metric, i.e.,*
(a) $d(P,Q)\geqslant 0$;
(b) $d(P,Q)=0$ *iff* $P=Q$;
(c) $d(P,Q)=d(Q,P)$;
(d) $d(P,Q)+d(Q,R)\geqslant d(P,R)$.

Proof. In (d) the students may need the hint: Show $\|X+Y\| \leqslant \|X\|+\|Y\|$. ∎

To obtain the ruler postulate, it is necessary to show first that if $R \in L_{PQ}$ then there is a *unique* value of t such that $R=P+t(Q-P)$. This makes the f given below well defined.

THEOREM 4. *If $L=L_{PQ}$, then f: $L \to \mathbb{R}$ given by $f(P+t(Q-P))=t\|Q-P\|$ is a coordinate function; i.e., f is one-to-one, onto, and $d(R,S)=|f(R)-f(S)|$.*

At this point we should note that B is between A and C if and only if $B=A+t(C-A)$ with $0<t<1$. Also, the line segment between A and C is $\overline{AC}=\{A+t(C-A)|0 \leqslant t \leqslant 1\}$.

To obtain the plane separation axiom we exploit the inner product. If $X=(x_1,x_2)$ we let $X^{\perp}$ denote $(-x_2,x_1)$. Note $\langle X,X^{\perp}\rangle=0$. The following lemma is needed here and in Section 4.

LEMMA 5. *$P \in L_{AB}$ if and only if $\langle P-A,(B-A)^{\perp}\rangle=0$.*

Proof. To show the *if* part, we first show that if $\langle Z,X^{\perp}\rangle=0$ with $Z=(z_1,z_2)$ and $X=(x_1,x_2)$ $\neq(0,0)$, then either $Z=(z_1/x_1)X$ or $Z=(z_2/x_2)X$. Hence $\langle P-A,(B-A)^{\perp}\rangle=0$ implies $P-A=t(B-A)$ and $P=A+t(B-A)$ for some $t \in \mathbb{R}$. ∎

THEOREM 6. *Given a line L, the set $\mathbb{S}-L$ is a union of two disjoint convex sets H^+ and H^- such that if $P \in H^+$ and $Q \in H^-$, then the segment $\overline{PQ}$ intersects L.*

Proof. Let $L=L_{AB}$ and set $H^+=\{X|\langle X-A,(B-A)^{\perp}\rangle>0\}$ and $H^-=\{X|\langle X-A,(B-A)^{\perp}\rangle<0\}$. Clearly $H^+ \cap H^-=\varnothing$ and, by Lemma 5, $\mathbb{S}-L=H^+ \cup H^-$. Convexity can be established by a straightforward calculation.

Finally, let $Y=(P-A)^{\perp}$ so that $\langle P-A,Y\rangle>0$ and $\langle Q-A,Y\rangle<0$. Then $0<t<1$, where $t=(\langle P-A,Y\rangle)/(\langle P-A,Y\rangle-\langle Q-A,Y\rangle)$, and $P+t(Q-P) \in \overline{PQ} \cap L_{AB}$ by Lemma 5. ∎

Recall that we say D is in the interior of $\angle BAC$ if B and D are on the same side of L_{AC} and if C and D are on the same side of L_{AB}. In our notation this is given in the following lemma.

LEMMA 7. *Given $\angle BAC$ and a point D, let $X=B-A$, $Y=C-A$ and $Z=D-A$. Then D is in the interior of $\angle BAC$ if and only if both*
(1) *$\langle Z,X^{\perp}\rangle$ and $\langle Y,X^{\perp}\rangle$ have the same sign, and*
(2) *$\langle Z,Y^{\perp}\rangle$ and $\langle X,Y^{\perp}\rangle$ have the same sign.*

3. The Cosine Function. Now we are ready to start the hard part—angle measurement. We would like to use the familiar formula $\langle X,Y\rangle=\|X\|\|Y\|\cos\theta$ and the inverse cosine function. To do this we need a definition of cosine that does not depend on geometry (similar triangles) or angle measurement. This will be done by defining $\cos^{-1}x$ as an integral. This in turn requires knowing that the improper integrals $\int_0^1(dt/\sqrt{1-t^2})$ and $\int_{-1}^0(dt/\sqrt{1-t^2})$ both converge. By symmetry and the additivity of the integral we need show only that $\int_{1/2}^1(dt/\sqrt{1-t^2})$ converges.

Let b be a real number with $1/2<b<1$. We integrate by parts:

$$\int_{1/2}^{b}\frac{dt}{\sqrt{1-t^2}}=\int_{1/2}^{b}\frac{1}{t}\frac{t\,dt}{\sqrt{1-t^2}}=\left.\frac{-\sqrt{1-t^2}}{t}\right|_{1/2}^{b}-\int_{1/2}^{b}\frac{\sqrt{1-t^2}}{t^2}\,dt.$$

Since $\sqrt{1-t^2}/t^2$ is continuous on $[1/2,1]$, $\lim_{b\to 1^-}\int_{1/2}^{b}(\sqrt{1-t^2}/t^2)\,dt$ exists. Thus

$$\lim_{b\to 1^-}\int_{1/2}^{b}\frac{dt}{\sqrt{1-t^2}}=2\sqrt{3/4}-\int_{1/2}^{1}\frac{\sqrt{1-t^2}}{t^2}\,dt,$$

and the improper integral converges.

We let $\int_0^1(dt/\sqrt{1-t^2})=p/2$ ($p=\pi$, but this is irrelevant) and define the inverse cosine function by

$$I(x)=\frac{p}{2}-\int_0^x \frac{dt}{\sqrt{1-t^2}} \quad \text{for } -1\leqslant x\leqslant 1.$$

$I(x)$ is differentiable for $-1<x<1$: $I'(x)=-1/\sqrt{1-x^2}$. Hence $I(x)$ is continuous for $-1<x<1$, and also at -1 and 1 by the definition of convergent improper integrals. Since $I'(x)<0$, $I(x)$ is strictly decreasing and hence one-to-one. Thus I is a bijection from $[-1,1]$ to $[0,p]$.

We denote the inverse function of $I(x)$ by $c(x)$. Thus c: $[0,p]\to[-1,1]$. Since $I'(x)\neq 0$, $c(x)$ has a derivative for $0<x<p$: $c'(x)=1/I'(c(x))=-\sqrt{1-c^2(x)}$. If we let $s(x)=\sqrt{1-c^2(x)}$, we obtain $s'(x)=c(x)$, $c'(x)=-s(x)$, $s''(x)=-s(x)$, and $c''(x)=-c(x)$.

We shall need the addition law for cosines. This will be obtained using uniqueness of solutions of differential equations.

LEMMA 8. *Suppose $0<b\leqslant p$ and that f: $[0,b]\to\mathbb{R}$ is continuous with $f''(x)=-f(x)$ for $0<x<b$. Then there exist unique real numbers A, B with $f(x)=Ac(x)+Bs(x)$ for $0\leqslant x\leqslant b$.*

Proof. Let $a\in\mathbb{R}$ with $0<a<b$. The following initial value problem has at most one solution for $0<x<b$:

$$y''(x)=-y(x), \qquad y(a)=f(a), \quad y'(a)=f'(a). \tag{3-1}$$

(If the general uniqueness theorem for initial value problems is not available, then an ad hoc argument will have to be given.)

Clearly if $y=Ac(x)+Bs(x)$ then $y''=-y$. Thus y solves Problem (3-1) if and only if

$$\begin{cases} Ac(a)+Bs(a)=f(a) \\ -As(a)+Bc(a)=f'(a). \end{cases} \tag{3-2}$$

Since $c(a)c(a)-(-s(a))s(a)=c^2(a)+s^2(a)=1\neq 0$, Equation (3-2) does have a unique solution for A and B. With these values of A and B, y solves Problem (3-1) and thus must equal $f(x)$, which also solves (3-1). Hence

$$f(x)=Ac(x)+Bs(x) \quad \text{for } 0<x<b.$$

By continuity this also holds at $x=0$ and $x=b$. ■

LEMMA 9. *If x,y, and $x+y$ are all in $[0,p]$, then $c(x+y)=c(x)c(y)-s(x)s(y)$.*

Proof. We treat y as a constant and x as a variable. If $y=0$ or if $y=p$ (so that $x=0$), the result follows easily. Thus we assume $0<y<p$ and let $f(x)=c(x+y)$. By Lemma 8 with $b=p-y$ we have $c(x+y)=Ac(x)+Bs(x)$. $f'(0)$ exists since f is differentiable on $(-y,p-y)$ and $0<y<p$. Thus $A=f(0)=c(y)$ and $B=f'(0)=c'(y)=-s(y)$. ■

4. The Angle Measurement Axioms. For convenience of notation we shall develop radian measure instead of degree measure. Suppose that A,B,C are not collinear and let $X=B-A$, $Y=C-A$. Then $X\neq 0$, $Y\neq 0$ and, by Lemma 2, $-1\leqslant\langle X,Y\rangle/\|X\|\,\|Y\|\leqslant 1$. Thus it makes sense to define $\mathrm{m}\angle BAC=I(\langle X,Y\rangle/\|X\|\,\|Y\|)$.

THEOREM 10. $0<\mathrm{m}\angle BAC<p$.

Proof. Since A,B,C are not collinear, $X\neq tY$ and $Y\neq 0$. Thus $-1<\langle X,Y\rangle/\|X\|\,\|Y\|<1$ by Lemma 2. Since $I(X)$ is one-to-one, $0<\mathrm{m}\angle BAC<p$. ■

THEOREM 11. *Let $\overrightarrow{AB}$ be a ray in the edge of a half-plane H, and let r be a real number with $0<r<p$. Then there exists a unique ray $\overrightarrow{AP}$ with $P\in H$ and $\mathrm{m}\angle PAB=r$.*

Proof. Let $X=(B-A)/\|B-A\|$. Let W be either $X^{\perp}$ or $-X^{\perp}$ so that $H=\{P|\langle P-A,W\rangle>0\}$. Let $P=A+c(r)X+s(r)W$, and let $B'=A+X$. Then $\angle PAB=\angle PAB'$ and $\mathrm{m}\angle PAB'=r$. Since $\langle P-A,W\rangle=s(r)>0$, then $P\in H$, giving existence.

Suppose $P'\in H$ with $m\angle P'AB'=r$. We may assume that $\|P'-A\|=1$ and let $Z=P'-A$. We claim $Z=\langle Z,X\rangle X+\langle Z,W\rangle W$. Now $\langle Z-\langle Z,W\rangle W,W\rangle=\langle Z,W\rangle-\langle Z,W\rangle=0$ so that $A+Z-\langle Z,W\rangle W\in L_{AB'}$ by Lemma 5. Thus $Z-\langle Z,W\rangle W=tX$ for some $t\in\mathbb{R}$, and t must be $\langle Z,X\rangle$. Now $r=m\angle P'AB'=I(\langle X,Z\rangle/\|X\|\,\|Y\|)=I(\langle X,Z\rangle)$ and $c(r)=\langle X,Z\rangle$. Since $\|Z\|=1$, we have $\langle Z,W\rangle=\pm\sqrt{1-\langle Z,X\rangle^2}=\pm\sqrt{1-c^2(r)}=\pm s(r)$. Since $P'\in H$ we know $\langle Z,W\rangle>0$ and $\langle Z,W\rangle=s(r)$. Thus $Z=c(r)X+s(r)W$ and $P'=P$. ∎

THEOREM 12. *If $\angle BAD$ and $\angle DAC$ form a linear pair (so that A is between B and C) then* $m\angle BAD+m\angle DAC=p$.

Proof. We may assume that $B-A=X, C-A=Y$, and $D-A=Z$ where $\|X\|=\|Y\|=\|Z\|=1$. Note that $Y=-X$. Since $I(-x)=p-I(x)$, $m\angle BAD+m\angle DAC=I(\langle X,Z\rangle)+I(\langle Z,Y\rangle)=I(\langle X,Z\rangle)+I(-\langle X,Z\rangle)=p$. ∎

LEMMA 13. *If D is in the interior of $\angle BAC$, then* $m\angle BAD<m\angle BAC$.

Proof. Let $X=B-A, Y=C-A, Z=D-A$ and assume $\|X\|=\|Y\|=\|Z\|=1$. Since $D\in\operatorname{Int}\angle BAC$, $\langle Z,X^{\perp}\rangle$ and $\langle Y,X^{\perp}\rangle$ have the same sign. Choose $W=\pm X^{\perp}$ so that $\langle Z,W\rangle>0$. As in the proof of Theorem 11, we have $Y=c(r)X+s(r)W$ and $Z=c(\rho)X+s(\rho)W$, where $r=m\angle BAC$ and $\rho=m\angle BAD$. Now $Z^{\perp}=\pm(s(\rho)X-c(\rho)W)$. Since D is interior to $\angle BAC$, $\langle X,Z^{\perp}\rangle$ and $\langle Y,Z^{\perp}\rangle$ have opposite signs. Thus the quantities

$$\langle X,s(\rho)X-c(\rho)W\rangle=s(\rho)$$

and

$$\langle c(r)X+s(r)W,\, s(\rho)X-c(\rho)W\rangle=c(r)s(\rho)-s(r)c(\rho)$$

have opposite signs. Hence $c(r)s(\rho)-s(r)c(\rho)<0$ and $c(r)/s(r)<c(\rho)/s(\rho)$. It is easy to show that $f(t)=c(t)/s(t)$ is decreasing, so $r>\rho$. ∎

THEOREM 14. *If $D\in\operatorname{Int}\angle BAC$ then* $m\angle BAC=m\angle BAD+m\angle DAC$.

Proof. Combining Theorem 12 and Lemma 13, we have $m\angle BAD+m\angle DAC<p$. Hence it is sufficient to prove that $c(m\angle BAC)=c(m\angle BAD+m\angle DAC)$, since $c(x)$ is one-to-one.

As before, we assume $X=B-A, Y=C-A, Z=D-A$ with $\|X\|=\|Y\|=\|Z\|=1$. Thus $m\angle BAC=I(\langle X,Y\rangle)$, $m\angle BAD=I(\langle X,Z\rangle)$, and $m\angle DAC=I(\langle Y,Z\rangle)$. Hence

$$\begin{aligned}c(m\angle BAD+m\angle DAC)&=c(I(\langle X,Z\rangle)+I(\langle Y,Z\rangle))\\&=\langle X,Z\rangle\langle Y,Z\rangle-\sqrt{1-\langle X,Z\rangle^2}\,\sqrt{1-\langle Y,Z\rangle^2}\,.\end{aligned}$$

Now

$$\langle X,Z^{\perp}\rangle\langle Y,Z^{\perp}\rangle=\pm\sqrt{1-\langle X,Z\rangle^2}\,\sqrt{1-\langle Y,Z\rangle^2}\,.$$

Since $D\in\operatorname{Int}\angle BAC$, $\langle X,Z^{\perp}\rangle$ and $\langle Y,Z^{\perp}\rangle$ have opposite signs and thus $\langle X,Z^{\perp}\rangle\langle Y,Z^{\perp}\rangle<0$. Hence $\langle X,Z^{\perp}\rangle\langle Y,Z^{\perp}\rangle=-\sqrt{1-\langle X,Z\rangle^2}\,\sqrt{1-\langle Y,Z\rangle^2}$. Thus

$$\begin{aligned}c(m\angle BAD+m\angle DAC)&=\langle X,Z\rangle\langle Y,Z\rangle+\langle X,Z^{\perp}\rangle\langle Y,Z^{\perp}\rangle\\&=\langle\langle X,Z\rangle Z+\langle X,Z^{\perp}\rangle Z^{\perp},Y\rangle\\&=\langle X,Y\rangle=c(I(\langle X,Y\rangle))=c(m\angle BAC).\ \blacksquare\end{aligned}$$

The final angular-measure result needed is the SAS theorem.

THEOREM 15. *Given $\triangle ABC$ and $\triangle DEF$ with $\overline{AB}\simeq\overline{DE}$, $\angle ABC\simeq\angle DEF$, and $\overline{BC}\simeq\overline{EF}$, then* $\triangle ABC\simeq\triangle DEF$.

Proof. First verify the law of cosines for $\triangle PQR$:

$$d(P,R)^2=d(P,Q)^2+d(Q,R)^2-2d(P,Q)d(Q,R)c(m\angle PQR),$$

and then use it repeatedly on $\triangle ABC$ and $\triangle DEF$. ∎

5. The Parallel Axiom. The easiest way to show that our model is Euclidean is to exhibit a rectangle, since the existence of such is equivalent to the parallel axiom [**2**, p. 353].

THEOREM 16. *If* $A=(0,0)$, $B=(1,0)$, $C=(1,1)$, *and* $D=(0,1)$, *then* $\square ABCD$ *is a rectangle.*

If the equivalence of the parallel postulate and the existence of rectangles is not available, then we must verify the parallel postulate directly.

THEOREM 17. *Given a line* L *and a point* $C \notin L$, *there exists a unique line* L' *through* C *parallel to* L.

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MATHEMATICAL EDUCATION

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ALTERNATIVES IN THE MATHEMATICS PREPARATION OF ELEMENTARY TEACHERS

JOAN R. LEITZEL, JAMES E. SCHULTZ, AND ARTHUR L. WHITE

Like many other universities, The Ohio State University provides mathematics preparation for prospective elementary school teachers in three distinct units: the mathematics content courses are taught in the Department of Mathematics, the course in methods of teaching mathematics to children is taught in the College of Education, the supervised teaching of mathematics is done in the area schools.

Two mathematics courses are required at Ohio State for the elementary education majors. These one-quarter, 5-credit-hour courses have mastery of high school level algebra and geometry as a prerequisite and stress problem-solving.*

The required methods course is a one-quarter, 3-credit-hour course, and the elective school experience designated specifically for the teaching of mathematics carries 2 credits.

The National Science Foundation supported at Ohio State a two-year comparative study of alternatives in the mathematics preparation of elementary teachers. This controlled study investigated the effects of varying the relative ratios of time spent by students in the areas of mathematics content, methods, and school experience. It also investigated the effects of different levels of integration in the instruction in the three areas of mathematics, methods, and school experience.

*The content of the two required mathematics courses is similar to that recommended in Courses 1,2,3 of Sequence 2 for level I preparation by CUPM, 1971, *Recommendations on Course Content for the Training of Teachers of Mathematics*.

In the first year of the study, three distinct programs were compared, with approximately 25 students in each. The first program, referred to as the "traditional program," modeled the existing Ohio State instruction, with students taking separate courses in mathematics and in methods. There was no related school experience for students in the traditional program; a 2-credit statistics and probability laboratory was taken in lieu of school participation. Students in this program had 10 hours of instruction each week with approximately 7 in mathematics, 3 in methods, 0 in school experience.

A second program, referred to as the "integrated program," used the Mathematics Methods Program developed at Indiana University to provide for integrated instruction in mathematics, methods, and school experience in a laboratory setting. The teacher from mathematics and the teacher from education worked as a team with these students to produce a highly integrated experience. The 10 instructional hours in the week included approximately 5 in mathematics, 3 in methods, and 2 in school experience, although the integrated nature of the instruction often made it difficult to differentiate among the three areas.

The third program, somewhat between the other two and referred to as the "combined program," used conventional materials for the mathematics and methods instruction. The teachers attempted to coordinate their courses, but the courses were distinct. The students were in the schools once a week for supervised teaching in mathematics, making the 10 hours each week split approximately into 6 hours in mathematics, 3 in methods, 1 in school experience.

Student attitudes and attitude changes were measured in four areas: toward mathematics, toward the teaching of mathematics, toward the teaching of elementary school children, toward elementary school children. Student achievement was measured in mathematics content, in methods of teaching mathematics, and in problem-solving. In addition, data were collected through interviews with students, classroom observations, instructor logs, and written student evaluations of their courses.

The results of the first-year comparisons showed no significant differences in student attitudes among the three programs. However, students in the two programs that included school experience felt significantly more positive about their programs than did the students in the traditional program. Students in the two programs using conventional materials for the mathematics component (the traditional and the combined programs) scored significantly higher in mathematics content than did students in the integrated program. Students in the integrated program scored significantly higher in methods of teaching mathematics than did students in the traditional program but not significantly higher than students in the combined program. On problem-solving there were no significant differences among the three programs.

In the second year of the project, comparisons were made between two programs, both developed to accommodate large numbers of students in instructional formats compatible with continued university funding. The first group of students again used the Mathematics Methods Program for an integrated approach in mathematics, methods, and school experience. This time these materials were supplemented in some mathematics topics and some methods topics. Classes were again taught in small sections, some by teaching associates.

The second program in the second year included both a laboratory attached to the mathematics instruction and school experience. It utilized some large lecture instruction and used conventional texts to cover the same topics as the other program. Again the students in the two programs were evaluated on the four attitude scales as well as in mathematics content, methods of teaching mathematics, and problem-solving. There were significant differences in changes in attitude toward teaching mathematics and toward teaching elementary children; there were significant differences in the performance on the examinations testing methods of teaching mathematics and problem-solving. All significant differences favored the second of the two programs.

The project staff reached several conclusions on the basis of statistical data and the experience of teachers in the project. Results indicated that highly integrated content-methods

instruction provided no measurable advantage for students but that coordination of separate courses in content and methods served to improve students' performance in both. Having experience in teaching mathematics concurrent with the mathematics content course seemed to improve students' attitudes toward their required courses. Furthermore, the school teaching experience, as well as the mathematics laboratory experience, appeared to increase the students' understanding of the mathematics they were studying.

Details of this study together with data from its evaluation are contained in the full report, which may be obtained from: Professor James E. Schultz, Department of Mathematics, The Ohio State University, Columbus, Ohio 43210.

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PROBLEMS AND SOLUTIONS

Edited by Vladimir Drobot

*Send all **proposed** problems, in duplicate if possible, to Professor Vladimir Drobot, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053. Please include solutions, relevant references, etc.*

An asterisk (*) indicates that neither the proposer nor the editors supplied a solution.

***Solutions** should be sent to the addresses given at the head of each problem set.*

A publishable solution must, above all, be correct. Given correctness, elegance and conciseness are preferred. The answer to the problem should appear right at the beginning. If your method yields a more general result, so much the better. If you discover that a Monthly *problem has already been solved in the literature, you should of course tell the editors; include a copy of the solution if you can.*

PROBLEMS DEDICATED TO EMORY P. STARKE

Solutions of these Problems Dedicated to E. P. Starke should be mailed to Professor A. P. Hillman, Department of Mathematics & Statistics, University of New Mexico, Albuquerque, N.M. 87131 (USA), by January 31, 1981. To facilitate consideration, type with double spacing in the format used below. If acknowledgment is desired, include a self-addressed card, label, or envelope.

S 34. *Proposed by O. Bottema, Delft, The Netherlands.*

In a plane, a non-self-intersecting pentagon $A_1A_2A_3A_4A_5$ is given. No three of the vertices A_i are collinear and (ijk) denotes the (signed) area of the (oriented) triangle $A_iA_jA_k$. Furthermore

$$(124)=a_1,\quad (235)=a_2,\quad (341)=a_3,\quad (452)=a_4,\quad (513)=a_5.$$

Determine the area of the pentagon $A_1A_2A_3A_4A_5$.

[The analogous problem, with (123), (234), (345), (451), and (512) being given, was solved by Gauss in 1823. The Canadian journal *Eureka* (vol. 3, no. 8, 1977, p. 240) gave a reproduction of his solution.]

SOLUTIONS OF PROBLEMS DEDICATED TO EMORY P. STARKE

8 (or More) in a Row

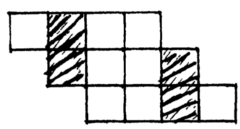
S 10 [1979, 306]. *Proposed by Richard K. Guy, University of Calgary, and J. L. Selfridge, Mathematical Reviews.*

When n-in-a-row (the generalization of tic-tac-toe) is played on a large board, it is easy to see that the first player has a winning strategy if $n=1$, 2, 3, or 4. There is a folk-theorem that Go-Moku ($n=5$) is also a first-player win, but nothing has been proved for $5 \leqslant n \leqslant 8$. Show that the second player can force a draw if $n \geqslant 9$, no matter how large the board is.

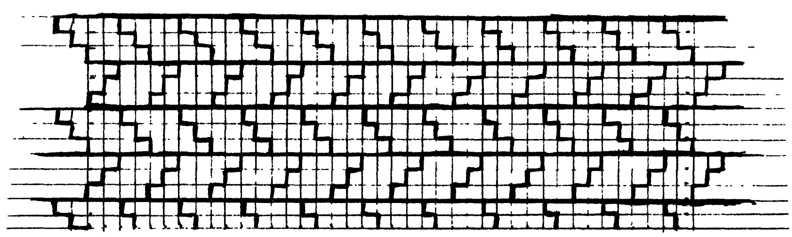
Solution by T. G. L. Zetters, Mathematisch Centrum, Amsterdam, Netherlands. There is a well-known proof that for 9 (or more) in a row the second player can force a draw; it uses a tiling of the plane with tiles of area 7 having the shape of a letter H.

THEOREM. *For* 8 (*or more*) *in a row, the second player can force a draw.*

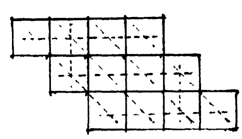
Proof. Tile the plane with tiles of area 12 and shape



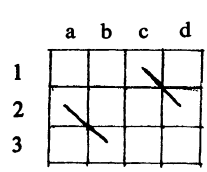
as follows:



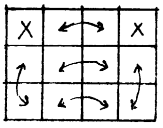
We claim that it is possible to play on each tile in such a way that the opponent never gets 4 in a row horizontally or 3 in a row diagonally or 2 in a shaded column vertically. The forbidden configurations are indicated by lines here.



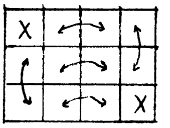
If I follow this strategy on each tile, my opponent cannot get more than 7 stones in a row diagonally or 6 in a row horizontally and vertically. Our tile is equivalent to a 3×4 rectangle, where rows, columns, and the indicated diagonals are forbidden.



Our strategy is as follows: If my opponent starts in column a, I answer b3; a first move in column b is answered by a2. The first move in the right hand part of the tile is answered symmetrically (i.e., by c1 or d2). Any move which is not a direct threat is answered by taking a point of the remaining diagonal pair. The remaining position is then—up to isomorphism—derived from the position



or



where columns b and c contain one of my stones (x indicates the opponent). The indicated strategy of coupled positions now suffices to stop my opponent.

(Note that the diagonals have been taken care of—by pairing the remaining cells, we see that the opponent cannot force a horizontal or vertical breakthrough.)

Proofs that the second player can force a draw for $n > 9$ were received from O. P. Lossers (Netherlands), Dana Richards, and the proposers.

Editor's Note. One of the proposers, Richard K. Guy, states that the H-tile strategy for forcing a draw in 9-in-a-row was first produced by Henry Pollak and Claude Shannon.

Tetrahedron Inequality

S 12 [1979, 392]. *Proposed by M. S. Klamkin, University of Alberta.*

If a,a_1; b,b_1; c,c_1 denote the lengths of the three pairs of opposite sides of an arbitrary tetrahedron, prove that $a+a_1$, $b+b_1$, $c+c_1$ satisfy the triangle inequality.

Solution by Roger B. Nelson, Lewis and Clark College. Without loss of generality, we may assume that the edges of one face are a, b, and c; while the edges emanating from the vertex opposite that face are a_1, b_1, and c_1. Hence the edges of the four triangular faces of the tetrahedron are (a,b,c); (a_1,b_1,c); (a_1,b,c_1); and (a,b_1,c_1). Each triangle satisfies the triangle inequality, so that

$$a<b+c,$$
$$a_1<b_1+c,$$
$$a_1<b+c_1,$$
$$a<b_1+c_1.$$

Adding and dividing by 2 gives $a+a_1<b+b_1+c+c_1$. The other two forms of the inequality are similarly obtained. Note that this result also holds for the degenerate tetrahedron consisting of four distinct coplanar points.

Also solved by K. Brown, L. Carlitz, F. S. Cater, J. DeMonte & K. Hourigan & I. Pawlyshyn, J. W. Di Paola, E. A. Dorsey, H. Eves, N. Franceschine III, Z. Franco, G. Gagola, L. Gerber, M. R. Gopal, J. W. Grossman, A. A. Jagers (Netherlands), M. Josephy (Costa Rica), V. Kannan (India), H. Kappus (Switzerland), I. Kastanas, L. Kuipers (Switzerland), M. L. Kung, M. K. Kwong, O. P. Lossers (Netherlands), H. J. Ludwig, R. T. Mahoney, N. A. Martin (Canada), M. Merriman, W. W. Meyer, M. D. Meyerson, B. M. O'Connor, V. Pambuccian (Romania), S. Rabinowitz, N. -S. Shou (Hong Kong), L. C. Singhal (India), W. Snow, F. B. Strauss, Z. D. Taborin (Yugoslavia), A. Venetoulias & A. Matsoukas (Greece), E. T. H. Wang (Canada), J. B. Zipperer, Jr., and the proposer.

ELEMENTARY PROBLEMS

Solutions of these Elementary Problems should be mailed in duplicate to Dr. J. L. Brenner, 10 Phillips Road, Palo Alto, CA 94303 (USA), by January 31, 1981. Please place the solver's name and mailing address on each (double-spaced) sheet. Include a self-addressed card or label (for acknowledgment).

E 2841. *Proposed by J. Michael Steele, Stanford University.*

Let $f(n)$ be a real valued function defined for every natural number n. Suppose $f(a+b+c) \leqslant f(a)+f(b)+f(c)$ for all a, b, c such that a/b and b/c are between $1/3$ and 3. Show that $\lim_{n\to\infty} f(n)/n$ exists and equals $\inf_n f(n)/n$.

E 2842. *Proposed by Jordi Dou, Barcelona, Spain.*

Let T be an isosceles right triangle. Let S be the circle such that the difference between the areas $T\cup S$ and $T\cap S$ is minimal. Show that the center of S divides the altitude on the hypotenuse of T in the golden ratio.

E 2843*. *Proposed by Peter Ungar, New York University.*

A set of nonoverlapping rectangles, each having its longer side equal to 1, is inside a circle of diameter $\sqrt{2}$. Show that the sum of their areas is $\leqslant 1$.

E 2844. *Proposed by Barry Powell, Kirkland, Washington.*

Give as elementary a proof as possible of the following. Let p_n be the sequence of primes. For infinitely many n, the difference $p_{n+1}-p_n$ exceeds $2\log p_n$. For infinitely many n, the difference $p_{n+1}-p_n$ is less than $\log p_n$. (More precise results can be obtained by advanced methods.)

E 2845. *Proposed by Douglas Hensley, Texas A & M University.*

Let $f(x)=(x^2-1)^{\frac{1}{2}}$, $x>1$. Prove that $f^{(n)}(x)>0$ for odd n and $f^{(n)}(x)<0$ for even $n>0$.

E 2846. *Proposed by D. Wiedemann, Institute for Defense Analyses.*

Let V be an n-dimensional vector space ($n>0$) over a field with characteristic $p\neq 0$. Let A be any affine map (linear plus a constant) from V to itself. Show there is an $x\in V$ and a positive integer $k\leqslant np$ such that the kth iterate of A takes x to itself.

SOLUTIONS OF ELEMENTARY PROBLEMS

Jacobi's Identities for Matrices

E 2735 [1978, 682]. *Proposed by I. P. Goulden and D. M. Jackson, University of Waterloo, Ontario.*

Let n be a fixed integer and define

$$f_k(x)=\sum_{r>0} x^{nr+k}/(nr+k)! \qquad (0\leqslant k\leqslant n-1).$$

For $P\subset S=\{0,1,\ldots,n-1\}$, let $F(P;x)=F(P)$ be the square matrix whose entries are indexed by elements of P and the (i,j)th entry is $f_{i-j}(x)$, $i,j\in P$. (We put $f_t(x)=f_k(x)$ if $t\equiv k \bmod n$.)

If n is even, show that $\det F(P)=\det F(S\setminus P)$ for all $P\subset S$. Generalize.

Solution by D. Ž. Djoković, University of Waterloo, Ontario. We use the *Jacobi identities*. Let F be the matrix $[g_{ij}]_0^{n-1}$; let P,Q be subsets of S; let $P'=S\setminus P, Q'=S\setminus Q$ be the complementary

subsets. Define $F(P,Q)$ as the submatrix of F based on the row-indices P and column-indices Q. Put $\varepsilon(P,Q)=\sum_{i\in P}i+\sum_{j\in Q}j$. Then if $|P|=|Q|$, $\det F(P,Q)=(-1)^{\varepsilon(P,Q)}\det F\cdot\det F^{-1}(Q',P')$.

Apply this identity to the matrix $F=[f_{i-j}(x)]$. Note that $F(S;x)=\exp Cx$, where C is the permutation matrix corresponding to a (circular) shift. Thus $[F(S;x)]^{-1}=F(S;-x)$, and (*) $\det F(P,Q;x)=(-1)^{\varepsilon(P,Q)}\det F(Q',P';-x)$, since $\det F(S,x)=\det e^{Cx}=1$. (Note $\operatorname{tr} C=0$.) In particular, if $P=Q$ and if n is even, then $F(P;x)=F(P';x)$. This follows from (*), using the additional observation that $f_k(-x)=(-1)^k f_k(x)$.

Jacobi's theorem, used above, appears on pp. 82–83 of: W. V. D. Hodge and D. Pedoe, *Methods of Algebraic Geometry*, vol. 1, Cambridge, 1968.

The Pascal Triangle Modulo a Prime

E 2775 [1979, 393]. *Proposed by Ko-Wei Lih, Academica Sinica, Nankang, Taipei, Taiwan, Republic of China*

If we replace even integers by 0 and odd integers by 1 in the ordinary Pascal triangle, we get the following modulo 2 Pascal triangle:

$$\begin{array}{c} 1 \\ 1\ \ 1 \\ 1\ \ 0\ \ 1 \\ 1\ \ 1\ \ 1\ \ 1 \\ 1\ \ 0\ \ 0\ \ 0\ \ 1 \\ \cdots\cdots\cdots \end{array}$$

Will 1101 or 1011 occur as a consecutive segment in any row of this modulo 2 Pascal triangle?

I. *Solution by J. G. Mauldon, Amherst College.* The given segment is impossible. More generally,

> if the Pascal triangle is reduced modulo the prime p, then the segment 1, 0, a, b, (with its reverse) occurs if and only if $a(2a+b)\equiv 0 \pmod p$. (1)

(1) is a consequence of the following lemma (with its obvious corollary), which is of interest in its own right.

LEMMA. *If the positive integers n and r have the base p decimal representation $n=\sum_0^M n_i p^i$ and $r=\sum_0^M r_i p^i$ with $0\leqslant n_i,\ r_i<p$, then*

$$\binom{n}{r}\equiv\prod_{i=0}^{M}\binom{n_i}{r_i} \pmod p. \tag{2}$$

COROLLARY. $\binom{n}{r}\equiv 0 \pmod p$ *if and only if, for some $i, r_i>n_i$.*

Deduction of (1). (i) *Necessity.* We write r,s,t,u for $r,r+1,r+2,r+3$, we denote by $\operatorname{seg}(n,r)$ the segment $\binom{n}{r}$, $\binom{n}{s}$, $\binom{n}{t}$, $\binom{n}{u}$, and we suppose $1,0,a,b\equiv\operatorname{seg}(n,r)$ with $a\not\equiv 0 \pmod p$. Then, adopting the natural notation and using the Corollary, we have $r_0\leqslant n_0<s_0$ so that $n_0=r_0$, and $t_0\leqslant n_0\leqslant s_0-1\leqslant p-2$ so that $t_0=0$ and hence $n_0=r_0=p-2$.

Writing

$$R=\textstyle\sum_1^M r_i p^{i-1},\quad N=\sum_1^M n_i p^{i-1},\quad\text{and}\quad K=\binom{N}{R+1}$$

we have, by the Lemma,

$$a\equiv\binom{n}{t}\equiv K\binom{p-2}{0}=K\quad\text{and}\quad b\equiv\binom{n}{u}\equiv K\binom{p-2}{1}=(p-2)K\equiv-2a \pmod p$$

from which we deduce (as required) that $2a+b\equiv 0 \pmod p$.

(ii) *Sufficiency*. If $p=2$, the segments 1000, 1001, and 1010 are easily discovered; so we suppose p is an odd prime and either $a\equiv 0$ or $b\equiv -2a \pmod p$. Since, without loss of generality, we may assume that a and b are positive integers, we find by using the Lemma that, if $a\equiv 0$, then $1,0,0,b\equiv \operatorname{seg}(bp+p-3,p-3)$ and, if $b\equiv -2a$, then $1,0,a,b\equiv \operatorname{seg}(ap+p-2,p-2)$. This completes the proof of sufficiency.

Proof of the Lemma by induction on M. The required inductive step is

$$\binom{n}{r}\equiv\binom{n_0}{r_0}\binom{N}{R} \pmod p \tag{3}$$

and this will now be proved.

The result is trivial if $r\geqslant n$, so we suppose $0<r<n$ and we define the integers s, S, and s_0 by the equations $n-r=s=pS+s_0$ with $0\leqslant s_0<p$, so that

$$\binom{n}{r}(pR)!(pS)!\prod_{1}^{r_0}(pR+i)\prod_{1}^{s_0}(pS+j)=(pN)!\prod_{1}^{n_0}(pN+k). \tag{4}$$

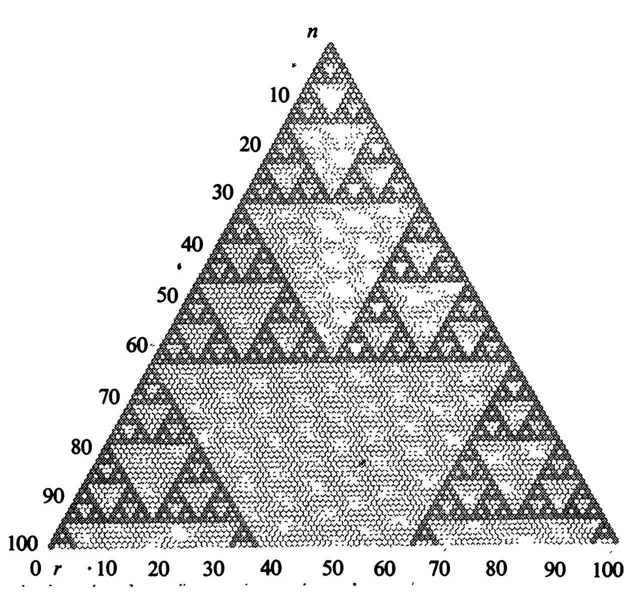
If $r_0>n_0$ then $N=R+S+1$ and we see that $\binom{n}{r}$ contains (as required) a factor p; so we may and shall suppose that $r_0+s_0=n_0$ and $R+S=N$. Canceling $p^{R+S}R!S!$ from each side of (4), we obtain

$$\binom{n}{r}\left(\prod_{I=1}^{R}\prod_{t=1}^{p-1}(pI-t)\right)\left(\prod_{J=1}^{S}\prod_{t=1}^{p-1}(pJ-t)\right)\prod_{1}^{r_0}(pR+i)\prod_{1}^{s_0}(pS+j)$$
$$=\binom{N}{R}\left(\prod_{K=1}^{N}\prod_{t=1}^{p-1}(pK-t)\right)\prod_{1}^{n_0}(pN+k). \tag{5}$$

The reduction modulo p of (5) yields $\binom{n}{r}r_0!s_0!\equiv\binom{N}{R}n_0!$, which implies (3) and (2), so completing the proof of the Lemma.

II. *Solution by Michael Mays, West Virginia University*. No. See [1], where it is noted that

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k}=\binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1};$$



and this contradicts the pattern

$$\begin{matrix} 1\ 0\ 1\ 1 \\ 1\ 1\ 0 \\ 0\ 1 \end{matrix}$$

References

1. V. E. Hoggatt, Jr., and Walter Hansell, The hidden hexagon squares, Fibonacci Quart., 9 (1971) 120, 133.

III. Several solvers furnished a pictorial argument. (See figure on p. 579.)

IV. *Comment by Heiko Harborth, Braunschweig, West Germany.* More general results appear in Harborth's article, *b-adic numbers in Pascal's triangle* modulo *b*, Fibonacci Quart., 16 (1978) 497–500.

Also solved by 73 other solvers, including the proposer.

Sums of Powers of a Number

E 2778 [1979, 393]. *Proposed by David J. Allwright, Cambridge University.*

Let k and r be integers with $r \geqslant 1$ and let z be a complex number with $|z|<1$. Calculate the sum S of $z^{\|N\|}$ as $N=(n_0,n_1,\ldots,n_r)$ ranges over all $(r+1)$-tuples of integers such that $n_0+n_1+\cdots+n_r=k$ and $\|N\|=|n_0|+|n_1|+\cdots+|n_r|$.

Solution by the proposer. The answer is

$$z^{|k|}\sum_{p=0}^{r}\binom{|k|+r}{r-p}\binom{r+p}{p}\left(\frac{z^2}{1-z^2}\right)^p.$$

Let $g(t) = \sum_{-\infty}^{\infty} t^n z^{|n|} = t(1-z^2)/[(t-z)(1-tz)]$. The desired sum is the coefficient of t^k in the Laurent expansion of $[g(t)]^{r+1}$. Since the value of the sum S is the same for k and $-k$, it is enough to evaluate it for nonnegative k. Thus, we have to evaluate $(2\pi i)^{-1}\int_K[t(1-z^2)/(t-z)\times(1-tz)]^{r+1}t^{|k|-1}dt$, where K is a circle of radius R, $|z|<R<|z|^{-1}$. Since the only singularity of the integrand is at $t=z$, this integral can be evaluated using standard techniques of calculus of residues.

Also solved by Michael Skalsky and Stewart S.-S. Wang.

Distinct Sums of the Residue Classes mod n

E 2781 [1979, 503]. *Proposed by James Propp, Harvard College.*

Let S be a set of n integers and $m=n(n+1)/2$. When $n\geqslant 3$, can $S+S$ constitute a complete residue set modulo m? (Here $S+S=\{a+b\,|\,a,\ b\in S\}$.)

Solution by Mark Pleszkoch, student, University of Virginia. Because addition is commutative, at most $m=n(n+1)/2$ sums can be formed. Thus, if a complete residue set mod m is to be formed, no two sums can be congruent. We will show that for $n\geqslant 3$, there exist two sums which are congruent, so $S+S$ cannot form a complete residue set.

Form the $n(n-1)$ ordered pairs (a,b), where $a,b\in S$ and $a\neq b$. For each ordered pair (a,b), consider the difference $a-b$. It can have $m-1$ possible values mod m (zero is excluded). But $n\geqslant 3$, so $n^2\geqslant 3n$, and $n(n-1)\geqslant n(n+1)/2=m>m-1$. Therefore, by the pigeonhole principle, two ordered pairs (a,b) and (c,d) must have the same difference mod$m$. But $a-b\equiv c-d$ (mod$m$) implies that $a+d\equiv c+b$ (mod$m$), where $a\neq b, c\neq d$, and $(a,b)\neq(c,d)$ guarantee that the two sums contain different terms. Thus $S+S$ cannot form a complete residue set mod$m$.

Also solved by M. Eisner, M. R. Gopal, E. M. Isaacs, H.-P. Ko, Nicholas A. Martin (Canada), L. E. Mattics, William Myers, and Western Maryland College Problem Seminar.

Bounds for $\Sigma_n^\infty k^{-2}$

2782 [1979, 503]. *Proposed by Robert E. Shafer, Berkeley, California.*

D. S. Mitrinović, in his reference book *Analytic Inequalities*, cites an inequality (p. 190, display 3.1.8) of Ostrowski:

$$\sum_{k=n}^{\infty}\frac{1}{k^2}<\frac{1}{n-\frac{1}{2}}\qquad\text{for } n \text{ a positive integer.}$$

Establish the generalization

$$2\arctan\frac{1}{2x-1}<\sum_{n=0}^{\infty}\frac{1}{(n+x)^2}<\frac{1}{x-\frac{1}{2}}\qquad\text{for } x>\tfrac{1}{2}.$$

Solution by J. Sutherland Frame, Michigan State University. For $x>\frac{1}{2}$, we have $(\Sigma\equiv\Sigma_0^\infty)$

$$\begin{aligned}2\arctan 1/(2x-1)&=2\sum\left[\arctan 1/(2n+2x-1)-\arctan 1/(2n+2x+1)\right]\\&=2\sum\arctan 1/2(n+x)^2<\sum(n+x)^{-2}<\sum 1/\left[(n+x)^2-\tfrac{1}{4}\right]\\&=\sum\left[1/\left(n+x-\tfrac{1}{2}\right)-1/\left(n+x+\tfrac{1}{2}\right)\right]=1/\left(x-\tfrac{1}{2}\right).\end{aligned}$$

Since the sum $S=\Sigma(n+x)^{-2}-1/x-1/2x^2$ has the continued fraction expansion [1]

$$2x^2S=\frac{1}{3x/1}+\frac{1}{5x/3}+\frac{1}{7x/6}+\frac{1}{9x/10}+\cdots,$$

a tighter inequality is

$$1/(6x^3+1.2x)<\sum(n+x)^{-2}-1/x-1/2x^2<1/6x^3,\quad x>0.$$

1. Math. Comp., 33, no. 146 (1979) 824–825.

Also solved by Charles Chouteau, Gustaf Gripenberg (Finland), Richard A. Groeneveld, G. A. Heuer, Thomas Jager, A. A. Jagers (Netherlands), L. Kuipers (Switzerland), Otto G. Ruehr, and the proposer.

The Functional Equation $\phi(z^2)=\phi(z)^2$

E 2783 [1979, 504]. *Proposed by William Knight and Bruce Lund, University of New Brunswick.*

Find all functions $\phi(z)$ such that ϕ is a one-to-one continuous map of the unit circle $\{z:|z|=1\}$ onto itself and $[\phi(z)]^2=\phi(z^2)$ for all z on the circle.

Solution by I. M. Isaacs, University of Wisconsin, Madison. The only such functions are $\phi(z)=z$ and $\phi(z)=z^{-1}$. This follows from the following slightly more general result: If $\phi(z^2)=\phi(z)^2$ is a continuous function from the unit circle into itself, then $\phi(z)=z^n$ for some integer n. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $\phi(e^{2\pi it})=e^{2\pi if(t)}$. For each t, $f(t)$ and $f(t+1)$ differ only by an integer, hence the continuous function $f(t+1)-f(t)$ assumes only integer values, i.e., it is an integer constant n. The condition on $\phi(z^2)$ implies that $f(2t)-2f(t)$ is an integer, so again it is an integer constant m. Define $g(t)=f(t)-nt+m$, so that $g(t+1)=g(t)$ and $g(2t)=2g(t)$. If $g(t)$ is not identically 0 then $g(a)=b\neq 0$ for some a and b. It follows that $g(2^ka)=2^kb$, so $g(t)$ is unbounded on $\mathbb{R}$, contradicting the fact that g is periodic and continuous. Thus $f(t)=nt-m$ and the result follows.

Also solved by A. Adelberg, I. C. Bivens, B. Brindza (Hungary), J. A. Cuenca and A. Castellán (Spain), L. Erlebach, T. Hermann (Hungary), T. Jager, A. A. Jagers (Netherlands), J. Leech, M. D. Meyerson, S. N. Noltie, D. Siegel, B. Ware, N. J. Wildberger, and the proposers.

ADVANCED PROBLEMS

Solutions of these Advanced Problems should be mailed in duplicate to Professor R. C. Lyndon, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109 (USA), by January 31, 1981. The solver's full post-office address should be on each sheet.

6288 [1980, 140] **(corrected)**. *Proposed by Jim Fickett and Arlan Ramsay, University of Colorado, Boulder.*

Let S be a convex set in real separable Hilbert space which is "fat" in the sense that, for each u of norm 1,

$$\sup\{(u,s)|s\in S\}-\inf\{(u,s)|s\in S\}\geqslant 1.$$

Must the interior of S be nonempty?

6304. *Proposed by I. J. Schoenberg, University of Wisconsin.*

If $0<t_1<t_2<1$, $r>0$, $s\geqslant 1$, set

$$t_1^1=t_1+t_1^r(t_2-t_1)^s$$
$$t_2^1=t_1+t_2^r(t_2-t_1)^s$$

and iterate to obtain

$$t_1^{(n+1)}=t_1^{(n)}+\left[t_1^{(n)}\right]^r\left(t_2^{(n)}-t_1^{(n)}\right)^s,$$
$$t_2^{(n+1)}=t_1^{(n)}+\left[t_2^{(n)}\right]^r\left(t_2^{(n)}-t_1^{(n)}\right)^s,$$

$n=0,1,\ldots$. Show that $\lim t_1^{(n)}=\lim t_2^{(n)}$ and if possible evaluate the common limit.

6305*. *Proposed by Clark Kimberling, University of Evansville.*

For $W=\{w_1,w_2\}$ inside the unit circle C in E^2, and $Z=\{z_1,z_2\}$ on C, define

$$d(W,Z)=\min\{|w_1-z_1|+|w_2-z_2|,\ |w_1-z_2|+|w_2-z_1|\},$$

and the *distance* from W to C as the average value of $d(W,Z)$ as Z ranges through all distinct pairs on C. Find any W which minimizes this distance. Extend this result to sets W and Z, each consisting of n points.

6306. *Proposed by Joseph Rotman, University of Illinois, Urbana.*

Let R be a commutative ring and M be a finitely generated flat R-module. Can M be imbedded as a submodule of a free R-module?

6307. *Proposed by H. Kestelman, University of London.*

Let M and Q be square matrices with spectral radius 1 such that $\{M^j\}$ and $\{Q^j\}$ converge to M_0 and Q_0, respectively. Show that if

$$A=\begin{bmatrix} M & S\\ 0 & Q\end{bmatrix}$$

then $\{A^j\}$ converges if and only if $M_0SQ_0=0$.

6308. *Proposed by David F. Findley, University of Tulsa.*

Let l_C^2 denote the Hilbert space of square summable complex sequences $a=(a_k)$, $k=0,1,2,\ldots$, with the usual inner product. Consider an $a\in l_C^2$ with $a_0\neq 0$ for which the convolution inverse a^{inv}, whose defining property is $a^{\mathrm{inv}}*a=(1,0,0,\ldots)$, also belongs to l_C^2. Define $a^{\mathrm{inv}}(N)=(a_0^{\mathrm{inv}},a_1^{\mathrm{inv}},\ldots,a_N^{\mathrm{inv}},0,0,\ldots)$ for $N=0,1,2\ldots$. Does $a^{\mathrm{inv}}(N)*a$ always converge to $(1,0,0,\ldots)$ in the norm of l_C^2?

6309. *Proposed by Michael F. Driscoll, Arizona State University.*

For any real-valued Lebesgue-integrable function f vanishing off $[0,1]$, define the function Tf by

$$\{Tf\}(y)=\int\{f(y-x)+f(1+y-x)\}f(x)\,dx$$

for $0\leqslant y\leqslant 1$ and zero elsewhere. Call f a *solution* if $Tf=f$ almost everywhere under Lebesgue measure. Clearly, f is a solution if $f=0$ a.e. or if f is a.e. equal to the characteristic (indicator) function of $[0,1]$.

(1) Show that these two are the only a.e.-nonnegative square-integrable solutions by finding all square-integrable solutions.

(2) Prove or disprove that there are solutions which are not square-integrable.

SOLUTIONS OF ADVANCED PROBLEMS

Continuity of Functions with Partial Derivatives

5888* [1972, 1141]. *Proposed by Stanley Rajnak, Kalamazoo College.*

Does there exist a real-valued function defined on $\mathbb{R}^2$ which has all partial derivatives of all orders at every point, but is not continuous on a dense set?

Solution by M. J. Pelling, University of Malaya, Kuala Lumpur, Malaysia. The answer is negative. G. P. Tolstov showed [*Izv. Akad. Nauk. SSSR., Ser. Mat,* 13 (1949), 425–446; *AMS Transl.*, 69 (1952)] that if a function $f(x,y)$ is continuous in each variable separately, then a partial derivative of first order existing everywhere must be in Baire Class 1. By the theory of such functions the continuity points of $\partial f/\partial x$, $\partial f/\partial y$ form dense G_δ sets so the set S of points where both $\partial f/\partial x$, $\partial f/\partial y$ are continuous is also a dense G_δ set. If $P\in S$ then $\partial f/\partial x, \partial f/\partial y$ are bounded in a neighborhood of P, $N(P)$ say, and by an elementary argument f will be continuous within $N(P)$. It follows that f must be continuous on a dense open set.

It is interesting to note however that the discontinuity points can form a set of positive planar measure—see Theorem 1 of M. J. Pelling, "On the singularity set of complex functions satisfying the Cauchy-Riemann equations," *Michigan Math. J.*, 26 (1979) 225–229, where such functions are constructed and are even harmonic.

$\Pi=83^3$

6044 [1975, 766; 1977, 392–394].

Editorial note. The published solutions to this problem include several conjectures by Bruckman (see p. 393). These have now been settled by Lagarias and Odlyzko; see this issue, pp. 561–564.

A Known Expected Value

6187 [1978, 53]. *Proposed by Ronald Evans, University of California, San Diego, La Jolla.*

Let $X_1, X_2, \ldots$ be a sequence of random numbers, uniformly distributed in $[0,1]$ and let N be minimal such that $\Sigma_{1\leqslant i\leqslant N}X_i^2>1$. Show that the expected value of N is

$$e^{\pi/4}\left(1+\int_0^1 e^{-\pi t^2/4}dt\right).$$

Editor's Note. Murray Klamkin, University of Alberta, points out that the result is a special case of known

results. It is obtainable from Equation (2.5) of M. S. Klamkin and J. H. van Lint, "An Asymptotic Problem in Renewal Theory," *Statistica Nederlandica,* 26 (1972) 191–196, by setting $x=1$ and $n=2$. Alternatively, it is obtainable from the last equation of D. J. Newman and M. S. Klamkin, "Expectations for Sums of Powers," this MONTHLY, 66 (1959) 293, by setting $x=\frac{1}{2}\pi^{1/2}$ and $n=2$.

A Formula for Expected Value

6245 [1978, 828]. *Proposed by C. L. Mallows, Bell Laboratories, Murray Hill, N.J.*

For $0<a<1$, $t\geqslant 0$, $b=1-a$, prove that

$$\frac{1}{\pi}\int_0^{\pi}\frac{(\sin u)^t}{(\sin au)^{at}(\sin bu)^{bt}}\,du=\frac{\Gamma(t+1)}{\Gamma(at+1)\Gamma(bt+1)}.$$

Solution by the proposer. If X is a positive stable random variable with characteristic index a, Williams (*Biometrika*, 64 (1977), 167–169) has shown that for $t>0$, $E(X^{-at})=\Gamma(t+1)/\Gamma(at+1)$. On the other hand, Kanter (*Ann. Prob.*, 3 (1975), 697–707), using Ibragimov and Chernin's integral representation of the density of X (*Theor. Prob. Appl.*, 4 (1959), 417–419), has shown that X has the same distribution as $f(u)^{1/a}w^{-b/a}$ where $f(u)=(\sin au)^a(\sin bu)^b/\sin u$, and where u and w are independent, with u uniform on $(0,\pi)$, and w having density e^{-w} on $(0,\infty)$. Since $E(w^{bt})=\Gamma(bt+1)$, the result follows. A direct proof would be interesting.

A Group Homomorphism

6246 [1979, 59]. *Proposed by L. Washington, University of Maryland, and W. Parry, Rutgers University.*

Let G be a compact Hausdorff topological group. Show that the only group homomorphism (not assumed continuous) from G to the integers is the trivial one.

Solution by Peter Nickolas, University of Queensland, Brisbane, Australia. The content of the Problem is precisely that of the Lemma in [2], where it was applied to an investigation of homomorphisms from locally compact groups into free products of groups with the discrete topology. This work generalizes part of Theorem 2 of [1], from which the solution to the Problem also follows easily.

References

1. R. M. Dudley, Continuity of homomorphisms, Duke Math. J., 28 (1961) 587–594.
2. Sidney A. Morris and Peter Nickolas, Locally compact group topologies on an algebraic free product of groups, J. Algebra, 38 (1976) 393–397.

Also solved by the proposers.

Probability of k Runs

6248* [1979, 59]. *Proposed by Milton P. Eisner, J. Sargeant Reynolds Community College, Richmond, Va.*

Let the set $S=\{1,2,\ldots,mn\}$, where m and n are positive integers, be partitioned randomly into n subsets each with m elements. For $0\leqslant k\leqslant n$, what is the probability $P(m,n,k)$ that exactly k of these subsets have the property of consisting of m consecutive integers?

Solution by B. R. Johnson, University of Victoria, Victoria, British Columbia. In addition to the solution for the proposed problem, a simple formula for the expected number of run subsets (subsets consisting of m consecutive integers) is also found.

View the experiment as randomly seeding mn distinguishable balls, numbered 1 to mn, into n ordered cells such that each cell receives exactly m balls. Then the sample space consists of

$(mn)!/(m!)^n$ equally likely sample points. Define the events

$$A_j: \text{Cell } j \text{ receives a run } (m \text{ consecutive integers}),\ 1 \leqslant j \leqslant n.$$

Then

$$P(m,n,k)=\sum_{r=k}^{n}(-1)^{r-k}\binom{r}{k}S_r, \quad \text{where } S_0=1 \quad \text{and}$$

$$S_r=\sum_{1\leqslant j_1<j_2<\cdots<j_r\leqslant n}P(A_{j_1}\cap A_{j_2}\cap\cdots\cap A_{j_r}), \qquad 1\leqslant r\leqslant n.$$

See Feller, page 106, or Parzen, page 76. Uniquely identify each distinct selection of r runs by an arrangement in a row of $m(n-r)$ stars and r bars according to the rule described below for the special case $m=3$, $n=4$, $r=2$:

Arrangement		Selection of r runs
|*|*	identifies	$\{3,4,5\}$, $\{9,10,11\}$
*||*****	identifies	$\{2,3,4\}$, $\{5,6,7\}$
|***|***	identifies	$\{1,2,3\}$, $\{7,8,9\}$
etc.		

From this identification it is easy to see that $A_{j_1}\cap A_{j_2}\cap\cdots\cap A_{j_r}$ has exactly

$$\binom{m(n-r)+r}{r}\cdot r!\cdot[m(n-r)]!/(m!)^{n-r}$$

elements. Therefore,

$$P(A_{j_1}\cap A_{j_2}\cap\cdots\cap A_{j_r})=[m(n-r)+r]!(m!)^r/(mn)!,$$

and

$$P(m,n,k)=\sum_{r=k}^{n}(-1)^{r-k}\binom{r}{k}\binom{n}{r}[m(n-r)+r]!(m!)^r/(mn)!.$$

The expected number of run subsets equals

$$\sum_{k=0}^{n}k\sum_{r=k}^{n}(-1)^{r-k}\binom{r}{k}\binom{n}{r}[m(n-r)+r]!(m!)^r/(mn)!$$

$$=\sum_{r=0}^{n}\binom{n}{r}[m(n-r)+r]!(m!)^r/(mn)!\sum_{k=0}^{r}k(-1)^{r-k}\binom{r}{k}$$

$$=n[m(n-1)+1]!m!/(mn)!$$

$$\left(\text{since } \sum_{k=0}^{r}k(-1)^{r-k}\binom{r}{k}=\begin{cases}1 & \text{if } r=1,\\ 0 & \text{otherwise}\end{cases}\right)$$

$$=mn\Big/\binom{mn}{m-1}.$$

References

1. W. Feller, An Introduction to Probability Theory and Its Applications, vol. 1, 3rd ed., Wiley, New York, 1968.

2. E. Parzen, Modern Probability Theory and Its Applications, Wiley, New York, 1960.

Also solved by Theodore S. Bolis, L. E. Clarke (England), H. F. Mattson, Jr., Allen J. Schwenk, Herbert S. Wilf, and N. Williams.

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR., AND LYNN A. STEEN
with the assistance of the mathematics departments of St. Olaf, Carleton, and Macalester Colleges
COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

Polyhedra Primer. By Peter Pearce and Susan Pearce. D. Van Nostrand, New York, 1978. viii + 134 pp. $6.95 (P). (Telegraphic Review, March 1979.)

Since the geometry of polyhedra is now one of my stronger mathematical interests, I was happy to be asked to review *Polyhedra Primer*. My own interest in the subject was not aroused by any of my formal mathematical training. In fact, none of the mathematics courses I took even alluded to the fact that this beautiful area of geometry exists. (I do not believe that I am very different, in this respect, from most mathematicians who received their education in the United States.) I came to know about this area of geometry when I discovered a relatively easy way to construct some of the simple polyhedra. Since then, studying the subject has occupied much of my time. How well I recall trying to read *Regular Polytopes* (the beautiful bible on this subject, by H. S. M. Coxeter) and trying to gain an understanding of the marvelous mathematical concepts connected with the symmetries of the models pictured there. And while I struggled with that and the various other reference books that were then available, it occurred to me that one of the most deceiving things I had learned from my high school geometry text was that the main purpose of taking the course was to learn about how to construct deductive proofs. My conversations with current high school teachers lead me to believe that, for the most part, the subject is still approached in the same way. I believe that the appeal to the human mind of the beauty of symmetry and form is very powerful—and to teach elementary geometry without emphasizing this beauty is to ignore one of the most motivating aspects of the subject.

It is for these reasons that I was delighted to see *Polyhedra Primer* (although for grammatical reasons I wish it had been called "Polyhedral Primer"). According to the authors, Peter and Susan Pearce, the goal of the book is to "teach the geometry of polyhedra." More specifically they say, "It can serve as a reference book while at the same time providing the reader with new ideas about the geometric organization of three-dimensional space. It is designed to enable the user to easily locate specific ideas and concepts." The topics covered include polygons, tessellations, finite polyhedra, space filling, open packings, and a chapter on the construction of geometric models.

Although there are minor flaws (more about that later), the format of the book is well organized, clear, and perceptually pleasing. The three-dimensional illustrations are drawn in perspective, with the edges shaded so that even a novice can perceive the polyhedron at first glance. An independent-study student of mine who experienced some difficulty understanding illustrations in other well-known books on polyhedra found that the illustrations in this book, because of the graduated widths used to represent the edges, were much easier to perceive as the three-dimensional objects they represent. More important, she was then able to return to the other more comprehensive books on polyhedra and interpret the illustrations encountered there in a meaningful way. She reported that she felt the organization of the book was clear and natural from an intuitive point of view. She found the descriptions and the logic easy to follow (with the exception of the naming of an eleven-sided polygon an "enneagon"). In short, she felt

it was tremendously helpful to her in her study of the more comprehensive books.

As I stated earlier, there are some flaws in the book. In what follows I will describe how the authors were made aware of these flaws and what they intend to do about them. I realize that to reveal this personal account is somewhat unusual in a book review, but inasmuch as it pertains directly to the quality of future editions of this book it seems appropriate to relate it here. I confess though that my goal is twofold, since the events illustrate so beautifully how three people are cooperating for the benefit of their audience. It is my hope that these events will encourage more cooperation of this type in the future.

On May 25, 1979, Branko Grünbaum and I were both returning to our respective homes on the West Coast after having attended the five-day Coxeter Symposium on Geometry, in Toronto. By chance, we were both aboard an American Airlines plane en route from Toronto to Chicago when an American DC 10 plane crashed at the Chicago airport. For us, one consequence was that we were held captive aboard our airplane on a runway at the Detroit airport. During the intervening hours before arriving in Chicago we discussed many topics, including the status of elementary geometry.

Grünbaum mentioned that he had already read the *Polyhedra Primer* book and, in fact, had written the authors to congratulate them "on the idea and execution" of their book and to offer some constructive suggestions for improving it.

After reading the book myself, and the correspondence between Grünbaum and the Pearces, I see no reason to alter Grünbaum's well-thought-out suggestions. It is plain that the best service to the readers of this review is to share them exactly as they were written. So I quote here (with permission) the pertinent excerpts from the letters that were exchanged between Grünbaum and the Pearces. First, Grünbaum wrote:

> May I point out a few errors that slipped into the "Primer" that could be corrected in later printings with only minor changes. Most of them are of the kind that will not seriously bother the informed, but that can easily pose needless worries to the type of reader for whom the "Primer" is intended.
>
> On page 22 the impression is created that in a tiling polygons must fit "... along matching edges." (It seems also clear that that was intended, since if tilings in which this condition is violated were admitted then there would be more than the eight "semiregular tilings" given on page 28, and even the enumeration of "regular tessellations" on page 27 would be incomplete.) However, on pages 35 and 36 tessellations violating the "matching edges" condition are presented without any explanation or warning.
>
> Page 26. "...There are only 14 such combinations..." In various forms this assertion recurs also on pages 30 and 32. Actually, there are 15 types of vertices—the one omitted on page 26 is $3\cdot4\cdot3\cdot12$, which occurs, for example, in the last tiling on page 31.
>
> Page 42. You say: "The plane can be subdivided periodically without the requirement of filling the whole plane. In such cases, regular polygons of 5, 9, 10 and 20 sides may be used." Fair enough, but why stress these numbers when *any* polygons could be used! The same comment applies to page 43.
>
> When tilings are defined on page 32 the word means tiling of the whole plane. But on page 44, for Euler's theorem, "tiling" seems to mean "tiling of a bounded region."
>
> If "semiregular polyhedron" is defined as on page 55, then one more polyhedron qualifies—the "pseudo-rhombicuboctahedron" with vertices $3\cdot4^3$, attributed by Coxeter to J. C. P. Miller.
>
> The definition of "dual polyhedron" given on page 74 is deficient in several respects. According to it, even many of the deltahedra on page 67 will not have duals, or else the duals will not be the expected ones. For less regular polyhedra the definition is even less appropriate.

The Pearces responded:

> It was with great delight that we received your letter of January 15, regarding our book, *Polyhedra Primer*. We appreciated both your compliments and your most constructive criticism....
>
> Your critical points are well taken, and we certainly will be able to correct these "errors" in the next printing. We were particularly embarrassed by the omission of the $3\cdot4\cdot3\cdot12$ vertex on page 26.
>
> Your comment that any regular polygons can be used to periodically subdivide the plane without the requirement of filling the whole plane is news to us. It is clear that this would be true now that you mention it, but we are having a little difficulty seeing it relative to the kind of pentagonal concentric patterns shown on page 42. Could you give us some references on this problem that would help clarify the matter.

Your point about our definition of "dual polyhedra" on page 74 is a good one. Can you help us with a better definition? Our definition obviously works for regular and semiregular polyhedra, but creates serious problems for lots of other interesting polyhedra.

Thank you for taking the time to respond so constructively to our little book. Again it is greatly appreciated.

Grünbaum's reply:

Concerning your questions:

I don't know of any reference for the patterns of regular polygons with an arbitrary number of sides. There are many ways in which one can construct patterns in which all polygons enter in an equivalent manner and are "boxed in" in their positions by contacts with their neighbors (for example, Figure 1). I am very intrigued by a pattern I found recently, in which all copies of the motif are equivalent and each copy has 6 neighbors touching it. It can be constructed with (essentially) any figure as motif, and in particular, with any regular polygon; its symmetry group is $p3$. Two typical examples (usually not thought of as fitting the $p3$ symmetry) involve squares and pentagons (see Figures 2 and 3).

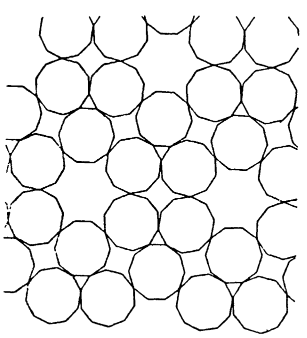


FIG. 1

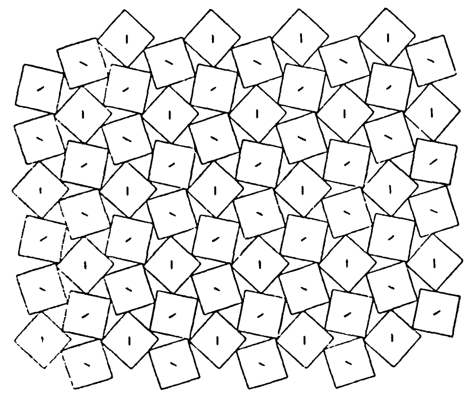


FIG. 2

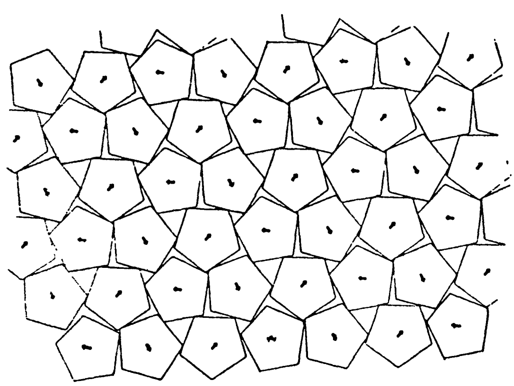


FIG. 3

The duality of polyhedra is a very difficult topic. There is a great and general confusion in the literature concerning terminology and definitions, and most assertions one reads are invalid—even those by respected geometers. I believe for the audience of the primer there is only one simple (and correct) solution —the one obtained, as you suggest, by limiting the scope of the definition. For example, the following would probably be appropriate on page 74 in a frame enclosing all three diagrams of that page:

> The dual polyhedron of a regular or semiregular polyhedron is obtained by joining a suitable point that is perpendicularly above the center of each face of the polyhedron to equivalent points above all neighboring faces. The points are chosen so that each new edge connecting two such points intersects an edge of the original polyhedron in a point at which they are both tangent to an enclosed sphere.

To be consistent, the third sentence on page 55 should read: "There are thirteen semiregular polyhedra besides the two infinite families of prisms and antiprisms."

What comes across so clearly in this correspondence is the genuine concern on the part of both Grünbaum and the Pearces for producing an elementary book about polyhedra that will be as interesting and accurate as possible. I heartily commend all three of them for their efforts and their splendid cooperation. I sincerely hope their example will be followed by other mathematicians who serve the educational needs of our community in different but equally important roles. The result of this particular cooperation will be an even more valuable and accurate version of *Polyhedra Primer* (owners of the current edition can refer to this review to correct their own copies).

I would like all high school teachers of geometry to have the *Polyhedra Primer* or at the very least to make it available to their students, many of whom will surely find its contents appealing. It is also conceivable that some college professors may find it illuminating. I can't help but reflect about an elementary article of mine that was recently rejected by one of the MAA journals because one of the referees said, "She uses ideas unfamiliar to most college professors (What, for example, is a platonic solid?)"!! Perhaps if I had sent a copy of *Polyhedra Primer* along when I submitted the manuscript, the article would have been accepted; it was subsequently published in the *Mathematical Gazette*. (In fairness to the referee I think it should be pointed out that he or she was probably correct. After all, where would most college professors have learned that?)

JEAN J. PEDERSEN, University of Santa Clara

MISCELLANEA

42. Two rationalizations.

Mathematics can overcome no prejudice, it can soften no stubbornness, it can moderate no partisan spirit; there is nothing moral that it can accomplish.

J. W. v. Goethe, 1829 (Maximen und Reflexionen, no. 608)

The entire mathematical arsenal that our modern sages command cannot establish facts. Practical people should always keep this in mind when they ask mathematicians for help.

M. Evgrafov, 1979 (Literaturnaya Gazeta, no. 49, Dec. 5, p. 12)

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, T?(13: 1), S, P, L. *An Invitation to Mathematics*. Norman Gowar. Oxford U Pr, 1979, vi + 206 pp, $11 (P); $26. [ISBN: 0-19-853001-3; 0-19-853002-1] A wealth of mathematical ideas is carefully introduced in a non-frightening tone with numerous diagrams and historical anecdotes to convey the nature of mathematics and its symbolism; an excellent resource for teachers of mathematics for liberal arts courses as well as exceptional supplementary reading for high school students. No exercises. JNC

GENERAL, T*(13: 1). *Mathematically Speaking*. Morton Davis. HarBrace J, 1980, xii + 484 pp, $16.95. [ISBN: 0-15-555190-6] Intended for intellectually curious liberal arts students. Written to capture the imagination with emphasis on broad ideas and their day-by-day practical importance. Topics include probability and statistics, games, voting, Markov chains, computers, and calculus. Sections include plenty of routine exercises plus sets called "Extras for Experts" and "Puzzles to Ponder." LCL

BASIC, T(13: 1). *Algebra for College Students*. Bernard Kolman, Arnold Shapiro. Acad Pr, 1980, xvi + 507 pp, $15.95. [ISBN: 0-12-417880-4] Light treatment of college algebra. Tops in user handholding. Liberally laced with progress checks, progress tests, warnings, "split-screen" presentations, charted solutions to problems, plus lists of terms and symbols, common errors, key ideas for review. Emphasis on word problems, including a separate chapter. Plenty of exercises, none challenging. Not patronizing and definitely written for the student. JK

PRECALCULUS, T(13). *Precalculus Mathematics: Algebra and Trigonometry*. Steven J. Bryant, Daniel Saltz. Goodyear, 1980, viii + 455 pp, $15.95. [ISBN: 0-87620-690-9] The title tells the story. Algebra covered does include logarithms, determinants, the binomial theorem, probability, and induction. Trigonometry includes inverse functions, DeMoivre's Theorem, and solution of triangles. To do all this in 455 pages means the pace is fast. AWR

HISTORY, P. *Les Principes des Mathématiques*. Louis Couturat. Albert Blanchard, 1980, viii + 310 pp, (P). Reprint of the author's original 1905 summary of Russell's *Principles of Mathematics*, plus a large appendix on Kant's philosophy of mathematics. LCL

FOUNDATIONS, T(17-18), P. *The Mathematical Theory of L Systems*. Grzegorz Rozenberg, Arto Salomaa. Pure and Appl. Math., V. 90. Acad Pr, 1980, xvi + 352 pp, $38. [ISBN: 0-12-597140-0] L systems are formal languages which differ from traditional systems in that (i) they allow parallel rewriting and (ii) the grammar is studied as a dynamical (temporal) process. They have been intensely studied since 1968 when they were originated to model biological processes. The authors hope to convince the reader that the theory of L systems contributes a totally new perspective to formal language theory through its richness in results and techniques and generality in scope. Their thorough survey emphasizes decidability problems. Many exercises. GHM

FOUNDATIONS, P. *Lecture Notes in Mathematics-769: Topological Model Theory*. Jörg Flum, Martin Ziegler. Springer-Verlag, 1980, x + 149 pp, $11.80 (P). [ISBN: 0-387-09732-5] Investigation of topological structures with the aid of formal languages (in this case L_t). The first part contains general model-theoretic results and the second part is devoted to concrete L_t-theories (topological-spaces, abelian groups, fields, vector spaces). LCL

COMBINATORICS, T(15-17: 1), S, P, L. *Graphs and Networks*. Bernard Carré. Clarendon Pr, 1979, xvi + 277 pp, $36.50; $19.50 (P). [ISBN: 0-19-859615-4; 0-19-859622-7] A readable introduction to the subject which also introduces algorithms and computational complexity. Topics include graphs, path problems, connectivity, independent sets, colorations and flows in networks. Gives many applications in the examples. Includes a good collection of exercises. CEC

LINEAR ALGEBRA, S*(13). *Systems of Linear Inequalities*. A.S. Solodovnikov. Trans: Lawrence M. Glasser, Thomas P. Branson. U of Chicago Pr, 1980, viii + 81 pp, $6 (P). [ISBN: 0-226-76786-8] The foreword claim that "No knowledge beyond what is covered in high school mathematics is prerequisite for the book" can in this case be accepted. This beautiful little book progresses at a steady pace from necessary analytic geometry through the geometry of convex sets to a final chapter on the duality theorem of linear programming. Nice supplemental reading on an often neglected topic. AWR

LINEAR ALGEBRA. *Applications of Linear Algebra, Second Edition*. Chris Rorres, Howard Anton. Wiley, 1979, ix + 295 pp, $6.95 (P). [ISBN: 0-471-05337-6] Nineteen self-contained applications which give a good indication of the diversity of linear algebra. New chapters on plane geometry, equilibrium of rigid bodies, the assignment problem, and computer graphics. The style allows these topics to be given as independent projects in a sophomore level course. (*First Edition*, TR, November 1977.) TLS

LINEAR ALGEBRA, T(13-14: 1, 2). *Introductory Linear Algebra with Applications, Second Edition*. Bernard Kolman. Macmillan, 1980, xvii + 535 pp, $17.95. [ISBN: 0-02-365970-X] New features include more applications (elementary applications scattered throughout the first five chapters; more extensive applications in the second half now include Markov chains and differential equations) and new material on polynomials and functions as vector spaces. (*First Edition*, TR, May 1976.) LCL

ALGEBRA, T(14-16), S, L. *Modern Algebra, An Introduction.* John R. Durbin. Wiley, 1979, xv + 329 pp, $15.95. [ISBN: 0-471-02158-X] A text for a course in modern algebra. The usual topics are covered, but more emphasis is placed on applications, including entire chapters devoted to symmetry and algebraic coding. One of the appendices summarizes various logical equivalences which should be helpful to students struggling to translate ideas into coherent written proofs. JEG

ALGEBRA, T(18: 1), S, P, L. *Lie Algebras.* Nathan Jacobson. Dover, 1979, ix + 331 pp, $5 (P). [ISBN: 0-486-63832-4] An unabridged and corrected paperback version of the original 1962 book, its publication testifies to the longevity and continuing usefulness of what is still one of the best available books for one interested in Lie algebras. JS

ALGEBRA, T(18: 1, 2), S, P. *Local Fields.* Jean-Pierre Serre. Grad. Texts in Math., V. 67. Springer-Verlag, 1979, viii + 241 pp, $24.80. [ISBN: 0-387-90424-7] A translation of *Corps Locaux* with updated bibliography and improved problem sets. Presents local class field theory from cohomological viewpoint. Occasional algebraic geometric interpretations, especially in non-singular curve theory. TLS

ALGEBRA, T(17-18), S, P, L. *Finite Groups, Second Edition.* Daniel Gorenstein. Chelsea, 1980, xvii + 519 pp, $22.50. [ISBN: 0-8284-0301-5] Errors from the *First Edition* (TR, August-September 1968) have been removed, plus a couple of sections whose contents are misleading from the perspective of 1978. LCL

FINITE MATHEMATICS, T(13), S. *Mathematics for Students of Business, Economics, and Social Science.* James Radlow. Duxbury Pr, 1979, xv + 805 pp, $17.95. [ISBN: 0-87872-210-6] Comprehensive coverage of finite mathematics topics for business and economics; includes flow charts but no material on series. Attempts to relieve math anxiety by stressing examples and applied problems. Provides mathematics tools for problem solution. Does not provide mathematics theory or conceptualization. Audience: business or economics students with weak mathematics background; person needing review of mathematics. WC

CALCULUS, T(13). *Calculus and Its Applications, Brief Edition.* Larry J. Goldstein, David C. Lay, David I. Schneider. P-H, 1980, xvii + 411 pp, $16.95. [ISBN: 0-13-112102-2] A short calculus aimed at social science students. No trigonometry; unusual introduction to e, then the natural log, both before integration is introduced. AWR

CALCULUS, T(13: 1). *Calculus for the Management and Social Sciences.* John Hegarty. Allyn, 1980, xii + 497 pp, $14.95 (P). [ISBN: 0-205-06886-3] A non-rigorous business-oriented approach using calculators, tables, graphs and numerous business applications to convince and motivate. Examples are worked out in detail; includes exponential and logarithmic functions. JNC

CALCULUS, T(13-14: 1, 2), L. *Calculus for the Managerial, Life, and Social Sciences.* Ernest F. Haeussler, Jr., Richard S. Paul. Reston, 1980, x + 573 pp, $16.95. [ISBN: 0-8359-0628-0] Somewhat more mathematically demanding (and less patronizing) than many of the recent texts designed for this audience, the book does not differ greatly from traditional calculus texts except for the absence of most proofs and the choice of applications. Contains more than the usual amount on functions of several variables, including the chain rule and implicit differentiation as well as a brief section on multiple integration. Good selection of exercises; answers, general index, and application index.JS

CALCULUS, T(13). *Applied Calculus, Second Edition.* Raymond F. Coughlin. Allyn, 1980, xvii + 427 pp, $18.95. [ISBN: 0-205-06910-X]; *Study Guide, Answers and Discussion of Problems,* 128 pp, (P). A short calculus with examples and problems drawn from wide range of disciplines. Includes a final chapter on trigonometry. (*First Edition,* TR, August-September 1976.) AWR

CALCULUS, T(13-14). *Calculus with Analytic Geometry.* John B. Fraleigh. A-W, 1980, xiii + 866 pp, $24.95. [ISBN: 0-201-03041-1] Intended for the standard three semester calculus sequence, this book makes a point of early introduction of trigonometric functions, allows for giving short shrift to conic sections for those so inclined, and includes some computer graphics--always put alongside an artist's sketch. AWR

REAL ANALYSIS, T(17-18: 1, 2), S, P. *Lecture Notes in Mathematics-667: Approximants de Padé.* Jacek Gilewicz. Springer-Verlag, 1978, xiv + 511 pp, $21.50 (P). [ISBN: 0-387-08924-1] Study of ordinary Padé approximants. Proceeds from a general treatment of the approximation of one function, the analytic extension of a function given by its Taylor Series, the extrapolation of the limit of a sequence, and the acceleration of convergence, to monotone functions and series and continued fractions. The last half deals with the algebraic theory of Padé approximants followed by several chapters devoted to numerical applications of the Padé method of approximation. Three appendices. Extensive bibliography. Index. RJA

REAL ANALYSIS, T(16: 1), S, L. *Differential Analysis: Differentiation, Differential Equations and Differential Inequalities.* T.M. Flett. Cambridge U Pr, 1980, vii + 359 pp, $39.50. [ISBN: 0-521-22420-9] Differential calculus in normed vector spaces at advanced undergraduate level. Chapter 1: Functions of a real variable, mean value and Taylor theorems. Chapter 2: Ordinary differential equations, existence theorems. Chapter 3: The Fréchet differential, applications to inverse and implicit function theorems, calculus of variations, Newton-Kantorovich theorem. Chapter 4: The Gateau and Hadamard differentials, constrained extras with equality, inequality constraints. Many examples and exercises, extensive historical notes, indices. RBK

COMPLEX ANALYSIS, P. *Lecture Notes in Mathematics-767: Families of Meromorphic Functions on Compact Riemann Surfaces.* Makoto Namba. Springer-Verlag, 1979, xii + 284 pp, $16.30 (P). [ISBN: 0-387-09722-8] Using a deformation theoretic point of view, the author attempts to provide a new look at meromorphic functions on compact Riemann surfaces. A major tool is the consideration of C(V), the set of all meromorphic functions on the surface V, as a Douady space. A detailed and consequently difficult treatment. TAV

DIFFERENTIAL EQUATIONS, S(14). *Hill's Equation.* Wilhelm Magnus, Stanley Winkler. Dover, 1979, viii + 129 pp, $3 (P). [ISBN: 0-486-63738-7] Theory of second order differential equations with real periodic coefficients. No problems. RBK

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Mathematics-712: Différentielles et Systèmes de Pfaff dans le Champ Complexe.* Ed: R. Gérard, J.-P. Ramis. Springer-Verlag, 1979, 364 pp, $19.60 (P). [ISBN: 0-387-09250-1] Papers originally presented in a seminar on differential equations at l'Institut de Récherche Mathématique Avancée. The main paper is "Etude de certains systèmes de Pfaff avec singularités" by Gérard and Sibuya. TLS

DIFFERENTIAL EQUATIONS, T(18: 1), P. *Lecture Notes in Mathematics-756: Elliptic Pseudo-Differential Operators--An Abstract Theory.* H.O. Cordes. Springer-Verlag, 1979, 331 pp, $18 (P). [ISBN: 0-387-09704-X] Emphasizes link between normal solvability of a linear operator and the structure of certain commutative C*-algebras. Appendices on operator algebras and commutative Banach algebras. RBK

DIFFERENTIAL EQUATIONS, T(14-16: 1), S, L. *A First Course in Differential Equations with Applications.* Dennis G. Zill. Prindle, 1979, viii + 535 pp. [ISBN: 0-87150-266-6] This text covers the usual topics in a first course in differential equations. There are many exercises and worked examples. The text is quite readable and seems to have been designed with the student in mind. JEG

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Mathematics-762: Group Theoretic Methods in Bifurcation Theory.* D.H. Sattinger. Springer-Verlag, 1979, 241 pp, $14 (P). [ISBN: 0-387-09715-5] Applies the techniques of group representations, especially for topological groups, to the study of bifurcations. Includes an appendix by Peter Olver on how to find the symmetry group of a differential equation using algebraic techniques. Many references, no index. TLS

NUMERICAL ANALYSIS, P. *Étude Numérique des Grands Systèmes.* J.L. Lions, G.I. Marchouk. Dunod (US Distr: SMPF, 14 E. 60th St., NY 10022), 1978, ix + 277 pp, 145 F. [ISBN: 2-04-010527-1] Collection of papers presented in Novosibirsk, June 9-12, 1976, on the theme of "Numerical solutions of large systems of functional equations with computers." Grouped under three topics: (1) new mathematical models; (2) approximate methods for mathematical physics, optimization and optimal control; (3) implementation of algorithms on large computers. RJA

NUMERICAL ANALYSIS, P. *Constructive Methods for Nonlinear Boundary Value Problems and Nonlinear Oscillations.* Ed: J. Albrecht, L. Collatz, K. Kirchgäsner. Birkhäuser Boston, 1979, 190 pp, $26.80 (P). [ISBN: 3-7643-1098-7] 15 papers from a conference held at Oberwolfach in November, 1978. Special numerical techniques, their analyses and examples for several nonlinear problems. RWN

FUNCTIONAL ANALYSIS, P. *Banach Modules and Functors on Categories of Banach Spaces.* Johann Cigler, Viktor Losert, Peter Michor. Lect. Notes in Pure and Appl. Math., V. 46. Dekker, 1979, xv + 282 pp, $29.50 (P). [ISBN: 0-8247-6867-1] Banach modules may be interpreted as generalizations of Banach spaces, where the field of scalars has been replaced by a Banach algebra or category of Banach spaces. JEG

FUNCTIONAL ANALYSIS, P. *Minimal Factorization of Matrix and Operator Functions.* H. Bart, I. Gohberg, M.A. Kaashoek. Operator Theory, V. 1. Birkhäuser Verlag, 1979, v + 227 pp, $17.50 (P). [ISBN: 3-7643-1139-8]

OPTIMIZATION, P. *Spiele auf Graphen.* Bernd Kummer. Int. Ser. Num. Math., V. 44. Birkhäuser Verlag, 1980, 92 pp, $18.50 (P). [ISBN: 3-7643-1077-4] Treats class of strategic games with complete information. Differs from usual treatments in that infinite numbers of moves are permitted and properties of the various solutions are central. Exercises. JD-B

OPTIMIZATION, P. *Finite Generalized Markov Programming.* P.J. Weeda. Math. Centre Tracts No. 92. Math Centrum, 1979, vii + 127 pp. [ISBN: 90-6196-158-0] The author treats a special case of the generalized Markov decision model, one with a finite state space and finite sets of interventions. A policy iteration method is employed to analyze the expected returns, and numerical (cutting set) methods are developed for specific applications. TAV

OPTIMIZATION, T(16-18: 1-3), S*, P*, L. *Mathematical Methods of Game and Economic Theory.* Jean-Pierre Aubin. Stud. in Math. and its Appl., V. 7. North-Holland, 1979, xxxii + 619 pp, $109.75. [ISBN: 0-444-85184-4] Optimization theory and convex analysis, game theory and mathematical economics, and non-linear analysis and control theory. Note the exorbitant price. FLW

OPTIMIZATION, P. *Anwendungen der Linearen Parametrischen Optimierung.* Klaus Lommatzsch. Math. Reihe, B. 69. Birkhäuser Verlag, 1979, x + 189 pp, $31.50. [ISBN: 3-7643-1058-8] Ten largely independent papers on linear parametric programming, by students of F. Nozicka (in one case jointly with him). JD-B

OPTIMIZATION, P. *Vector-Valued Optimization Problems in Control Theory.* M.E. Salukvadze. Trans: John L. Casti. Math. in Sci. and Eng., V. 148. Acad Pr, 1979, x + 219 pp, $27.50. [ISBN: 0-12-616750-8] The book begins with a survey of optimization problems with vector criteria. Historical development continues, emphasizing emergence of the question of designing control systems that are optimal relative to several performance indices. Attention focuses throughout on the work of Soviet scholars. Applications are drawn from flight control, control of rockets, and (in a final chapter) control of a nuclear reactor installation. AWR

ANALYSIS, T*(16), S*, L*. *Fourier Analysis.* Larry Baggett, Watson Fulks. Anjou Pr, 1979, viii + 183 pp, $22. [ISBN: 0-88446-001-0] A satisfying treatment of Fourier series, summability, transform theory, applications and generalizations. An excellent choice for a course or seminar emphasizing the central position of Fourier analysis in classical and applied analysis. TAV

ANALYSIS, P. *Integral Representations of Functions and Imbedding Theorems.* Oleg V. Besov, Valentin P. Il'in, Sergey M. Nikol'skii. V.H. Winston. *Volume I,* 1978, viii + 345 pp, $19.95 [ISBN: 0-470-26540-X]; *Volume II,* 1979, viii + 311 pp, $19.95. [ISBN: 0-470-26593-0] Continuation of study of spaces of functions satisfying differential-difference conditions on open subsets of E^n. Bibliography. No index. RBK

ALGEBRAIC GEOMETRY, P. *Lectures on Torus Embeddings and Applications (Based on joint work with Katsuya Miyake).* Tadao Oda. Springer-Verlag, 1978, xi + 175 pp, $9.90 (P). [ISBN: 0-387-08852-0] This typewritten set of notes explores the combinatorial aspects of toroidal embeddings. The setting is highly algebraic, though much geometry is in evidence. TLS

DIFFERENTIAL GEOMETRY, T(18: 1, 2), S, P. *Complex Manifolds Without Potential Theory, Second Edition.* Shiing-shen Chern. Springer-Verlag, 1979, 152 pp, $12 (P). [ISBN: 0-387-90422-0] A reissue (*First Edition,* TR, May 1968) which adds as an appendix Chern's lecture notes on the geometry of characteristic classes. While the style is always quite compact and at first glance not quite complete, upon understanding one must admit that the explanation is always best possible. TLS

GEOMETRY, S(14-16). *The Decomposition of Figures into Smaller Parts.* Vladimir Grigor'evich Boltyanskii, Izrail' Tsudikovich Gohberg. Trans: Henry Christoffers, Thomas P. Branson. U of Chicago Pr, 1980, vi + 74 pp, $6 (P). [ISBN: 0-226-06357-7] Delightful volume dealing with some classical problems in convexity and combinatorial geometry. Specifically, it concentrates on decomposition, covering, and illumination problems in the Euclidean and Minkowskian plane. Although careful and rigorous, it is accessible to gifted high school students and college undergraduates. Unfortunately, no index. SS

GEOMETRY, P. *Hilbert's Third Problem: Scissors Congruence.* C.-H. Sah. Pitman, 1979, 188 pp, $18.95 (P). [ISBN: 0-273-08426-7] An interesting account of the developments that have come from Hilbert's original problem. While many sophisticated techniques are used the primary concepts are never far away. Hence the uninitiated are welcome. Meant to complement Boltyanskii's book. Unfortunately the exposition is not quite up to the mathematics. TLS

TOPOLOGY, T(18: 1, 2), S, P. *Lecture Notes in Mathematics-738: Differentiable Periodic Maps, Second Edition.* P.E. Conner. Springer-Verlag, 1979, iv + 181 pp, $9.80 (P). [ISBN: 0-387-09535-7] A rewriting of Conner-Floyd stressing the technique of bordism to the study of periodic maps on closed manifolds. The updated text and bibliography both show the tremendous amount of progress that has been made since the first edition. TLS

TOPOLOGY, T(17-18: 1, 2), S, P. *Lecture Notes in Mathematics-761: Homotopy Equivalences of 3-Manifolds with Boundaries.* Klaus Johannson. Springer-Verlag, 1979, 303 pp, $18 (P). [ISBN: 0-387-09714-7] Main emphasis is on exotic homotopy equivalences--i.e., restrictions to the boundary cannot be deformed into the boundary--and a classification theorem is proved for them. Appendix. References. Index. RJA

TOPOLOGY, P. *The Symplectic Cobordism Ring. I.* Stanley O. Kochman. Memoirs No. 228. AMS, 1980, ix + 206 pp, $9.60 (P). [ISBN: 0-8218-2228-4] The first of three papers which will investigate the ring of cobordism classes of closed smooth manifolds with a symplectic structure on their stable normal bundle. LCL

PROBABILITY, P. *Large Deviations and Asymptotic Efficiencies.* P. Groenboom. Math. Centre Tracts, No. 118. Math Centrum, 1980, 123 pp, Dfl. 15 (P). [ISBN: 90-6196-190-4]

PROBABILITY, P. *Lecture Notes in Mathematics-724: Additive and Cancellative Interacting Particle Systems.* David Griffeath. Springer-Verlag, 1979, v + 108 pp, $9 (P). [ISBN: 0-387-09508-X] Notes based on a course in stochastic interacting particle systems using a graphical representation approach. JEG

PROBABILITY, P. *Binary Time Series.* Benjamin Kedem. Lect. Notes in Pure and Appl. Math., V. 52. Dekker, 1980, x + 140 pp, $23.50 (P). [ISBN: 0-8247-6920-1] Assuming a time series with stochastically dependent observations and a large amount of data, the author provides a number of "counting method" approaches to analyze the data. Many new ideas appear in this work, and so the author fills several gaps in the literature. TAV

PROBABILITY, S(15-17). *Problems in Probability Theory, Mathematical Statistics and Theory of Random Functions.* Ed: A.A. Sveshnikov. Dover, 1978, ix + 481 pp, $6.50 (P). [ISBN: 0-486-63717-4] Reprint of the 1968 English translation of the 1965 Russian edition (TR, October 1968). Each section contains basic formulas, solutions of typical problems, and a problem set for which there are answers and/or solutions in the back. RSK

PROBABILITY, P. *Lecture Notes in Mathematics-714: Calcul Stochastique et Problèmes de Martingales.* J. Jacod. Springer-Verlag, 1979, x + 539 pp, $25.30 (P). [ISBN: 0-387-09253-6] A rather extensive presentation of stochastic processes. After an introductory section there are sections on transformation rules when the space, probability, or filtration are changed. The last section gives applications to martingales, especially stochastic differential equations in the theory of Markov processes.TLS

STATISTICS, P. *Asymptotic Optimality Theory for Testing Problems with Restricted Alternatives.* T.A.B. Snijders. Math. Centre Tracts, No. 113. Math Centrum, 1979, xi + 265 pp, Dfl. 32 (P). [ISBN: 90-6196-183-1] Deals with a class of testing problems for exponential families, giving special attention to testing problems for contingency tables. Main results are concerned with asymptotically most stringent and everywhere asymptotically most stringent tests. RSK

STATISTICS, T(13: 1). *Statistics.* Robert S. Witte. HR&W, 1980, xv + 315 pp, $15.95. [ISBN: 0-03-055231-1] A non-mathematical treatment of the usual topics and some tests for ranked data. FLW

STATISTICS, T(13: 1). *Applied Elementary Statistics.* Richard I. Levin, David S. Rubin. P-H, 1980, x + 579 pp, $17.95. [ISBN: 0-13-040113-7] Presupposes no college level mathematics. Treats the usual introductory topics plus some non-parametric tests and, briefly, time series and index numbers. FLW

STATISTICS, T(13: 1). *Basic Statistics for Business and Economics.* Leonard J. Kazmier. McGraw, 1979, xiv + 457 pp, $16.95. [ISBN: 0-07-033445-5] Somewhat terse presentation of a wide variety of topics. In addition to standard introductory material it includes chapters on analysis of variance, the chi-square test, statistical decision analysis, regression and correlation analysis, multiple regression and correlation analysis, time-series analysis and forecasting, index numbers, and nonparametric statistics. RSK

STATISTICS, T(13: 1). *Elements of Statistics, An Introduction to Probability and Statistical Inference, Third Edition.* Donald R. Byrkit. D. Van Nostrand, 1980, xii + 482 pp, $16.95. [ISBN: 0-442-25771-6] Revision of the author's 1975 *Second Edition* (*First Edition,* TR, February 1974). Contains the usual topics, including analysis of variance, correlation and regression, nonparametric tests, etc., but the coverage is more detailed than in most texts. Treats, for example, both the Scheffé and Newman-Keuls *post hoc* multiple comparison tests. RSK

STATISTICS, T(14: 1), S. *Principles of Statistics.* M.G. Bulmer. Dover, 1979, iv + 252 pp, $3.50 (P). [ISBN: 0-486-63760-3] Reprint of the 1967 *Second Edition* (TR, June-July 1968). Classical and somewhat dated intermediate level introduction, requiring some calculus. Contains several interesting historical and philosophical discussions. RSK

STATISTICS, T(14-17: 1), S*, P, L*. *Randomization Tests.* Eugene S. Edgington. Statistics, V. 31. Dekker, 1980, xii + 287 pp, $29.50. [ISBN: 0-8247-6878-7] The use of randomization tests to determine the significance of results when only nonrandom samples are available. Presupposes an introductory course in statistics. Includes the needed programs. FLW

STATISTICS, P. *Inference and Linear Models.* D.A.S. Fraser. McGraw, 1979, xii + 297 pp, $34.50. [ISBN: 0-07-021910-9] Theoretical examination of aspects of statistical inference, with applications and illustrations, emphasizing the utility of a variation-based model. RSK

COMPUTER PROGRAMMING, S??. *32 BASIC Programs for the TRS-80 (Level II) Computer.* Tom Rugg, Phil Feldman. Dilithium Pr, 1980, xvii + 267 pp, $15.95 (P). [ISBN: 0-918398-27-4] Addressed to persons who, knowing nothing about programming, buy a Radio Shack TRS-80 computer. Programs are supplied for balancing one's checkbook, calculating gas mileage, practicing arithmetic, playing games, and solving certain mathematical problems. No explanation of how to write your own programs. AWR

COMPUTER SCIENCE, P. *Algol 68 Transput.* J.C. van Vliet. Math Centrum, 1979. *Part I: Historical Review and Discussion of the Implementation Model.* Math. Centre Tracts, No. 110, ix + 59 pp, Dfl. 8 (P) [ISBN: 90-6196-178-5]; *Part II: An Implementation Model.* Math. Centre Tracts, No. 111, ii + 218 pp, Dfl. 27 (P). [ISBN: 90-6196-179-3] *Part I*--describes the input/output facilities for Algol 68 and the development, fundamentals and features of a standard model for their implementation. *Part II*--a precise definition in terms of some implementable primitives of a standard for the input/output facilities of Algol 68. Based upon the work of the "Task Force on Transport." RWN

COMPUTER SCIENCE, T(15-16: 1), S, P, L. *The Denotational Description of Programming Languages, An Introduction.* Michael J.C. Gordon. Springer-Verlag, 1979, 160 pp, $9 (P). [ISBN: 0-387-90433-6] Demonstrates the usefulness of the techniques of the denotational method for the description of the semantics of programming languages. Based on the work of Strachey and Scott. Presumes familiarity with higher level languages. RWN

COMPUTER SCIENCE, T(15-16: 1, 2), S. *The Art of Digital Design: An Introduction to Top-Down Design.* David Winkel, Franklin Prosser. P-H, 1980, xiii + 498 pp, $24.95. [ISBN: 0-13-046607-7] The text emphasizes the structured approach of design problems, rather than the study of hardware ("we must let the problem requirements guide us to suitable hardware"). It systematically leads from logic fundamentals through digital design with microcomputers, with emphasis on solving digital problems using hardwired structures of the complexity of medium- and large-scale integration. Assumes some elementary knowledge of a high level language and the use of assembly language; assumes no prior knowledge of electronics. LCL

COMPUTER SCIENCE, T*(15-18: 1, 2), S, L. *Principles of Database Systems.* Jeffrey D. Ullman. Computer Sci Pr, 1980, 379 pp, $19.95. [ISBN: 0-914894-13-7] Includes descriptive material, relevant data structures, relation-based query languages. Principal ideas are associated to relations and concurrency. Relational, network, and hierarchical data models. Query optimization and security. Chapter exercises and bibliographic notes. Bibliography. Index. RJA

SYSTEMS THEORY, S(16-17), P, L. *Connectivity, Complexity, and Catastrophe in Large-Scale Systems.* John Casti. Wiley, 1979, xiii + 203 pp, $34.50. [ISBN: 0-471-27661-8] The seventh volume in a series published by the International Institute for Applied Systems Analysis, whose purpose is to make information about systems analysis available to a (world-)wide public and to encourage further research and investigation. By way of examples and illustrations, this volume provides an overview of recent attempts at coming to grips mathematically with such properties as connectivity, complexity, and stability in large scale systems. Unusually strong emphasis on models based on the tools and techniques of algebra and topology (e.g., semigroups, homology theory), as opposed to analysis. Useful as a source book and reference for a modelling course. LCL

APPLICATIONS (ARTIFICIAL INTELLIGENCE), P. *Computer Vision Systems.* Ed: Allen R. Hanson, Edward M. Riseman. Acad Pr, 1978, xxvii + 390 pp, $29. [ISBN: 0-12-323550-2] Papers from the workshop held June 1-3, 1977 at the University of Massachusetts in Amherst. Four sections of papers:

(1) Issues and Research Strategies; (2) Segmentation; (3) Theory and Psychology; (4) Systems. Introductory overview to the workshop topic and organization. List of contributions. Each section is preceded by a simple outline of the topic and a sentence which describes the contents of each of the papers of the section. Text is very nicely arranged. RJA

APPLICATIONS (CONTROL THEORY), P. *Lecture Notes in Mathematics-680: Mathematical Control Theory.* Ed: W.A. Coppel. Springer-Verlag, 1978, 257 pp, $14.30 (P). [ISBN: 0-387-08941-1] Proceedings of invited lectures from the Conference on Mathematical Control Theory held at the Australian National University in Canberra, August 23-September 2, 1977. RJA

APPLICATIONS (ELECTRICAL ENGINEERING), T(16-18: 1, 2), S, L. *Digital Signal Processing and Time Series Analysis, Pilot Edition.* Enders A. Robinson, Manuel T. Silvia. Holden-Day, 1978, v + 405 pp, $12.95 (P). [ISBN: 0-8162-7264-4] Format of a mathematics text. Assumes knowledge of advanced calculus. Introductory chapters on complex variables and numerical analysis. Includes chapters on the transfer function, Fourier transform of digital signals, analog and digital systems, digital filters, the kepstrum, random processes, spectral estimation, seismic and speech deconvolution. Appendix. Bibliography. Many problems and answers. RJA

APPLICATIONS (PHYSICS), T(15-17: 1, 2). *Simple Quantum Physics.* Peter Landshoff, Allen Metherell. Cambridge U Pr, 1979, viii + 177 pp, $24.95; $7.95 (P). [ISBN: 0-521-22498-5; 0-521-29538-6] A brief first course in quantum mechanics. Includes several appendices on mathematical topics (power series solution, delta function, Fourier transform), and a concluding section on hints for the problems. LAS

APPLICATIONS (PHYSICS), S(16-18), L*. *Worked Problems in Applied Mathematics.* N.N. Lebedev, I.P. Skalskaya, Y.S. Uflyand. Trans: Richard A. Silverman. Dover, 1979, xi + 429 pp, $6 (P). [ISBN: 0-486-63730-1] Unabridged republication of the original 1965 English edition titled *Problems of Mathematical Physics.* Over 600 problems, each with a hint for solution and, where appropriate, an answer. Detailed solutions are given for about 90 problems which illustrate specific methods or are particularly difficult or important. Extensive bibliography now out-of-date. Worthwhile sourcebook.JK

APPLICATIONS (PHYSICS), P. *Lecture Notes in Physics-99: Voraussage--Wahrscheinlichkeit--Objekt.* Michael Drieschner. Springer-Verlag, 1979, xi + 308 pp, $17.60 (P). [ISBN: 0-387-90248-X] Attempt to formulate à priori the structure of physical theory, to give a transcendental basis for physics. Defends the thesis that the structure of quantum mechanics (QM) is determined a priori simply by the requirement that it serve as a theory of experience. Cornerstone of the analysis is the principal role given to temporality, i.e., to prediction, which is seen as a constitutive part of any objective theory of reality. Other basic concepts of QM (viz., probability and object) are derived from prediction, and a detailed analysis is then given of the axioms and traditional problems of QM in terms of this framework. Written in nontechnical language. GHM

APPLICATIONS (PHYSICS), T(13: 1), S, L. *Principles of Mathematical Modeling.* Clive L. Dym, Elizabeth S. Ivey. Comp. Sci. and Appl. Math. Acad Pr, 1980, xiii + 260 pp, $17.95. [ISBN: 0-12-226550-5] An elementary treatment of diverse physical models (pendulum, linear oscillator, traffic flow, growth and decay, linear programming, diffraction), preceded by three chapters on tools (dimensional analysis, scaling, approximation), supplemented by cookbook appendices on the elementary transcendental functions, and simple first and second order differential equations. LAS

APPLICATIONS (PHYSICS), S(18), P, L. *The Logico-Algebraic Approach to Quantum Mechanics, Volume II: Contemporary Consolidation.* Ed: C.A. Hooker. Reidel, 1979, xx + 466 pp, $24 (P); $69. [ISBN: 90-277-0709-X] Anthology of key papers published since 1968 examining the formal, or abstract, structure of elementary quantum theory. (*Volume I* collected historical papers, 1935-68, emphasizes the mathematical and physical development of quantum theory.) In this approach physical theory is analyzed by extracting the logical and algebraic structures underlying the theory. The formal properties of these abstract structures are then developed, leading back to a clarification of the physical theory. GHM

APPLICATIONS (PHYSICS), P. *Geometric Quantization and Quantum Mechanics.* Jędrzej Śniatycki. Appl. Math. Sci., V. 30. Springer-Verlag, 1980, ix + 230 pp, $14 (P). [ISBN: 0-387-90469-7] Revised and expanded notes from two seminar series at the University of Calgary during 1976-77. JAS

APPLICATIONS (PHYSICS), L. *Concise Dictionary of Physics and Related Subjects, Second Edition, Revised and Enlarged.* J. Thewlis. Pergamon Pr, 1979, ix + 370 pp, $52. [ISBN: 0-08-023048-2] Over 6,000 one-sentence definitions supplemented with appendices giving SI units, the latest fundamental physical constants, and the periodic table of elements. LAS

APPLICATIONS (PHYSICS), T(18: 1), P. *Lecture Notes in Mathematics-733: Modern Differential Geometric Techniques in the Theory of Continuous Distributions of Dislocations.* Frederick Bloom. Springer-Verlag, 1979, xii + 206 pp, $12.50 (P). [ISBN: 0-387-90528-4] Applies the techniques of Riemannian geometry and distribution theory to the study of dislocations and anelasticity. Good bibliography. TLS

Reviewers Whose Initials Appear Above

Richard J. Allen, St. Olaf; William Carlson, St. Olaf; Judith N. Cederberg, St. Olaf; Clifton E. Corzatt, St. Olaf; John Dyer-Bennet, Carleton; Jay E. Goldfeather, Carleton; Roger B. Kirchner, Carleton; Richard S. Kleber, St. Olaf; Joseph Konhauser, Macalester; Loren C. Larson, St. Olaf; George H. Mills, St. Olaf; R.W. Nau, Carleton; A. Wayne Roberts, Macalester; John Schue, Macalester; Seymour Schuster, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; Timothy L. Strotman, St. Olaf; T.A. Vessey, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY PAUL A. HAEDER, University of Nebraska at Omaha

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036

PERSONAL ITEMS

U.S. Naval Academy: Dr. *Gail A. Kaplan*, Southampton College, has been appointed Assistant Professor. Ms. *Shirley M. Stolavski*, Prince George Community College, has been appointed Instructor. Assistant Professor *Dominic Leung* has left to work at the Electromagnetic Compatibility Analysis Center. Assistant Professor *Donald Hartig* has accepted an appointment as Assistant Professor at Cal. Poly. State University. Assistant Professors *Richard L. Davis* and *Thomas J. Sanders* have been promoted to Associate Professors. Associate Professor *Earl G. Swafford* has retired.

University of Illinois: Dr. *Bruce A. Reznick*, University of California, Berkeley, has been appointed Assistant Professor. Dr. *Haskell P. Rosenthal* has been appointed as Professor at the University of Texas. Associate Professor *Robert E. Bohrer* has been promoted to Professor. Assistant Professor *Amassa C. Fauntleroy* has been promoted to Associate Professor. Professor *Kenneth I. Appel* has been awarded a Fulkerson Prize.

Tennessee Technological University, Cookeville: Professor *Jacob T. B. Beard, Jr.*, University of Texas, Arlington, has been named Chairman of the Department of Mathematics and Computer Science. Associate Professor *James D. Harris* has accepted an appointment in the Department of Computer Science and Information Systems at Pittsburg State University, Pittsburg, Kansas. Associate Professor *S. B. Khleif* has been promoted to Professor. Professor and Chairman *Ralph Boles* has retired with the title of Professor Emeritus.

Miami University, Oxford, Ohio: Visiting Assistant Professor *Clifton Ealy* has been appointed Associate Professor at Northern Michigan University. Associate Professor *Dennis Burke* has been promoted to Professor. Assistant Professor *Mark Smith* has been promoted to Associate Professor.

Frostburg State College, Frostburg, Maryland: *Edward T. White*, formerly Assistant Professor at Allegheny Community College, Cumberland, Maryland, and *Terry L. King*, formerly Instructor at Theil College, Pannsylvania, have been appointed Instructors. Professor *Richard C. Weimer* has been appointed Head of the Department of Mathematics. Professor *Walter J. Rissler* has retired with the title of Professor Emeritus.

University of Nebraska at Omaha: Professor *Paul A. Haeder* has retired with the title of Professor Emeritus. He was awarded the UNO Great Teacher Award at the 1980 commencement exercises. Associate Professor *Charles Downey* has been appointed Chairman of the Department of Mathematics and Computer Science.

Iowa State University: Dr. *Peter H. Chang*, University of Western Ontario and Dr. *Harland W. Stech*, Virginia Polytechnic Institute, have been appointed Assistant Professors. Dr. *Gary M. Lieberman*, Stanford University, has been appointed Instructor. Associate Professor *Irvin R. Hentzel* has been promoted to Professor. Instructors *Richard L. Epstein* and *Elgin H. Johnston* have been promoted to Assistant Professors.

Emory University, Atlanta, Georgia: Dr. *Ronald J. Gould*, San Jose State University, has been appointed Assistant Professor. Assistant Professor *William T. Franke* has been appointed Assistant Professor at Oxford College of Emory University.

Northern Michigan University, Marquette: Assistant Professor *Agnes Andreassian* and Instructor *Isao Ozaki* have accepted appointments at Central College, Iowa, and Louisiana State University, respectively. Associate Professors *Theodore Eisenberg*, *Jane O. Swafford*, and *Robert B. McNeil* have been promoted to Professors.

Dartmouth College, Hanover, New Hampshire: *Vera Pless*, University of Illinois at Chicago Circle, has been a Visiting Professor. *Thomas Tucker*, Colgate University, has been a Visiting Associate Professor. Assistant Professor *Charles Dwight Lahr* has been promoted to Associate Professor with tenure.

Fairleigh Dickinson University, Teaneck, New Jersey: Associate Professor *Gilbert Steiner* has been promoted to Professor. Assistant Professor *Norman Landis* has been promoted to Associate Professor. Instructor *Tsung Dow Huang* has been promoted to Assistant Professor.

Clemson University: Recent appointments to the staff are Associate Professor *Robert E. Jamison*, formerly Assistant Professor at LSU, Associate Professor and Associate Head *Richard D. Ringeisen*, formerly of Purdue University. Professor *W. H. Ruckle* is a Visiting Professor at Western Washington State University, Bellingham. The following are visiting faculty members: Visiting Professor *Thomas A. Carnevale*, Essex Community College, New Jersey; Adjunct Associate Professors *Larry Q. Eifler*, University of Missouri at Kansas City, *D. R. Shier*, National Bureau of Standards, and *Martin Van Wyk*, Simpson College, Indianola, Iowa; Adjunct Professor *Paul Ohme*, Mississippi College; and Visiting Assistant Professor *Vernon Liberty*, University of Oklahoma. Dr. *C.V. Aucoin* has been named Chairman of the UMAP Editorial Board and to the Executive Council of CBMS. Dr. *J.V. Brawley* is a Lecturer in MAA's Visiting Lecturers Program. Dr. *J.W. Kenelly* is President of the Southeastern Section, MAA, and the Chairman of the College Board's Committee on Calculus Development. Dr. *S.M. Lukawecki* has been named by the MAA to be the regional contest chairperson for the State of South Carolina. Dr. *T.G. Proctor* is Chairman of the SIAM Education Committee and is a Lecturer in SIAM's Visiting Lecturer Program.

University of Pittsburgh: Dr. *C.D. Levermore*, formerly of Livermore Laboratory, has been appointed Assistant Professor. Assistant Professor *Peter Lipow* has accepted an appointment as Associate Professor at the University of the Pacific. Associate Professor *T.A. Porsching* has been promoted to Professor. Assistant Professors *T. A. Metzger* and *Jacob Burbea* have been promoted to Associate Professors. Professor *Joseph Lehner* has retired with the title of Professor Emeritus.

Illinois State University, Normal: Instructor *John Bradburn* has returned to Elgin Community College, Elgin, Illinois. Assistant Professor *Charles Weaver* has been employed by a computer company in Champaign, Illinois. Members on leave and their respctive assignments are: *Robert Speiser*, University of Minnesota; *Kwang-Chul Ha*, Kona National University in Seoul; *Lawrence Egan*, Associate Editor, Mathematical Reviews, Ann Arbor, Michigan.

University of California, Berkeley: Professor *Calvin C. Moore* has accepted the position of Director of the Center for Pure and Applied Mathematics for 1979-80. Visiting scholars are: *Alain Etcheberry*, Universidad Simon Bolivar; *Ted Hill*, Georgia Institute of Technology; *Robert Osserman*, Stanford University; *Sehie Park*, Seoul National University; and *David Rush*, University of California, Riverside.

Louisiana State University: Professor *Richard D. Anderson* has retired with the title of Boyd Professor Emeritus. Professors *Dan R. Scholz* and *Luther I. Wade* have retired as Professors Emeriti.

Recent appointments to the staff at San Jose State University are Assistant Professors *David Posner*, *Veril Phillips*, and *Marilyn Roth*.

Lt. Colonel *Walter M. Patterson*, III, Deputy Director of Studies and Analysis, Military Airlift Command, has been appointed Associate Professor of Mathematics at Lander College, Greenwood, South Carolina.

Professor *Yousef Alavi* has been chosen as the first recipient of Western Michigan University's new Distinguished Service Award.

Professor *Charles H. Cunkle*, Slippery Rock State College, Pennsylvania, has retired with the title of Professor Emeritus.

Professor *R.H. Bing*, past President of the MAA and AMS, received an honorary D.Sc. at commencement ceremonies at Kenyon College on May 25, 1980.

Dr. *Paul Blanchard*, University of Southern California, and Dr. *John McFall*, University of Waterloo, have been appointed Assistant Professors at Boston University.

Dr. *Helena S. Wisnieski*, City University of New York, has been appointed Assistant Professor in the Computer and Decision Sciences Department of Seton Hall University.

Associate Professor *Eileen L. Poiani*, Saint Peters College, Jersey City, New Jersey, has been promoted to Professor.

Professor *Floyd F. Helton*, University of the Pacific, Stockton, California, has retired after 38 years of full-time college and university teaching.

Dr. *Sheldon P. Gordon*, Suffolk Community College, Seldon, New York, has been named the 1980 recipient of the New York State Mathematics Association for Two Year Colleges award for Outstanding Contributions to College Mathematics.

Assistant Professor *J. Gopal Danaraj*, the Cleveland State University, has accepted a position at the NASA research center in Langley, Virginia.

Associate Professor *Franz X. Hiergeist*, West Virginia University, has been promoted to Professor.

Father *Robert R. Dobbins*, Fordham University, has been appointed Assistant Professor at Iona College, New Rochelle, New York.

Associate Professor *R.R. Davidson*, University of Victoria, British Columbia, has been promoted to Professor.

Pamela Coxson, Ph.D., University of Southern California, has been appointed Assistant Professor at the University of Delaware.

Professor *Frederick Mosteller*, Harvard School of Public Health, Department of Biostatistics, has been awarded the Lazarsfeld Prize for Applied Social Science Research (1979).

Associate Professor *Verne Sanford*, Valparaiso University, Indiana, has been promoted to Professor.

Associate Professor *Arun K. Agarwal*, Grambling State University, Louisiana, has been promoted to Professor.

Professor *Kenneth S. Williams*, Carleton University, Ottawa, Canada, has been appointed Chairman of the Department of Mathematics and Statistics for a three year term beginning July 1, 1980.

Associate Professor *Warren Page*, New York City Community College, has been promoted to Professor.

Professor Emeritus *Alfred E. White*, Kansas State University, died on May 31, 1980. at the age of 101! He was a charter member of the Kansas section, which was the first organized section of MAA.

Professor Emeritus *Cecil G. Phipps*, Tennessee Technologocal University, died on September 5, 1979, at the age of 84. He was a member of the Association for forty nine years.

Mrs. *Nelontine M. Larsen*, a former Assistant Professor at the University of Nebrska at Omaha, died in Lincoln, Nebraska, on March 12, 1980, at the age of 55. She was a member of the Association for seventeen years.

Associate Professor *Walter J. Carpenter*, North Georgia College, died on July 30, 1979. He was a member of the Association for twenty one years.

Mr. *Robert M. Pinkerton*, retired, died on October 3, 1978, at his home in Kinsdale, Virginia. He was a member of the Association for fifty years.

Professor Emeritus *Ralph D. James*, Ganges, British Columbia, died on May 19, 1979. He was a member of the Association for forty seven years.

Dr. *Louis O. Kattsoff*, Boston College, died in June, 1979. He was a member of the Association for forty three years.

Dr. *Joseph F. Thomson*, retired, died on October 11, 1979 in New Orleans, at the age of 75. He was a member of the Association for forty two years.

Mr. *Kenneth L. Calder*, Raleigh, North Carolina, died on March 11, 1980, at the age of 84. He was a member of the Association for two years.

Professor *Russell N. Bradt*, University of Kansas, died on April 8, 1980, at the age of 57. He was a member of the Association for thirty one years.

Dr. *James F. Gray*, Saint Mary's University, San Antonio, Texas, died on June 4, 1980, at the age of 58. He was a member of the Association for twenty nine years.

OPERATIONS RESEARCH SOCIETY OF AMERICA PRESENTS GEORGE E. KIMBALL MEDAL TO PROFESSOR BERNARD OSGOOD KOOPMAN

Bernard Osgood Koopman, Professor Emeritus of Mathematics at Columbia University, was awarded the George E. Kimball Medal of the Operations Research Society of America on May 6, 1980 at the Shoreham Hotel, Washington, D.C. The presentation was made by David M. Boodman, member of the Kimball Selection Committee, for Koopman's "outstanding service to the profession of operations research for over four decades" in his work as theoretician, practitioner, and representative of the profession.

Professor Koopman has served his country as a member of the Weapons Systems Evaluation Group of the Department of Defense; as the United States/United Kingdom Liaison Officer; and subsequently as Director of the Weapons Systems Evaluation Division of The Institute of Defense Analysis; and as a member of the advisory panel to NATO where he still serves.

Professor Koopman was President of the Operations Research Society of America in 1956 and 1957.

A REQUEST FOR COPIES OF PUBLICATIONS ON STATISTICAL DISTRICUTIONS AND THEIR APPLICATIONS TO VARIOUS FIELDS

A Dictionary and Bibliography of Statistical Distributions is now in its final stages of preparation. Both discrete and continuous distributions are included, whether univariate or multivariate. Further, a Display of Literature on Statistical Distributions in Scientific Work will be organized during the forthcoming program to be held at Trieste, Italy during July 10-August 1, 1980. One or two copies of relevant literature in the form of reprints, reports, computer programs, theses, books, etc., would be most welcomed as soon as possible. (No need to send the materials again if they have already been sent.) Please send these to: Professor G. P. Patil, Department of Statistics, 318 Pond Laboratory, The Pennsylvania State University, University Park, Pennsylvania 16802, USA.

THE JOHN VON NEUMANN THEORY PRIZE WINNERS NAMED

On May 6, 1980 Professors David Gale, Harold W. Kuhn and Albert W. Tucker were awarded the 1980 John von Neumann Theory Prize by the Operations Research Society of America and The Institute of Management Sciences, it was announced by Donald P. Gaver, prize committee chairman.

The three winners belong to what is known as the "Princeton School" and much of their work has appeared in Princeton's annals of Mathematical Studies. Professor Gale now teaches at the University of California, Berkeley. Dr. Kuhn is the professor of Mathematical Economics at Princeton University. Dr. Tucker is the Albert Baldwin Dod Professor of Mathematics Emeritus, Princeton University.

The $2,000 prize is awarded for fundamental theoretical contributions which have stood the test of time and have had large impact on the development of operations research and management science theory. The individually-authored research papers of these three scholars, together with those formally identified as their collaborators, form a distinctive and substantial portion of the foundational literature of linear programming, the theory of games, and non-linear programming.

The John von Neumann Theory Prize was established in 1975 by the Operations Research Society of America and The Institute of Management Sciences.

THE 1980 WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The 41st Annual William Lowell Putnam Mathematical Competition will be held at participating institutions on Saturday, December 6, 1980. This competition is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship and is administered by The Mathematical Association of America. All colleges and universities in Canada and the United States may register eligible undergraduates. Registration forms will be mailed to institutions that participated in the 40th competition by September 22, 1980. Other institutions that wish to enter undergraduates should request registration forms from Professor L. F. Klosinski, Director; The William Lowell Putnam Mathematical Competition; University of Santa Clara; Santa Clara, CA 95053. Completed registrations must be received by the Director no later than October 17, 1980.

Further details are given in the Announcement Brochure that is mailed with the registration material. Reports of previous competitions, including examination questions and outlines of solutions, are in past issues of this *MONTHLY*; the most recent of these reports were in the issues of November 1979, March 1979, January 1978, November 1976, and November 1975.

COMPUTER-BASED INSTRUCTION IN MATHEMATICS

A Consortium Interest Survey of those interested in or using computers or researching their use in mathematics instruction in high school and college courses is being distributed by Dr. Ronald H.

Wenger, Associate Professor of Mathematics and Associate Dean of the College of Arts and Sciences at the University of Delaware. Please write to him for a copy of this brief survey or to tell him of your own activities. His address is: 123 Memorial Hall, University of Delaware, Newark, Delaware, 19711 (302/738-2351).

SURVEY ARTICLE ON EQUATIONAL LOGIC

In a special 1979 issue, the Houston Journal of Mathematics published a survey article entitled "Equational Logic" by Walter Taylor. This is a survey of existing work of many authors in equational logic or varieties of algebras. The primary interest is in equations for general algebraic systems. The exposition is self-contained and no proofs are included.

The objective is to make more mathematicians aware of this subject and to provide a readable introduction to its examples, its theorems, and what they mean in a fairly broad context. At the same time the author provides a reasonably complete survey of the literature which will be helpful to specialists. The survey contains 83 pages and has over 700 references.

Individual copies, at the cost of US $6.00 each (prepaid), can be ordered from:

Houston Journal of Mathematics
University of Houston, Central Campus
Houston, Texas 77004

NEW FOLDER DESCRIBES "BASIC" AND "FORTRAN' VIDEOTAPED COURSES

Now offered by Colorado State University's Engineering Renewal & Growth Program is a four-page folder describing in detail two color videotaped courses on computer programming. "Introductory FORTRAN" course consists of ten 25-minute lectures. The Course is designed to provide a working knowledge of FORTRAN and an initiation to the power of computer operating systems. The "Introductory BASIC" course consists of seven 15-minute lectures which focus on developing proficiency in the BASIC language. For free folder, contact: W. L. Somervell, Jr., ERG Director, Colorado State University, Christman Field, Bldg. 1000, Ft. Collins, Colorado 80523.

VISITING LECTURER PROGRAM IN STATISTICS

THE VISITING LECTURER PROGRAM IN STATISTICS is continuing into its eighteenth successive year. This year's program again is available to Canadian schools. The program is sponsored jointly by the principal statistical organizations in North America, the American Statistical Association, the Biometric Society and the Institute of Mathematical Statistics. Leading teachers and research workers in statistics—from universities, industry and government—have agreed to participate as lecturers. Lecture topics include subjects in experimental and theoretical statistics, as well as in such related areas as probability theory, information theory and stochastic models in the physical, biological and social sciences.

The purpose of the program is to provide information to students and college faculty about the nature and scope of modern statistics, and to provide advice about careers, graduate study, and college curricula in statistics. Inquiries should be addressed to: Barry C. Arnold, Chair., Visiting Lecturer Program in Statistics, Department of Statistics, University of California, Riverside, CA 92521.

NEED INFORMATION ON CALCULATORS?

The Calculator Information Center at The Ohio State University has received funding from the National Institute of Education for another year of operation. The Center, in operation since 1977, collects and disseminates information on calculator uses. Educators using calculators can contribute to the information exchange by calling 614-422-8509 between 8 and 4 (Eastern time zone) or by writing: Marilyn N. Suydam, Director, Calculator Information Center, 1200 Chambers Road - Room 201, Columbus, Ohio 43212.

The Center disseminates information primarily through two types of bulletins. *Reference* Bulletins cite journal articles and other types of publications; *Information* Bulletins contain suggestions on such topics as selecting calculators, using calculators at various instructional levels, and conducting a workshop. If you would like a copy of the latest reference bulletin (and would like your name added to the mailing list), contact the Center at the above address.

CONFERENCE ANNOUNCEMENT

The VIII Annual Ohio Developmental Education Conference entitled "Intelligence Can be Taught!?" will be held November 5-7, 1980 at the Carrousel Inn, Cincinnati, Ohio. For information and registration forms contact Dr. Tanya Ludutsky or Dr. Phyllis Sherwood, Raymond Walters General & Technical College, 9555 Plainfield Road, Cincinnati, Ohio 45236 (513-745-4202).

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

KENTUCKY SECTION ANNUAL MEETING

The sixty-third annual meeting of the Kentucky section of the Mathematical Association of America was held at Western Kentucky University, Bowling Green, Kentucky, on April 11-12, 1980.

Invited speakers were Professor *Doris Schattschneider*, Moravian College and editor of the Mathematics Magazine, and Professor *Joan Leitzel*, Ohio State University. On Friday night, Professor *Schattschneider* spoke on *Mathematical Symmetry and Art*. On Saturday afternoon, she spoke on *Will It Tile? Try the Conway Criterion*. Professor *Leitzel*'s address was entitled *Current National Trends in Developmental Mathematics*.

On Friday evening, the movies *Adventures in Perception* and *Maurits Escher: Painter of Fantasies* were shown.

Contributed papers given on Saturday were:

X-amining Sets and Logic, Professor *James B. Barksdale*, Western Kentucky University
Remediation and Microcomputers, Professor *Roger Geeslin*, University of Louisville
Some Aspects of Knot Theory, Professor *Steve Kaplan*, University of Kentucky
A Time-To-Go Algorithm for a Guided Missile Using Acceleration Information, Professor *Randy York*, Western Kentucky University
Microcomputer-Generated Graphics, Professor *Tom Jenkins*, University of Louisville
Another Look at Formal Power Series, Professor *Paul Eakins*, University of Kentucky
Using Systems of Linear Congruences to Cipher and Decipher Codes, Professor *Pauline Lowman*, Western Kentucky University; and
Abelian Groups and Compact Abelian Groups, Professor *Ed Enochs*, University of Kentucky.

Total attendance was 125, of whom 78 were members.

Professor *Ed Enochs*, University of Kentucky, was elected Vice-Chairman for the 1980-82 term and Professor *Joe Smith*, Northern Kentucky University, was re-elected Secretary-Treasurer for 1980-83.

The sixty-fourth meeting will be held at Jefferson Community College, Louisville, on April 3-4, 1981.

JOE K. SMITH, *Secretary*

REPORT OF THE KANSAS SECTION MAA MEETING

The sixty-fifth annual meeting of the Kansas Section of the Mathematical Association of America was held on April 11-12, 1980, at Kansas State University, Manhattan, Kansas. Section Chairman *Elaine Tatham* presided. Approximately 90 people attended.

Invited addresses were *Preparation for Careers in Applied Mathematics* and *Mathematical Models and Existence Theorems* by *Dorothy L. Bernstein* and *Some Problems Neither My Computer Nor I Can Solve . . . Yet* and *Floating Point Numbers—Angels or Demons* by *Richard V. Andree*.

The following contributed papers were presented:

Indices and Logarithms by *Wanda Reves*, Fort Hays State University
Bordered Hessians by *Gary N. McGrath*, Pittsburg State University
Mathematics in Software Quality Assurance by *David Foley*, Wichita State University
Some Geometry Problems Solved with the Help of Complex Numbers by *Lucio Arteaga*, Wichita State Univ.
Factorization Structures by *Austin Melton*, Kansas State University
On the Genus of a Knot by *Thirza Mossman*, Kansas State University
Fourier-Stieltjes Coefficients of Measure on the Torus by *William M. Self*, Pittsburg State Univesity
Student Achievement and Retention in Liberal Arts Mathematics Taught Using Small Groups by *David K. Urion*, Marymount College.

The four top performers in the 1978 Putnam Examination from the Kansas Section were awarded one year memberships in the MAA. They are: *Bruce Leban*, Kansas University; *Daniel Grubb*, Kansas State U; *Eric Krohn*, Kansas State U; *Douglas Burkholder*, McPherson College.

ROBERT THOMPSON, *Secretary*

REPORT OF MARYLAND-D.C.-VIRGINIA SECTION MEETING

The annual Spring meeting of the Maryland-District of Columbia-Virginia section of the Mathematical Association of America was held Saturday, April 12, 1980, at the University of Richmond in Richmond, Virginia. The program was structured about the topics, *What's Happening in Pure Mathematics?* and *A Mathematical Program Especially for Students*. In addition, the keynote speaker, Professor *Leonard Gillman*, Treasurer of the MAA, spoke on *Optimal Strategies for Sports*.

J. R. HANSON, *Secretary*

MEETING OF THE IOWA SECTION MAA

The 67th regular meeting of the Iowa Section, MAA, was held on the campus of Simpson College, Indianola, Iowa on April 18 and 19, 1980. The meeting was held jointly with the Iowa sections of the Society for Industrial and Applied Mathematics and the American Statistical Association as well as the Mathematics Section of the Iowa Academy of Science. Chairperson *Frank Kosier* presided over the Friday session while *Herbert Hethcote* and *Stuart Klugman* of SIAM and ASA, respectively, presided

over the Saturday sessions. There were 43 persons present on Friday, 31 of whom were members of the section. On Saturday 61 people attended in the morning and 50 in the afternoon, of whom 42 and 36, respectively, were MAA members.

The program, arranged by *Arnold Adelberg, Herbert Hethcote,* and *Stuart Klugman,* consisted of the contributed papers, invited addresses, and the panel discussion listed below along with the Governor's report, presented by *James Cornette,* and the business session.

At the business meeting *A. M. Fink* of Iowa State University was elected to the office of Chairperson Elect. Following a lengthy discussion the following motion was passed.

> Motion: The Iowa Section of the MAA shall no longer function as the Mathematics Section of the Iowa Academy of Science. The Executive Committee of the Iowa Section of the MAA is instructed to make every effort to negotiate in a constructive manner our holding meetings at the time and place of the Academy meeting, consistent with our interest in joint meetings with SIAM and ASA.

Thanks were extended to *James Van Deventer* and Simpson College for their services.

Papers, invited addresses, and panel discussion:

Survey Sampling, WAYNE FULLER, Iowa State University

Estimating Remainders by Geometry, DONALD E. SANDERSON, Iowa State University

NP-Completeness, DAVID L. REINER, Grinnell College

Invited Address: *Alternative Rings - from the Cayley Numbers to the Present,* ERWIN KLEINFELD, University of Iowa

The Khachian Algorithm for Solving Linear Programs, A. M. FINK, Iowa State University

Invited Address: *Mathematics and Computing Science in an Industrial Research Laboratory,* NORMAN L. SCHRYER, Bell Telephone Laboratories, Murray Hill, New Jersey

Panel Discussion: *Current Employment Opportunities in the Mathematical Sciences.* JOHN HUNT, John Deere & Co. Technical Center; HAL SCHOEN, University of Iowa; REG YODER, Bankers Life Insurance Co.; JOHN J. GOEBEL, Iowa State University; HERBERT HETHCOTE, University of Iowa, Moderator.

Public Utility Rates: Natural Gas Usage and Weather, EUGENE RASMUSSEN, Iowa State Commerce Commission, Des Moines

Some Open Questions in Three-Dimensional Geometric Approximations, JAMES HURT, John Deere & Co. Technical Center, Moline, Illinois

Finding the Source of a Signal, RUSSELL LENTH, University of Iowa

Interdisciplinary Approach to Programming Courses, H. M. WALKER, Grinnell College

Recruiting a Center for Iowa (with Calculus), S. R. PORTER, Grinnell College.

E. J. PEAKE, *Secretary-Treasurer*

NEBRASKA SECTION SPRING MEETING

The fifty-seventh annual meeting of the Nebraska Section was held at Doane College, Crete, Nebraska on Friday and Saturday, April 18-19, 1980. (The eleventh annual meeting had been held at Doane College in 1934.) Professor *Mildred Gross*, Chairman of the Section, presided at both sessions. Professor *Dorothy Bernstein*, President of the MAA, brought greetings and discussed programs and projects.

Officers for 1980-1981 were elected as follows: Chairman—*Alexander Mehaffey*, Jr., University of South Dakota, Vermillion, South Dakota 57078; Chairman Elect—*Randall K. Heckman*, Kearney State College, Kearney, Nebraska 68847; Contest Chairman—*Stanley D. Luke*, Nebraska Wesleyan University, Lincoln, Nebraska 68504.

Henry M. Cox will serve as Secretary-Treasurer (second year of a three-year term) and, with *Thomas Shores*, edit the Section Newsletter. Professor *Gary Meisters*, Governor for the Section, reported at the Business Meeting.

Papers were presented as follows:

A Mini-Course in the Teaching of the History of Mathematics, ALEXANDER MEHAFFEY, JR., University of South Dakota

Some Results on Finite Semigroups, JOHN C. MEAKIN, University of Nebraska-Lincoln

On Maximal Intervals of Existence and Uniqueness for Solutions of Boundary Value Problems, ALLAN PETERSON, University of Nebraska-Lincoln

Dancing Curves: A Dynamic Demonstration of Geometric Principles, CHRISTOPHER F. MASTERS, Doane College

A Small College's Experiences with Applications in the Mathematics Curriculum, DOROTHY L. BERNSTEIN, Brown University, President of the Mathematical Association of America

An Improbable Result in Probability, BERNARD J. PORTZ, Creighton University

Some New Combinatorial Designs, SPYROS S. MAGLIVERAS, University of Nebraska-Lincoln

How Continued Fractions Help to Solve Diophantine Equations, GARY H. MEISTERS, University of Nebraska-Lincoln

Efficient Calculation of Continued Fraction Expansions, DALE M. MESNER, University of Nebraska-Lincoln

A New Approach to the Problem of Uniqueness of Multivariable Trigonometric Series, VIVIAN MOSCA, University of South Dakota

L'Hospital's Rule as a Test for Series Convergence, ALAN V. LAIR, University of Nebraska-Lincoln

Report on the 1980 Nebraska-South Dakota Annual High School Mathematics Examination, STANLEY D. LUKE, Nebraska Wesleyan University

Mathematical Modeling and Existence Theorems, DOROTHY L. BERNSTEIN, Brown University, President of the Mathematical Association of America.

HENRY MIOT COX, *Secretary*

MAA SPRING MEETING ALLEGHENY MOUNTAIN SECTION

The annual meeting of the Allegheny Mountain Section, MAA, was held April 25-26, 1980, at West Virginia Wesleyan College, Buckhannon, West Virginia. There were ninety members and forty-three students present at the meeting.

Invited lectures were given by Joseph Gallian, University of Minnesota-Duluth, entitled *The Marvelous Mathieu Groups;* by MORTON GURTIN, Carnegie Mellon University, on *The Dynamics of Population;* and by RICHARD D. ANDERSON, Louisiana State University, *Algorithmically Defined Functions.*

There were two panel discussions which were of great interest to the members present at the meeting. *Nicholas Ford,* Penn State-Fayette Campus moderated a panel on the High School Math Contest. Members of the panel were *Frank Kocher,* Penn State-University Park, *Francis H. S. Hall,* Penn State-Fayette Campus, *Richard Genung,* Meadville High School, and *Georgiana Esch,* Altoona High School. A panel discussion moderated by *E. Robert Anderson,* West Virginia Wesleyan College on Mathematics and/or Computer Science Programs was an important part of the Saturday program. The other members of the panel were *Dave Mader,* Parkersburg Community College, and *Richard McDermot,* Allegheny College.

The student portion of the program was highlighted by a swap session Friday evening and a Saturday morning discussion of *Reflections via Escher* under the direction of *Barbara Faires,* Westminster College.

The contributed talks given by the faculty were:

Integrating Mathematical Modeling into the Curriculum, JOSEPH E. WIEST, West Virginia Wesleyan College
A UMAP Module on the Force of Interest, MICHAEL MAYS, West Virginia University
The Construction of Finite State Automata Conistent with Given Data, WILLIAM REYNOLDS, Carlow College
A Demonstration of Microcomputer Graphics as an Instructional Tool, ROY E. MYERS, Penn State-New Kensington Campus
Primes, Permutations, and Combinations, JOHN B. LANE, Edinboro State College

The student contributed talks were:

Applications of Fourier Series in Optics, MARK COMELLA, St. Vincent
Automorphism Groups of Codes, DONNA FREEMAN, Allegheny College
Countable Compactifications, CAROL HARTLEY, Allegheny College
A Decision Model for the Allocation of Funds for Mathematical Research in the U.S., DAVID HUDAK nd PAUL WHITESIDE, St. Vincent.
Knight's Tour of the Chess Board (Solution by Computer Model), ARTHUR KAUPE, St. Vicenet
Problems from Actuarial Exams, BERT PAUL, Westminster
Markov Analysis of Beach Morphology Changes, DALE WEISS, Allegheny College.

The business meeting chaired by *Allan Krall,* Pennsylvania State University, was held Saturday morning. High school math contest reports were given by *Nicholas Ford,* Penn State-Fayette Campus for Western Pennsylvania; and by *I. Dee Peters,* West Virginia University. Reports were given by *Ben Haytock,* Allegheny College, High School Lecture program; *Melvin Woodard,* Indiana University of Pennsylvania, Governor's report; and *John Milsom,* Butler County Community College, Putnam exam report.

Student membership in MAA will be presented to the four highest places for the Putnam exam in Allegheny Mountain Section. The high school students in western Pennsylvania and West Virginia who achieved the best scores for the high school math contest in their respective states will receive a copy of *Math Gems I.*

A report was given by *Richard Anderson* for the national office. Officers elected were *Donald Platte,* Mercyhurst College, Second Vice-Chairman; *John Atkins,* Bethany College, Secretary-Treasurer; and *Barbara Faires,* Westminster College, Coordinator of the Student Program. *Francis H. S. Hall,* Penn State-Fayette Campus, moved up to First Vice-Chairman.

JOHN ATKINS, *Secretary-Treasurer*

APRIL MEETING OF THE OHIO SECTION

The Ohio Section of MAA held its sixty-fourth annual Spring meeting at Wittenberg University, Springfield, Ohio, April 25-26, 1980. One hundred and thirty people registered for the meeting. Section Chairman *D.O. Koehler* presided; *H.W. Vayo* was the Program Chairman. The invited address: *Groups—A Key to Patterns in Science,* was presented by *J. Sutherland Frame,* Michigan State University.

The following contributed papers were also presented:

Centralizer of the Fixed-Point Algebra of a Modular Automorphism Group, N. AZARNIA, Miami University-Hamilton
Number of Subgroups of the Additive Rationals, R. BAER, Miami University-Hamilton
Cardinal Pairs and the Size of Sets, T. DILLON, Youngstown State University
The Numerical Solution to Some Singular Initial Value Problems, J. HINTZ, University of Akron
Reducibility of Linear Systems, J. JONES and A.R. TOBIN, Air Force Inst. of Tech., Dayton, and T. MAZUMDAR, Wright State University
Sums of Cubes, D. KULLMAN, Miami University
Moment Matching Samples, C. LOONEY, University of Toledo
The Numerical Approximation of Analytic Functions in the Complex Domain, T. PRICE, University of Akron.

The meeting agenda also included meetings of The Executive Committee and of *Ad hoc* committees. Additional program highlights included: *Heroes of Mathematics' Lecture, On R.L. Moore,* by G. YOUNG, Case-Western Reserve University; retiring chairman's address, *Evolution and Mathematical Models of Evolution,* by D.O. KOEHLER, Miami University; and the following special sessions - Student Papers, moderated by M. COX (Miami University), B. DAVIS (University of Toledo), and J.D. FAIRES (Youngstown State University) - Job Market Assessment, moderated by G. YOUNG (Case-Western Reserve University) - and Computer Graphics, moderated by T. HERN (Bowling Green State University), C. LONG (Bowling Green

State University), and F. SCHUURMANN (Miami University).

The officership and committee chairmanships for academic year 1980-81 include: *Executive Committee*-D.L. DEEVER (Otterbein College), Section Chairman; C.A. LONG (Bowling Green State University), Section Chairman-Elect; D.O. KOEHLER (Miami University), Section Past-Chairman; G. MAVRIGIAN (Youngstown State University), Secretary-Treasurer; S.W. HAHN (Wittenberg University), Sectional Governor; and D.J. HORWATH (John Carroll University), Program Committee Chairman. *Program Committee*—D.J. HORWATH, Chairman; J.P. LEITZEL (Ohio State University); and A.G. POORMAN (Ashland College). *Nominations Committee*—M.D. WETZEL (Denison University), Chairman; W.H. BEYER (University of Akron); and J.A. MURTHA (Marietta College). *Ad Hoc Committee Chairmen*—Committee on Cooperation Among Colleges and Universities: K.E. ELDRIDGE (Ohio University) and J.D. FAIRES (Youngstown State University), co-chairmen. Committee on Curriculum: H.L. PUTT (University of Akron). Committee on Teacher Training and Certification: H.W. BROCKMAN (Capital University). Committee on Computing: Z. KARIAN (Denison University). Public Informations Officer and Newsletter Editor: R.A. LITTLE (Baldwin-Wallace College). Representative to The Two-Year College Mathematics Journal: C.P. YANG (Miami University-Middletown). High School Mathematics Competition Supervisor: L.J. SCHNEIDER (John Carroll University).

GUS MAVRIGIAN, *Secretary-Treasurer*

SOUTHWESTERN SECTION MEETING

The 40th annual meeting of the Southwestern Section MAA was held at NORTHERN ARIZONA UNIVERSITY on 25,26 April, 1980. Forty four people registered their attendance.

The invited address was given by National Treasurer Professor *Leonard Gillman* of The University of Texas at Austin. It was entitled *Optimal Strategies in Sports*. Professor *Gillman* also gave a talk after the banquet Friday night. It was entitled *We Don't Know How to Teach Our Way Out of a Paper Bag*.

Professors *M. Ratliff*, *Allan Gray*, and *E. Walter* each presided over one of the three sessions for contributed papers. The following papers were contributed.

Families of Periodic Orbits and the Restricted Three Body Problem, LARRY PERKO, Northern Arizona U
On a Conjecture of Everett Walter, ROBERT WISNER, New Mexico State University
On the Construction of Heron Parallelograms and Triangles, FRANCES BAKER, Vassar College
On Estimating the Slope of a Straight Line, GRAYDON BELL, Northern Arizona University
Beyond $\sum_{j=o}^{n} \binom{n}{j} = 2^n$. JUSTIN G. MACCARTHY, Deming, New Mexico
Triangle Geometry Via Grassmann Algebra, A. SWIMMER, Arizona State University
Vector Identities via Quaternions, WILLIAM SCHULZ, Northern Arizona University
Some Remarks on Mathematics in Africa, JAMES NYMAN, University of Texas at El Paso
Periodic Wavetrains in Reaction-Diffusion Systems, RICHARD SMOCK, University of Arizona
M-Matrices Whose Inverses are Stochastic, RONALD SMITH, Northern Arizona University
The Construction of Integral Cevians, CHARLES MOORE, Northern Arizona University
Searching for Outliers in Sets of Irradiated Fuel Elements, THOMAS BEMENT, Los Alamos Scientific Laboratory
A Coherent Theory of Divisibility Rules, ROBERT WISNER, New Mexico State University.

DENNIS BONNETT, Chairman of the Southwestern Section, acted as meeting coordinator.

A. SWIMMER, *Secretary-Treasurer*

APRIL MEETING OF THE NORTH CENTRAL SECTION

The North Central Section of the MAA held its Spring Meeting at Gustavus Adolphus College on April 25-26, 1980. Approximately one hundred people registered for the meeting.

The Invited Addresses were: *Strange Attractors in Dynamical Systems*, by GEORGE SELL, University of Minnesota and *Infinitesimal Calculus-Past and Present*, by K.D. STROYAN, University of Iowa.

Contributed Papers were:

The Four-Colored Theorem & Mathematical Proof: A Rebuttal to Tymoczko; MARK LUKER, University of Minnesota, Duluth
The 1,2,3 Distribution: An Example for Elementary Statistics; ROBERT DUMONCEAUX, St. Johns University
A 'Distribution' Principle for Maximum/Minimum Problems; GEORGE BRIDGMAN, Hamline University
The Behavior of Quadratics over Z_n Given That n is not Prime, PETER MUMFORD, Student, Gustavus Adolphus
The Approximation of Factorial Fragments; SYLVAN BURGSTAHLER, University of Minnesota, Duluth
Factoring Differential Operators; RON MATHSEN, North Dakota State University
A Generalization of the Classical Hilbert Basis Theorem; FRANCIS HANNICK, Mankato State University
Automorphisms and Fixed Points in Linear Algebra; NICHOLAS MILLER, Gustavus Adolphus College
A Game of Deception; BERT FRISTEDT, University of Minnesota.

President *Hubert Walczak* presided at the Business Meeting. Announcement was made that the section will hold a Summer Seminar at St. Marys College, Winona, MN, during June 1981. The topic of the Seminar will be *Operations Research*. New officers elected were: *Roger Kirchner*, Carleton College, President-Elect; *Harlan Hewitt*, North Hennepin Community College, Member at Large.

CHARLES V. HEUER, *Secretary-Treasurer*

ILLINOIS SECTION SPRING MEETING

The 59th annual meeting of the Illinois Section of the Mathematical Association of America met on April 25-26, 1980, at John A. Logan College in Carterville, Illinois. Dr. *Robert E. Tarvin*, President of the college, gave the welcome. The following presentations were made:

The Lighthouse, A.K. RIGLER, University of Missouri-Rolla
Circles, Fractions, & Pythagoras, PAUL D. HUMKE, Western Illinois University
Ideas for Attracting More Women into College Mathematics Courses, BARBARA JUISTER, Elgin Community College
Finite Handles, TERANCE PERCIANTE, Wheaton College
Celestial Mechanics & The Three Body Problem, JULIAN PALMORE, University of Illinois
Applications of Mathematics to Some Problems in Ecology, MEREDITH POTTER, Rockford College
Remedial College Mathematics, TERRY POPP, John A. Logan College
The Leibnitz Letter-Fact or Fiction, RAY G. LANGEBARTEL, University of Illinois

Professor *Paul R. Halmos* was the keynote speaker at the Friday night banquet. He spoke on *How to be a Mathematician*. At the annual business meeting on Friday, Professor *Gordon Mock* of Western Illinois University was introduced as the newly elected Governor of the Section.

HOWARD C. SAAR, *Secretary-Treasurer*

SPRING MEETING OF THE TEXAS SECTION

The Annual Spring Meeting of the Texas Section was held at East Texas State University in Commerce, Texas on April 11-12, 1980. Professor *Bill Anderson* of East Texas State University was in charge of arrangements. Registered attendance was 164.

Invited Speakers included Professor *Leonard Gillman* of The University of Texas at Austin, Treasurer of the MAA, who spoke on *Optimal Strategies in Sports*, Professor *Margaret Waid* of the University of Delaware, currently visiting The University of Texas at Austin, who spoke on *Mathematical Models of Nerve Signal Transmission through Axons* and Professor *Bernard Madison*, Chairman of Mathematics at the University of Arkansas and Chairman of the Panel on Basic Skills of the MAA Committee on Placement Examinations, who discussed the MAA Placement Testing Program. Professor *Gillman* was the official MAA representative to the meeting.

A panel discussion on *Developing Computer Science Programs* was presented by Professors *Richard Alo* of Lamar University, *Bill Pervin* of The University of Texas at Dallas, *Grady Early* of Southwest Texas State University and *Robert Tennison* of the University of Texas at Arlington. Professor *Alo* served as moderator. An *Ad Hoc* Committee of the Texas Association of Academic Administrators in the Mathematical Sciences gave a report on its study of the short supply of public school mathematics teachers in Texas and the resulting changes proposed in mathematics teacher certification. This group also reported on the *Texas Assessment Program* (designed to insure minimum achievement in mathematics in public schools).

Honored with a Resolution of Commendation was *Richard Mifflin*, student at Rice University, who ranked in the top five in the *William Lowell Putnam Mathematical Competition*. Also honored with a Resolution of Commendation were *Pang Chen* and *Nic McPhee*, high school students from Wichita Falls, Texas and their teacher, *Tillie Himstedt*, for their achievement in the Texas Mathematical Olympiad. *Pang Chen* was winner of the Texas Olympiad and *Nic McPhee* received a high score. In a special ceremony *Pang* was presented with the first place trophy. All three were guests of the Section at the Section Banquet.

Officers and Directors for 1980-81 are as follows: Chair, BILL ANDERSON, East Texas State U; Secretary-Treasurer, GLEN MATTINGLY, Sam Houston State U; Chair-Elect, ROBERT TENNISON, The University of Texas at Arlington; Immediate Past Chair, DALTON TARWATER, Texas Tech U; Level-I Director, DAVID SANCHEZ, San Antonio College; Level-II Director, R. VICK MORGAN, Sul Ross State U; Level-III Director, BENNIE WILLIAMS, The University of Texas at Arlington; Director-at-Large, LANDON COLQUITT, Texas Christian U; Arrangements Chair, RAY TEBBETTS, San Antonio College; Arrangements Chair-Elect, RICHARD ALO, Lamar U; State Contest Director, J.R. BOONE, Texas A & M U; Governor, JOHN MOHAT, North Texas State University.

Contributed papers were as follows:

The College Board Mathematics Level I Test and Placement at UT Austin, JOHN R. DURBIN, U-Texas-Austin
Geometrical and Functional Insights Concerning the Consequences of Relativity Theory for a Liberal Arts Mathematics Course, CHRIS BOLDT, Eastfield College
Vectors and Spherical Trigonometry, ALI R. AMIR-MOEZ, Texas Tech University
The Creation, Care, and Feeding of the Math-Anxious, JAMES W. DANIEL, The University of Texas-Austin
Mathematics and Esperanto, MICHAEL JONES, Monthly Mathematics Bulletin
Further Results on the Summation of Certain Infinite Series by Using the Gamma Function, RUSSELL COWAN, Lamar University
A Note on the Summability of Split Series, TOM KEAGY, The University of Texas at Tyler
The Cesaro Method Compared to Certain Summability Processes With Respect to Absolute Summability, DAVID F. DAWSON, North Texas State University
Approximating Curves by Piecewise Linear Curves, RUSSELL BILYEU, North Texas State University
Another Interesting Property of the Variation Function, FRANK N. HUGGINS, The Univ. of Texas-Arlington
Mean Value Theorems for Integrals, W. VANCE UNDERHILL, East Texas State University
Semi Open Sets and Semi Compactness, CHARLES DORSETT, Texas A & M University
Hyperspace and the Normal Moore Space Problem, CARL P. PIXLEY, Southwest Texas State University
On the Distribution of Quadratic Residues, MONTIE G. MONZINGO, Southern Methodist University
Impulse Functions on Projective Geometries, DON E. EDMONDSON, The University of Texas at Austin
A Way to Get Some Routine Three-Space Surface Perspectives from a Plotter, JOE ALLISON, Eastfield Col.
Graphing Complex Roots of a Quadratic Equation, JOE HUBER, Pan American University
Computer Generated Cardboard Models of $z = f(x,y)$, NORMAN W. NAUGLE, Texas A & M University

ANNOUNCEMENT OF ALLENDOERFER, FORD AND POLYA AWARDS

At its meeting on January 28, 1977, in St. Louis, Missouri, the Board of Governors authorized a number of awards to authors of expository articles published in the *MONTHLY*, to be named after *Lester R. Ford*, Sr., *MATHEMATICS MAGAZINE*, to be named after *Carl B. Allendoerfer*, and the *TWO-YEAR COLLEGE MATHEMATICS JOURNAL*, to be named after *George Polya*. A maximum of two *Carl B. Allendoerfer* Awards, five *Lester R. Ford* Awards, and two *George Polya* Awards will be made annually; each award is in the amount of $100. The articles are to be selected by committees appointed by the President of the Association for this purpose and the Chairman of the Committee on Publications is to be an *ex-officio* member of each of these committees.

The recipients of the *Carl B. Allendoerfer* Awards for 1979 were selected by a committee consisting of *Roy Dubisch*, Chairman; *Edwin F. Beckenbach, ex-officio;* and *Thomas W. Tucker*. The recipients of *Allendoerfer* Awards for articles published in 1979 were the following:

Some Unsolved Problems in Plane Geometry, VICTOR L. KLEE, Jr., MATHEMATICS MAGAZINE 52 (1979), 131-145

The Three Crises in Mathematics: Logicism, Intuitionism and Formalism, ERNST SNAPPER, MATHEMATICS MAGAZINE 52 (1979), 207-216.

The recipients of the *Lester R. Ford* Awards for 1979 were selected by a committee consisting of *Lida K. Barrett*, Chairman; *Edwin F. Beckenbach, ex-officio; Branko Grünbaum*. The recipients of *Ford* Awards for articles published in 1979 were the following:

Bonnesen-Style Isoperimetric Inequalities, ROBERT OSSERMAN, MONTHLY 86 (1979), 1-29

Hermann Grassman and the Creation of Linear Algebra, DESMOND FEARNLEY-SANDER, MONTHLY 86 (1979), 809-817

The Game of Hex and the Brouwer Fixed Point Theorem, DAVID GALE, MONTHLY 86 (1979), 818-826

Nonstandard Set Theory, KAREL HRBACEK, MONTHLY 85 (1979), 659-677

Nonlinear Conservation Equations, CATHLEEN S. MORAWETZ, MONTHLY 85 (1979) 284-287.

The recipients of the George Polya Awards for 1979 were selected by a committee consisting of *Kay Dundas*, Chairman; *Edwin F. Beckenbach, ex-officio;* and *Warren Page*. The recipients of *Polya* Awards for articles published in 1979 were the following:

The Discovery of a Generalization, HUGH OUELLETTE and GORDON BENNETT, TWO-YEAR COLLEGE MATHEMATICS JOURNAL 10 (1979) 11-15

Pictures, Probability, and Paradox, ROBERT NELSON, TWO-YEAR COLLEGE MATHEMATICS JOURNAL 10 (1979) 182-190.

DAVID P. ROSELLE, *Secretary*

1980 CONTRIBUTING MEMBERS AND SPECIAL GIFTS

The Association is deeply indebted to the generosity of the 91 members listed below who have elected to be Contributing Members, Sponsors or Patrons for 1980 by making contributions beyond the normal dues.

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CALENDAR OF FUTURE MEETINGS

Sixty-fourth Annual Meeting, San Francisco, California, January 9–11, 1981.
Sixty-fifth Annual Meeting, Cincinnati, Ohio, January 15–17, 1982.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May 15–16, 1981.

EASTERN PENNSYLVANIA AND DELAWARE, Saturday before Thanksgiving.

FLORIDA, Bethune Cookman College, Daytona Beach, March 6–7, 1981.

ILLINOIS, Illinois State University, Normal, May 1–2, 1981.

INDIANA, De Pauw University, Greencastle, October 18, 1980.

INTERMOUNTAIN

IOWA, third weekend in April. Deadline for papers February 1.

KANSAS, Benedictine College, Atchison, April 1981 (probably April 25).

KENTUCKY, Jefferson Community College, Louisville, April 3–4, 1981.

LOUISIANA-MISSISSIPPI, Mississippi State University, Mississippi State, February 13–14, 1981.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Goucher College, Towson, Maryland, November 14–15, 1980.

METROPOLITAN NEW YORK, spring. Deadline for papers two weeks before meeting.

MICHIGAN, first Friday and Saturday in May. Deadline for papers six weeks before meeting.

MISSOURI, Northwest Missouri State University, Maryville, April 10–11, 1981.

NEBRASKA, University of South Dakota, Vermillion, April 11–12, 1981.

NEW JERSEY, Union College, Cranford, October 25, 1980.

NORTH CENTRAL, North Dakota State University, Fargo, October 24–25, 1980.

NORTHEASTERN, Saturday after Thanksgiving, and third week in June in odd-numbered years.

NORTHERN CALIFORNIA, first or second Saturday in February.

OHIO, John Carroll University, University Heights, October 17–18, 1980.

OKLAHOMA-ARKANSAS, Oklahoma Christian College, Oklahoma City, March 27–28, 1981.

PACIFIC NORTHWEST, second Saturday in June. Deadline for papers six weeks before meeting.

ROCKY MOUNTAIN, Colorado College, Colorado Springs, May 1–2, 1981.

SEAWAY, Daemen College, Buffalo, New York, November 7–8, 1980.

SOUTHEASTERN, University of Alabama, Birmingham, April 10–11, 1981.

SOUTHERN CALIFORNIA, first or second Saturday in March.

SOUTHWESTERN, usually in April. Deadline for papers two weeks before meeting.

TEXAS, Friday and Saturday in early April. Deadline for papers March 1.

WISCONSIN, University of Wisconsin, La Crosse, late March–early April 1981.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Toronto, January 3–8, 1981.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES, Sheraton National Hotel, Arlington, Virginia, October 9–13, 1980.

AMERICAN MATHEMATICAL SOCIETY, San Francisco, California, January 7–10, 1981.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION

ASSOCIATION FOR COMPUTING MACHINERY, Nashville, Tennessee, October 27–29, 1980.

ASSOCIATION FOR SYMBOLIC LOGIC, San Francisco, California, January 9–10, 1981.

ASSOCIATION FOR WOMEN IN MATHEMATICS, San Francisco, California, January 7–11, 1981.

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHÉMATIQUES

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, St. Louis, Missouri, April 22–25, 1981.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Four Seasons Sheraton, Toronto, Canada, May 4–6, 1981.

PI MU EPSILON

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Indianapolis, Indiana, October 30-November 1, 1980.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Stouffer's Greenway Plaza Hotel, Houston, Texas, November 6–8, 1980.

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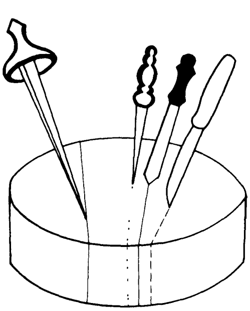

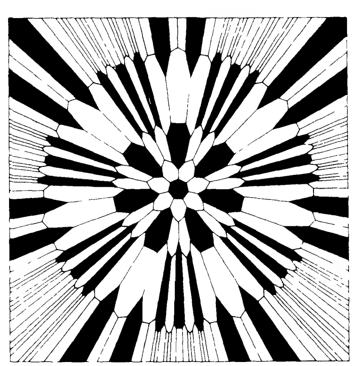

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THE AMERICAN
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(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

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NOTICE TO AUTHORS

Research papers per se are unsuitable; see statement of policy (Vol. 85, p. 1).

Please follow the format in current issues of the MONTHLY. Manuscripts must be legibly typewritten or reproduced from typewritten copy, double spaced with wide margins, and on only one side of the paper. Three copies should be submitted to the appropriate editor and one kept by the author as protection against loss. The author's *full* mailing address must appear at the beginning of the paper, in a line following the author's name. (This is a change from the style previously used in this MONTHLY.) All authors are asked to supply subject classifications with their manuscripts, if possible from the "1980" classification that appears in the 1978 and later Mathematical Reviews Indexes.

Estimated median time from submission to publication: Main Articles 15 months, Progress Reports 6 months, Math. Notes 24 months, Research Problems 6 months, Classroom Notes 14 months, Math. Education 10 months.

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THE MATHEMATICAL SCIENCES AND WORLD WAR II

MINA REES
Graduate Center, The City University of New York, 33 West 42 Street, New York, NY 10036

I shall present an account of some of the activities in mathematics that were carried on during World War II and comment on their impact on the development of the mathematical sciences in the United States after the war. Most of this memoir will be concerned with aspects of mathematical activity with which I had personal contact because of my role as executive assistant to Warren Weaver, who was Chief of the Applied Mathematics Panel of the Office of Scientific Research and Development during the war, and with war-related developments that came within my purview because of my responsibilities as head of the mathematical research program of the newly established Office of Naval Research (ONR) after the war.[1]

The Mathematical Environment in the United States Before World War II

I want first to try to set the wartime work in context by speaking briefly about the mathematical environment in the United States in the 1930's and early forties. Applied Mathematics was not strongly represented at American universities, although Richard Courant, who had come to this country in 1934, had drawn together an able group at New York University, and William Prager, with effective support from R. G. D. Richardson, then Dean of the Graduate School at Brown, in 1941 established at Brown a Program of Advanced Instruction and Research in Applied Mechanics.[2] As Professor Prager said in 1972:

> In the early thirties, American applied mathematics could, without much exaggeration, be described as that part of mathematics whose active development was in the hands of physicists and engineers rather than professional mathematicians. This is not to imply that there were no professional mathematicians genuinely interested in the applications, but that their number was extremely small. Moreover, with a few notable exceptions, they were not held in high professional esteem by their colleagues in pure mathematics, because there was a widespread belief that you turned to applied mathematics if you found the going too hard in pure mathematics. As a distinguished evaluation committee...put it [in 1941]: "In our enthusiasm for pure mathematics, we have foolishly assumed that applied mathematics is something less attractive and less worthy."[3]

The situation in mathematical statistics was somewhat similar. By 1940 only a handful of universities in the United States were offering serious work in this field. Harold Hotelling was at Columbia and Jerzy Neyman was at Berkeley. S. S. Wilks, who had earned his Ph.D. at Iowa under H. L. Rietz, had been appointed at Princeton in 1933 to develop work in mathematical statistics. However, he "did not give a formal course in statistics at Princeton until 1936, owing to a prior commitment that the university had made with an instructor in the department of economics and social institutions who had been sent off at university expense to develop a course on 'modern statistical theory' two years before; and owing to the need for resolution by the university's administration of an equitable division of responsibility for the teaching of statistics between that department (which . . . had been solely responsible for all teaching of statistics) and the department of mathematics. Wilks . . . in the spring of 1937 . . . gave an undergraduate course [in statistics], quite possibly the first carefully formulated college undergraduate course in mathematical statistics based on one term of calculus."[4]

In 1962, Mina Rees received the first of the MAA's Awards for Distinguished Service to Mathematics; a summary of her career and her many honors up to that date appears on pages 185–187 of volume 69 of this MONTHLY. At that time she had recently become Dean of Graduate Studies at the City University of New York; she retired in 1972 as President Emeritus of the Graduate School and University Center. We welcome this opportunity to publish her reminiscences of the war years.—*Editors*

On the other hand, U.S. research in what we now call "core mathematics" had been assuming increased importance on the international scene in the twenties and thirties. Moreover, it had a substantial flowering just before America felt herself inevitably drifting toward active participation in the war. For, with the coming of Hitler in 1933, many of the world's leading mathematicians had sought asylum in the United States and had greatly enriched the quality and quantity of mathematical activity in this country. In 1940, *Mathematical Reviews* was established by the American Mathematical Society, with two of the notable refugees, Otto Neugebauer and William Feller, assuming editorial responsibility, a step that fundamentally changed the reliance of American (and world) mathematicians on *Zentralblatt für Mathematik*, which had been for a decade the world's reviewing journal for mathematics.

With the passage of time, it became increasingly clear that war was inevitable. In the developing mobilization of mathematicians in support of the war effort, some enlisted or were drafted, some remained at their colleges or universities and participated in the training programs in mathematics that the armed services were setting up, and some left their universities to assume specific war-related activities.

Where did the mathematicians go who left their universities to assume noncombat war-related tasks, and what was the nature of their work?

There were many working for the armed services, some in uniform, like Herman Goldstine at Aberdeen and J. H. Curtiss in the Navy's Bureau of Ships, and others as civilians, like E. J. McShane at Aberdeen and F. J. Weyl in the Navy's Bureau of Ordnance. A number of mathematicians were attached to various Air Commands as members of Operations Research teams, like G. Baley Price in the Eighth Air Force; and others were associated with British and Canadian research efforts. There was the Navy's Operations Research Group, directed by the MIT physicist Philip M. Morse. Another group of mathematicians was working on war tasks in industry. For mathematicians, Bell Telephone Laboratories was, perhaps, the most familiar of the industrial laboratories, but a number of industrial groups (e.g., RCA, Westinghouse, Bell Aircraft) with war contracts employed mathematicians professionally. There was a group of mathematicians in cryptanalysis and another group in the Manhattan Project, which had been set up to develop the atomic bomb.

In addition, a large number of mathematicians were employed in the various parts of the Office of Scientific Research and Development (OSRD), a civilian establishment in the Executive Office of the President.

The OSRD had several parts: one devoted to medical research; one devoted to fuse research, a project of highest priority and secrecy; and the third and largest, the National Defense Research Committee (NDRC), which comprised groups of scientists and engineers concerned with submarine warfare, radar, electronic countermeasures, explosives, rocketry, etc. One of the divisions was the Radiation Laboratory at MIT. NDRC had been set up in 1940, even before the United States entered the war, to provide scientific assistance to the military forces. There was initially no mathematics division. By 1942 the demands for analytical studies had increased rapidly. As Warren Weaver observed in his autobiography:

> As the war went on, the emphasis [by NDRC] on the design and production of hardware necessarily tapered off somewhat, for the practical reason that by then a brand-new device simply could not be conceived of, designed, built in pilot model, tested, improved, standardized, and put into service in time to affect the conduct of the war.[5]

The Establishment of the Applied Mathematics Panel

By the Fall of 1942, Vannevar Bush, who headed OSRD, decided to reorganize NDRC to enable it to perform its remaining tasks more competently and to incorporate into the reorganization a new unit, the Applied Mathematics Panel (AMP).[6] The task assigned to the Panel, as it was called, was to help with the increasingly complex mathematical problems that

were assuming importance and with those other problems that were relatively simple mathematically but needed mathematicians to formulate them adequately. Warren Weaver agreed to serve as Chief.

Weaver, who had been Professor and Chairman of the Mathematics Department of the University of Wisconsin, was, in 1940, Director of the Division of Natural Sciences of the Rockefeller Foundation. In the original NDRC, he was head of a Fire Control Section whose most important assignment was to develop an anti-aircraft director that would serve as an essential component in the system that was needed to protect Britain from German bombing; and he was, personally, deeply involved in this development. However, in February 1942, when the AA director developed under his guidance was accepted by the Army (as the M-9 Director),[7] Weaver became available for his new assignment.[8a]

Many of the mathematicians who left their universities to work on war-related problems were employed, during the war, under contract with the new Applied Mathematics Panel. But many others were attached to projects that were being carried forward under other parts of NDRC, such as those I have already mentioned. A. H. Taub, for example, was attached to the explosives division. Much interesting and important applied mathematics was going on there and in many other divisions of NDRC. But AMP was set up to provide additional mathematical assistance, aiding the military services and other divisions of OSRD when they were asked to do so, provided they considered that they had a reasonable chance to do something useful. By the end of the war, AMP had undertaken almost 200 studies, nearly one-half of which represented direct requests from the armed services.

The general policy of the Panel was based on recommendations made by a group of mathematicians known as the Committee Advisory to the Scientific Officer. The Panel consisted of Richard Courant, G. C. Evans, T. C. Fry (Deputy Chief), L. M. Graves, Marston Morse, Oswald Veblen, S. S. Wilks, and, of course, Warren Weaver as Chairman. I was a civil servant and technical aide to the Chief. Among other technical aides were I. S. Sokolnikoff and S. S. Wilks, who were my colleagues on the Board of Editors of the *Summary Technical Report of the Applied Mathematics Panel*. The Panel (with its own office in New York) set up contracts with eleven universities, including Princeton, Columbia, New York University, the University of California (Berkeley), Brown, Harvard, and Northwestern, and had responsibility for the work of the Mathematical Tables Project (established originally as a scientific program by the National Bureau of Standards and administered during its first five years by the Works Project Administration).[8b] Many of the country's ablest mathematicians were employed on these university contracts, and many moved from their homes in order to participate. Two economists, W. Allen Wallis, who was to become Chancellor of the University of Rochester, and Milton Friedman, who was to win a Nobel prize in economics, operated as statisticians. John von Neumann, who had come to Princeton in 1930 and moved to the Institute for Advanced Study in 1933, was also one of those involved with the Panel. But his role, not only during the war but after its conclusion, was unique; for he was a consultant or other participant in so many government or learned activities that his influence was very broadly felt. It was during the war that the seminal book *Theory of Games and Economic Behavior* reached the printer, evolving from von Neumann's early work with some of the basic ideas and from his collaboration, beginning in 1940, with the economist Oskar Morgenstern. Moreover, as a consultant to the Aberdeen Proving Grounds, which sponsored the work at the University of Pennsylvania, where the ENIAC, the first electronic digital computer, was being developed, von Neumann had a profound influence on the design of electronic computers even in their initial stages. And his perceptions of the most urgent directions in computer development were greatly affected by the needs of the Manhattan Project. Until the time of his death in 1957, von Neumann continued to have great influence on the development of computers and of game theory. (Since I had no direct contact during the war either with the Manhattan Project or with cryptanalysis, I shall not discuss mathematical contributions to these fields, although I am sure they are of interest. The

work of the Manhattan Project is, perhaps, better known than that of the cryptologists and the cryptanalysts who played a critical role in the Allied victory).

Wartime Computing and the Post-War Computer Program

Mathematicians had been alerted as early as 1940 to the fact that we were on the threshold of a new computer age when George Stibitz, surely one of the most powerful of the early digital computer designers, demonstrated, at the summer meetings of the mathematical organizations at Dartmouth in 1940, a machine he had designed at Bell Telephone Laboratories. As the *Bulletin of the American Mathematical Society* reported (46 (1940) 841): "The Bell Telephone Laboratories exhibited a machine for computing with complex numbers. The recording instrument at Hanover was connected by telegraph with the computing mechanism in New York. This machine was available to members from 11 A.M. to 2 P.M. each day of the meeting." Dr. Stibitz's paper was entitled "Calculating with Telephone Equipment." In fact, as the pressure for machine computation developed during the war, telephone relays proved to be the most reliable components available in the earliest days of automatically sequenced calculators. The focus at that time was on getting machines into operation that would immediately solve important problems and provide a significant advance over the desk calculators that were being very skillfully used wherever scientific workers were trying to get answers to pressing problems.

Aberdeen was heavily engaged in ballistic computations and, as I mentioned above, was supporting machine development at the University of Pennsylvania. The Navy's Bureau of Ordnance, also in acute need of computation, had its major machine development at Harvard, where (with IBM support) Howard Aiken had a machine in operation before the end of the war. The earliest operating large-scale computers (which had telephone relays as their principal components) did not have the speed of the automatically sequenced electronic computers developed somewhat later, but they made important contributions to the military needs during wartime and to the swelling interest of mathematicians and engineers in the potential of automatically sequenced machines. Before the end of the war, there was an awakening realization among mathematicians that a new focus in numerical analysis would be needed as the machines became more important in scientific work. It would be false to give the impression that there was a widespread concern among the country's leading mathematicians about what would be needed in numerical analysis or, indeed, about what would happen in computer development. But some of the men and women who had had wartime experience did develop an interest in this emerging field. As the speed and capacity of machines increased after the war's end, the scope of mathematical problems that would require attention if the machines were to be properly used expanded significantly and, partly under the stimulation of the Office of Naval Research, these problems aroused the interest of increasing numbers of mathematicians.

Although automatically sequenced electronic computers were not available before the end of the war, the needs of the war played a decisive role in their initial development and the military services continued their interest and provided much of the financing for the post-war developments. In 1946 the ENIAC, the first electronic computer, became operational at the Moore School; in 1947 it was moved to Aberdeen. By that time, the activities leading to the establishment of the National Applied Mathematics Laboratories of the National Bureau of Standards were already under way. These Laboratories were jointly supported by those agencies of the federal government that had a stake in developing or using large-scale automatic computing facilities. ONR was one of the supporting agencies. The Laboratories would, when they were established, include a Computing Laboratory, a Machine Development Laboratory, a Statistical Engineering Laboratory, all in Washington; and, a little later, an Institute for Numerical Analysis, located on the UCLA campus. An Applied Mathematics Executive (later Advisory) Council, consisting of some of the country's most active scientists in the field, as well

as representatives from the various government agencies, was formed to serve as a forum before which practically all major undertakings in the computer field were thrashed out with decisive effects on their scope and orientation. It was here that a reasonable national level of research in this new field was set, taking account of the current state of electronics and relevant theories and the scope of required and probable applications. The needs of the Census Bureau were pressing, and military programs in the computer field played a large role. The work of the code-makers and code-breakers was, to a certain extent, incorporated informally, as were developments at Los Alamos. The existence of all these pressures and the support of government agencies, as well as the impressive performance of the National Bureau of Standards, were largely responsible for the establishment of U.S. leadership in computer technology. These developments took place during 1946–1953. At that time, commercial companies began to make major commitments to the production of computers, making them generally available. Many of the people who supported this effort had been trained in the code-making and code-breaking establishment.

An Overview of the Work of the Applied Mathematics Panel

Fluid Mechanics, Classical Dynamics, the Mechanics of Deformable Media, and Air Warfare. Since the Applied Mathematics Panel represented the largest group of mathematicians organized under government auspices to provide mathematical assistance wherever it was needed during the war, it may be of interest to give a brief overview of the nature of the studies carried on by the Panel from its founding in late 1942 until its dissolution at the end of 1945.[9]

Most AMP studies were concerned with the improvement of the theoretical accuracy of equipment by suitable changes in design or by the best use of existing equipment, particularly in such fields as air warfare. It often happened that a considerable development of basic theory was needed. The following illustrations are taken from the work at New York University, Brown, and Columbia.

At New York University, the work in gas dynamics was principally concerned with the theory of explosions in the air and under water and with aspects of jet and rocket theory. New results were obtained in the study of shock fronts associated with violent disturbances of the sort that result from explosions. A request by the Bureau of Aeronautics for assistance in the design of nozzles for jet motors gave rise to an extended study of gas flow in nozzles and supersonic gas jets. In this field, as in every part of the work of the Applied Mathematics Panel, one result of the work was to provide men (alas, there were not many women) who were broadly and deeply informed in a number of important and difficult fields and who were therefore often called upon as consultants. I have a vivid remembrance of a visit in the company of Richard Courant and Kurt Friedrichs to the rocket work going on at the California Institute of Technology. The Caltech people were having trouble with the launching of their rockets, and they were eager for advice. When I talked about that visit fairly recently with Professor Friedrichs, he was characteristically modest; but when we left Pasadena back in 1944, the Caltech people had new experiments planned, at least partially inspired by suggestions they had received. And the outcome, whether or not significantly affected by Friedrichs's suggestions, was successful.

Because so many questions were raised by wartime agencies about the mathematical aspects of the dynamics of compressible fluids, a Shock Wave Manual was prepared at NYU and published in its first version in 1944 by the Applied Mathematics Panel. It was one of the major documents of continuing mathematical interest to grow out of the Panel's work. Its successor, the book *Supersonic Flow and Shock Waves*,[10] was published in 1948. Its preface stated:

> The present book originates from a report issued in 1944 under the auspices of the Office of Scientific Research and Development. Much material has been added and the original text has been almost entirely rewritten. The book treats basic aspects of the dynamics of compressible fluids in mathematical form; it attempts to present a systematic theory of non-linear wave propagation, particularly in relation to gas dynamics. Written in the form of an advanced text book, it accounts for classical as well as some recent developments, and, as the authors hope, it reflects some progress in the scientific penetration of the subject

> matter. On the other hand, no attempt has been made to cover the whole field of non-linear wave propagation or to provide summaries of results which could be used as recipes for attacking specific engineering problems . . .
>
> Dynamics of compressible fluids, like other subjects in which the non-linear character of the basic equations plays a decisive role, is far from the perfection envisaged by Laplace as the goal of a mathematical theory. Classical mechanics and mathematical physics predict phenomena on the basis of general differential equations and specific boundary and initial conditions. In contrast, the subject of this book largely defies such claims. Important branches of gas dynamics still center around special types of problems, and general features of connected theory are not always clearly discernible. Nevertheless, the authors have attempted to develop and to emphasize as much as possible such general viewpoints, and they hope that this effort will stimulate further advances in this direction.

After the war, the NYU group continued its interest in a number of the problems worked on during the war with support from all the military services. J. J. Stoker's studies of water waves, in particular, were continued. And, with the growth of computers, the group greatly expanded its work in fields related to computer applications.

At Brown, the work focused on problems in classical dynamics and the mechanics of deformable media. The mathematical output of the Brown group was substantial; but I think it is worth quoting a paragraph from a letter from William Prager, the head of the Brown group, to Churchill Eisenhart, written in June 1978. He says:

> While the Applied Mathematics Group at Brown University worked on numerous problems suggested by the military services, I believe that its essential service to American Mathematics was to help in making Applied Mathematics respectable . . . The fact that the Program of Advanced Instruction and Research in Applied Mechanics, the forerunner of Brown's Division of Applied Mathematics, relied heavily on the financial support available under a war preparedness program illustrates the influence of the war on the development of the mathematical sciences in the U.S.

It is certainly true of the post-war programs at Brown and at NYU that they drew great strength from the importance of their work to the war effort and from the interest of the military services in their continuing vitality after the war.

At Harvard, the work in underwater ballistics produced a polished account of the water entry problem and, like all the other projects, it provided a group of expert advisers, in this case for the Navy. Moreover, it gave applied mathematics in the United States an important, newly active participant, Garrett Birkhoff.

The three projects I have thus far mentioned were all concerned with what can be described as classical applied mathematics. The largest of the so-called "Applied Mathematics Groups," the one at Columbia, had a different kind of assignment. For several years, its work was devoted primarily to studies in aerial warfare, the most extensive analyses being devoted to air-to-air gunnery. At the time of its establishment in 1943, this group was headed by E. J. Moulton; during its last year, from the beginning of September 1944 to the end of August 1945, Saunders Mac Lane was its "Technical Representative."

The final summary of the work done by the Applied Mathematics Group at Columbia under the AMP contract, as well as related work done elsewhere in the United States and abroad, was reported in the *Summary Technical Report of the Applied Mathematics Panel*[11] under the following headings: (1) Aeroballistics—the motion of a projectile from an airborne gun; (2) Theory of deflection shooting; (3) Pursuit curve theory—important because the standard fighter employed guns so fixed in the aircraft as to fire in the direction of flight, and important also in the study of guided missiles that continually change direction under radio, acoustical, or optical guidance unwillingly supplied by the target; (4) The design and characteristics of own-speed sights—devices designed for use in the special case of pursuit curve attack on a defending bomber; (5) Lead computing sights—which assume that the target's track relative to the gun mount is essentially straight over the time of flight of the bullet; (6) The basic theory of a central fire control system; (7) The analytical aspects of experimental programs for testing airborne fire control equipment; (8) New developments, such as stabilization and radar.

That part of the program of the Applied Mathematics Panel that was concerned with the use of rockets in air warfare was primarily the responsibility of Hassler Whitney, who served as a member of the Applied Mathematics Group at Columbia. He not only integrated the work carried on at Columbia and Northwestern in the general field of fire control for airborne rockets but maintained effective liaison with the work of the Fire Control Division of NDRC in this field and with the activities of many Army and Navy establishments, particularly the Naval Ordnance Test Station at Inyokern, the Dover Army Air Base, the Wright Field Armament Laboratory, the Naval Bureau of Ordnance, and the British Air Commission.

All these studies were concerned with the best use of equipment or with changes in equipment that could be effected in time to be of use in World War II. Two studies in air warfare carried out under AMP auspices came closer to having general tactical scope than did most of the other work done by the Panel. In 1944, the Panel responded to a request from the Army Air Force (AAF) asking for collaboration "in determining the most effective tactical application of the B-29 airplane" by setting up three contracts: one at the University of New Mexico, to carry on large-scale experiments; a second at Mt. Wilson Observatory, to carry on small-scale optical studies; and a third at Princeton, to provide mathematical support for the whole undertaking.[12] At Mt. Wilson the staff was concerned principally with the defensive strength of single B-29's against fighter attack, and the effectiveness of fighters against B-29's. One indirect result of the optical studies was a set of moving pictures showing the fire-power variation of formations as a fighter circles about them. Warren Weaver reports that, concerning such pictures, the President of the Army Air Forces Board remarked that he "believed these motion pictures gave the best idea to air men as to the relative effect of fire power about a formation yet presented." Certain of these pictures were flown to the Marianas and viewed by General LeMay and by many gunnery officers at the front.[13] The extent to which the claim can be made here for the power of mathematics may be limited, but the study was an effective one.

Probability and Statistics. Another part of the Panel's work in the analytical studies of aerial warfare was concerned with flak analysis and fragmentation-and-damage studies. These were based on probability studies of damage to an aircraft or group of aircraft from one or more shots from anti-aircraft guns, with some attention to related problems arising in air-to-air bombing or in air-to-air or ground-to-air rocket fire. Probability considerations arose in a wide array of Panel studies, as did statistical problems. Indeed, the need for the use of statistics and probability theory was so great that there were four contracts concerned with such problems. To quote S. S. Wilks:

> The methodology of research varied from formal mathematical analysis, at one extreme, to synthetic processes and statistical experiments or models at the other. Formal analysis is the more precise and hence satisfying process, but the difficulties of formulating the problem in analytical terms and then (worse) of finding numerical solutions increase rapidly with the complexity of the bombing situation. For example, it is very easy to deduce almost all the probability consequences regarding the problem of aiming a single bomb at a rectangular target, but very few deductions can be made directly from the equations which describe the dropping of a train of as few as three bombs on a rectangular target. Since the problem of dropping a train of three bombs is itself extremely simple, compared to many common bombing operations, it is apparent that formal mathematical processes cannot alone be depended upon to carry the burden, but they are powerful when used in conjunction with synthetic methods and statistical models.[14]

By the end of the war the major effort of three of the four statistical research groups was being spent on nineteen studies dealing with probability and statistical aspects of bombing problems.

The other major fields in which statistical work was being carried on were the development of statistical methods in inspection, research, and development work; the development of new fire effect tables (work that was continued after the war under a contract between Princeton and the Navy); and miscellaneous studies relating to such things as spread angles for torpedo salvos, land mine clearance, and search problems.

Statistical Methods in Inspection, Research, and Development: The Genesis of Sequential Analysis. The first of these major fields, the development of statistical methods in inspection, research, and development, was assigned to the largest of the statistical research groups, the one at Columbia (SRG-C). W. Allen Wallis, the Director of Research of this group, said in a recent speech[15] that this was surely the most extraordinary group of statisticians ever organized, taking into account both number and quality, and that it was a model that has not been equaled of an effective statistical consulting group. I can certainly attest that it was a tremendously productive group and an exciting one to be associated with. The great bulk of its work was in consulting or in the investigation of problems of a predominantly statistical or probabilistic nature. It developed a variety of useful materials, both theoretical and practical, that have become established parts of statistics. The most striking of these is sequential analysis, called by Wallis "one of the most powerful and seminal statistical ideas of the past third of a century." He reports that the 1975 and 1976 volumes of *Current Index to Statistics* each lists between 50 and 55 articles that include the term "sequential analysis" in their titles, and he asserts that sequential analysis continues to be one of the dominant themes in statistical research.

The importance of sequential analysis during the war is attested by Warren Weaver. He writes in his summary of AMP's work:

> During the war, it was recognized by the Services that the statistical techniques which were developed by the Panel for Army and Navy use, on the basis of the new theory of sequential analysis, if made generally available to industry, would improve the quality of products produced for the Services. In March 1945, the Quartermaster General wrote to the War Department liaison officer for NDRC a letter containing the following statement: "By making this information available to Quartermaster contractors on an unclassified basis, the material can be widely used by these contractors in their own process control and the more process quality control contractors use, the higher quality the Quartermaster Corps can be assured of obtaining from its contractors. For, by and large, the basic cause of poor quality is the inability of the manufacturer to realize when his process is falling down until he has made a considerable quantity of defective items . . . With thousands of contractors producing approximately billions of dollars worth of equipment each year, even a 1% reduction in defective merchandise would result in a great saving to the Government. Based on our experience with sequential sampling in the past year, it is the considered opinion of this office that savings of this magnitude can be made through wide dissemination of sequential sampling procedures." On the basis of this and similar requests, the Panel's work on sequential analysis was declassified, and the reports . . . were published. The Quartermaster Corps reported in October 1945 that at least 6,000 separate installations of sequential sampling plans had been made and that in the few months prior to the end of the war new installations were being made at the rate of 500 per month. The maximum number of plans in operation simultaneously was nearly 4,000.[16]

The story of the genesis of sequential analysis is given below chiefly because the tale is an interesting one but also because of the importance of the results at the time of their discovery and their continuing importance. The following account is excerpted from a letter sent to Warren Weaver by Allen Wallis in March 1950 in response to a question asked by Weaver in January of that year:

> Late in 1942 or early in 1943 you assigned us the task of evaluating an approximation developed by (Navy) Captain Garret L. Schuyler that was supposed to simplify a complicated British formula for calculating the probability of a hit by anti-aircraft fire on a directly approaching dive bomber. Schuyler's approximation was no good. Ed Paulson worked on the problem for us and was able to give rather simple formulas bounding the correct probability . . .
>
> [Paulson and I worked up] material on comparing two proportions which is now presented in Chapter 7 of *Techniques of Statistical Analysis*. When I presented this result to Schuyler, he was impressed by the largeness of the samples required for the degree of precision and certainty that seemed to him desirable in ordnance testing. Some of these samples ran to many thousands of rounds. He said that when such a test program is set up at Dahlgren [U.S. Naval Proving Ground] it may prove wasteful. If a wise and seasoned ordnance expert like Schuyler were on the premises, he would see after the first few thousand or even few

hundred rounds that the experiment need not be completed . . . he thought it would be nice if there were some mechanical rule which could be specified in advance stating the conditions under which the experiment could be terminated earlier than planned . . .

. . . Several days after I returned to New York I got to thinking about Schuyler's comment . . .

This was early in 1943, after Milton Friedman had joined SRG but before he had been able to move his family to New York. He was commuting from Washington to New York for two or three days each week. He and I regularly had lunch together, and one day I brought up Schuyler's suggestion. We discussed it at some length, and came to realize that some economy in sampling can be achieved merely by applying an ordinary single-sampling test sequentially. That is, it may become impossible for the full sample to lead to rejection, or for it to lead to acceptance, in which case there is no sense in completing the full sample. The fact that a test designed for its optimum properties with a sample of predetermined size could be still better if that sample size were made variable naturally suggested that it might pay to design a test in order to capitalize on this sequential feature; that is, it might pay to use a test which would not be as efficient as the classical tests if a sample of exactly N were to be taken, but which would more than offset this disadvantage by providing a good chance of terminating early when used sequentially. Milton explored this idea on the train back to Washington one day, and cooked up a rather pretty but simple example involving Student's t-test.

When Milton returned to New York we spent a great deal of time at lunches over this matter . . . We finally decided to bring in someone more expert in mathematical statistics than we . . . We decided to turn the whole thing over to Wolfowitz.

The next day we talked with Jack but were totally unable to arouse his interest . . .

We got Wald over the next morning and explained the idea to him . . . We presented the problem to Wald in general terms for its basic theoretical interest . . .

At this first meeting Wald was not enthusiastic and was completely non-committal . . .

The next day Wald phoned that he had thought some about our idea and was prepared to admit that there was sense in it. That is, he admitted that our idea was logical and was worth investigating. He added, however, that he thought nothing would come of it; his hunch was that tests of a sequential nature might exist but would be found less powerful than existing tests. On the second day, however, he phoned that he had found that such tests do exist and are more powerful, and furthermore he could tell us how to make them. He came over to the office and outlined his sequential probability ratio to us. This is the ratio of the probability under the null-hypothesis, with which I had been puttering around, to the probability under the alternative hypothesis—or rather, the reciprocal of this ratio. He found the critical levels by an inverse probability argument, showing that the same critical levels result no matter what assumption is made about the *a priori* distribution . . .

While it later developed that there had been previous work related to sequential analysis, you can see from the foregoing account that Wald's development did not actually grow out of preceding work . . .

. . . While Wald was still preparing his monograph on the theory,[17] we started to work on a book on applications. We were understaffed at that time, and other work had higher priority. Finally, we arranged with Harold Freeman of MIT to take on the job as a special assignment. He wrote the first version of *Sequential Analysis of Statistical Data: Applications*. While he was working on this, he was called in by the Boston office of the Quartermaster Corps for advice on acceptance inspection, and it seemed to him that sequential analysis was eminently suitable for their problem. He therefore gave a series of lectures to the staff, including the top officer, a Colonel Rogow, who had come to the Quartermaster Corps from Sears Roebuck and who after the war became president of Eversharp . . . Rogow encountered considerable opposition in introducing sequential analysis, particularly from the Army Ordnance Department . . . but he achieved an amazingly quick revolution in the QMIS. Actually, sequential analysis deserves only a small part of the credit for the total improvement achieved. Much of the improvement was due simply to better methods of inspecting given items, better methods of reporting, etc. Nevertheless, sequential analysis became the opposite of a scapegoat: something to which all the credit could be attached, so that it would not be necessary to say that they were simply doing what could have been done twenty years sooner.

The Navy interest in sequential analysis came first from John Curtiss. I gave him Wald's basic formulas at lunch one day . . . He was quick to perceive the usefulness of sequential analysis in sampling inspection work. Curtiss was the first to suggest to me that the decision criteria be transformed from levels of the likelihood ratio to levels for the actual count of defectives, to be shown as a function of sample size. This was an adaptation of the standard tables of acceptance and rejection numbers used by Army Ordnance and taken by them from the Bell Laboratories. At SRG we later thought of the graphical presentation of these acceptance and rejection numbers.

The Effect of Wartime Pursuits on Mathematicians and Statisticians

The foregoing account will, I think, justify Wallis's claim for the importance of sequential analysis and his pride in the fact that it originated in the Statistical Research Group at Columbia. He makes another claim for that Group—that it contributed definitively to the subsequent careers of a substantial number of men who were to become leaders in statistics in the next three decades. One may say more generally, I think, that for a number of mathematicians, whether their work was in AMP or elsewhere, what they did during the war had a substantial impact on their subsequent careers. Herman Goldstine became a computer authority, Barkley Rosser became a versatile applied mathematician, John Curtiss committed himself for a considerable period to the building and administration of the Applied Mathematics Laboratories of the National Bureau of Standards. And there are many others whose careers were essentially changed.

As to other claims made by Wallis for the Statistical Research Group at Columbia, these, too, apply more generally. I have already emphasized the consulting role played by many Panel mathematicians; and the quality of the members of all the groups was truly noteworthy. In particular, the Applied Mathematics Group at Columbia, like the Statistical Research Group there, was distinguished by the quality and number of its members. However, its work was very diverse and constrained by the needs of wartime problems. Thus, in spite of its wartime importance, the work of AMG-C did not serve as a basis for a mathematical field of growing importance as did the work of the Applied Mathematics Group at New York University and that at Brown. But, during and after the war, the work at AMG-C was much appreciated. The Naval Ordnance Development Award was conferred on the Group for distinguished service to the research and development of Naval Ordnance; and the military services used the Group as consultants on a wide variety of problems.

Military Evaluations of Contributions of Mathematicians

In a conversation with Warren Weaver in June 1978, shortly before his death, I asked him how he assessed the view of the military of the value of AMP's work. He said that, initially, their attitude toward the Panel was a pretty restrained one. There were few people in the Army who had had enough training to have any concept of what could be done, a principal exception being Major (now General) Simon of Aberdeen. Many of the Army aviators had had more scientific training than the men in the other branches of the Army, and many of the Navy people were eager for help; so the Navy and what later became the Air Force were among the "first believers."

Problems were usually forwarded to the Applied Mathematics Panel after a responsible person in the services had written to Warren Weaver saying that they had a problem and, though they were not at all sure that the Panel could help, they would like to get together to discuss it. Then a group from the Panel would go down to Washington for a meeting that usually brought in some "high brass." Fortunately, some of the early problems were easy to solve. One particular one was concerned with the determination of the kind of barrage of torpedoes to lay down against a big Japanese vessel to maximize the probability of hitting the ship. The Navy had no idea how fast the vessels concerned could accelerate in a straight line, how rapidly they could turn, etc., but they did have good photographs of large numbers of Japanese vessels. The people at NYU quickly provided the information that, in 1887, Lord Kelvin had established that the waves following a ship moving in a straight line are confined to a sector of semi-angle 19°28′ regardless of the ship's size and speed, provided the speed is constant. The ship's speed is indicated by the spacing of cusps along the bow waves.[18]

Since the photographs of the Japanese ships were almost always taken in turns it was desirable to extend Lord Kelvin's analysis to turning ships. We found that this could be done rather simply and that we could get the data we needed from a picture of the wavelets. In a test

of the mathematical results in an experimental run of a new destroyer, the agreement of theory and observation was extremely good—within a few percent for both speed and turning radius.[19] The Navy found this result impressive. The method developed by the Applied Mathematics Panel was adopted by the Navy's Photographic Interpretation Center, which incorporated much of the research in an official handbook. This and similar experiences won over the armed services to the notion that mathematics could be of great help to them.

There were, of course, many problems to which we could make no useful contribution. But there were also some important successes, as illustrated in the following account given in Warren Weaver's Summary.[20]

> In January 1944, Brigadier General Robert W. Harper, AC/AS (Training), wrote in a letter to Dr. Vannevar Bush, Director of OSRD, that "the problems connected with flexible gunnery are probably the most critical being faced by the Air Forces to-day. It would be difficult to state the importance of this work or the urgency of the need; the defense of our bomber formations against fighter interception is a matter which demands increasing coordinated expert attention."...
>
> The immediate proposal contained in General Harper's letter was that the Applied Mathematics Panel should recruit and train competent mathematicians who had the "versatility, practicality, and personal adaptability requisite for successful service in the field"; it was planned that these men, after two months' training in this country, would be assigned to the Operations Research Sections in the various theaters to devote their attention to aerial flexible gunnery problems. The Panel was in a position to carry out this program because it had already been drawn into studies of rules for flexible gunnery training and because it had access to many of the ablest young mathematicians in the country. The assignment was completed promptly [and was much appreciated by the Air Forces].

In June 1944, General Harper, in a letter to Dr. Bush, paid tribute to OSRD for the outstanding work done in training the ten mathematicians for Operational Research Groups and stated that the demands for more such men had come in at such a rate that it was deemed necessary to train eight additional mathematicians.[21] The recruitment of these men proved more difficult than in the earlier training assignment because so many "competent and willing mathematicians had already entered upon war work." (See Note 21.) However, the task was successfully completed. One of those recruited in this second group, Dr. John W. Odle, reports:

> [The] training was extremely valuable to me and was directly applicable to my subsequent assignment in the flexible gunnery subsection of the Operations Research Group at the Eighth Air Force in England. Without the general orientation and the specialized instruction that I received ... I would have been woefully lost in a field of endeavor that was completely new and unfamiliar to me ... The training certainly opened up immense new vistas to me. In fact, that introduction to OR, and my later wartime experiences as a practitioner, completely changed the course of my career.[22]

Some Effects of Wartime Work on Mathematics

This and other wartime programs that put American mathematicians in touch with operations research activities being carried on in the field, as well as those being pursued in the United States, had an effect after the war's end. Two post-war efforts to increase interest in nonmilitary uses of operations research should be mentioned. The first is a speech by Philip M. Morse, head of the Operations Research Group of the U.S. Navy during the war, who was the Josiah Willard Gibbs lecturer at the meeting of the American Mathematical Society in December 1947. He spoke on the subject "Mathematical Problems in Operations Research," basing his paper on several mathematical problems that arose in operations research during World War II.[23] The paper emphasized the potentials for use of operations research in peacetime applications, in particular, in business and industry. The second post-war effort to increase interest in the peacetime uses of operations research that I shall mention was an undertaking of the National Research Council. In April 1951 the Council published a brochure prepared by its Committee on Operations Research, entitled "Operations Research with Special Reference to Nonmilitary Applications," which sought to introduce the methods of operations research into business and industry in the United States.

In the ensuing years, universities in the United States developed a variety of ways in which to handle the interest of students and potential employers in the availability of instruction in operations research. In some universities, departments of operations research were established in the liberal arts college. In others, the subject was taught in the business school and, usually, in the engineering school. The patterns have great variety.

One of the most prominent fields of operations research, linear programming, was started in 1946 and was a natural continuation of Air Force planning activities that had developed during the war. Extraordinary coordination had been required during the war to ensure that the economy had the capability to relinquish men, materiel, and productive capacity from the civilian to the military sector on a schedule that permitted necessary training of men, deployment in combat theaters, supply and maintenance, and a wide spectrum of other requirements. Time was a critical factor.

George Dantzig, when he returned to the Office of the Air Controller after completing his Ph.D. in 1946, was requested to mechanize this planning, since it seemed likely that electronic computers with very large capacity and great speed would soon become available. He realized that the complex wartime procedures were unsuitable for high-speed computation. He found that the equations to be satisfied in order to achieve the required degree of combat readiness at a stated time were so complicated that he could not see how to impose the additional requirement of minimum cost. Finally, he saw that the goal of the complex procedures used during the war could be achieved by using inequalities instead of equations. By the end of 1947, he had described the problem mathematically, formulated a method of solution, and recognized that there was a wide range of applications. Mathematically, the problem is to find a solution of a system of linear equations and linear inequalities that minimizes a linear form.

Dantzig arranged to have the Mathematical Tables Project of the National Bureau of Standards test the method he proposed (the simplex method) on the diet problem formulated by George Stigler in 1945,[24a] carrying out the computations by hand. The solution required nearly 17,000 multiplications and divisions, which were carried out by five statistical clerks using desk computers in 21 working days. This was the first life-size computation to be performed by the simplex method, and it established that the method would be practicable for virtually all problems once appropriate electronic computing machines became available.[24b]

Although Fourier,[25] in the 1820's, and Kantorovich,[26] in 1938 and subsequently, had also realized the importance of the subject and devised methods in many ways similar to those of Dantzig for solving these problems, Fourier died in 1830 without developing his ideas, and Kantorovich published his results in a monograph that was unknown outside of Russia until it came to the attention of T. C. Koopmans in the middle 1950's and was translated into English through his efforts. Thus the contemporary development of linear programming stems directly from the Air Force beginning. This development was of first importance both to economic theory and to phases of practice in business and industry that were central to operations.

In addition to Dantzig's Air Force colleagues, the Washington mathematical community furnished active support. The National Bureau of Standards provided research and computing assistance, and the Office of Naval Research gave support for related university research. In this respect, special mention should be made of the Princeton project under A. W. Tucker, which catalyzed the interest of academic mathematicians. Tucker and his former students, David Gale and Harold Kuhn, were active in developing and systematizing the underlying theory of linear inequalities. Their main efforts were in game theory, whose equivalence with linear programming had been conjectured by von Neumann as early as October 1947, when he met George Dantzig for the first time and learned from him of his efforts in linear programming.[27]

The role of catalyst for economists was played by T. C. Koopmans, who had, in fact, anticipated some aspects of linear programming concepts in research in transportation theory he had undertaken during the war.[28] He recognized the importance of Dantzig's work and identified the implications of linear programming for the whole theory of resource allocation.

Koopmans and Kantorovich shared a Nobel prize in economics for work involving linear programming. Other Nobel Laureates in economics associated with the subject include Kenneth Arrow, Ragnar Frisch, Wassily Leontieff, Paul Samuelson, and Herbert Simon.

The Navy's interest in linear programming was based on a recognition of its potential contributions to the Navy's logistics operations. ONR's Logistics Program was set up in 1947, and a separate Logistics Branch of the Mathematical Sciences Division was established in 1949.

Summary and Conclusion

In 1968, the National Academy of Sciences published a report[29] that comments on the development of new fields that "combine the use of numerical data . . . with mathematical models to provide guidance for managerial action and judgment." It says, in part:

> During World War II, the use of simple mathematical models and mathematical thinking to study the conduct of military operations became a recognized art, as first scientists and later mathematicians, lawyers, and people with other backgrounds demonstrated its effectiveness. After the war, attempts to apply the same attitudes and approaches to business and industrial operations and management were pressed forward rather successfully. Combined with techniques and thinking drawn from, or suggested by, classical economics, this line of development has now led to an active field [variously called management science, operations research, cost-benefit analysis, optimization theory, mathematical programming, etc.] . . .
>
> Whatever the title, the flavor of what is done is the same, combining the use of numerical data about operating experience so characteristic of early military applications with mathematical models to provide guidance for managerial action and judgment. This field was created by scientists accustomed to the use of mathematics; both its spirit and its techniques have always been thoroughly mathematical in character. This mathematical approach is steadily penetrating the practice of management and operation.
>
> A number of the leading schools of business administration have concluded that mathematics is important both as a tool and as a language for management, and that training for the professional class of managers should include a substantial dose of this field of many names. Therefore, calculus, linear algebra, and computer programming either must be prerequisite for entrance or must be taken early in the graduate training program . . .
>
> This field is pervasively mathematized and computerized, but it is far from being strictly a mathematical science. The pattern of its problems is frequently described as formulating the problem, constructing a mathematical model, deriving a solution from the model, testing the model and the solution, establishing control over the solution, and implementing the solution. Only one of the six steps is completely mathematical; the others involve the actual problem in an essential way. In these other steps, of course, there are many applications, some of them crucial, of statistics and computer science. The mathematical step, especially when dealing with management rather than operational problems, often draws on concepts and results from the field of optimized allocation, control, and decision.
>
> A good practitioner combines the characteristics of most professional consulting and of most effective application of mathematics: abundant common sense, willingness to produce half-answers in a half-hour, recognition of his key roles as problem formulator and contributor to long-run profits (rather than as problem solver or researcher). Yet for all this, and in an alien environment, he must retain his skill as a mathematician.

Under the stimulus of government support, the development of these new fields at a time of expanding availability and greater sophistication in computers has brought about a great increase in the mathematization of many aspects of business and industry.

With the increasing mathematization of society, the Association for Computing Machinery came into being in 1947; the Industrial Mathematics Society, in 1949; the Operations Research Society of America and the Society for Industrial and Applied Mathematics, in 1952; and the Institute of Management Sciences, in 1953. Courses, or components of courses, dealing with mathematics for the behavioral sciences were offered by the mathematics departments of a number of liberal arts colleges with the encouragement of the Mathematical Association of America's Committee on the Undergraduate Program in Mathematics, while, in some universities, separate courses in mathematics were taught in the economics department, the school of industrial management, the engineering school, and so on. In many universities, separate

departments have been established with names like Computer Science, Operations Research, Systems Science and Mathematics, and Applied Mathematics. Thus, as the uses of mathematics have expanded in new directions, many institutions have adopted new organizational arrangements to accommodate the new content, much of which reflects developments in the mathematical sciences that grew out of military requirements in World War II.

Notes

1. See: Mina S. Rees, Mathematics and the government: The post-war years as augury of the future, in The Bicentennial Tribute to American Mathematics, 1776–1976, Mathematical Association of America, 1977, pp. 101–116.

2. After a stay in Turkey, Prager had been appointed Professor at Brown in 1941.

3. William Prager, Quart. Appl. Math., 30 (1) (1972) 1.

4. Churchill Eisenhart, Dictionary of Scientific Biography, vol. 14, Scribner, New York, 1976, pp. 383, 384.

5. Warren Weaver, Scene of Change, Scribner, New York, 1970, p. 87.

6. In the spring of 1942, a presentation was made to James B. Conant, Chairman of NDRC, and Vannevar Bush, Director of OSRD, by Marshall Stone and Marston Morse, as representatives of the American Mathematical Society. The discussion was based on a carefully prepared memorandum that described wartime activities considered appropriate for members of the American Mathematical Society. The establishment of the Applied Mathematics Panel may have been influenced by this presentation, but the American Mathematical Society was not consulted about the nature of the work to be undertaken by AMP, nor about its staffing pattern, and there were initial complaints about what was perceived as too little use of distinguished "pure" mathematicians in the work of the Panel.

7. The M-9 director was spectacularly successful during the buzz bomb attacks on Britain in 1944, working in combination with automatic radar tracking developed by the Radiation Laboratory and the proximity fuse developed by the fuse section of OSRD. General Sir Frederick A. Pile, who was in charge of the British Anti-Aircraft Command at that time, wrote to General George Marshall in August 1944: "The equipment you have sent us is absolutely first class. . . As the troops get more expert with [it] I have no doubt very few bombs will reach London." His prediction proved to be correct.

8a. This account is adapted from Warren Weaver's autobiography (see Note 5), pp. 78–87.

8b. The location of contracts established by the Applied Mathematics Panel, with the names of the "Technical Representatives," follows:

> *Applied Mathematics Groups*: NYU, R. Courant; Columbia, E. J. Moulton, S. Mac Lane, A. Sard; Brown, R. G. D. Richardson; Institute for Advanced Study, J. von Neumann; Princeton, M. M. Flood; Northwestern, E. J. Moulton, W. Leighton; Carnegie Institution of Washington, Pasadena, W. S. Adams; Harvard, Garrett Birkhoff; University of New Mexico, E. J. Workman.
>
> *Statistical Research Groups*: Columbia, H. Hotelling; University of California (Berkeley), J. Neyman; Columbia, J. Schilt; Princeton, S. S. Wilks.
>
> *Computation*: The Franklin Institute, H. B. Allen; The National Bureau of Standards, Arnold Lowan.

9. This account draws freely on Warren Weaver's Summary that appears in each of the three volumes of the Summary Technical Report of the Applied Mathematics Panel, NDRC, Washington, D.C., 1946. This was published with a confidential classification, but the whole of the report has now been declassified.

10. Richard Courant and K. O. Friedrichs, Supersonic Flow and Shock Waves, Interscience, New York, 1948.

11. Summary Technical Report of the Applied Mathematics Panel (see Note 9), vol. 2, pp. 9–124.

12. Ibid., vol. 2, pp. 197–220.

13. Ibid., vol. 2, p. 3.

14. Ibid., vol. 3, Probability and Statistical Studies in Warfare Analysis, p. ix.

15. This was an invited address delivered on August 14, 1978, at a meeting of the American Statistical Association. It was entitled "The Statistical Research Group, 1942–1945." It is to be published in revised form by the Journal of the American Statistical Association, 8 June 1980.

16. Warren Weaver (see Note 9), p. 5.

17. Abraham Wald, Sequential Analysis, Wiley, New York, 1947. The basic work on this volume was done at SRG-C and was published as a restricted report in 1943 by the Applied Mathematics Panel.

18. Lord Kelvin (Sir W. Thomson), On the waves produced by a single impulse in water of any depth, or in a dispersive medium, Proc. of the Royal Society of London, Ser. A, 42 (1887) 80–85.

19. J. J. Stoker, Water Waves: The Mathematical Theory with Applications, Interscience, New York, 1957, pp. 229, 230.

20. Warren Weaver (see Note 9), pp. 3, 4.

21. Saunders Mac Lane, Summary Report on AMP Study 103, AAF Training Program, Columbia University Division of War Research—Applied Mathematics Group, 24 August 1945.

22. John W. Odle, Letter to Dr. Churchill Eisenhart, 21 May 1979.

23. Bull. Amer. Math. Soc., 54 (1948) 602–621.

24a. George J. Stigler, The cost of subsistence, Journal of Farm Economics, 27 (2) (May 1945) 303–314.

24b. New results, which would provide a possibly significant improvement on this method of solution, were reported in January 1979 by a Russian mathematician. These results were unknown in America until early summer 1979. See L. G. Hačijan, A polynomial algorithm in linear programming, Soviet Math. Dokl., 20 (1979) 191–194.

25. Jean Baptiste Joseph Fourier, Solution d'une question particulière du calcul des inégalités, in Oeuvres de Fourier, Tome Second, Gauthier-Villars, Paris, 1826, pp. 315–328 (including notes by G. Darboux). Darboux comments in the Preface to the second volume of the Oeuvres that Fourier's enthusiasm for the problem seemed to be somewhat exaggerated.

26. L. V. Kantorovich, Mathematical Methods of Organizing and Planning Production, Leningrad S.U. Press, Leningrad, 1939.

27. George B. Dantzig, Linear programming and its progeny, Naval Research Reviews, Office of Naval Research, Washington D.C., June 1966, p. 6.

28. T. C. Koopmans, Exchange ratios between cargoes on various routes (Memorandum for the Combined Shipping Adjustment Board, Washington D.C., 1942, pp. 1–12), in Scientific Papers of Tjalling C. Koopmans, Springer, New York, 1970, pp. 77–86.

29. The Mathematical Sciences: A Report, National Academy of Sciences, publication #1681, 1968, pp. 113, 114.

THE DIRICHLET PROBLEM FOR HARMONIC FUNCTIONS

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1. Introduction. The classical potential theory in the m-dimensional Euclidean space R^m can be roughly characterized as the theory of the Laplace operator

$$\Delta = \sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2}.$$

The function h is said to be **harmonic** on the open set $V \subset R^m$, if h has continuous second partial derivatives on V and $\Delta h(x) = 0$ for all $x \in V$. The set of all harmonic functions on V will be denoted by $\mathcal{H}(V)$. Note that linear functions are harmonic on R^m.

If U is a nonempty bounded open subset of R^m and f is a real-valued function defined on the boundary ∂U of U, **the classical Dirichlet problem** on U is that of finding a function h harmonic on U such that $\lim_{y \to x} h(y) = f(x)$ for all $x \in \partial U$. The set U is said to be **regular**, provided the classical Dirichlet problem has a solution for all continuous functions on ∂U. The existence of

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nonregular sets was recognized in the first decades of this century. It turns out, however, that even in the case of a nonregular set U there is a possibility to assign to every element of a wide class of functions defined on ∂U a harmonic function on U. This harmonic function tends to the "correct" boundary values in most boundary points for all continuous boundary conditions and coincides with the classical solution, if it exists. Various special constructions of such a **generalized solution of the Dirichlet problem** were described in the literature (see comments at the end of this note). Each of those methods presented a "reasonable" generalization of the classical Dirichlet problem.

This note mainly deals with the question whether a generalized solution submitted to some natural requirements is **uniquely determined**. We wish to emphasize the fact that in the frame of modern potential theory, the corresponding problem was cleared up only a few years ago. We show that some of the recent deep contributions can be presented in an accessible way to a nonspecialist, at least in the context of the classical potential theory. It turns out that basic facts of **the Choquet theory** provide another insight to some classical notions and problems of potential theory as shown at the end of this note.

In the next section we recall the Perron-Wiener-Brelot method of obtaining harmonic functions corresponding to very general boundary functions.

2. The PWB Solution of the Dirichlet Problem. An extended real-valued function u on an open set $V \subset R^m$ is **hyperharmonic** on V, if u is lower finite, lower semi-continuous and

$$u(x) \geqslant (\operatorname{vol} B)^{-1} \int_B u(y)\,dy,$$

whenever $B \subset V$ is a closed ball with the center x (here vol B denotes the Lebesgue measure of B). If, moreover, u is identically $+\infty$ on no component of V, the function u is called **superharmonic** on V. The minimum of two superharmonic functions is obviously superharmonic. Remark that a function h defined on V is harmonic if and only if both h and $-h$ are superharmonic on V. In other words, the mean value property characterizes harmonic functions among continuous functions.

In what follows, U will denote an arbitrary nonempty open set with compact closure $\bar{U}$ in R^m and $H(U)$ will stand for the set of all functions continuous on $\bar{U}$ and harmonic on U. Thus the restriction of $H(U)$ to ∂U consists precisely of those functions for which the classical Dirichlet problem has a solution. Note that the space $H(U)$ is uniformly closed.

Let f be any extended real-valued function defined on ∂U. The set $\mathbf{U}_f$ of all lower bounded hyperharmonic functions on U such that $\liminf_{y \to x} u(y) \geqslant f(x)$ for all $x \in \partial U$ is called **the upper class** determined by f. The functions

$$\bar{H}^U f = \inf\{u; u \in \mathbf{U}_f\}, \quad \underline{H}^U f = \sup\{v; v \in -\mathbf{U}_{(-f)}\}$$

are called **the upper solution** and **the lower solution** for the generalized Dirichlet problem for f, respectively. On every component of U, each of the functions $\bar{H}^U f$ and $\underline{H}^U f$ is either identically $+\infty$, identically $-\infty$, or harmonic and the inequality $\bar{H}^U f \geqslant \underline{H}^U f$ holds on U. If $\bar{H}^U f = \underline{H}^U f$ and both functions are harmonic on U, then f is said to be **resolutive** and $H^U f = \bar{H}^U f$ is called **the Perron-Wiener-Brelot generalized solution of the Dirichlet problem** for f. If $f \in C(\partial U)$, the space of continuous functions on ∂U, then f is resolutive and $H^U(\alpha f + \beta g) = \alpha H^U f + \beta H^U g$, whenever α, β are real numbers and $f, g \in C(\partial U)$. The operator $f \mapsto H^U f$ is positive on $C(\partial U)$ in the sense that $H^U f \geqslant 0$ on U whenever $f \geqslant 0$ on $\partial U, f \in C(\partial U)$. Of course, $H^U f$ coincides with the classical solution, if it exists for f. In other words, we have $H^U(h|_{\partial U}) = h|_U$ for every $h \in H(U)$.

A point $x \in \partial U$ is termed a **regular point** of U if $\lim_{y \to x} H^U f(y) = f(x)$ for all $f \in C(\partial U)$; otherwise x is an **irregular point.** The set of all regular points will be denoted $\partial_r U$. Clearly, U is regular if and only if $\partial U = \partial_r U$. The set of irregular points is negligible in a potential-theoretic sense and appears to be exceptional in many situations. For example, the following **uniqueness**

theorem holds: *If* $h_1, h_2 \in \mathcal{H}(U)$ *are bounded and* $\lim_{y\to z} h_1(y) = \lim_{y\to z} h_2(y)$ *for every* $z \in \partial_r U$, *then* $h_1 = h_2$ *on* U.

(All assertions stated above may be found in introductory texts on potential theory, cf. **[11]**, **[16]**, **[24]**, **[36]**, **[40]**.)

3. **The Keldyš Theorem.** In connection with the investigation of unicity of a "reasonable" generalized solution of the classical Dirichlet problem, we adopt the following definition.

The operator A: $C(\partial U) \to \mathcal{H}(U)$ is called a **Keldyš operator** on U, if

(α) A is linear;
(β) A is positive;
(γ) $A(h|_{\partial U}) = h|_U$ for every $h \in H(U)$.

We know that $H^U: f \mapsto H^U f$ is a Keldyš operator. One of the remarkable and important results of potential theory reads as follows:

THEOREM. *There is exactly one Keldyš operator on U.*

We shall present below a proof of this theorem which is elementary in the sense that it uses only basic results included in standard textbooks such as **[11]**, **[16]**, **[24]**. (Comments concerning the corresponding bibliography and the historical development are postponed to the end of this note.) More specifically, the following two assertions are supposed to be accepted by the reader. Throughout the rest of the paper, $V \subset R^m$ will be a fixed bounded open set containing $\overline{U}$.

PROPOSITION 1. *Let w be a nonnegative superharmonic function on V. Then there exists a superharmonic function s on V such that* $0 \leqslant s \leqslant w$, $s = w$ *on* $(V \setminus \overline{U}) \cup \partial_r U$ *and s is harmonic on U.*

Proof. See **[16]**, pp. 135, 205].

PROPOSITION 2. *Let s be a finite nonnegative superharmonic function on V. Then there are nonnegative continuous superharmonic functions* s_k *on V such that*

$$s = \sum_{k=1}^{\infty} s_k. \tag{1}$$

Proof. See **[16]**, pp. 116, 118].

COROLLARY. *Let w be a finite nonnegative superharmonic function on V. Then there is an increasing sequence* $\{g_n\}$ *of nonnegative functions belonging to* $H(U)$ *such that* $g_n \leqslant w$ *on* $\overline{U}$ *for every n and*

$$\lim_{n\to\infty} g_n(x) = w(x), \qquad x \in \partial_r U.$$

Proof. Apply Proposition 1 to get s which corresponds to given w. Write s in the form (1) with s_k as in Proposition 2, and put

$$t_n = \sum_{k=1}^{n} s_k.$$

Observe that t_n is superharmonic and continuous on V. Being a sum of superharmonic functions, $s - t_n$ is superharmonic on V. Since s is harmonic on U, the function $-t_n = (s - t_n) - s$ is also superharmonic on U. We see that t_n is harmonic on U. To complete the proof, define g_n as the restriction of t_n to $\overline{U}$.

Proof of Theorem. Suppose that A is a Keldyš operator on U. We are going to show that $A = H^U$.

Denote by W the set of all functions w of the form

$$w = \min(h_1, \ldots, h_k) \tag{2}$$

where k is a positive integer and all h_j are nonnegative harmonic functions on V. Then W is obviously a min-stable subset of the system of superharmonic functions on V. It is easy to verify that the sum of two functions from W belongs to W. Since $W-W=\{u-v;\ u,v\in W\}$ is a vector lattice containing constants and separating points of V, we conclude, by the well-known lattice version of the Stone-Weierstrass theorem that $W_1=(W-W)|_{\partial U}$ is uniformly dense in $C(\partial U)$.

To establish the equality $A=H^U$ it is sufficient to prove that

$$Aw=H^U w, \qquad w\in W. \tag{3}$$

(We write Aw instead of $A(w|_{\partial U})$ and similarly for H^U.) Indeed, if (3) holds, then A and H^U coincide on W_1 by linearity. Fix a point $x\in U$. Then $f\mapsto Af(x)$ and $f\mapsto H^Uf(x)$ are positive linear functionals having the same values on the dense subspace $W_1\subset C(\partial U)$. It follows that $Af(x)=H^Uf(x)$ for all $f\in C(\partial U)$, and we conclude that $A=H^U$.

Fix now $w\in W$ and observe that

$$Aw\leqslant w \quad \text{on } U. \tag{4}$$

To see this write w in the form (2). Then $w\leqslant H_j$ on V and $h_j|_{\bar U}\in H(U)$, so that (β) and (γ) from the definition of the Keldyš operator yield $Aw\leqslant Ah_j=h_j$ on U for every $j=1,\ldots,k$. Consequently, $Aw\leqslant \min(h_1,\ldots,h_k)=w$ and (4) is verified.

By Corollary there are $g_n\in H(U)$ such that $0\leqslant g_n\leqslant w$ on $\bar U$ and $g_n(x)\nearrow w(x)$ for every $x\in\partial_r U$. Choose $z\in\partial_r U$ and $\epsilon>0$. For n large enough we have $g_n(z)\geqslant w(z)-\epsilon$. Since $g_n=Ag_n\leqslant Aw$ on U, we conclude that

$$\liminf_{y\to z} Aw(y)\geqslant g_n(z)\geqslant w(z)-\epsilon.$$

This together with (4) gives

$$\lim_{y\to z} Aw(y)=w(z)=\lim_{y\to z} H^U w(y).$$

By the above-mentioned uniqueness theorem the equality (3) holds and the proof is complete.

4. The Choquet Theory and the Dirichlet Problem. In the rest of this note we shall look at the space $H(U)$ from another point of view. At first we summarize some basic results of the Choquet theory (cf. **[1]**, **[5]**, **[13]**, **[35]** where further references can be found).

Let X be a metrizable compact topological space and $C(X)$ be the Banach space of continuous functions on X. Suppose that P is a closed linear subspace of $C(X)$ which separates the points of X and contains the constant functions. For every $x\in X$ we shall use the symbol M_x to denote the set of all positive (Radon) measures on X such that $f(x)=\mu(f)$ for all $f\in P$. The elements of M_x are called **representing measures** for x. Obviously, the Dirac measure ϵ_x concentrated at x belongs to M_x. The set

$$\mathrm{Ch}_P X=\{x\in X;\ M_x=\{\epsilon_x\}\}$$

is called **the Choquet boundary** of X (with respect to P). Note that $\mathrm{Ch}_P X$ is of type G_δ. One of the important results of the Choquet theory says that every point $x\in X$ can be represented by means of a measure carried by $\mathrm{Ch}_P X$. More precisely, *for every $x\in X$ there is $\nu_x\in M_x$ such that $\nu_x(X\setminus \mathrm{Ch}_P X)=0$.* Such a measure ν_x need not be uniquely determined. The space P is said to be **simplicial** if for every $x\in X$ there exists a unique representing measure which is carried by $\mathrm{Ch}_P X$.

A point $x\in X$ is an **exposed point** if there is a function $h\in P$ such that $h(x)=0$ and $h>0$ on $X\setminus\{x\}$. It is easy to see that every exposed point belongs to the Choquet boundary. It is known that if P is *simplicial*, the converse assertion holds: *Every point of* $\mathrm{Ch}_P X$ *is an exposed point.*

Now we shall apply the results above to the following situation: $X=\bar U$ and $P=H(U)$.

LEMMA. $\mathrm{Ch}_{H(U)}\bar U\subset\partial_r U$.

Proof. For every $x\in U$, the mapping $f\mapsto H^Uf(x)$ is a positive Radon measure on ∂U. This measure will be denoted μ_x and called the harmonic measure. Clearly, $x\in\partial U$ is a regular point

of U if and only if

$$\lim_{y \to x} \mu_y = \epsilon_x \tag{5}$$

(in the sense of the weak convergence of measures).

In view of the mean value property of harmonic functions, no point of U belongs to $\mathrm{Ch}_{H(U)} \overline{U}$. Suppose that $x \in \mathrm{Ch}_{H(U)} \overline{U}$. We know that $x \in \partial U$. Fix a sequence $\{x_n\}$ of points of U tending to x and write μ_n instead of μ_{x_n}. Since μ_n are probability measures, there is a measure μ and a subsequence $\{\mu_{n_j}\}$ converging to μ (see, e.g., [**16**, p. 26]). But

$$h(x_{n_j}) = \mu_{n_j}(h), \qquad h \in H(U),$$

so that, letting $j \to \infty$, we get $h(x) = \mu(h)$. Consequently, $\mu \in M_x$ and $\mu = \epsilon_x$ because $x \in \mathrm{Ch}_{H(U)} \overline{U}$. The same argument shows that every convergent subsequence of $\{\mu_n\}$ tends to ϵ_x; thus $\mu_n \to \epsilon_x$ as $n \to \infty$. Now (5) is obvious and $x \in \partial_r U$.

THEOREM. *The space $H(U)$ is simplicial and*

$$\mathrm{Ch}_{H(U)} \overline{U} = \partial_r U. \tag{6}$$

Proof. Fix a point $x \in \partial_r U$. We know that there is a measure $\nu_x \in M_x$ such that $\nu_x(\overline{U} \setminus \mathrm{Ch}_{H(U)} \overline{U}) = 0$. We are going to show that $\nu_x = \epsilon_x$. This in turn implies that $x \in \mathrm{Ch}_{H(U)} \overline{U}$, since otherwise we would have $\nu_x(\overline{U} \setminus \mathrm{Ch}_{H(U)} \overline{U}) = 1$.

Choose $w \in W$. By Corollary, there is an increasing sequence $\{g_n\}$ of nonnegative functions of $H(U)$ such that $g_n \to w$ pointwise on $\partial_r U$. It follows from the Lemma that the measure ν_x is carried by $\partial_r U$ so that

$$\lim_{n \to \infty} \nu_x(g_n) = \nu_x(w).$$

On the other hand, $\nu_x(g_n) = g_n(x) \to w(x) = \epsilon_x(w)$. Hence ϵ_x and ν_x coincide on W. But $W_1 = (W - W)_{\partial U}$ is dense in $C(\partial U)$; thus $\nu_x = \epsilon_x$.

Suppose that $y \in \overline{U}$ and η_y, τ_y are representing measures for y carried by $\mathrm{Ch}_{H(U)} \overline{U}$. Let w and g_n have the same meaning as above. Since $\eta_y(g_n) = \tau_y(g_n)$ $(= g_n(y))$, we obtain $\eta_y = \tau_y$ by a similar reasoning as in the first part of the proof. It follows that $H(U)$ is simplicial.

COROLLARY (Keldyš lemma). *If $z \in \partial_r U$, then there is a function $h \in H(U)$ such that $h(z) = 0$ and $h > 0$ on $\overline{U} \setminus \{z\}$.*

Using this corollary we can prove a modification of the Keldyš theorem. At first we introduce the following definition.

The operator T: $C(\partial U) \to \mathcal{H}(U)$ is called a ***K*-operator** on U, if

(β') T is monotone (i.e., $Tf_1 \leqslant Tf_2$ on U whenever $f_1\, f_2$ on ∂U);

(γ) $T(h_{|\partial U}) = h_{|U}$ for every $h \in H(U)$.

Obviously, every Keldyš operator is a K-operator since conditions (α) and (β) imply (β^1).

THEOREM. *There is exactly one K-operator on U.*

Proof. Suppose that T is a K-operator on U and fix $z \in \partial_r U$. We shall prove that

$$\lim_{y \to z} Tf(y) = f(z), \qquad f \in C(\partial U). \tag{7}$$

Then, as we already know, Tf necessarily coincides with $H^U f$. Choose $f \in C(\partial U)$ and suppose that h is the function from the Keldyš lemma. Fix $\epsilon > 0$ and a neighborhood N of z such that $f(y) < f(z) + \epsilon$ for every $y \in \partial U \cap N$. Since $h > 0$ on $\overline{U} \setminus \{z\}$, we can find $c > 0$ such that

$f-f(z)\leqslant ch$ on $\partial U\setminus N$. Defining $\phi=f(z)+\epsilon+ch$, we have $\phi\in H(U)$ and $f\leqslant\phi$ on ∂U. Conditions (β') and (γ) yield $Tf\leqslant T\phi=\phi$, so that

$$\limsup_{y\to z} Tf(y)\leqslant \lim_{y\to z}\phi(y)=f(z)+\epsilon.$$

Similarly,

$$\liminf_{y\to z} Tf(y)\geqslant f(z)-\epsilon;$$

hence (7) holds and the proof is complete.

Let us define

$$S(U)=\left\{s\in C(\bar{U}); s|_U \text{superharmonic}\right\}$$

and put for $f\in C(\partial U)$

$$\begin{aligned}
\bar{P}^U f&=\inf\{s|_U; s\in S(U), s|_{\partial U}\geqslant f\},\\
\underline{P}^U f&=\sup\{t|_U; t\in -S(U), t|_{\partial U}\leqslant f\},\\
\bar{D}^U f&=\inf\left\{v|_U; v\in C(\bar{U}), v|_U\in C^2(U), \Delta v\leqslant 0, v|_{\partial U}\geqslant f\right\},\\
\underline{D}^U f&=\sup\left\{w|_U; w\in C(\bar{U}), w|_U\in C^2(U), \Delta w\geqslant 0, w|_{\partial U}\leqslant f\right\}.
\end{aligned}$$

As an application of the last theorem we obtain the following result.

THEOREM. *The equalities*

$$H^U f=\bar{P}^U f=\underline{P}^U f=\bar{D}^U f=\underline{D}^U f$$

hold on U for every $f\in C(\partial U)$.

Proof. Note that the condition $\Delta v\leqslant 0$ implies superharmonicity of v ([**16**, p. 57]). It is clear that

$$\underline{D}^U f\leqslant \underline{P}^U f\leqslant \underline{H}^U f\leqslant \bar{H}^U f\leqslant \bar{P}^U f\leqslant \bar{D}^U f. \tag{8}$$

If $v\in S(U), h\in H(U)$ and $v\geqslant h$ on ∂U, then $v\geqslant h$ on U. This follows from the minimum principle for superharmonic functions ([**16**, p. 59]). Consequently, $\bar{D}^U(h|_{\partial U})=h|_U=\underline{D}^U(h|_{\partial U})$ for every $h\in H(U)$. Both operators $\bar{D}^U$ and $\underline{D}^U$ are obviously monotone. Since there is exactly one K-operator on U, $\bar{D}^U=\underline{D}^U$. The rest follows from (8).

5. Notes and Comments. Historical development of the Dirichlet problem is described in the following books and articles: **[38], [30], [21], [9], [12], [28]**. The question of unicity of a reasonable generalization of the classical Dirichlet problem was raised and partially answered by A. F. Monna in 1935–1939 (for a detailed analysis of Monna's contribution to the subject and for the corresponding references, see **[4], [31]**). The Keldyš lemma was proved in **[19]** by a very complicated method based on fine reasonings connected with Wiener's test of regularity. The Keldyš theorem (providing the complete solution of Monna's problem) is established in **[20]**. Our proof of the unicity result for K-operators is essentially a reproduction of the original Keldyš proof. Various conditions securing the unicity of a generalized solution are studied in **[18], [32], [39], [10]**. M. Brelot showed in **[10]** that a weakened form of the Keldyš lemma leads to a satisfactory proof of the Keldyš theorem. He gave a nice and relatively simple proof of such a weakened form and observed that linearity and positivity can be replaced by monotonicity of the operator in question. Methods of functional analysis are used in connection with the Keldyš theorem in **[10], [23], [24], [17], [26]**. A. F. Monna emphasizes in **[31]** the fact that the fundamental theorem about a unique extension of the classical Dirichlet problem ("the Keldyš theorem" in our terminology) is not treated in modern textbooks on classical potential theory such as **[16], [36]**.

A series of papers deals with diverse aspects of the Keldyš theorem in the context of the axiomatic potential theory (which includes a wide class of partial differential equations of elliptic and parabolic types): **[10]**, [7], **[14]**, **[25]**, **[26]**, [27], **[28]**, and [37], where the notion of K-operator is introduced and investigated.

The identification of the Choquet boundary with $\partial_r U$ was established for the classical case in [2] using the Keldyš lemma. Simpliciality is implicitly contained in **[10]** and some special cases for the classical potential theory are studied in **[15]** and **[17]**. In the frame of harmonic spaces, partial results on simpliciality and on the relation between $\mathrm{Ch}_{H(U)}\overline{U}$ and $\partial_r U$ are obtained in [3], **[22]**, [7], **[14]**. Final and general results in this direction are proved in [5] (cf. **[6]**). Introducing $\bar{P}^U$ and $\underline{P}^U$ we come close to the original Perron's definition [33]. Note that the equality of H^U with these solutions is not at all evident and seems to be observed for the first time in **[8]** as a consequence of the Keldyš lemma. Our last theorem shows that solutions investigated, e.g., in [29], [34], give the same result as the PWB-method. Identity of $\bar{P}^U$ and $\underline{P}^U$ is proved under general conditions in **[26]**.

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AN EXTENSION OF EGOROV'S THEOREM

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The importance of the classical theorem of Egorov [2] in measure theory is well appreciated: it provides the key to passing from the almost everywhere convergence of a sequence of measurable functions to the almost uniform convergence (and hence the convergence in measure) of this sequence. In its usual statement Egorov's Theorem is ordinarily given for a finite measure space, and simple examples can be given to show that Egorov's Theorem may fail for infinite measure spaces. However, it is known that Egorov's Theorem remains valid if the sequence is dominated by an integrable function.

In this note we give necessary and sufficient conditions for a sequence of measurable functions to be almost uniformly convergent. It will be seen that almost uniform convergence can take place under somewhat less stringent restrictions than are usually required. The formulation we shall give makes almost uniform convergence appear much like convergence in measure; we consider this to be an advantage.

The arguments are entirely standard; indeed, they are primarily reformulations of the steps ordinarily employed in measure theory to prove Egorov's and related theorems. It will be noted that, although we will be concerned only with real-valued functions, the proofs hold for functions with values in a Banach space.

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1. The Finiteness and Vanishing Restrictions. Let $(X, \mathbf{X}, \mu)$ be a measure space. (In general we will follow the notation and terminology in [1] and [3].)

1.1. DEFINITION. If (f_n) is a sequence of $\mathbf{X}$-measurable functions on X and $\alpha>0$, $n\in\mathbf{N}$, then we define

$$E_n(\alpha)=\bigcup_{i,j=n}^{\infty}\{x\in X:|f_i(x)-f_j(x)|>\alpha\}. \tag{1}$$

If f is an $\mathbf{X}$-measurable function on X, we define

$$E_n^f(\alpha)=\bigcup_{i=n}^{\infty}\{x\in X:|f_i(x)-f(x)|>\alpha\}. \tag{2}$$

It is evident that the sets $E_n(\alpha)$ and $E_n^f(\alpha)$ belong to $\mathbf{X}$ and that, if $m\leqslant n$, then $E_n(\alpha)\subseteq E_m(\alpha)$ and $E_n^f(\alpha)\subseteq E_m^f(\alpha)$. Moreover, if $0<\alpha<\beta$, then we have $E_n(\beta)\subseteq E_n(\alpha)$ and $E_n^f(\beta)\subseteq E_n^f(\alpha)$. In addition, since

$$\begin{aligned}\{x\in X:|f_i(x)-f_j(x)|>\alpha\}\subseteq\{x\in X:|f_i(x)-f(x)|>\alpha/2\}&\\ \cup\{x\in X:|f_j(x)-f(x)|<\alpha/2\},&\end{aligned}$$

it is readily seen that

$$E_n(\alpha)\subseteq E_n^f(\alpha/2). \tag{3}$$

The next result gives an inclusion in the opposite direction.

1.2. LEMMA. *If $f(x)=\lim_{k\to\infty}f_k(x)$ for all $x\in X$, then $E_n^f(\alpha)\subseteq E_n(\alpha)$, for all $\alpha>0$, $n\in\mathbf{N}$.*

Proof. If $x_0\notin E_n(\alpha)$, then we have $|f_i(x_0)-f_j(x_0)|\leqslant\alpha$ for all $i, j\geqslant n$. Passing to the limit as $j\to\infty$, we infer that $|f_i(x_0)-f(x_0)|\leqslant\alpha$. Thus it follows that $x_0\notin E_n^f(\alpha)$.

1.3. DEFINITION. (a) We say that the sequence (f_n) satisfies the *finiteness restriction* [*with respect to* f] if for all $\alpha>0$ there exists a natural number n_α such that the set $E_{n_\alpha}(\alpha)$ [respectively, $E_{n_\alpha}^f(\alpha)$] has finite μ-measure.

(b) We say that the sequence (f_n) satisfies the *vanishing restriction* [*with respect to* f] if for all $\alpha>0$ we have

$$\lim_{n\to\infty}\mu(E_n(\alpha))=0 \qquad \left[\text{respectively, } \lim_{n\to\infty}\mu(E_n^f(\alpha))=0\right].$$

It follows from the above inclusions that if (f_n) satisfies the finiteness [respectively, vanishing] restriction with respect to f, then (f_n) satisfies the finiteness [respectively, vanishing] restriction. On the other hand, if (f_n) satisfies the finiteness [respectively, vanishing] restriction and if $f(x)=\lim_k f_k(x)$ for all $x\in X$ (or for almost all $x\in X$), then (f_n) satisfies the finiteness [respectively, vanishing] restriction with respect to f.

In Definition 1.3 it is *not* assumed that f is the limit of the sequence (f_n) in any particular sense, although some types of convergence of (f_n) to f are of special interest.

It is readily seen that if the sequence (f_n) converges [to f] uniformly on X, then for all $\alpha>0$ there exists n_α such that $E_{n_\alpha}(\alpha)=\varnothing$ [respectively, $E_{n_\alpha}^f(\alpha)=\varnothing$].

1.4. PROPOSITION. *If the sequence (f_n) converges almost uniformly* [*to* f], *then it satisfies the vanishing restriction* [*with respect to* f].

Proof. Let $\alpha>0$ be given. If $\varepsilon>0$, then there exists $B_\varepsilon\in\mathbf{X}$ such that $\mu(B_\varepsilon)<\varepsilon$ and the sequence (f_n) converges [to f] uniformly on $X-B_\varepsilon$. Consequently, there is an n_α such that $E_{n_\alpha}(\alpha)\subseteq B_\varepsilon$ [respectively, $E_{n_\alpha}^f(\alpha)\subseteq B_\varepsilon$] whence $\mu(E_{n_\alpha}(\alpha))<\varepsilon$ [respectively, $\mu(E_{n_\alpha}^f(\alpha))<\varepsilon$]. Thus $\lim_n\mu(E_n(\alpha))=0$ [respectively, $\lim_n\mu(E_n^f(\alpha))=0$].

We now prove the converse assertion.

1.5. Proposition. (a) *If the sequence* (f_n) *satisfies the vanishing restriction with respect to* f, *then it converges almost uniformly to* f.

(b) *If* (f_n) *satisfies the vanishing restriction, then there exists a measurable function* f *such that* (f_n) *converges almost uniformly to* f.

Proof. (a) Let $\varepsilon > 0$ be given. For each $k \in \mathbf{N}$ there exists n_k such that $\mu(E^f_{n_k}(1/k)) < \varepsilon/2^k$. Let $B_\varepsilon \in \mathbf{X}$ be defined by

$$B_\varepsilon = \bigcup_{k=1}^{\infty} E^f_{n_k}(1/k)$$

so that we have $\mu(B_\varepsilon) < \varepsilon$. Further, if $j \geqslant n_k$, then we have

$$|f_j(x) - f(x)| \leqslant 1/k \tag{$*$}$$

for all $x \notin E^f_{n_k}(1/k)$. Since $E^f_{n_k}(1/k) \subseteq B_\varepsilon$, it follows that if $x \notin B_\varepsilon$ then $(*)$ holds provided $j \geqslant n_k$. But this implies that (f_n) converges to f uniformly on $X - B_\varepsilon$.

(b) In this case we show, as in (a), that the sequence (f_n) is uniformly Cauchy on $X - B_\varepsilon$. Hence there is a function g_ε on $X - B_\varepsilon$ to which (f_n) converges. Let $B_0 = \bigcap B_{1/n}$ so that $\mu(B_0) = 0$, and let $f(x) = 0$ for $x \in B_0$ and $f(x) = g_{1/n}(x)$ for $x \notin B_{1/n}$. It is readily seen that f is consistently defined and is the desired function.

The next result enables one to pass from the finiteness restriction to the vanishing restriction.

1.6. Proposition. *If the sequence* (f_n) *satisfies the finiteness restriction with respect to* f, *and if* $f(x) = \lim_k f_k(x)$ *for almost all* $x \in X$, *then the sequence satisfies the vanishing restriction with respect to* f.

Proof. For convenience let $C_n(\alpha) = X - E^f_n(\alpha)$, so that

$$C_n(\alpha) = \bigcap_{i=n}^{\infty} \{x \in X : |f_i(x) - f(x)| \leqslant \alpha\}$$

and let $C = \{x \in X : f(x) = \lim_k f_k(x)\}$. We note that

$$C = \bigcap_{\alpha > 0} \bigcup_{n=i}^{\infty} C_n(\alpha).$$

Therefore $C \subseteq \bigcup_{n=1}^{\infty} C_n(\alpha)$ for all $\alpha > 0$. Now, by hypothesis, $X - C$ is contained in a μ-null set. Therefore we have

$$\mu\left(\bigcap_{n=1}^{\infty} E^f_n(\alpha)\right) = 0.$$

But $(E^f_n(\alpha))_n$ is a decreasing sequence of measurable sets and, since $\mu(E^f_n(\alpha)) < +\infty$ for $n \geqslant n_\alpha$, we deduce that

$$\lim_{n\to\infty} \mu(E^f_n(\alpha)) = 0.$$

Therefore (f_n) satisfies the vanishing restriction with respect to f.

We combine the preceding results in a unified statement.

1.7. Theorem. *Let* (f_n) *be a sequence of measurable functions and let* f *be a measurable function. Then the following assertions are equivalent.*

(i) *The sequence* (f_n) *converges almost uniformly to* f.
(ii) *The sequence* (f_n) *satisfies the vanishing restriction with respect to* f.
(iii) *The sequence* (f_n) *satisfies the finiteness restriction with respect to* f *and*

$$f(x) = \lim_{n\to\infty} f_n(x) \quad \mu\text{-almost everywhere}.$$

Since the finiteness restriction is automatically satisfied when $\mu(X)<\infty$, this result can be considered to be an extension of Egorov's Theorem.

2. Domination Conditions. We shall provide a sufficient condition for the finiteness restriction that is often applicable.

2.1. DEFINITION. Let g be a nonnegative $\mathbf{X}$-measurable function on X. We define the *distribution function* of g by

$$\omega_g(\alpha)=\mu(\{x\in X:g(x)>\alpha\})\quad \text{for} \quad \alpha>0.$$

We say that g has a *finite distribution function* in case $\omega_g(\alpha)<+\infty$ for all $\alpha>0$.

It is clear that, in general, ω_g is a decreasing function defined on $(0,+\infty)$ to $[0,+\infty]$. As an example of a function with finite distribution function, we cite a nonnegative function that is integrable over $(X, \mathbf{X}, \mu)$. Indeed, for such a function we have

$$\alpha\omega_g(\alpha)\leqslant\int_X g<+\infty.$$

The restriction to integrable functions is not necessary, however. For example, let $X=[0,\infty)$ with Lebesgue measure, and let g_1 be any nonnegative function such that $\lim_{x\to\infty}g_1(x)=0$. Then g_1 has a finite distribution function.

2.2. PROPOSITION. *Suppose that g is a nonnegative measurable function with a finite distribution function. Suppose that the sequence (f_n) satisfies the condition*

$$|f_i(x)-f_j(x)|\leqslant g(x)\qquad [\textit{respectively}, |f_i(x)-f(x)|\leqslant g(x)] \tag{$*$}$$

for all $x\in X$ and all $i,j\in\mathbf{N}$. Then the sequence (f_n) satisfies the finiteness restriction [with respect to f].

Proof. Indeed, if $(*)$ holds for all $i,j\in\mathbf{N}$, then

$$\{x\in X:|f_i(x)-f_j(x)|>\alpha\}\subseteq\{x\in X:g(x)>\alpha\}.$$

Therefore, we have $\mu(E_n(\alpha))\leqslant\omega_g(\alpha)<\infty$ for all $n\in\mathbf{N}$, $\alpha>0$.

The above proposition immediately yields the dominated form of Egorov's Theorem: If g is a nonnegative integrable function such that $|f_i(x)|\leqslant g(x)$ for $x\in X, i\in\mathbf{N}$, and if $f(x)=\lim f_i(x)$ for almost all $x\in X$, then the convergence is almost uniform on X. (See [**4**, p. 221].)

We now turn to another characterization of almost uniform convergence based on functions that "dominate" the sequence in an appropriate sense.

2.3. DEFINITION. If (f_n) is a sequence of $\mathbf{X}$-measurable functions on X and $n\in\mathbf{N}$, we define

$$\phi_n(x)=\sup\{|f_i(x)-f_j(x)|:i,j\in\mathbf{N},i\geqslant n,j\geqslant n\}.$$

If f is an $\mathbf{X}$-measurable function on X, we define

$$\phi_n^f(x)=\sup\{|f_i(x)-f(x)|:i\in\mathbf{N},i\geqslant n\}.$$

It is evident that the functions ϕ_n and ϕ_n^f are $\mathbf{X}$-measurable and that if $m\leqslant n$, then

$$\phi_n(x)\leqslant\phi_m(x)\qquad\left[\text{respectively, } \phi_n^f(x)\leqslant\phi_m^f(x)\right]$$

for all $x\in X$. It is also evident that, if $x\in X$, then the sequence $(f_i(x))$ is a Cauchy sequence if and only if $\lim_{n\to\infty}\phi_n(x)=0$ and that $\lim_{n\to\infty}f_n(x)=f(x)$ if and only if $\lim_{n\to\infty}\phi_n^f(x)=0$.

2.4. LEMMA. *If $\alpha>0$ and $n\in\mathbf{N}$, then*

$$E_n(\alpha)=\{x\in X:\phi_n(x)>\alpha\}\quad \textit{and}\quad E_n^f(\alpha)=\{x\in X:\phi_n^f(x)>\alpha\}.$$

Proof. If $x \in E_n(\alpha)$ then there exist $i,j \in \mathbf{N}$ with $i \geqslant n, j \geqslant n$ such that $|f_i(x)-f_j(x)|>\alpha$, whence $\phi_n(x)>\alpha$, and conversely. Similarly for $E_n^f(\alpha)$.

2.5. COROLLARY. *The sequence (f_n) satisfies the finiteness restriction [with respect to f] if and only if for every $\alpha>0$ there exists a natural number n_α such that the set $\{x \in X : \phi_{n_\alpha}(x)>\alpha\}$ [respectively, $\{x \in X : \phi_{n_\alpha}^f(x)>\alpha\}$] has finite μ-measure.*

2.6. COROLLARY. *The sequence (f_n) satisfies the vanishing restriction [with respect to f] if and only if, for all $\alpha>0$, then*

$$\lim_{n\to\infty} \mu\{x \in X : \phi_n(x)>\alpha\}=0 \qquad \left[\textit{respectively, } \lim_{n\to\infty} \mu\{x \in X : \phi_n^f(x)>\alpha\}=0\right].$$

Some of these results can be used to reformulate the propositions in Section 1. We shall content ourself with the following statement.

2.7. THEOREM. *Let (f_n) be a sequence of measurable functions and let f be a measurable function. Then the following assertions are equivalent.*

(i) *The sequence (f_n) converges almost uniformly to f.*
(ii) *For every $\alpha>0$ there is a natural number n_α such that $\mu\{x \in X : \phi_{n_\alpha}^f(x)>\alpha\}<+\infty$, and $\lim_{n\to\infty}\phi_n^f(x)=0$ for μ-almost all x.*
(iii) *The sequence (ϕ_n^f) converges in measure to* 0.

Thus, almost uniform convergence of the (f_n) to f is equivalent to convergence in measure to 0 of the sequence (ϕ_n^f).

3. **Convergence in Measure and Mean.** We shall now connect the ideas in the previous sections with the notions of convergence in measure and convergence in mean. First we give some negative results.

3.1. *Example.* If (f_n) converges in measure [respectively, in mean] to f, then it does *not* follow that (f_n) satisfies the finiteness or the vanishing restrictions. Indeed, let f_n be the characteristic function of the interval $[n, n+1/n]$, with Lebesgue measure on R.

3.2. *Example.* The analog of Proposition 1.6 for convergence in measure [respectively, in mean] does *not* hold. Indeed, let (g_n) be a sequence of characteristic functions on the interval $X=[0,1]$ that converges in measure to the 0-function, but which does not converge at any point of X [1, p. 68]. Then the finiteness restriction is satisfied (for Lebesgue measure on X), but we do not have almost uniform convergence, and hence (g_n) does not satisfy the vanishing restriction.

However, there are some positive results.

3.3. PROPOSITION. *If (f_n) converges in measure (or in mean) to f, and if (f_n) satisfies the vanishing restriction, then the sequence (f_n) converges almost uniformly to f.*

Proof. It follows from Proposition 1.5(b) that there exists a measurable function g to which (f_n) converges almost uniformly, and therefore in measure. Consequently, $f(x)=g(x)$ for almost all $x \in X$, whence it follows that (f_n) converges almost uniformly to f.

The next definition is generalization of the notion of monotone convergence of a sequence (f_n) to a limit function f.

3.4. DEFINITION. A sequence (f_n) of measurable functions is said to be *M-convergent* to a measurable function f if, for all $\alpha>0$, we have

$$\{x \in X : |f_j(x)-f(x)|>\alpha\} \subseteq \{x \in X : |f_i(x)-f(x)|>\alpha\}$$

whenever $i \leqslant j$ $(i,j \in \mathbf{N})$.

3.5. PROPOSITION. *If the sequence (f_n) converges in measure [respectively, in mean] to f and is*

M-convergent, then the sequence (f_n) satisfies the vanishing restriction and the convergence to f is almost uniform.

Proof. If the sequence is M-convergent to f, then

$$E_n^f(\alpha)=\{x\in X:|f_n(x)-f(x)|>\alpha\}.$$

But since (f_n) converges in measure to f, we have $\lim\mu(E_n^f(\alpha))=0$. Thus the vanishing restriction is satisfied.

The next result is a version of a well-known theorem due to F. Riesz [5]. The proof is virtually the same.

3.6. PROPOSITION. *If a sequence (f_n) is Cauchy in measure [respectively, in mean], then it has a subsequence (f_{n_k}) that satisfies the vanishing restriction, and which converges almost uniformly to a measurable function f. Moreover, (f_n) converges in measure to f.*

Proof. Since (f_n) is Cauchy in measure, we can choose a subsequence $(g_k)=(f_{n_k})$ such that if

$$A_k=\{x\in X:|g_{k+1}(x)-g_k(x)|>1/2^k\},$$

then $\mu(A_k)<1/2^k$. Let $F_k=\bigcup_{j=k}^{\infty}A_j$ so that $F_k\in\mathbf{X}$ and $\mu(F_k)<1/2^{k-1}$. If $i\geqslant j\geqslant k$ and if $x\notin F_k$, then

$$|g_i(x)-g_j(x)|\leqslant|g_i(x)-g_{i-1}(x)|+\cdots+|g_{j+1}(x)-g_j(x)|$$
$$\leqslant\frac{1}{2^{i-1}}+\cdots+\frac{1}{2^j}<\frac{1}{2^{j-1}}\leqslant\frac{1}{2^{k-1}}.$$

Hence if $\alpha>0$ is given, let K_1 be such that if $k\geqslant K_1$, then $1/2^{k-1}<\alpha$. It follows that, if $i\geqslant j\geqslant k$, then

$$|g_i(x_0)-g_j(x_0)|>\alpha\quad\text{implies that}\quad x_0\in F_k.$$

Thus we have $E_n(\alpha)\subseteq F_k$ for all $n\geqslant k$.

Now let $\varepsilon>0$ be given and let $K\geqslant K_1$ be chosen such that $1/2^{K-1}<\varepsilon$. If $n\geqslant K$ and if $i\geqslant j\geqslant n$, then we have $E_n(\alpha)\subseteq F_K$ so that $\mu(E_n(\alpha))<\mu(F_K)<1/2^{K-1}<\varepsilon$. Since $\varepsilon>0$ is arbitrary, it follows that (g_k) satisfies the vanishing condition. Therefore (g_k) converges almost uniformly (and therefore in measure) to a measurable function f.

3.7. PROPOSITION. *If the sequence (f_n) satisfies the condition $\Sigma_{n=1}^{\infty}\|f_n-f\|<\infty$, then the sequence (f_n) satisfies the vanishing restriction with respect to f and converges almost uniformly to f.*

Proof. If $\alpha>0$ and if

$$B_n=\{x\in X:|f_n(x)-f(x)|>\alpha\},$$

then we have $B_n\in\mathbf{X}$ and $\mu(B_n)\leqslant(1/\alpha)\|f_n-f\|$. Moreover, we have $E_n^f(\alpha)=\bigcup_{i=n}^{\infty}B_i$ so that

$$\mu(E_n^f(\alpha))\leqslant\frac{1}{\alpha}\sum_{i=n}^{\infty}\|f_i-f\|.$$

Since the series $\Sigma\|f_n-f\|$ converges, we deduce that $\lim\mu(E_n^f(\alpha))=0$, so that (f_n) satisfies the vanishing restriction.

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THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

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The following results of the fortieth William Lowell Putnam Mathematical Competition, held on December 1, 1979, have been determined in accordance with the governing regulations. This annual contest is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship, left by Mrs. Putnam in memory of her husband, and is held under the auspices of the Mathematical Association of America.

The first prize, five thousand dollars, was awarded to the Department of Mathematics of the **Massachusetts Institute of Technology**, Cambridge, Massachusetts. The members of its winning team were Daniel V. D'Eramo, Miller Puckette, and Michael Roberts; each was awarded a prize of two hundred fifty dollars.

The second prize, two thousand five hundred dollars, was awarded to the Department of Mathematics of the **California Institute of Technology**, Pasadena, California. The members of its team were Christopher S. Bretherton, Peter Shor, and John Stembridge; each was awarded a prize of two hundred dollars.

The third prize, one thousand five hundred dollars, was awarded to the Department of Mathematics of **Princeton University**, Princeton, New Jersey. The members of the team were David D. Chambliss, Mark P. Kleiman, and Charles H. Walter; each was awarded a prize of one hundred fifty dollars.

The fourth prize, one thousand dollars, was awarded to the Department of Mathematics of **Stanford University**, Stanford, California. The members of its team were Richard Beigel, Ken Olum and Stephen Omohundro; each was awarded a prize of one hundred dollars.

The fifth prize, five hundred dollars, was awarded to the Department of Mathematics of the **University of Waterloo**, Waterloo, Ontario, Canada. The members of its team were Michael H. Albert, Bradd Hart, and Geoffrey Mess; each was awarded a prize of fifty dollars.

The five highest-ranking individual contestants, in alphabetical order, were **Randall Dougherty**, University of California, Berkeley; **Richard Mifflin**, Rice University; **Mark G. Pleszkoch**, University of Virginia; **Miller Puckette**, Massachusetts Institute of Technology; and **Charles H. Walter**, Princeton University. Each of these students was designated a Putnam Fellow by the Mathematical Association of America and awarded a prize of five hundred dollars by the Putnam Prize Fund.

The next five highest-ranking individuals, in alphabetical order, were *Edward Branagan, Jr.*, Case Western Reserve University; *Raymond A. Coley*, Princeton University; *Mark P. Kleiman*, Princeton University; *Michael Raship*, Harvard University; and *Peter Shor*, California Institute of Technology. Each of these students was awarded a prize of two hundred fifty dollars.

The following teams, named in alphabetical order, received honorable mention: *University of California, Berkeley*, with team members Randall Dougherty, Lin Goldstein, David Magagnosc; *Case Western Reserve University*, with team members Edward Branagan, Jr., Paul Herdeg, Daniel Stock; *Harvard University*, with team members Daniel Bloch, Victor Milenkovic, and David Montana; *University of Maryland, College Park*, with team members Ravi Boppana, Eric Kuritzky, and David Phillips; and *Washington University, St. Louis*, with team members Kevin Keating, Nathan Schroeder, and David Williams.

Honorable mention was achieved by the following thirty-two individuals, named in alphabetical order: *Michael H. Albert*, University of Waterloo; *Michael Barall*, Princeton University; *David A. Barrington*, Amherst College; *Robert F. Chamberlain*, Rensselaer Polytechnic Institute; *Jeffrey W. Clark*, Yale University; *Daniel V. D'Eramo*, Massachusetts

Institute of Technology; *Branda Fahey*, Harvard University; *Daniel Freed*, Harvard University; *Paul Herdeg*, Case Western Reserve University; *Irwin Jungreis*, Cornell University; *Kevin Keating*, Washington University, St. Louis; *Daniel Knierim*, University of California, Davis; *Eric Kuritzky*, University of Maryland, College Park; *Geoffrey Mess*, University of Waterloo; *Renato Mirollo*, Columbia University; *David Montana*, Harvard University; *Jacob Nemchyonok*, Princeton University; *Ken Olum*, Stanford University; *Stephen Omohundro*, Stanford University; *Lawrence Penn*, Harvard University; *Lloyd Rawley*, Michigan State University; *Tim Redmond*, University of California, Santa Barbara; *Ehud Reiter*, Harvard University; *Niles D. Ritter*, University of Southern California; *Lorenzo Sadun*, Massachusetts Institute of Technology; *Nathan Schroeder*, Washington University, St. Louis; *Brian Sheppard*, Harvard University; *Patrick Smith*, McGill University; *Mark Spivakovsky*, Harvard University; *Gregory Taylor*, Rensselaer Polytechnic Institute; *William Titus*, Harvard University; *David S. Witte*, University of Wisconsin, Madison.

The other individuals who achieved ranks among the top 100, in alphabetical order of their schools, were: University of Arizona, *Michael A. Filaseta*; Beloit College, *Stephen J. Curran*; Brigham Young University, *Thomas C. Hales*, *Gary R. Lawlor*; Brown University, *Jeffrey E. Piazza*; California Institute of Technology, *Christopher S. Bretherton*, *Sekhar Chivukula*, *Lance J. Dixon*, *Vitaly Kupisk*, *John R. Stembridge*; University of California, Berkeley, *Lin Goldstein*; University of California, Davis, *Glen Y. Kishi*; University of California, Los Angeles, *Robert M. English*; University of California, Santa Barbara, *Dean C. Wills*; Carleton College, *Ross D. Willard*; Case Western Reserve University, *Wayne R. Dannels*, *Scott R. Fluhrer*; Columbia University, *Boris A. Datskovsky*; Drexel University, *Stephen P. Yankovich*; Harvard University, *Daniel L. Bloch*, *Paul Feit*, *Michael P. Mattis*, *Victor J. Milenkovic*, *Ron K. Unz*, *Robert L. Zako*; University of Illinois, *Christopher R. Eddy*; The Johns Hopkins University, *Marc A. Drexler*, *Christina U. Grot*; Massachusetts Institute of Technology, *Mike Roberts*; McGill University, *Satish Anjilvel*, *Marc T. Tessier-Lavigne*; Michigan State University, *Richard A. Ostrander*; University of North Carolina, *Peter N. Heller*, *Edward J. Rak*; Northwestern University, *Charles E. Bantz*; University of Pennsylvania, *Hal M. Switkay*; Princeton University, *Kenneth W. Regan*, *Joseph S. Weening*; Rensselaer Polytechnic Institute, *Christopher J. Keavney*; Rice University, *Robert S. Eberle*; St. Louis University, *Michael K. May*; Stanford University, *Richard Beigel*, *Robert J. Holt*; University of Toronto, *Nicholas A. Martin*, *Guy E. Moorhouse*; Vanderbilt University, *Hartwig P. Arenstorf*; University of Virginia, *Charles E. Gilbert*; Washington University, *Eric D. Mjolsness*, *Karl F. Narveson*; University of Waterloo, *William P. Hughes*, *Guy W. Hulbert*, *Duncan J. Murdoch*; University of Wisconsin, *Ronald C. Pruitt*.

There were 2141 individual contestants from 338 colleges and universities in Canada and the United States in the competition of December 1, 1979. Teams were entered by 258 institutions.

The Questions Committee of the fortieth competition consisted of L. A. Zalcman (Chairman), E. J. Barbeau, and K. B. Stolarsky; they proposed the problems listed below and were most prominent among those suggesting solutions.

PROBLEMS

Problem A-1

Find positive integers n and $a_1, a_2, \ldots, a_n$ such that

$$a_1+a_2+\cdots+a_n=1979$$

and the product $a_1a_2\cdots a_n$ is as large as possible.

Problem A-2

Establish necessary and sufficient conditions on the constant k for the existence of a continuous real valued function $f(x)$ satisfying $f(f(x))=kx^9$ for all real x.

Problem A-3

Let $x_1, x_2, x_3, \ldots$ be a sequence of nonzero real numbers satisfying

$$x_n = \frac{x_{n-2}x_{n-1}}{2x_{n-2}-x_{n-1}} \text{ for } n=3,4,5,\ldots.$$

Establish necessary and sufficient conditions on x_1 and x_2 for x_n to be an integer for infinitely many values of n.

Problem A-4

Let A be a set of $2n$ points in the plane, no three of which are collinear. Suppose that n of them are colored red and the remaining n blue. Prove or disprove: there are n closed straight line segments, no two with a point in common, such that the endpoints of each segment are points of A having different colors.

Problem A-5

Denote by $[x]$ the greatest integer less than or equal to x and by $S(x)$ the sequence $[x],[2x],[3x],\ldots$. Prove that there are distinct real solutions α and β of the equation $x^3-10x^2+29x-25=0$ such that infinitely many positive integers appear both in $S(\alpha)$ and in $S(\beta)$.

Problem A-6

Let $0 \leqslant p_i \leqslant 1$ for $i=1,2,\ldots,n$. Show that

$$\sum_{i=1}^{n} \frac{1}{|x-p_i|} \leqslant 8n\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}\right)$$

for some x satisfying $0 \leqslant x \leqslant 1$.

Problem B-1

Prove or disprove: there is at least one straight line normal to the graph of $y=\cosh x$ at a point $(a, \cosh a)$ and also normal to the graph of $y=\sinh x$ at a point $(c, \sinh c)$.

[At a point on a graph, the normal line is the perpendicular to the tangent at that point. Also, $\cosh x=(e^x+e^{-x})/2$ and $\sinh x=(e^x-e^{-x})/2$.]

Problem B-2

Let $0<a<b$. Evaluate

$$\lim_{t\to 0}\left\{\int_0^1 [bx+a(1-x)]^t dx\right\}^{1/t}.$$

[The final answer should not involve any operations other than addition, subtraction, multiplication, division, and exponentiation.]

Problem B-3

Let F be a finite field having an odd number m of elements. Let $p(x)$ be an irreducible (i.e., nonfactorable) polynomial over F of the form

$$x^2+bx+c, \qquad b, c \in F.$$

For how many elements k in F is $p(x)+k$ irreducible over F?

Problem B-4

(a) Find a solution that is not identically zero, of the homogeneous linear differential equation

$$(3x^2+x-1)y''-(9x^2+9x-2)y'+(18x+3)y=0.$$

Intelligent guessing of the form of a solution may be helpful.

(b) Let $y=f(x)$ be the solution of the *nonhomogeneous* differential equation

$$(3x^2+x-1)y''-(9x^2+9x-2)y'+(18x+3)y=6(6x+1)$$

that has $f(0)=1$ and $(f(-1)-2)(f(1)-6)=1$. Find integers a, b, c such that $(f(-2)-a)(f(2)-b)=c$.

Problem B-5

In the plane, let C be a closed convex set that contains (0,0) but no other point with integer coordinates. Suppose that $A(C)$, the area of C, is equally distributed among the four quadrants. Prove that $A(C) \leq 4$.

Problem B-6

For $k=1,2,\ldots,n$ let $z_k = x_k + iy_k$, where the x_k and y_k are real and $i=\sqrt{-1}$. Let r be the absolute value of the *real part* of

$$\pm\sqrt{z_1^2+z_2^2+\cdots+z_n^2}\ .$$

Prove that $r \leqslant |x_1|+|x_2|+\cdots+|x_n|$.

SOLUTIONS

In the 12-tuples $(n_{10}, n_9, \ldots, n_0, n_{-1})$ following each problem number below, the entry n_i for $10 \geqslant i \geqslant 0$ is the number of entrants, among the top 205, who achieved i points for the problem, and n_{-1} is the number of those not submitting a solution.

A-1. (102, 29, 10, 0, 0, 1, 0, 5, 5, 0, 34, 19)

We see that $n=660$ and that all but one of the a_i equal 3 and the exceptional a_i is a 2 as follows. No a_i can be greater than 4 since one could increase the product by replacing 5 by $2\cdot 3$, 6 by $3\cdot 3$, 7 by $3\cdot 4$, etc. There cannot be both a 2 and a 4 or three 2's among the a_i since $2\cdot 4<3\cdot 3$ and $2\cdot 2\cdot 2<3\cdot 3$. Also, there cannot be two 4's since $4\cdot 4<2\cdot 3\cdot 3$. Clearly, no a_i is a 1. Hence the a_i are 3's except possibly for a 4 or for a 2 or for two 2's. Since $1979=3\cdot 659+2$, the only exception is a 2 and $n=660$.

A-2. (49, 11, 6, 6, 0, 1, 0, 4, 11, 71, 28, 18)

The condition is $k \geqslant 0$. If $k \geqslant 0$, one sees that $f(x)=\sqrt[4]{kx^3}$ satisfies $f(f(x))=kx^9$. For the converse, we note that $f(f(x))=kx^9$ for all real x with $k \neq 0$ implies that f takes on all real values since kx^9 does and implies that f is one-to-one since $f(a)=f(b)$ leads to $ka^9=f(f(a))=kb^9$ and hence $a=b$. But a *continuous* one-to-one function f from the real numbers $\mathbb{R}$ onto itself must be strictly monotonic. Also, if f is monotonic, either always increasing or always decreasing, $f(f(x))$ will always be increasing and so cannot equal kx^9 if $k<0$.

A-3. (48, 3, 13, 9, 1, 2, 1, 22, 7, 13, 18, 68)

The condition will be seen to be that $x_1=x_2=m$ for some integer m. Let $r_n=1/x_n$. Then $r_n=(2x_{n-2}-x_{n-1})/x_{n-2}x_{n-1}=2r_{n-1}-r_{n-2}$ and the r_n form an arithmetic progression. If x_n is a nonzero integer when n is in an infinite set S, the r_n for n in S satisfy $-1 \leqslant r_n \leqslant 1$ and all but a finite number of the other r_n are also in this interval due to being nested among r_n with n in S; this can only happen if the r_n are all equal since the terms of an arithmetic progression are unbounded if the common difference $r_{n+1}-r_n$ is not zero. Equality of the r_n implies that $x_1=x_2=m$, an integer. Clearly, this condition is also sufficient.

Alternatively, let the r_n form the arithmetic progression defined above. If x_i and x_j are integers with $i \neq j$, then r_i and r_j and the common difference $(r_i-r_j)/(i-j)$ are rational. It follows that r_1 and r_2 are rational and hence that $r_1=a/q$ and $r_2=(a+d)/q$ with a, d, and q integers. Then $x_n=1/r_n=q/[a+(n-1)d]$. Since q has only a finite number of integral divisors, x_n can be an integer for an infinite set of n's only if $d=0$. This gives the same condition as in the first solution.

A-4. (27, 6, 5, 5, 0, 0, 0, 4, 11, 14, 42, 91)

There are a finite number (actually $n!$) of ways of pairing each of the red points with a blue

point in a 1-to-1 way. Hence, there exists a pairing for which the sum of the lengths of the segments joining paired points is minimal. We now show that for such a pairing no two of the n segments intersect.

Let red points R and R' be paired with B and B', respectively, and assume that segments RB and $R'B'$ intersect. The triangle inequality implies that the sum of the lengths of these segments exceeds the sum of the lengths of segments RB' and $R'B$. Then interchanging B and B' would give us a new pairing with a smaller sum of segment lengths. This contradiction proves the existence of a pairing with nonintersecting segments.

A-5. (5, 3, 3, 1, 0, 0, 4, 6, 58, 23, 35, 67)

Let $f(x)=x^3-10x^2+29x-25$. Then the table

x	1	2	3	5	6
$f(x)$	-5	1	-1	-5	5

shows that $f(x)=0$ has three real solutions a, b, c with $1<a<2, 2<b<3, 5<c<6$. The number of integers that the set $\{1,2,\dots, n\}$ has in common with $S(a)$, $S(b)$, and $S(c)$ is $[n/a]$, $[n/b]$, and $[n/c]$, respectively. Since

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}>\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1,$$

one sees that

$$\lim_{n\to\infty}\left\{\left[\frac{n}{a}\right]+\left[\frac{n}{b}\right]+\left[\frac{n}{c}\right]-n\right\}=\infty$$

and hence that an infinite number of positive integers appear in more than one of $S(a)$, $S(b)$, $S(c)$. This implies that some pair of these sets must have an infinite intersection.

A-6. (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 38, 167)

For $k=0,1,\dots,2n-1$ let I_k be the open interval $(k/2n,[k+1]/2n)$. Among the $2n$ intervals I_k there exist n not containing any of the p_i and we place an x_j at the center of each of these n intervals. Let $|x_j-p_i|=d_{ij}$ and

$$B=8n\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}\right).$$

For fixed i, the d_{ij} satisfy $d_{ij}\geqslant 1/4n$, at most two of them do not satisfy $d_{ij}\geqslant 3/4n$, at most four do not satisfy $d_{ij}\geqslant 5/4n$, etc. Hence

$$\sum_{j=1}^{n}\frac{1}{d_{ij}}\leqslant 2\sum_{h=0}^{n-1}\frac{4n}{1+2h}=B.$$

(This inequality can be improved.) Thus we have

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\frac{1}{d_{ij}}\right)=\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\frac{1}{d_{ij}}\right)\leqslant nB.$$

So clearly there is a value of j for which $\sum_{i=1}^{n}(1/d_{ij})\leqslant B$ and the x_j for such a j can serve as the desired x.

B-1. (71, 59, 13, 4, 1, 0, 0, 0, 6, 5, 30, 16)

We assume that there is such a common normal and obtain a contradiction. This assumption implies

$$-\frac{a-c}{\cosh a-\sinh c}=\cosh c=\sinh a. \tag{I}$$

Since $\cosh x > 0$ for all real x and $\sinh x > 0$ only for $x > 0$, (I) implies $a > 0$. Using the fact that $\sinh x < \cosh x$ for all x and (I), one obtains

$$\sinh c < \cosh c = \sinh a < \cosh a.$$

This, $a > 0$, and the fact that $\cosh x$ increases for $x > 0$ imply that $c < a$. Thus the leftmost expression in (I) is negative and cannot equal $\cosh c$. This contradiction shows that no common normal exists.

B-2. (89, 7, 4, 1, 0, 1, 7, 0, 0, 4, 68, 24)

Let $u = bx + a(1-x)$; then the definite integral becomes

$$I(t) = \frac{1}{b-a}\int_a^b u^t\,du = \frac{b^{t+1}-a^{t+1}}{(1+t)(b-a)}.$$

Using standard calculus methods for evaluating limits of indeterminate expressions, one finds that

$$[I(t)]^{1/t} \to e^{-1}(b^b/a^a)^{1/(b-a)} \text{ as } t \to 0.$$

B-3. (24, 14, 17, 12, 5, 4, 9, 6, 6, 6, 7, 95)

Let $r = (m-1)/2$. We show that $q(x) = p(x) + k$ is irreducible over F for r elements k of F. Since m is odd, the characteristic of F is not 2, $1+1=2\neq 0$, $2^{-1}b$ is an element h of F, the $2r+1$ elements of F can be expressed in the form $0, f_1, -f_1, \ldots, f_r, -f_r$, and $\{0, f_1^2, \ldots, f_r^2\}$ is the set of the $r+1$ distinct squares in F. Now

$$q(x) = (x+h)^2 - (h^2 - c - k)$$

is irreducible over F if and only if it has no zero in F, i.e., if and only if $h^2 - c - k$ is not one of the $r+1$ squares f^2 in F. Hence k must be one of the r elements left when the $r+1$ elements of the form $h^2 - c - f^2$ are removed from the $2r+1$ elements of F.

B-4. (21, 3, 0, 4, 0, 3, 0, 52, 37, 1, 7, 77)

(a) Trial of e^{mx} shows that $y = e^{3x}$ satisfies the homogeneous equation. Trial of a polynomial $x^d + \cdots$ shows that d must be 2 and then trial of $x^2 + px + q$ shows that $y = x^2 + x$ is a solution. Any linear combination $he^{3x} + k(x^2+x)$, with at least one of the constants h and k not zero, is an answer.

(b) It is easy to see that $y = 2$ satisfies the nonhomogeneous equation and hence $f(x)$ is of the form $2 + he^{3x} + k(x^2+x)$. Now $f(0) = 1$ gives us $2 + h = 1$ or $h = -1$. Then $[f(-1)-2][f(1)-6] = 1$ leads to

$$-e^{-3}(2 + 2k - e^3 - 6) = 1, \quad (2k-4)e^{-3} = 0, \quad k = 2.$$

Hence $f(x) = 2 - e^{3x} + 2(x^2+x)$, $f(-2) = 6 - e^{-6}$, $f(2) = 14 - e^6$. Therefore we let $a = 6$, $b = 14$, and $c = 1$.

We note that if one stops guessing after obtaining an answer $g(x)$ to (a), the standard substitutions $y = g(x)z$ followed by $z' = w$ will reduce the nonhomogeneous equation to a linear equation which can be solved by a well-known method.

B-5. (16, 7, 4, 3, 1, 4, 0, 1, 24, 15, 49, 81)

A support line for C is a straight line touching C such that one side of the line has no points of C. There is a support line containing $(0,1)$; let its slope be m. If $m \geqslant 1/2$, the part of the area of C in the fourth quadrant is no more than 1 and we are done. Similarly, if $m \leqslant -1/2$. So we assume that $-1/2 < m < 1/2$ and assume the analogous facts for support lines containing $(1,0)$, $(0,-1)$, and $(-1,0)$. At least one of the angles of the quadrilateral formed by these four support

lines is not acute; we may take this angle α to be at a vertex (h,k) in the first quadrant. Then $\alpha \geqslant \pi/2$ implies that $h+k \leqslant 2$, and this in turn implies that the area of C in the first quadrant does not exceed 1. Hence $A(C) \leqslant 4$.

B-6. (22, 4, 3, 0, 0, 0, 1, 0, 10, 4, 42, 119)

Let $X=(x_1,\ldots,x_n)$ and $Y=(y_1,\ldots,y_n)$. Also let $a+bi$ be either square root of $z_1^2+\cdots+z_n^2$. Then $ab=X\cdot Y=x_1y_1+\cdots+x_ny_n$ and

$$a^2-b^2=\|X\|^2-\|Y\|^2=(x_1^2+\cdots+x_n^2)-(y_1^2+\cdots+y_n^2).$$

The Cauchy–Schwarz inequality tells us that $|X\cdot Y| \leqslant \|X\|\cdot\|Y\|$ and hence $|a|\cdot|b| \leqslant \|X\|\cdot\|Y\|$. Therefore, the assumption that $|a|>\|X\|$ would imply that $|b|<\|Y\|$. This and $a^2=\|X\|^2-\|Y\|^2+b^2$ would yield $a^2<\|X\|$ and thus the contradiction $|a|<\|X\|$. Hence the assumption is false and $r=|a|\leqslant\|X\|$. Since $\|X\|^2\leqslant(|x_1|+\cdots+|x_n|)^2$, this implies the desired $r\leqslant|x_1|+\cdots+|x_n|$.

MATHEMATICAL NOTES

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HOW TO CUT A CAKE FAIRLY

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In this note we prove that a cake can be divided fairly among n people, although each may have a different opinion as to which parts of the cake are most valuable. It can be done even if "fair" means that all people must receive their first choices!

In a simpler version of the problem, a division is regarded as "fair" if all people ("players") are satisfied that each has received at least $1/n$ of the cake. For this version, there is a simple and practical solution, attributed by Steinhaus [1] to Banach and Knaster. Martin Gardner describes the case $n=3$ in his newest book [2]:

> "One person moves a large knife slowly over a cake. The cake may be any shape, but the knife must move so that the amount of cake on one side continuously increases from zero to the maximum amount. As soon as any one of the three believes that the knife is in a position to cut a first slice equal to $1/3$ of the cake, he/she shouts 'Cut!' The cut is made at that instant, and the person who shouted gets the piȩce. Since he/she is satisfied that he/she got $1/3$, he/she drops out of the cutting ritual. In case two or all three shout 'Cut!' simultaneously, the piece is given to any one of them.
>
> "The remaining two persons are, of course, satisfied that at least $2/3$ of the cake remain. The problem is thus reduced to the previous case . . .
>
> "This clearly generalizes to n persons."

Gardner then describes the more difficult version of the problem, in which a division is regarded as "fair" only if all players consider their own pieces to be at least as valuable as any of the others—essentially, all players get their first choices. The procedure described above doesn't always meet this test, because the player who claims the first piece may have a change of mind after seeing the remaining pieces. When $n=3$, we propose a new procedure to meet this objection:

A referee moves a sword from left to right over the cake, hypothetically dividing it into a small left piece and a large right piece. Each player holds a knife over what he considers to be the midpoint of the right piece. As the referee moves his sword, the players continually adjust their knives, always keeping them parallel to the sword (see Fig. 1). When any player shouts "cut," the cake is cut by the sword and by whichever of the players' knives happens to be the middle one of the three.

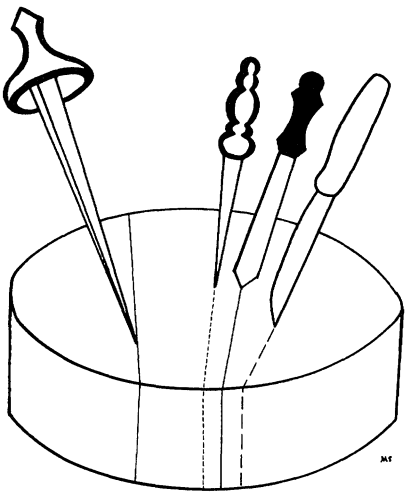

FIG. 1.

The player who shouted "cut" receives the left piece. He must be satisfied, because he knew what all three pieces would be when he said the word. Then the player whose knife ended nearest to the sword, if he didn't shout "cut," takes the center piece; and the player whose knife was farthest from the sword, if he didn't shout "cut," takes the right piece. The player whose knife was used to cut the cake, if he hasn't already taken the left piece, will be satisfied with whichever piece is left over. If ties must be broken—either because two or three players shout simultaneously or because two or three knives coincide—they may be broken arbitrarily.

This procedure does not generalize to larger n. John Selfridge, John Conway, and Richard Guy, in their research on the fair division of wine, have discovered a more elegant algorithm for $n=3$, but it, too, fails to generalize. In this note we shall be content with a nonconstructive existence theorem valid for all n.

One existence theorem, operating on quite different principles, has already appeared in this MONTHLY [3]. Dubins and Spanier (in an article with the same title as this note) assumed that each player's preferences are defined by a nonatomic measure over the cake. They proved that, given any finite number of measures (including those of the players and those of the kibitzers as well), there is a partition of the cake into n parts that are equal according to all of the measures. This was one of several results illustrating the power of Lyapunov's Theorem and other measure-theoretic techniques. Unfortunately, their result depends on a liberal definition of a "piece" of cake, in which the possible pieces form an entire σ-algebra of subsets. A player who hopes only for a modest interval of cake may be presented instead with a countable union of crumbs.

In this note we shall adhere more closely to the original model by imposing a rigid structure on the ways in which the cake may be cut. In particular, we shall insist that it be cut by $(n-1)$ planes, each parallel to a given plane. The possible cuts can then be represented by numbers in the interval $[0,1]$; and the possible divisions of the cake, by vectors $x=(x_1,x_2,\ldots,x_{n-1})$ such that $0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{n-1} \leqslant 1$. By convention let $x_0=0$ and $x_n=1$, so that the ith piece is the interval $[x_{i-1},x_i]$. The possible divisions form a compact set in $\mathbb{R}^{n-1}$, which we call the *division simplex*,

$$S=\{(x_1,\ldots,x_{n-1})|0 \leqslant x_1 \leqslant \cdots \leqslant x_{n-1} \leqslant 1\}.$$

See Fig. 2. S has the shape of an $(n-1)$-simplex with vertices at $v_1=(1,1,\ldots,1), v_2=(0,1,\ldots,1),\ldots,v_n=(0,0,\ldots,0)$. The vertex v_i represents the division in which the ith piece is the whole cake, and the face opposite v_i, which we shall call S_i, consists of divisions for which the ith piece is empty.

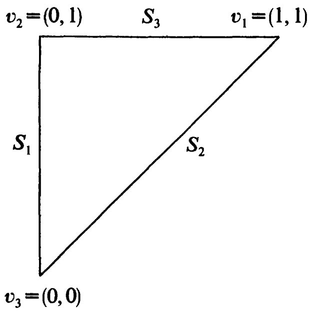

FIG. 2. When $n=3, S\subseteq R^2$.

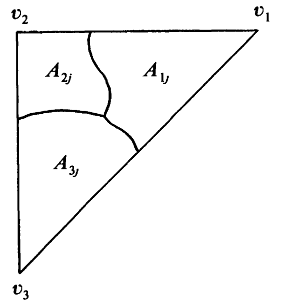

FIG. 3. Preferences of the jth player.

We allow great generality in the player's preferences. We assume that the choice for the jth player is based on a real-valued evaluation function f_j, which gives the value of the ith piece in terms of $x_1,\ldots,x_{n-1}$ (and i). Thus the value of the ith piece to the jth player is denoted by $f_j(x,i)$. Intuitively, one expects $f_j(x,i)$ to depend only on x_{i-1} and x_i, but the added generality comes at no extra cost.

For a given x, we say that player j *prefers* the ith piece if $f_j(x,i)\geqslant f_j(x,k)$ for all k. For some divisions a player may be indifferent to two or more "preferred" pieces. The division is fair if each player can be given a preferred piece.

We assume that each f_j is a continuous function of x. We must also assume that no player ever prefers an empty piece of cake.

THEOREM. *Under these assumptions, there is a division x and a way to assign the pieces to the players such that all players prefer their assigned pieces.*

Proof. For each i,j, let A_{ij} be the set of divisions $x\in S$ for which the jth player prefers the ith piece. From the continuity of the functions f_j we know that each A_{ij} is closed. For each j, the sets A_{ij} cover S. The assumption that no player prefers an empty piece implies that A_{ij} has empty intersection with the face S_i for each i,j. The sets A_{ij} provide all the information we need about the players' preferences, so we shall not refer again to the functions f_j. See Fig. 3.

Define $B_{ij}=\bigcap_{k\neq i}(S-A_{kj})$. Thus B_{ij} is the set of divisions for which the jth player prefers *only* the ith piece. Typically B_{ij} is the interior of A_{ij}, but that is not necessary. Each B_{ij} is open (relative to S). Note that, for a given j, the sets B_{ij} do *not* cover S; the uncovered part $(S-\bigcup_i B_{ij})$ consists of divisions for which the jth player is indifferent to two or more acceptable pieces.

Now define $U_i=\bigcup_j B_{ij}$. Thus U_i is the set of divisions for which *some* player prefers *only* the ith piece. Note that each U_i is open (as always, relative to S) and that U_i does not intersect S_i. We now divide the proof into two cases.

Usual Case: *The sets U_i cover S.* In this case we rely on a topological lemma.

LEMMA. *Suppose an $(n-1)$-simplex S is covered by n open sets $U_1,\ldots,U_n$, such that no U_i intersects the corresponding face S_i. Then the common intersection of $U_1,\ldots,U_n$ is nonempty.*

To see how this lemma proves the theorem (in the usual case), choose a division x in the common intersection of the U_i's. Since $x\in U_i$ for each i, every piece will be the unique acceptable piece for some player. Since there are exactly enough pieces to go around, all players can take their own first-choice pieces.

Proof of Lemma. We continue to regard S as a subset of the vector space $\mathbb{R}^{n-1}$, and use v_i and S_i as before. Write ∂S for the boundary of S. For each i, and for each $x \in S$, let $d_i(x)$ be the distance from x to the closed set $(S-U_i)$, and define $D(x)=\Sigma_i d_i(x)$. Since $x \in U_i$ for some i, some $d_i(x)$ is positive and so is $D(x)$. We may therefore define $f: S \to S$ by

$$f(x)=\sum_i \frac{d_i(x)}{D(x)} v_i.$$

The restriction of f to ∂S is a function $f_0: \partial S \to \partial S$ that takes each face S_i to itself. Hence we may define maps $f_t: \partial S \to \partial S$ by $f_t(x)=tx+(1-t)f(x)$. These maps define a homotopy between f_0 and the identity on ∂S. Therefore f_0 cannot be extended to a map from S to ∂S. In particular, since f is an extension of f_0, its image must intersect the interior of S.

Finally, if x is any point in f^{-1} (int S), then x is in each U_i.

Unusual Case: The sets U_i do not cover S. This case is unusual because it depends on a coincidence: if x is not in any U_i, then it is not in any B_{ij} for any j, so it must leave *every* player indifferent to two or more acceptable pieces. But this is not impossible. For example, if all players have identical preferences, the "coincidence" is certain to occur.

Our strategy in this case is to modify the players' preferences. We will approximate the sets A_{ij} by sets A'_{ij}, which are more orderly and for which the "coincidence" does not occur. By applying the lemma we shall find a division that would be fair, if the players' preferences were described by the sets A'_{ij}. As the approximations improve, these approximate solutions will converge to a division that is fair according to the actual preferences.

We start by choosing irrational numbers $\alpha_1, \ldots, \alpha_n$, one for each player, that are linearly independent over the rationals. We say that a number is *related* to α_j if its difference from α_j is rational. Numbers related to α_j are dense in $\mathbb{R}$, but no number can be related to both α_j and α_k if $j \neq k$.

Let M be a (large) integer.

For each j, construct A'_{ij} as follows. Divide S into cells by all planes of the form $\{x | x_k = (L/M)+\alpha_j\}$ for $k=1,\ldots,n$ and for all integers L. A cell, together with its boundary, is part of A'_{ij} if i is the smallest subscript for which A_{ij} intersects the cell. See Fig. 4.

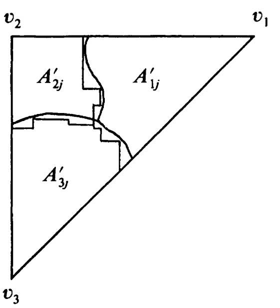

FIG. 4. Approximating the preferences.

The important properties of A'_{ij} are (1) every point on the boundary of A'_{ij} has some coordinate related to α_j, and (2) A'_{ij} approximates A_{ij} in the sense that every point A'_{ij} is within $\sqrt{n}/M$ of some point of A_{ij}.

Now for each i,j, define $B'_{ij} = \cap_{k \neq i}(S-A'_{kj})$—this *is* equal to the interior of A'_{ij}—and define $U'_i = \cup_j B'_{ij}$. As before, the sets U'_i are open and (if M has been chosen large enough) U'_i does not intersect S_i. To prove that the sets U'_i cover S, note that if x is not in any U'_i, it must not be in any B'_{ij}, so it must be on the boundary of some A'_{ij} for every j. That means that x must have a coordinate related to each of the α_j's. But this is impossible, because x has only $(n-1)$ coordinates, and each may be related to only one α_j.

Therefore, we may apply the lemma to find a point in the common intersection of the sets U_i'. We call it x_M. If the cake is divided according to x_M, the pieces can be assigned to the players in such a way that if the jth player receives the ith piece, then x_M is contained in A_{ij}' and is within $\sqrt{n}/M$ of A_{ij}.

As M increases, we can generate a sequence of divisions $\{x_M\}$. Since S is compact, we can find a subsequence that converges to some division $x \in S$; and by reducing to another subsequence if necessary, we can guarantee that the assignment of pieces to players is the same for each division x_M in the subsequence. Cut the cake according to x and assign the pieces to the players as for these x_M. Then if the jth player receives the ith piece, x must be arbitrarily close to A_{ij}. Since A_{ij} is closed, this implies that $x \in A_{ij}$, and the jth player prefers the assigned piece.

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3. L. E. Dubins and E. H. Spanier, How to cut a cake fairly, this MONTHLY, 68 (1961) 1–17.

NAPOLEON'S THEOREM AND THE PARALLELOGRAM INEQUALITY FOR AFFINE-REGULAR POLYGONS

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A well-known theorem, first proved in [2] but credited to Napoleon, reads as follows:

Construct equilateral triangles outwardly on the sides of any triangle. Their centers form the vertices of an equilateral triangle.

A lesser-known theorem of Thébault [11] (which is easily obtained as a corollary of Van Aubel's theorem [6], [1]) states:

Construct squares outwardly on the sides of any parallelogram. Their centers form the vertices of a square.

Clearly these theorems are related, and we may conjecture that they are part of a sequence of theorems leading from some m-gons to regular m-gons. In what sense, however, is a parallelogram, rather than the general quadrilateral, the successor of an arbitrary triangle? An answer to this question is provided by the following observations:

(a) Any triangle is the image of an equilateral triangle under an affine transformation.

(b) A quadrilateral is a parallelogram if and only if it is the affine image of a square.

These suggest the following result, which, in spite of its simplicity, appears to be new for $m \geqslant 5$, since even Thébault's result is not mentioned in survey articles [1], [7], [8], and [9]. Throughout the paper all subscripts will be taken modulo m.

THEOREM 1. *Let $\mathcal{P} = P_0P_1 \cdots P_{m-1}$ be a simple plane m-gon and construct regular m-gons on the sides of $\mathcal{P}$, one set outwardly and one set inwardly. Their centers form the vertices of m-gons $\mathcal{Q}$ and $\mathcal{Q}'$, and the centroids of $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{Q}'$ coincide. If $\mathcal{P}$ is the affine image of a regular m-gon, then*:

1. *The m-gons $\mathcal{Q}$ and $\mathcal{Q}'$ are regular.*
2. *The difference of the areas of $\mathcal{Q}$ and $\mathcal{Q}'$ is $4\cos^2(\pi/m)$ times the area of $\mathcal{P}$.*
3. *The sum of the squares of the edges of $\mathcal{Q}$ and $\mathcal{Q}'$ is $4\cos^2(\pi/m)$ times the sum of the squares of the edges of $\mathcal{P}$.*

Conversely, if $\mathcal{Q}$ (or $\mathcal{Q}'$) is regular, then $\mathcal{P}$ is affine-regular.

Proof. The coincidence of the centroids was shown in [**8**]. To prove the remaining results we shall represent points in the plane by complex numbers. If $\theta=2\pi/m$ then the vertices of a regular m-gon $\mathcal{R}$ are given by

$$R_k=\cos k\theta+i\sin k\theta,\qquad k=0,1,\dots,m-1.$$

Now for some constants a,b,c,d, the general affine transformation of the plane (modulo a translation) maps the point $x+iy$ into $(a+ic)x+(b+id)y$; so if $\mathcal{P}$ is affine-regular its vertices may be taken as

$$P_k=(a+ic)\cos k\theta+(b+id)\sin k\theta,\qquad k=0,1,\dots,m-1.$$

Let Q_k be the center of the regular m-gon on P_kP_{k+1} and let $M_k=\frac{1}{2}(P_k+P_{k+1})$ be the midpoint of P_kP_{k+1}. Then M_kQ_k is perpendicular to P_kP_{k+1} and $Q_k-\frac{1}{2}(P_k+P_{k+1})$ is a real multiple of $i(P_k-P_{k+1})$. Since the angle $M_kQ_kP_k$ is $\frac{1}{2}\theta$, it is easy to see that

$$Q_k=\tfrac{1}{2}(P_k+P_{k+1})+\tfrac{1}{2}i\cot\tfrac{1}{2}\theta(P_k-P_{k+1}).$$

The points Q_k are outside $\mathcal{P}$ if the affinity is direct and inside if it is opposite. The situation is reversed simply by interchanging the roles of P_k and P_{k+1} to get the vertices Q_k'. Thus

$$\begin{aligned}
\left(2\sin\tfrac{1}{2}\theta\right)Q_k&=\left(\sin\tfrac{1}{2}\theta+i\cos\tfrac{1}{2}\theta\right)P_k+\left(\sin\tfrac{1}{2}\theta-i\cos\tfrac{1}{2}\theta\right)P_{k+1}\\
&=\left(\sin\tfrac{1}{2}\theta+i\cos\tfrac{1}{2}\theta\right)[(a+ic)\cos k\theta+(b+id)\sin k\theta]\\
&\quad+\left(\sin\tfrac{1}{2}\theta-i\cos\tfrac{1}{2}\theta\right)[(a+ic)\cos(k+1)\theta+(b+id)\sin(k+1)\theta]\\
&=(a+ic)\left\{\sin\tfrac{1}{2}\theta[\cos k\theta+\cos(k+1)\theta]+i\cos\tfrac{1}{2}\theta[\cos k\theta-\cos(k+1)\theta]\right\}\\
&\quad+(b+id)\left\{\sin\tfrac{1}{2}\theta[\sin k\theta+\sin(k+1)\theta]+i\cos\tfrac{1}{2}\theta[\sin k\theta-\sin(k+1)\theta]\right\}\\
&=2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta\left\{(a+ic)\left[\cos\left(k+\tfrac{1}{2}\right)\theta+i\sin\left(k+\tfrac{1}{2}\right)\theta\right]\right.\\
&\quad\left.+(b+id)\left[\sin\left(k+\tfrac{1}{2}\right)\theta-i\cos\left(k+\tfrac{1}{2}\right)\theta\right]\right\}\\
&=2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta\left\{\left[(a+d)\cos\left(k+\tfrac{1}{2}\right)\theta+(b-c)\sin\left(k+\tfrac{1}{2}\right)\theta\right]\right.\\
&\quad\left.+i\left[(a+d)\sin\left(k+\tfrac{1}{2}\right)\theta-(b-c)\cos\left(k+\tfrac{1}{2}\right)\theta\right]\right\}.
\end{aligned}$$

If $a+d=0$ and $b-c=0$, then the vertices Q_k coincide at the origin which yields a trivial regular m-gon. Otherwise let

$$e=\left[(a+d)^2+(b-c)^2\right]^{1/2},\quad \cos\alpha=(a+d)/e,\quad \sin\alpha=(b-c)/e$$

to get

$$Q_k=e\cos\tfrac{1}{2}\theta\left[\cos\left(k\theta+\tfrac{1}{2}\theta-\alpha\right)+i\sin\left(k\theta+\tfrac{1}{2}\theta-\alpha\right)\right],\qquad k=0,\dots,m-1,$$

so the Q_k are the vertices of a regular m-gon of circumradius $e\cos\frac{1}{2}\theta$. Similarly, we find $\mathcal{Q}'$ to be regular with circumradius $e'\cos\frac{1}{2}\theta$ where

$$e'=\left[(a-d)^2+(b+c)^2\right]^{1/2}.$$

Now the area of a regular m-gon of circumradius r is fr^2 where $f=\frac{1}{2}m\sin\theta$; so the area of $\mathcal{R}$ is f and the area of its image $\mathcal{P}$ is $f(ad-bc)$. Also

$$\begin{aligned}
\operatorname{area}(\mathcal{Q})-\operatorname{area}(\mathcal{Q}')&=f\left[(e\cos\theta)^2-(e'\cos\theta)^2\right]\\
&=f\cos^2\theta\left[(a+d)^2+(b-c)^2-(a-d)^2-(b+c)^2\right]\\
&=4f\cos^2\theta(ad-bc),
\end{aligned}$$

which yields part 2. This was given for triangles in [**10**].

Direct computation yields

$$\sum_{k=m-1}^{m-1} |P_{k+1}-P_k|^2 = 2m\sin^2\tfrac{1}{2}\theta(a^2+b^2+c^2+d^2),$$

while it is clear that the edge-squares of $\mathfrak{Q}$ and $\mathfrak{Q}'$ are, respectively,

$$4\sin^2\tfrac{1}{2}\theta\cos^2\tfrac{1}{2}\theta\left[(a+d)^2+(b-c)^2\right] \quad\text{and}\quad 4\sin^2\tfrac{1}{2}\theta\cos^2\tfrac{1}{2}\theta\left[(a-d)^2+(b+c)^2\right],$$

so part 3 is immediate. This was given for triangles in [4].

The centers of the constructed regular m-gons in the statement are the vertices of isosceles triangles with vertex angle $2\pi/m$. In this form parts 1 and 3 are valid for affine-regular star m-gons of density d provided the isosceles triangles are given vertex angles $2\pi d/m$.

To prove the converse, let the vertices of $\mathfrak{Q}$ be given by $Q_k = Re^{ik\theta}$, $k=0,\ldots,m-1$. Then P_{k+1} is the image of P_k in a rotation of $-\theta$ about Q_k, which is equivalent to the product of the translation $-Q_k$, a rotation of $-\theta$ about the origin, and the translation Q_k. Thus

$$P_1 = (P_0 - Re^{i\theta})e^{-i\theta} + Re^{i\theta} = P_0e^{-i\theta} + R(e^{i\theta}-1),$$
$$P_2 = (P_1 - Re^{2i\theta})e^{-i\theta} + Re^{2i\theta} = P_0e^{-2i\theta} + R(e^{2i\theta} - e^{i\theta} + 1 - e^{-i\theta}),$$

and in general

$$\begin{aligned} P_k &= P_0e^{-ki\theta} + R(e^{ki\theta} - e^{(k-1)i\theta} + \cdots - e^{-(k-1)i\theta}) \\ &= P_0e^{-ki\theta} + R\frac{e^{ki\theta}-e^{-ki\theta}}{1+e^{-i\theta}} \\ &= P_0(\cos k\theta - i\sin k\theta) + R\left(\tan\tfrac{1}{2}\theta + i\right)\sin k\theta \\ &= P_0\cos k\theta + \left[R\tan\tfrac{1}{2}\theta + i(R-P_1)\right]\sin k\theta, \end{aligned}$$

which shows that $\mathfrak{P}$ is affine-regular.

It will certainly come as a surprise to the reader that the apparently weaker part 3 is also a sufficient condition for $\mathfrak{P}$ to be affine-regular. We shall establish this result in Theorem 3 but we shall first turn to an apparently unrelated subject.

The parallelogram law says: If $\mathfrak{P}$ is a parallelogram, e is the sum of the squares of its four edges, and f is the sum of the squares of its two diagonals, then $e=f$.

This equality, known to the ancient Greeks in a slightly different form, has become well known since 1935 when Jordan and von Neumann showed that a Banach space that satisfies it is a Hilbert space. However, the parallelogram law is but a special case of the theorem [5]:

The parallelogram inequality: If $\mathfrak{P}$ is a set of four points, not necessarily coplanar, labeled as if they were the vertices of a parallelogram, then $e \geq f$, and equality holds if and only if $\mathfrak{P}$ is the vertex set of a parallelogram.

In [3] we extended the parallelogram inequality to parallelepipeds as follows:

Let $\mathfrak{P}$ be a set of 2^n points, $n \geq 2$, in a Euclidean space of any number of dimensions, labeled as if they were the vertices of an n-dimensional parallelepiped. Let e be the sum of the squares of the $n2^{n-1}$ "edges," and f the sum of the squares of the 2^{n-1} "diagonals." Then $e \geq f$, and equality holds if and only if $\mathfrak{P}$ is the vertex set of an n-dimensional parallelepiped.

The result we shall need for our converse theorem is the following extension of the parallelogram inequality to affine-regular m-gons. (Note that if $m=4$ the left-hand side counts each diagonal twice.) Other such results, with metric hypotheses and affine conclusions, do not seem to exist in the literature.

THEOREM 2. *Let $\mathfrak{P} = P_0P_1\cdots P_{m-1}$, $m \geq 4$, be an m-gon in a Euclidean space of arbitrary*

dimension n. If at least two of the P_k are distinct, then

$$\sum_{k=0}^{m-1} |P_kP_{k+2}|^2 \leqslant 4\cos^2\frac{\pi}{m}\sum_{k=0}^{m-1}|P_kP_{k+1}|^2.$$

Equality holds if and only if $\mathcal{P}$ is a plane affine-regular m-gon.

Proof. Let the cartesian coordinates of P_k be $(x_k, y_k, \dots)$, $k=0,1,\dots,m-1$. Since $\sum_{k=0}^{m-1}|P_kP_{k+1}|^2=0$ implies that all the P_k coincide, we may assume $\sum_{k=0}^{m-1}|P_kP_{k+1}|^2=1$. The set of polygons satisfying this condition corresponds to a closed and bounded set in space of mn dimensions, so the continuous function $\sum_{k=0}^{m-1}|P_kP_{k+2}|^2$ attains its maximum. Proceeding by the method of Lagrange multipliers, we consider the function

$$f(\mathcal{P}) = \sum_{k=0}^{m-1} |P_kP_{k+2}|^2 - \lambda \sum_{k=0}^{m-1}|P_kP_{k+1}|^2.$$

For $k=0,1,\dots,m-1$,

$$\begin{aligned} 0=\tfrac{1}{2}\frac{\partial f}{\partial x_k} &= (x_k-x_{k-2})+(x_k-x_{k+2})-\lambda[(x_k-x_{k-1})+(x_k-x_{k+1})] \\ &= u_{k-1}+(2-\lambda)u_k+u_{k+1}, \end{aligned}$$

where $u_k = x_{k-1}-2x_k+u_{k+1}$, the symmetric second difference of the x_k. Differentiating with respect to the y_k, etc., produces similar systems.

This system of homogeneous linear equations will have a nontrivial solution if and only if the circulant determinant of its coefficient matrix vanishes. The factors of this determinant are easily seen to be

$$1+(2-\lambda)e^{ij\theta}+e^{2ij\theta}$$

so that

$$\lambda=(1+2e^{ij\theta}+e^{2ij\theta})/e^{ij\theta}=4\cos^2\tfrac{1}{2}j\theta, \qquad j=0,1,\dots,m-1.$$

Now $j=0$ corresponds to the solution $u_0=u_1=\cdots=u_{m-1}=0$, which implies the excluded case $x_0=x_1=\cdots=x_{m-1}$ (since x_0 follows x_{m-1}, a constant first difference implies a zero first difference). Since the value of λ is that of the maximum we seek, we choose the next largest value, viz., $j=1$, $\lambda=4\cos^2\frac{1}{2}\theta$. (The solutions for $j=d$, $d=2,\dots,[(m-1)/2]$ correspond to the maximum over star-polygons of density d.) In this case the solution is a linear combination of the real and imaginary parts of

$$\mathbf{u}=(u_0,u_1,\dots,u_{m-1})=(1,e^{i\theta},e^{2i\theta},\dots,e^{(m-1)i\theta}).$$

Since the difference of a geometric sequence is a geometric sequence with same ratio, it follows that $\mathbf{x}=(x_0,x_1,\dots,x_{m-1})$, after subtracting the same number from each coordinate, has the form $\mathbf{x}=\mathrm{Re}(x_0e^{i\alpha}\mathbf{u})$ where x_0 and α are real. If we let

$$\mathbf{c}=(1,\cos\theta,\dots,\cos(m-1)\theta) \quad\text{and}\quad \mathbf{s}=(0,\sin\theta,\dots,\sin(m-1)\theta),$$

then $\mathbf{x}=(x_0\cos\alpha)\mathbf{c}-(x_0\sin\alpha)\mathbf{s}$ and similarly for y, etc. This shows that the coordinate matrix of the vertices has rank two and that the vertices belong to an affine-regular m-gon.

We now return to the converse of Theorem 1.3.

THEOREM 3. *Let $\mathcal{P}$ be a simple plane m-gon, and let $\mathcal{Q}$ and $\mathcal{Q}'$ be as in Theorem 1. Let e be the sum of the squares of the edges of $\mathcal{Q}$ and $\mathcal{Q}'$, and let f be the sum of the squares of the edges of $\mathcal{P}$. Then $e \geqslant (4\cos^2(\pi/m))f$, and equality holds if and only if $\mathcal{P}$ is affine-regular.*

Proof. For $k=0,\dots,m-1$, let A_k be the interior angle of $\mathcal{P}$ at P_k and let p_k, q_k, and q_k' denote the lengths of $P_{k-1}P_k$, $Q_{k-1}Q_k$, and $Q_{k-1}'Q_k'$. It is easy to see that

$$\left(4\sin^2\tfrac{1}{2}\theta\right)q_k^2=p_k^2+p_{k+1}^2+2p_kp_{k+1}\cos(\theta-A_k)$$

and

$$\left(4\sin^2\tfrac{1}{2}\theta\right)q'^2_k=p_k^2+p_{k+1}^2+2p_kp_{k+1}\cos(\theta+A_k),$$

so

$$\begin{aligned}4\sin^2\tfrac{1}{2}\theta(q_k^2+q'^2_k)&=2(p_k^2+p_{k+1}^2)+(2p_kp_{k+1}\cos A_k)2\cos\theta\\&=2(p_k^2+p_{k+1}^2)+(p_k^2+p_{k+1}^2-|P_{k-1}P_{k+1}|^2)2\cos\theta\\&=4\cos^2\tfrac{1}{2}\theta(p_k^2+p_{k+1}^2)-2\cos\theta\,|P_{k-1}P_{k+1}|^2.\end{aligned}$$

Summing over k we get

$$\left(4\sin^2\tfrac{1}{2}\theta\right)e=\left(8\cos^2\tfrac{1}{2}\theta\right)f-2\cos\theta\sum_{k=0}^{m-1}|P_{k-1}P_{k+1}|^2. \tag{*}$$

By Theorem 2, $\Sigma_{k=0}^{m-1}|P_{k-1}P_{k+1}|^2\leqslant f$, with equality if and only if $\mathcal{P}$ is affine regular. Substituting, we get

$$\left(4\sin^2\tfrac{1}{2}\theta\right)e\geqslant 4\cos^2\tfrac{1}{2}\theta(2-2\cos\theta)f.$$

Since $4\sin^2\frac{1}{2}\theta=2-2\cos\theta$, we have the result.

Note that (*) is valid if regular m-gons are constructed on the sides of a polygon of any number of sides; this result is due to Glaisher [**4a**]. In particular, if we choose $m=4$ so $\cos\theta=0$, we have the following:

COROLLARY 4. *If squares are erected outwardly and inwardly on the sides of any simple plane polygon* $\mathcal{P}$ *and* e *and* f *are as in Theorem* 3, *then* $e=2f$.

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SOME PIGEONHOLE PRINCIPLE RESULTS EXTENDED

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A standard example of the use of the Pigeonhole Principle is the following. Suppose a player (e.g., tennis or chess) practices on d consecutive days, playing at least one game a day and a total of no more than b games where $d<b<2d$. The assertion is that for each $i\leqslant 2d-b-1$ there is a block of consecutive days on which, in total, exactly i games are played. Proofs of this fact can be found in [**1**, pp. 16, 22], [**2**], [3, p. 74], [**4**], or [**5**, p. 22]. A natural question is whether the

bound $2d-b-1$ is best possible, and more generally one asks for which integers $i>2d-b-1$ is there always a succession of days during which i games are played. We present a new result that answers some of these questions and includes the standard results. The result shows also that the bounds on b, $d<b<2d$, are reasonable since they include all interesting cases.

DEFINITIONS. Fix positive integers d and b $(d<b)$. If $\mathbf{g}=(g_1,g_2,\ldots,g_d)$ is a sequence of positive integers, $\mathbf{g}$ is said to be a *game sequence* and is called *admissible* if $\sum_{i=1}^{d} g_i \leqslant b$. The game sequence $\mathbf{g}$ is said to have *property* k, k a positive integer, if there are indices m and n such that $\sum_{i=m}^{n} g_i = k$.

Let A be the set of integers i such that every admissible game sequence has property i. The standard Pigeonhole Principle results assert that $\{1,2,\ldots,2d-b-1\}\subseteq A$, but in general more is true.

We now see why the lower bound on b, $d<b$, is reasonable: if $b<d$, there are no admissible game sequences and A is empty, and if $b=d$, $(1,1,\ldots,1)$ is the only admissible game sequence and $A=\{1,2,\ldots,d\}$.

Let $\mathbf{a}=(a_1,a_2,\ldots,a_d)$ be a strictly increasing sequence of positive integers. Then $\mathbf{a}$ is called a *sum sequence* and is *admissible* if $a_d\leqslant b$. A sum sequence is said to have *property* k if either an entry equals k or two entries differ by k. There is clearly a one-to-one correspondence between game sequences (admissible game sequences, game sequences with property k) and sum sequences (admissible sum sequences, sum sequences with property k, respectively); let $\mathbf{a}$ be the partial sums of a game sequence $\mathbf{g}$, or conversely let $\mathbf{g}$ be the consecutive differences of a sum sequence $\mathbf{a}$. We shall study the problem mainly from the point of view of sum sequences.

One important sum sequence $\mathbf{a}(k,d)$ is the sequence

$$(1,2,\ldots,k-1,2k,2k+1,\ldots,3k-1,4k,\ldots),$$

which corresponds to the game sequence

$$(1,1,\ldots,1,k+1,1,\ldots,1,k+1,\ldots)$$

with blocks of $k-1$ consecutive ones. Note that $\mathbf{a}(k,d)$ does not have property k and, as we shall see, it is the minimal such sequence. We can determine the entries of $\mathbf{a}(k,d)$ quite precisely; namely, if $d=mk+r$, $0\leqslant r<k$, then $\mathbf{a}(k,d)$ contains m subsequences of the form

$$(2qk+1,2qk+2,\ldots,(2q+1)k-1,(2q+2)k)$$

for $q=0,1,\ldots,m-1$, plus a final sequence

$$(2mk+1,2mk+2,\ldots,2mk+r).$$

Note that the last entry in $\mathbf{a}(k,d)$ is $2mk+r=2d-r$.

LEMMA 1. *If $\mathbf{a}'=(a_1',a_2',\ldots,a_d')$ is a sum sequence that does not have property k then $\mathbf{a}(k,d)\leqslant\mathbf{a}'$; i.e., if $\mathbf{a}(k,d)=(a_1,a_2,\ldots,a_d)$, then $a_i\leqslant a_i'$, $i=1,2,\ldots,d$.*

Proof. Suppose the lemma is not true and that i is the smallest index such that $a_i'<a_i$. Since both sequences are strictly increasing and $\mathbf{a}(k,d)$ increases only by unity from term to term, except for every kth term, it must be that $i=jk$ for some value $1\leqslant j\leqslant m$. Thus $a_i'=a_{jk}'\leqslant 2jk-1$ $=a_{jk}-1=a_i-1$ and the components $\{a_1',a_2',\ldots,a_{jk}'\}$ take values from the $2jk-2$ numbers in $S=\{1,2,\ldots,2jk-1\}-\{k\}$. We define a pairing of elements of S by pairing x and x^* if $x^*=x+k$ and $[x/k]$ is even. Note that $\{a_1',a_2',\ldots,a_{jk}'\}$ contains at most one element from each pair and that S contains elements from at most $jk-1$ pairs. By the Pigeonhole Principle we obtain a contradiction.

Thus $\mathbf{a}(k,d)$ is the smallest sum sequence without property k, and we use this fact to determine the elements of A.

THEOREM 2. *Given positive integers d,b $(d<b)$, and k, k is in A if and only if $2d-r>b$ where $d=mk+r$, $0\leqslant r<k$.*

Proof. If $\mathbf{a}(k,d)=(a_1,a_2,\dots,a_d)$, then $a_d=2d-r$. Suppose $k\notin A$. Then there is an admissible sum sequence, $\mathbf{a}'=(a_1',a_2',\dots,a_d')$, that does not have property k, and $2d-r=a_d\leqslant a_d'\leqslant b$ by Lemma 1. Conversely, if $2d-r=a_d\leqslant b$, then $\mathbf{a}(k,d)$ is an admissible sum sequence without property k, and $k\notin A$.

COROLLARY 3. *If $b\geqslant 2d$, A is empty.*

Proof. In this case we have $b\geqslant 2d\geqslant 2d-r$ for all nonnegative r.
Hence it is reasonable to assume $d<b<2d$.

Many facts follow directly from Theorem 2. The first part of the next theorem includes the standard results; the results of the second part have also been studied as they are equivalent to a generalization of a problem in [**1**, p. 23] or [**5**, p. 22].

THEOREM 4. *Let d and b be positive integers with $d<b<2d$.*
(i) $\{1,2,\dots,2d-b\}\subseteq A$.
(ii) $\{b-d+1,b-d+2,\dots,d\}\subseteq A$.
(iii) $\{d+1,d+2,\dots\}\subseteq A^c$, *the complement of A.*
(iv) $2d-b+1\in A^c$ *if and only if* $d\equiv b$ (modulo $2d-b+1$), *and thus the interval in* (i) *cannot be extended.*
(v) $b-d\in A^c$ *if and only if $b>3d/2$ and thus the interval in* (ii) *cannot be extended.*

Proof. If $i\leqslant 2d-b$, let $d=mi+r$, $0\leqslant r<i$. Then $2d-r\geqslant 2d-(i-1)\geqslant 2d-(2d-b-1)=b+1$. Thus $\{1,2,\dots,2d-b\}\subseteq A$.

If $b-d+1\leqslant i\leqslant d$, let $i=d-j$ where $0\leqslant j\leqslant 2d-b-1$. If $d=mi+r=m(d-j)+r$ with $0\leqslant r<i=d-j$, then $r=d-m(d-j)\leqslant d-(d-j)=j\leqslant 2d-b-1$, and $2d-r\geqslant 2d-(2d-b-1)=b+1$. Thus $\{b-d+1,\dots,d\}\subseteq A$.

If $i>d$, then $d=0\cdot i+d$ and $2d-r=d<b$ whence $\{d+1,\dots\}\subseteq A^c$.

Let $i=2d-b+1$ and $d=mi+r$, $0\leqslant r<i$. Then $2d-r>2d-i+1=b$ if and only if $r<i-1$. Thus $2d-b+1\in A^c$ if and only if $r=i-1$ or, equivalently, $d\equiv 2d-b$ (modulo $2d-b+1$) and the result of (iv) follows. This congruence is satisfied, for example, when d is odd and $b=(3d+1)/2$.

If $b>3d/2$, then $2d-b<b-d$ and $d=(b-d)+(2d-b)$ with $0\leqslant 2d-b<b-d$. Thus $2d-r=b$ and $b-d\in A^c$. If $b\leqslant 3d/2$, then $b-d\leqslant 2d-b$ and we find that the intervals of (i) and (ii) include all integers from 1 to d.

COROLLARY 5. $\{1,2,\dots,d\}\subseteq A$ *if and only if* $b\leqslant 3d/2$.

Thus the interval of most interest is $\{2d-b+1,2d-b+2,\dots,b-d\}$, and when this interval is nonempty (i.e., when $b>3d/2$) we denote it by A'. For any particular values of d and b Theorem 2 tells us which elements of A' are in A and which are in A^c; however, it would be nice to know more about A' in general. For example, does A' always intersect both A and A^c and, if so, what are the intersection patterns?

By Corollary 5 we know that, when $b>3d/2$, A' meets A^c. The next results show that A' also meets A except in a few cases, which we characterize.

THEOREM 6. *Let $b=2d-1$. Then $i\in A$ if and only if i divides d.*

Proof. Since $b=2d-1$, $2d-r>b$ if and only if $r=0$.

COROLLARY 7. *If d is prime and $b=2d-1$, then $A=\{1,d\}$.*

Now we sharpen the results of Theorem 4.

THEOREM 8. *Suppose $b>3d/2$, and let $n=[d/2]$. Then*
(i) $\{n+1,n+2,\dots,b-d\}\subseteq A^c$, *and*
(ii) $n\in A$ *unless $d=2n+1$ and $b=4n+1$.*

Proof. Suppose $d=2n+1$ and $b=3n+j$, $2\leqslant j\leqslant n+1$. Pick $n+i\in\{n+1,n+2,\dots,n+j-1$

$=b-d\}$, $1\leqslant i\leqslant j-1$. Since $d=2n+1=(n+i)+(n-i+1)$, $0\leqslant n-i+1<n+i$, $2d-r=3n+i+1\leqslant b=3n+j$. Suppose $d=2n$ and $b=3n+j$, $1\leqslant j\leqslant n-1$. Picking $n+i$, $1\leqslant i\leqslant j$, we proceed as before $d=(n+i)+(n-i)$, $0\leqslant n-i<n+i$, and $2d-r=3n+i\leqslant b$. Thus $\{n+1,n+2,\ldots,b-d\}\subseteq A^c$.

When $d=2n$, clearly $n\in A$. Suppose $d=2n+1$. Then $n\in A$ if and only if $4n+1>b$.

In summary we have the following intersection properties for A'.

COROLLARY 9. *If $b>3d/2$, then A' meets both A and A^c except in the following cases:*
(i) *if $d=2n+1$ and $b=3n+2$, then $A'=\{n+1\}\subseteq A^c$.*
(ii) *if d is prime and $b=2d-1$, then $A'=\{2,3,\ldots,d-1\}\subseteq A^c$.*

Note that $n\in A'$ unless we are in case (i) of the corollary.

Thus the results of Theorems 6 and 8 again narrow down the interval of interest to an initial segment of A', namely, $\{2d-b+1,\ldots,n-1\}$ (where $n=[d/2]$), in the case that $(3d+1)/2<b<2d-1$. We could narrow down this interval as well, for we are observing the following pattern of intersections of A and A^c with A': let $i_k=[d/k]$, $k=2,3,\ldots,l$, where l is the least integer such that $[d/l]\leqslant 2d-b$, and consider the subinterval of A', $S_k=\{i_k+1,i_k+2,\ldots,i_{k-1}\}$. For every $i\in S_k$ we have $d=(k-1)i+r$, $0\leqslant r<i$, and, as i increases in S_k, the remainder r strictly decreases since the quotient, $k-1$, remains fixed. Thus some initial segment of S_k belongs to A^c and the remaining final segment belongs to A. For example, $S_2=\{n+1,n+2,\ldots,d\}$ and has as initial segment in A^c either $\{n+1,\ldots,b-d\}$ or the empty set, depending upon the relative size of b and $3d/2$. One could calculate the dividing point of S_k for larger values of k, but such general results begin to have diminishing value.

We close with one unsettled question. Is there, in general, a simple characterization of the least integer in A^c?

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NOTE ON A RESULT OF NIVEN ON IMPOSSIBLE TESSELLATIONS

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Recently, I. Niven [1] proved the following interesting result:

THEOREM. (a) *If α and β are any positive real numbers, it is impossible to tile the plane with any collection of strictly convex polygons each of which has 7 or more sides, area greater than α, and perimeter less than β.*

(b) *If any one of the conditions on the polygons is removed, then it is possible to tile the plane with polygons meeting the remaining requirements.*

It was pointed out that there is no congruence requirement whatsoever on the polygons and that the theorem is not restricted to tilings that are edge-to-edge. An edge-to-edge tiling of the plane by polygons is one in which every two polygonal tiles have (i) no point in common, or (ii) exactly one point in common, which is a vertex of each of the two polygonal tiles, or (iii) a line segment in common that is a complete edge (or side) of each of the tiles. Also, to avoid using the Isoperimetric Theorem (which is not so elementary), the following result was used:

LEMMA. *If β is any positive real number, a polygon with perimeter less than β has area less than β^2.*

In this note, we give a proof that avoids the angle considerations in Niven's argument and is also shorter. Instead of requiring that the perimeter be bounded, we require that the length of the longest chord (i.e., the diameter) of each polygon be less than β. This change of hypothesis results in a statement that is logically equivalent to Niven's Theorem, since putting an upper bound on the perimeter p of a convex polygon is equivalent to putting an upper bound on the diameter d of the polygon and vice versa. This follows from the elementary inequalities $2d<p<4d$.

Proof of Lemma (with "perimeter" replaced by "diameter"). The polygon fits inside a square of side β. It follows from the last statement that the perimeter of the polygon is less than that of the square, or $p<4d$. Also, the diameter of a polygon of perimeter p is smaller than a segment of length $p/2$, or $2d<p$.

Proof of part (a) *of Theorem (with "perimeter" replaced by "diameter").* Suppose that such a tessellation exists. On it, consider three concentric circles, C_1, C_2, and C_3, with radii $\gamma-\beta, \gamma$, and $\gamma+\beta$, respectively. Subsequently, we will choose γ sufficiently large to yield a contradiction.

Let Q denote the union of all polygons that lie completely within C_2. By the diameter condition, Q covers C_1. By the Lemma, the number f of polygons in Q satisfies

$$f \geqslant \pi(\gamma-\beta)^2/\beta^2. \tag{1}$$

Let $\mathfrak{F}$ be the set of all polygons that border on Q, not including those that intersect Q at only one vertex. By the diameter condition, these polygons all lie completely between C_1 and C_3. By the area condition, the number $|\mathfrak{F}|$ of polygons in $\mathfrak{F}$ must satisfy

$$|\mathfrak{F}| \leqslant \frac{\pi(\gamma+\beta)^2-\pi(\gamma-\beta)^2}{\alpha}. \tag{2}$$

Now consider Q as constituting a planar representation of a graph $\mathcal{G}$ and let v, f, and e denote its respective number of vertices, finite regions, and edges. Note that this use of f is consistent with (1). Since each polygon has 7 or more sides,

$$2e \geqslant 7f, \tag{3}$$

and, by (1),

$$e \geqslant 7\pi(\gamma-\beta)^2/2\beta^2. \tag{4}$$

By the convexity condition, each vertex of $\mathcal{G}$ lies on at least 3 edges of $\mathcal{G}$, except possibly for some on the boundary of Q. These exceptional vertices, if they exist, must lie on 2 edges of $\mathcal{G}$. Let B denote the set of them and let b be their number. Then,

$$2e \geqslant 3v-b. \tag{5}$$

Each vertex in B must belong to two adjacent polygons in $\mathfrak{F}$, whether the tessellations are edge-to-edge or not. On the other hand, each such pair of polygons corresponds to at most one vertex in B since the polygons in $\mathfrak{F}$ form a ring around Q. Hence, $b \leqslant |\mathfrak{F}|$ and then, by (2),

$$b \leqslant 4\pi\gamma\beta/\alpha. \tag{6}$$

By Euler's formula,

$$1=v+f-e.$$

Then, from (3) and (5),

$$1\leqslant(2e+b)/3+2e/7-e=b/3-e/21,$$

and then, from (4) and (6),

$$1\leqslant 4\pi\gamma\beta/3\alpha-\pi(\gamma-\beta)^2/6\beta^2.$$

If γ is sufficiently large, we get a contradiction in the last inequality.

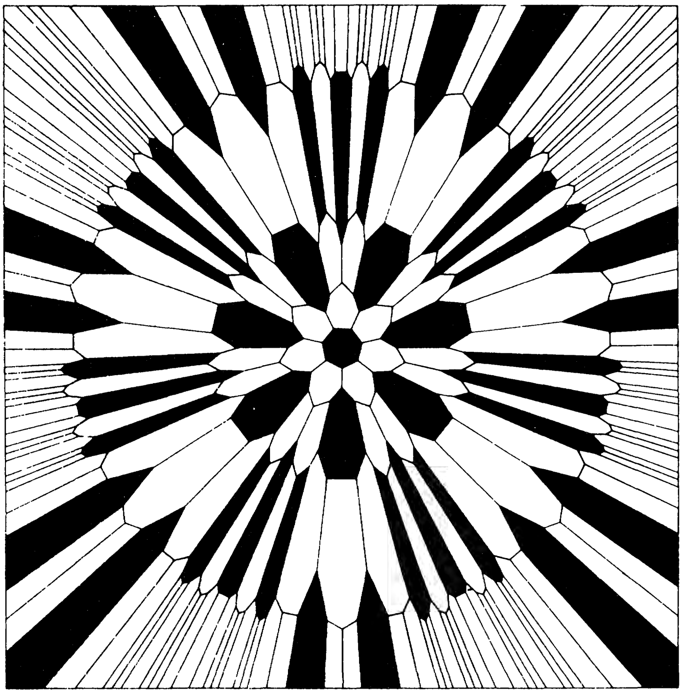

FIG. 1

Proof of part (b) *of Theorem (with "perimeter" replaced by "diameter").* The counterexamples in **[1]** also apply here. Finally, it is to be noted that Martin Gardner **[2]** published a diagram (see Fig. 1) that serves as a counterexample if both the area and the diameter conditions are removed.

Acknowledgment. We are grateful to Martin Gardner for permission to use his figure.

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UNSOLVED PROBLEMS

EDITED BY RICHARD GUY

In this department the MONTHLY *presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada T2N 1N4.*

LOOPS IN DUALS

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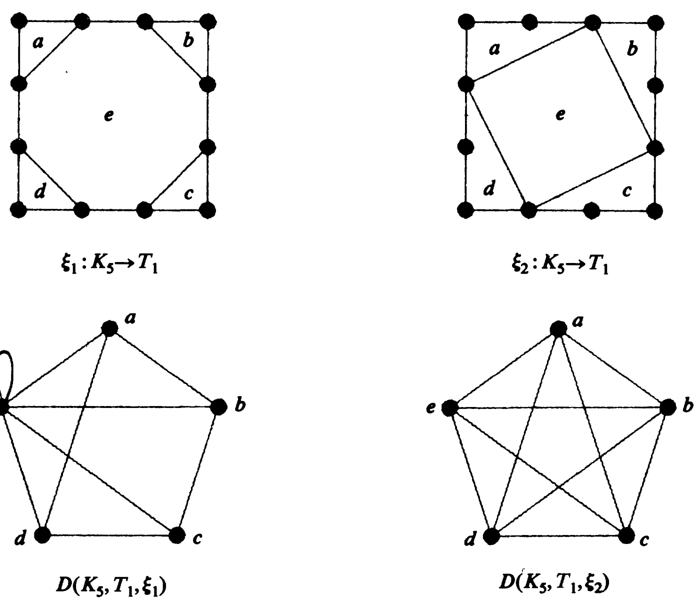
Let G be a graph imbedded in the surface S by the map ξ. Define the **dual graph** $D = D(G, S, \xi)$ to be the graph with vertex set

$$V(D) = \{r \mid r \text{ is a region of the imbedding } \xi\}$$

and edge set

$$E(D) = \{(v, w)_e \mid e \in E(G) \text{ and } v, w \text{ are the regions of } \xi(G) \text{ on either side of } \xi(e)\}.$$

Note that the subscripting allows for multiple edges in $E(D)$; also loops may occur in $E(D)$ corresponding to cases in which $v = w$. Fig. 1 illustrates this notion, which is the obvious generalization of planar dual.



$\xi_1 : K_5 \to T_1$ $\xi_2 : K_5 \to T_1$

$D(K_5, T_1, \xi_1)$ $D(K_5, T_1, \xi_2)$

Imbeddings of K_5 and Their Duals

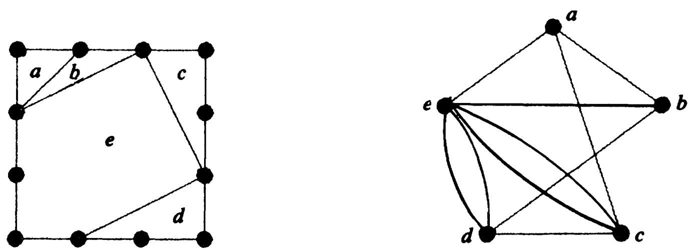
FIG. 1

For the torus T_1 the graph $D(K_5, T_1, \xi_1)$ has 2 loops and the graph $D(K_5, T_1, \xi_2)$ is isomorphic to K_5. In fact, K_5 has 6 nonisomorphic duals with respect to T_1 [3]. An **imbedding** of a graph G in a surface S (a compact 2-manifold without boundary) is a realization of the structure of G in S. The regions of such an imbedding are the connected components of $S - G$. An imbedding is a

disc imbedding if each of its regions is homeomorphic to the interior of a disc.

Question 1. Given a graph G and an integer n, does there exist a disc imbedding of G whose dual has n loops?

The imbedding of K_5 on T_1 pictured in Fig. 2 has a loop-free dual, but the dual has multiple edges. This example leads to a generalization of the first problem.



$\xi_3 : K_5 \to T_1$ $D(K_5, T_1, \xi_3)$

Loop Free Dual of K_5

FIG. 2

Question 2. Given a graph G and integers m and n, does there exist a disc imbedding of G whose dual has m loops and n multiple edges?

In the following we restrict attention to 2-connected graphs.

By T_n for $n \geqslant 0$ we will denote the orientable surface of genus n. By P_n for $n \geqslant 1$ we will denote the nonorientable surface of genus n. The **orientable genus** of a graph G, denoted by $\gamma(G)$, is the least n such that G can be imbedded on T_n. The **nonorientable genus** of a graph G, denoted by $\tilde{\gamma}(G)$, is the least integer n such that G can be imbedded on P_n. Youngs **[8]** has shown that:

THEOREM. *Every imbedding of a graph on its orientable (nonorientable) genus surface is a disc imbedding.*

The orientable and nonorientable genus of several infinite families of graphs has been determined—most notably, the complete graphs and the complete bipartite graphs. The solution of these and many other imbedding problems are discussed in **[6]**.

Surprisingly little, however, is known about how graph-theoretic properties are reflected in imbeddings. A notable exception is the well-known result that 3-connected planar graphs have unique imbeddings in the plane. Even in this case the face pattern of the imbedding is usually not known until after an imbedding is completed. Except for graphs which triangulate a surface or decompose a surface in a fairly regular way, very little can be said about what the boundaries of the regions of an imbedding will look like. But we do at least know how many regions a graph will determine for any disc imbedding on a particular surface. If G is disc imbedded on a surface S determining $R(G,S)$ regions on S, then

$$R(G,S) = \beta(G) - \delta(G) + 1$$

where

$$\beta(G) = |E(G)| - |V(G)| + 1$$

is the first Betti number of G and either $\delta(S) = 2\gamma(S)$ if S is orientable or $\delta(S) = \tilde{\gamma}(S)$ if S is nonorientable. The **maximum orientable genus** of G is

$$\gamma_m(G) = \max\{n \mid G \text{ can be disc imbedded on } T_n\}$$

and the **maximum nonorientable genus** $\tilde{\gamma}_m(G)$ is defined similarly. It follows from the result for $R(G,S)$ that

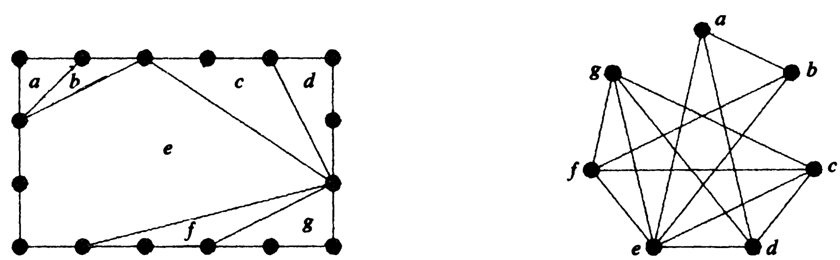
$$\gamma_m(G) \leqslant [\beta(G)/2] \text{ and } \tilde{\gamma}_m(G) \leqslant \beta(G).$$

A graph for which

$$\gamma_m(G) = [\beta(G)/2]$$

is called upper imbeddable [5]. In 1965 Edmonds [1] proved that every graph G has a disc-imbedding with a single region but that unless $\beta(G)$ is odd or equal to 2 his proof did not determine whether the surface involved was orientable or nonorientable. This result answers Question 1 affirmatively for $n=0$. Edmonds also asked whether a structural characterization of graphs with disc imbeddings with a single region on an orientable surface is possible. Recently, Jungerman [4] answered this question.

In 1946 Tutte [7] generalized some results of Tait which relate imbeddings to colorings of graphs. Tutte proved that if a cubic graph has a Tait coloring (edge 3-coloring) then this graph has a disc-imbedding on a connected surface in which the boundary of each region is a cycle of the graph. Tutte also proved that any connected bicubical graph has such an imbedding. These results of Tutte are solutions to special cases of Questions 1 and 2. Tutte actually proved more than is required by Questions 1 and 2—even if not more than is hoped for—as imbeddings exist which have duals that contain neither loops nor multiple edges and still do not have each region of the imbedding bounded by a cycle of the graph. Fig. 3 is an example of such an imbedding. It is unknown whether or not every graph has an imbedding on some surface such that the boundary of each region of the imbedding is a cycle of the graph. Results in this direction can be found in [2].



$\xi_4 : G \to T_1$ $\qquad$ $D(G, T_1, \xi_4)$

Loop and Multiple Edge Free Dual

FIG. 3

This work was supported by a University of Maine Faculty Summer Research Grant.

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EDITED BY DEBORAH TEPPER HAIMO AND FRANKLIN TEPPER HAIMO

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FUNCTIONS WITH ZERO RIGHT DERIVATIVES ARE CONSTANT

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The purpose of this note is to establish the following elementary theorem.

1. *Theorem. Let f be a continuous real function on an interval I. If the derivative of f from the right exists and is* 0 *at each interior point of I, then f is constant on I.*

A place where this theorem can be useful is found on page 148 of W. Feller's advanced book on probability theory **[1]**. In establishing an integration-by-parts formula for Stieltjes integrals, Feller showed that the derivative from the right of a certain function is 0. His reader is instructed to complete the proof of the constancy of the function by verifying that the left derivative is 0 also. But this turns out to be much harder, due to the possibility of discontinuities in the integrator function of an integral. The theorem above shows that such a verification is unnecessary.

To prove Theorem 1, we shall use three lemmas. Two of these are weak versions of Rolle's Theorem, and the other is their "mean-value corollary."

2. LEMMA. *Let f be a continuous real function on a compact interval $[a,b]$, with $f(a)=f(b)=0$. If f has a right derivative, $f'_+(x)$, at every point x in the open interval (a,b), then there exists a point c in (a,b) at which $f'_+(c)\leqslant 0$.*

Proof. Assume $f'_+(x)$ exists for each x in (a,b). If f is constant, then the lemma is trivially true. So suppose f is not identically zero. Since f is continuous, f assumes either a positive maximum or a negative minimum value.

Case 1. If f has a positive maximum, then this occurs at some point c in (a,b). We find that $(f(x)-f(c))/(x-c)\leqslant 0$ when $c<x\leqslant b$, so taking the limit as $x\to c$ from the right yields $f'_+(c)\leqslant 0$.

Case 2. If f has a negative minimum, then this occurs at some point q in (a,b). Now look at $f'_+(x)$ when $a<x<q$. If $f'_+(x)\leqslant 0$ for all x in (a,q), then the lemma is true. If not, choose a point p in (a,q) such that $f'_+(p)>0$. Then for all x sufficiently near p on the right we have $(f(x)-f(p))/(x-p)>0$, from which we obtain $f(x)>f(p)$. Now let M denote the maximum value of f on $[p,q]$. Clearly $M>f(p)\geqslant f(q)$, so f assumes the value M at some point c interior to $[p,q]$. Now the argument in Case 1 shows that $f'_+(c)\leqslant 0$. This completes the proof.

3. LEMMA. *Let f be continuous on $[a,b]$, with $f(a)=f(b)=0$. If f has a right derivative at each point in (a,b), then there exists a point d in (a,b) at which $f'_+(d)\geqslant 0$.*

Proof. Apply Lemma 2 to $-f$.

4. COROLLARY. *Let f be continuous on $[a,b]$. If f has a right derivative at each point in (a,b), then there exist points c and d in (a,b) such that*

$$f'_+(c)\leqslant \frac{f(b)-f(a)}{b-a}\leqslant f'_+(d).$$

Proof. Apply Lemmas 2 and 3 to the function

$$g(x)=f(x)-f(a)-\frac{f(b)-f(a)}{b-a}(x-a).$$

Proof of Theorem 1. Take any points $a<b$ in the given interval I. Apply Corollary 4 to f on $[a,b]$. Since $f'_+(x)\equiv 0$ on (a,b), we find that $f(b)=f(a)$. This suffices to prove that f is constant.

Added in proof. It has come to the author's attention that Theorem 1 is a corollary of the more general result that, if any one of the four Dini derivates of f is zero on I except for a countable set, then f is constant on I [**2**, pp. 365, 382–383]. This seems not to be very well known, as witness the Feller reference below.

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COMPLETE SOLUTIONS OF LINEAR DIFFERENCE EQUATIONS

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1. Introduction. The solution of a linear homogeneous difference equation of order k

$$y_n=a_1y_{n-1}+a_2y_{n-2}+\cdots+a_ky_{n-k} \qquad (n\geqslant k) \tag{1}$$

with (complex) constant coefficients $a_1,\dots,a_k$ involves the characteristic polynomial

$$f(t)=t^k-a_1t^{k-1}-\cdots-a_{k-1}t-a_k, \tag{2}$$

which provides solutions $y_n=t_i^n$ for every one of the zeros $t_1,\dots,t_k$ of f. If the t_i are distinct, the standard Vandermonde-determinant argument shows that these k solutions are linearly independent and span the vector space of all solutions. The case of multiple zeros is more complicated, and most textbooks relegate all or part of it to exercises, or to the "it can be shown" category.

This note provides a simple method for dealing with the general case in full detail. (The same idea works on differential equations, as we show in Section 4.) The basic philosophy is the same as that in Henrici [**2**], but we completely by-pass the long string of well laid out, detailed sub-results that lead to the final conclusion. Instead, we simply use the well-known fact that a polynomial cannot have more zeros, counting multiplicities, than its degree. For example, if p,q,r are distinct, then the matrix

$$M=\begin{bmatrix}1 & p & p^2\\ 1 & q & q^2\\ 1 & r & r^2\end{bmatrix}$$

is nonsingular, because any column vector $v=[b_0,\ b_1,b_2]^T$ such that $Mv=0$ has the property that p,q,r all are zeros of $b_2x^2+b_1x+b_0$; hence $b_0=b_1=b_2=0$.

2. Finding Solutions. Since the values of $y_0,\dots,y_{k-1}$ can be chosen freely, a linearity argument shows that the set of all solutions of (1) is a vector space of dimension k.

Suppose f has distinct zeros $t_1,\dots,t_l$, of multiplicities $m_1,\dots,m_l$ $(m_i>0)$. Then

$$f(t_i)=0,\quad f'(t_i)=0,\dots,f^{m_i-1}(t_i)=0 \qquad (1\leqslant i\leqslant l). \tag{3}$$

For every t we have

$$t^n - a_1 t^{n-1} - \cdots - a_k t^{n-k} = t^{n-k} f(t). \tag{4}$$

Now differentiate both sides of (4) j times, where $0 \leqslant j \leqslant m_i - 1$, and set $t = t_i$; then by (3) the terms on the right vanish, and we have

$$(n)_j t_i^{n-j} - a_1 (n-1)_j t_i^{n-j-1} - \cdots - a_k (n-k)_j t_i^{n-k-j} = 0,$$

where we use the abbreviation $(x)_j = x(x-1) \cdot \cdots \cdot (x-j+1)$. Hence, we have the following set of solutions of (1):

$$y_n = (n)_j t_i^{n-j} \qquad (0 \leqslant j \leqslant m_i - 1; 1 \leqslant i \leqslant l). \tag{5}$$

These form a total of $\sum_{i=1}^{l} m_i = k$ solutions. In the next section we show that these are linearly independent. It will follow, then, that (5) is a basis of the solution space of (1).

3. Showing Linear Independence. Let M be the k-by-k matrix whose row vectors consist of the first k elements y_n $(0 \leqslant n \leqslant k-1)$ of the sequences (5). It suffices, then, to show that the rows of M are linearly independent. Instead, we prove the equivalent assertion that the columns of M are linearly independent.

Indeed, suppose $b_0, \ldots, b_{k-1}$ are any k numbers; form the polynomial

$$g(t) = b_0 + b_1 t + \cdots + b_{k-1} t^{k-1}, \tag{6}$$

and assume that

$$M[b_0, \ldots, b_{k-1}]^T = 0. \tag{7}$$

We show $g(t) \equiv 0$, which is equivalent to $b_0 = \cdots = b_{k-1} = 0$.

Consider a typical row of M, i.e., a sequence of elements (5), for fixed i,j, and with $0 \leqslant n \leqslant k-1$. Its inner product with $[b_0, \ldots, b_{k-1}]^T$ is then zero:

$$0 = \sum_{n=0}^{k-1} b_n (n)_j t_i^{n-j} = \frac{d^j}{dt^j} g(t)\Big|_{t=t_i} = g^{(j)}(t_i). \tag{8}$$

This holds for all j with $0 \leqslant j \leqslant m_i - 1$, and therefore t_i is a zero of g of multiplicity at least m_i. Since the t_i are distinct, g has at least $\sum m_i = k$ zeros but is of degree at most $k-1$ (see (6)); hence $g \equiv 0$. ■

4. Differential Equations. Consider the differential equation

$$L(y) \equiv y^{(k)} - a_1 y^{(k-1)} - \cdots - a_k y = 0 \qquad \left(y^{(i)} = \frac{d^i y}{dx^i} \right) \tag{9}$$

for (complex) constants $a_1, \ldots, a_k$; with characteristic polynomial (2). The distinct zeros are again $t_1, \ldots, t_l$, with respective multiplicities $m_1, \ldots, m_l$. Then for every t

$$L e^{tx} = e^{tx} f(t). \tag{10}$$

Differentiate both sides of (10) j times *with respect to* t (where $0 \leqslant j \leqslant m_i - 1$), and set $t = t_i$. Then the terms on the right vanish, and because L and d/dt commute, we have

$$L(x^j e^{t_i x}) = 0 \qquad (1 \leqslant i \leqslant l; 0 \leqslant j \leqslant m_i - 1).$$

The functions

$$x^j e^{t_i x} \qquad (1 \leqslant i \leqslant l; 0 \leqslant j \leqslant m_i - 1) \tag{11}$$

are therefore k solutions of (9). We show that they are linearly independent. Define the k-by-k matrix M as follows. For each of the k functions (11) calculate its derivatives of orders 0 through $k-1$, evaluated at $x = 0$, and place these into one row of M. It suffices to show that the rows of M are linearly independent. Since we have

$$\frac{d^n}{dx^n} x^j e^{t_i x}|_{x=0} = \binom{n}{j} j!\, t_i^{n-j} = (n)_j t_i^{n-j},$$

comparison with (5) shows that our matrix M is identical with that in Section 3; hence M is nonsingular.

Another elegant independence proof, for differential equations, appears in [1].

The matrix M is nothing new, of course: its determinant is $W(0)$, the Wronskian at 0. Recall that $W(x) \neq 0$ iff $W(0) \neq 0$, for all linear differential equations (with not necessarily constant coefficients).

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ANOTHER APPROACH TO RIEMANN-STIELTJES INTEGRALS

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We offer here an alternative definition of the Riemann-Stieltjes integral that has some advantages over the classical definition. Let F be a nondecreasing function on a closed interval $[a,b]$ and let f be a bounded function on $[a,b]$. There are two standard approaches to pre-measure-theoretic Stieltjes integrals, which we now describe; the technical terms will be discussed afterwards.

To obtain the Darboux-Stieltjes integral, for each partition P of $[a,b]$ one defines an upper sum $U_F(f,P)$ and a lower sum $L_F(f,P)$. The upper integral is

$$U_F(f) = \inf\{\, U_F(f,P) : P \text{ partitions } [a,b]\},$$

and the lower integral is

$$L_F(f) = \sup\{\, L_F(f,P) : P \text{ partitions } [a,b]\}.$$

If the upper and lower integrals agree, then f is said to be Darboux-Stieltjes integrable and the Darboux-Stieltjes integral is defined to be the common value:

$$\mathrm{DS}\int_a^b f\,dF = U_F(f) = L_F(f).$$

To obtain the Riemann-Stieltjes integral, infinitely many Riemann-Stieltjes sums $S_F(f,P)$ are defined for each partition P of $[a,b]$. Then f is Riemann-Stieltjes integrable provided a number r exists with the following property. To each $\epsilon > 0$ there corresponds a $\delta > 0$ such that

$$|S_F(f,P) - r| < \epsilon$$

for all Riemann-Stieltjes sums corresponding to partitions P having mesh less than δ. In this case, the Riemann-Stieltjes integral is the number r:

$$\mathrm{RS}\int_a^b f\,dF = r.$$

We now turn to the undefined notation $U_F(f,P)$, $L_F(f,P)$, $S_F(f,P)$ and the definition of "mesh." We deliberately omitted these because our approaches to the Darboux-Stieltjes and the Riemann-Stieltjes integrals are exactly as described above. The only difference will be in the definitions of these four objects. We need a good deal of notation. For $t \in (a,b]$, $F(t^-)$ will denote the left-hand limit of F at t: $F(t^-) = \lim_{x \to t^-} F(x)$. Likewise $F(t^+)$ signifies the right-hand limit at t for $t \in [a,b)$. Also we decree $F(a^-) = F(a)$ and $F(b^+) = F(b)$. A partition P is a finite

ordered subset of $[a,b]$ of the form

$$P=\{a=t_0<t_1<\cdots<t_n=b\}.$$

For a nonvoid subset S of $[a,b]$ we abbreviate

$$m(f,S)=\inf\{f(x):x\in S\} \quad \text{and} \quad M(f,S)=\sup\{f(x):x\in S\}.$$

Standard Definitions.

$$U_F(f,P)=\sum_{k=1}^{n} M(f,[t_{k-1},t_k])\cdot[F(t_k)-F(t_{k-1})],$$

$$L_F(f,P)=\sum_{k=1}^{n} m(f,[t_{k-1},t_k])\cdot[F(t_k)-F(t_{k-1})],$$

$$S_F(f,P)=\sum_{k=1}^{n} f(x_k)\cdot[F(t_k)-F(t_{k-1})] \quad \text{where } x_k\in[t_{k-1},t_k],$$

$$\operatorname{mesh}(P)=\max\{t_k-t_{k-1}:k=1,2,\ldots,n\}.$$

Our definitions below are a little more complicated, but they explicitly take into account the jump effects of F.

Recommended Definitions. Let

$$J_F(f,P)=\sum_{k=0}^{n} f(t_k)\cdot[F(t_k^+)-F(t_k^-)]$$

and define

$$U_F(f,P)=J_F(f,P)+\sum_{k=1}^{n} M(f,(t_{k-1},t_k))\cdot[F(t_k^-)-F(t_{k-1}^+)],$$

$$L_F(f,P)=J_F(f,P)+\sum_{k=1}^{n} m(f,(t_{k-1},t_k))\cdot[F(t_k^-)-F(t_{k-1}^+)],$$

$$S_F(f,P)=J_F(f,P)+\sum_{k=1}^{n} f(x_k)\cdot[F(t_k^-)-F(t_{k-1}^+)]$$

where $x_k\in(t_{k-1},t_k)$. Also define

$$F\text{–mesh}(P)=\max\{F(t_k^-)-F(t_{k-1}^+):k=1,2,\ldots,n\}.$$

Nothing is lost with these recommended definitions in the following sense. Recall that, in the usual theory, a Riemann-Stieltjes integrable function is Darboux-Stieltjes integrable and the integrals agree. In general, a function can be Darboux-Stieltjes integrable without being Riemann-Stieltjes integrable, but the definitions are equivalent if F is continuous. *If a function is Darboux-Stieltjes integrable in the usual sense, then it is Darboux-Stieltjes integrable and Riemann-Stieltjes integrable in the recommended sense and all three integrals agree.* In fact, the latter two definitions are equivalent for all F. If f is integrable in either sense, we will say f is *F-integrable*.

In the usual theory, simple and unfortunate examples appear early. In fact, f will not be integrable with respect to F if they share a point of discontinuity. For example, $\int_a^b f\,dF$ is meaningless if $f=F=J_u$ where

$$J_u(t)=\begin{cases}0 & \text{for } t<u\\ 1 & \text{for } t\geqslant u,\end{cases}$$

$a<u\leqslant b$. For reasons such as this, mathematicians abandon the Riemann-Stieltjes integral in favor of the Lebesgue-Stieltjes integral where the above integral has value 1. But Lebesgue-Stieltjes theory may not be accessible to undergraduates and other persons who run into these

notions in, for example, the study of probability. With the recommended definitions we have:

THEOREM. *Let* (u_n) *be a sequence in* $[a,b]$ *and let* (c_n) *be a summable sequence of positive numbers. Let* $F=\sum_{n=1}^{\infty}c_nJ_{u_n}$. *Then every bounded function* f *on* $[a,b]$ *is* F*-integrable and*

$$\int_a^b f\,dF=\sum_{n=1}^{\infty}c_nf(u_n).$$

The next result applies to any nondecreasing F.

THEOREM. *If* f *is piecewise continuous or bounded piecewise monotonic, then* f *is* F*-integrable.*

This theorem relies on the useful fact that *if* f *is* F*-integrable on* $[a,b]$ *and* $g(x)=f(x)$ *except for finitely many* x *in* $[a,b]$, *then* g *is* F*-integrable.* This result does not hold in the standard theory.

Most of the basic results in the standard theory carry over without difficulty or change. Integration by parts is a slightly delicate matter. In the standard theory, the formula

$$\int_a^b F_1\,dF_2+\int_a^b F_2\,dF_1=F_1(b)F_2(b)-F_1(a)F_2(a)$$

holds provided either of the integrals exists. This formula fails dramatically with $F_1=F_2=J_u$ where $a<u\leqslant b$. So of course the formula fails for Lebesgue-Stieltjes integrals too. Edwin Hewitt [1] gave the correct formulation for Lebesgue-Stieltjes integrals. His result specializes to the following, which can be proved without reference to the Lebesgue-Stieltjes theory.

THEOREM. *Suppose that* F_1 *and* F_2 *are nondecreasing functions on* $[a,b]$ *and define*

$$F_1^*(t)=\tfrac{1}{2}[F_1(t^-)+F_1(t^+)] \quad \text{and} \quad F_2^*(t)=\tfrac{1}{2}[F_2(t^-)+F_2(t^+)]$$

for all $t\in[a,b]$. *Then we have*

$$\int_a^b F_1^*\,dF_2+\int_a^b F_2^*\,dF_1=F_1(b)F_2(b)-F_1(a)F_2(a).$$

As mentioned before, it is not difficult to work out the proofs using the recommended definitions. The following lemma is needed twice: *If* $\delta>0$, *there exists a partition* P *such that* F-mesh $(P)<\delta$. A detailed presentation of this approach and extensions to Riemann-Stieltjes integrals on $(-\infty,\infty)$ appears in [2].

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MATHEMATICAL EDUCATION

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ADVANCED ORDINARY DIFFERENTIAL EQUATIONS IN AN ENGINEERING ENVIRONMENT: A COMPUTER-ASSISTED APPROACH

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1. Introduction. A deeper look at ordinary differential equations—one characterized by the

PROBLEMS AND SOLUTIONS

EDITED BY VLADIMIR DROBOT

Send all ***proposed*** *problems, in duplicate if possible, to Professor Vladimir Drobot, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053. Please include solutions, relevant references, etc.*

An asterisk (*) indicates that neither the proposer nor the editors supplied a solution.

Solutions *should be sent to the addresses given at the head of each problem set.*

A publishable solution must, above all, be correct. Given correctness, elegance and conciseness are preferred. The answer to the problem should appear right at the beginning. If your method yields a more general result, so much the better. If you discover that a MONTHLY *problem has already been solved in the literature, you should of course tell the editors; include a copy of the solution if you can.*

SOLUTIONS OF PROBLEMS DEDICATED TO EMORY P. STARKE

Matrix with Non-negative Entries

S 13 [1979, 392]. *Proposed by H. Kestelman, University College, London.*

A non-negative real matrix A with spectral radius 1 has the property that for some pair p,q, the p,q element of A^j tends to 0 as $j\to\infty$. Show that, for some r,s, the r,s element of A^j is 0 for all positive integers j.

Solution by Man Kam Kwong, Northern Illinois University, DeKalb, Illinois. Suppose the proposition is false, i.e., for every i,k there is a j with $e_i^TA^je_k>0$ (e_r is the column vector with rth component 1 and all others 0). It follows that for every integer $m, e_m^TA^ne_q\to 0$ as $n\to\infty$: for if s is chosen so that $e_p^TA^se_m>0$ we have

$$(e_p^TA^se_m)(e_m^TA^ne_q)\leqslant e_p^TA^{s+n}e_q\to 0$$

as $n\to\infty$. Similarly, $e_p^TA^ne_m\to 0$ as $n\to\infty$. From this we deduce that for every g,h, $e_g^TA^ne_h\to 0$ and so $v^TA^nv\to 0$ as $n\to\infty$ for every vector v. But v exists with $Av=\lambda v$, $v^Tv=1$ and $|\lambda|=1$; this implies $|v^TA^nv|=1$ for all n and completes the proof by contradiction.

The condition "with spectral radius 1" can easily be relaxed to "with spectral radius at least 1."

Also solved by A. A. Jagers (Netherlands), Marvin Marcus, and the proposer.

Beckenbach's Monotonic Integral Functional

S 15. [1979, 503]. *Proposed by Joel L. Brenner, Palo Alto, California.*

Let $f(t)>0$ for $a\leqslant t\leqslant b$ and u be a fixed real number. Show that the functional $[\int f^{s+u}/\int f^s]^{1/u}$ increases with s.

For $u=1$, this theorem is due to E. F. Beckenbach (this MONTHLY, 1950, p. 1).

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, Netherlands. (i) For $u>0$ we substitute $g=f^u$ and $s=tu$. Then

$$[\int f^{s+u}/\int f^s]^{1/u}=[\int g^{t+1}/\int g^t]^{1/u}.$$

The latter expression is clearly an increasing function by Beckenbach's result and the fact that $1/u>0$.

(ii) For $u<0$ the assertion can be proved by substitution of $u=-v$ and $s=t+v$. Since

$$[\int f^{s+u}/\int f^{s}]^{1/u}=[\int f^{t+v}/\int f^{t}]^{1/v},$$

the assertion follows from (i).

Also solved by Robert Breusch, F. S. Cater, Doug Hensley, A. A. Jagers (Netherlands), L. Kuipers (Switzerland), Amram Meir (Canada), Ingram Olkin, James Theiler, and the proposer.

ELEMENTARY PROBLEMS

Solutions of these Elementary Problems should be mailed in duplicate to Dr. J. L. Brenner, 10 Phillips Road, Palo Alto, CA 94303 (USA), by February 28, 1981. Please place the solver's name and mailing address on each (double-spaced) sheet. Include a self-addressed card or label (for acknowledgment).

E 2847. *Proposed by Emeric Deutsch, Polytechnic Institute of New York.*

For the $n\times n$ real matrix $A=[a_{ij}]$, define

$$\|A\|_\infty=\max_i \sum_{j=1}^{n} |a_{ij}|;\ \|A\|^2=\text{max proper value of } A^*A;\ |||A|||^2=\sum_{i,j=1}^{n} |a_{ij}|^2.$$

Set $\operatorname{Re}A=\frac{1}{2}(A+A^*)$. Find

$$\text{(i)}\quad \max_{A\geqslant 0, A\neq 0}\|A\|_\infty/\|\operatorname{Re}A\|;\quad \text{(ii)}\quad \max_{A\geqslant 0, A\neq 0}\|A\|_\infty/|||\operatorname{Re}A|||.$$

Here A^* denotes the transpose of A; $A\geqslant 0$ means A has no negative element.

E 2848. *Proposed by J. Fickett, Texas A & M University.*

Prove that the regular tetrahedron has minimum diameter among all tetrahedra that circumscribe a given sphere. (The diameter is the length of a longest edge.)

E 2849. *Proposed by Emeric Deutsch, Polytechnic Institute of New York.*

Let $f_1,\ldots,f_n$ be n real linearly independent square integrable functions defined on an interval I. Let $G=(g_{ij})$ denote their Gram matrix, i.e., $g_{ij}=\int f_i(x)f_j(x)dx$. Let Q denote the matrix obtained by interchanging simultaneously m pairs of rows of G $(2m\leqslant n)$. Show that the eigenvalues of Q are all real.

E 2850. *Proposed by Clark Kimberling, University of Evansville.*

Let $a_1=1$, $b_1=2$, $c_1=3$. Beginning with $n=1$ and continuing for $n\geqslant 2$, let $S(n)$ be the set of all a_i, b_i, c_i, for which $i\leqslant n$, and let

$$a_{n+1}=\text{least positive integer not in } S(n);$$
$$b_{n+1}=\text{least positive integer not in } S(n) \text{ and not } = a_{n+1};$$
$$c_{n+1}=a_{n+1}+b_{n+1}.$$

Let d_k be the increasing sequence of all n for which $b_n=a_n+2$. Prove:

(i) $\lim_{k\to\infty}d_k/k=6$;

(ii) If B is any integer, then $(d_k-6k)/2=B$ for infinitely many k.

E 2851. *Proposed by Peter Ungar, New York University.*

Suppose all three matrices $A, B, A+B$ have rank 1. Prove that either all the rows of A and B are multiples of one and the same row vector v or else all the columns of A and B are multiples of one and the same column vector w.

E 2852. *Proposed by Jan Mycielski, University of Colorado.*

For any positive integer n let $\omega(n)$ be the product of all positive integers k such that n is a power of k. Prove that

$$\prod_{n=1}^{\infty}\left(\frac{\omega(n)}{n+1}\right)^{1/n}=1.$$

SOLUTIONS OF ELEMENTARY PROBLEMS

Covering $V-\{0\}$ with Hyperplanes in F_q

E 2785 [1979, 592]. *Proposed by Stephen M. Gagola, Jr., Texas A & M University.*

A *flat* X in a vector space V over a field F is defined to be a coset of a maximal subspace of B. Assume that F is finite with q elements. If V has dimension n and $V-\{0\}$ is the union of m flats, prove that $m \geqslant n(q-1)$.

Solution by Hugh M. Edgar, San Jose State University. The elements $(\beta_1, \beta_2, \beta_3, \ldots, \beta_n)$ of V, in which all components are nonzero, require the use of $(V: V_1)-1=q-1$ flats coming from the maximal subspace $V_1=\{(0,\alpha_2,\alpha_3,\ldots,\alpha_n)|\forall\alpha_2\in F_q,\ldots,\forall\alpha_n\in F_q\}$. The same sort of argument applies to each of the remaining $(n-1)$ analogously defined maximal subspaces $V_2, V_3, V_4, \ldots, V_n$. (The lower bound obtained is sharp.)

O. P. Lossers (Netherlands) found proofs in R. E. Jamison, *Covering finite fields*, J. Combin. Theory Ser. A, 22 (1977) 253–266, and A. E. Brouwer and A. Schrijver, *The blocking number of an affine space*, ibid, 24 (1978) 251–253.

Also solved by Dalton E. Orr and the proposer.

The Two Consecutive Integers $2x^2-1$, $2x^2$

E 2786[1979, 592]. *Proposed by Walter Stromquist, George Washington University.*

The consecutive integers 31 and 32 have these properties: the larger one is twice a square, and the sum of the digits in both numbers is a square.

(a) How many pairs of consecutive integers have the same properties?
(b) Would there exist such a pair if we used base 3 instead of decimal notation?
(c) Does such a pair exist in any odd base other than 3?

Solution by Lorraine L. Foster, California State University, Northridge. I offer a slightly more general result: Express the positive integers (Z^+) in base $b>1$. Then: (a) if $b=2m$, $m\geqslant 1$, there are infinitely many pairs n, $n+1$, with the desired properties; (b) if $b\equiv 1 \pmod 4$ there are no such pairs. If $b\equiv 3 \pmod 4$ and $\mu(b)\neq 0$, there are no such pairs; (c) if $b=75$, there are infinitely many such pairs.

Let T_b denote the sum of the digits of n and $n+1$ in base b.

(a) Define $n+1=2m^2(2m)^{2k}$ where $k=(4m-2)r^2-1$, $r\in Z^+$. Clearly, $T_b=(4m-2)^2r^2$.

(b) Suppose $n+1$, n have the desired properties, $n+1=2m^2=\sum_{j=0}^{t} d_j b^j$, $0\leqslant d_j<b$, $d_t\neq 0$, $s=\sum_{j=0}^{t} d_j$. Suppose that $b\equiv 1 \pmod 4$. Clearly $n+1\equiv s \pmod 4$. Also $T_b\equiv 2s-1\equiv -1\not\equiv\square \pmod 4$, a contradiction. Hence suppose that $b\equiv 3 \pmod 4$, $\mu(b)\neq 0$. Define $s'=\sum_{j=0}^{t}(-1)^j d_j\equiv n+1 \pmod 4$. Clearly $2s\equiv 2s'\equiv 0 \pmod 4$. If $d_0>0$, we have a contradiction as above. If

$d_0=0$, then $b|2m^2$, $b|m$, and $n+1$ ends in an *even number* ($2v$, say) *of zero digits*. Hence $T_b=2s-1+2v(b-1)\equiv -1 \pmod 4$, again a contradiction.

(c) Let $n+1=2(15)^2(75)^{2k}$ where $k=((125+148r)^2-85)/148\in Z^+$, $r\geqslant 0$. Then $T_{75}=(125+148r)^2$.

D. L. Shell found the examples $(b,x)=(27,90)$, $(847,7007)$, $(867,1071)$, $(891,11187)$. The proposer noted the example $(54675,14580)$.

Also solved by Duane Broline, L. Kuipers (Switzerland), L. E. Mattics, Manfred Pyka, R. S. Stacey, and the proposer.

Solutions of $x=(\log x)^k$

E 2787 [1979, 592]. *Proposed by James V. Whittaker, University of British Columbia.*

Show that if $k\geqslant 3$, then the equation $(\log x)^k=x$ for $x\geqslant 1$ has just two solutions r_k and s_k, where $r_k\to e$ and $s_k\to\infty$ as $k\to\infty$.

Solution by Ron Adin (Haifa), J. D. Faires, Youngstown State University, A. A. Jagers Techniche Hogeschool Twente, Enschede, Netherlands, M. Josephy, Universidad de Costa Rica, O. G. Ruehr, Michigan Technological University, Hugh Edgar & Edgar Simons, San Jose State University, the University of Hartford Problems Group, and the University of Santa Clara Problems Ring. Let $t=\log x$ and consider the equivalent equation $1/k=\log t/t=F(t)$. Note that $F'=(1-\log t)/t^2$, so that F is monotone increasing for $t\in(0,e)$ and decreasing for $t>e$. Since $F(1)=0$, $F(e)=e^{-1}>1/3$, and $F\to 0$ as $t\to\infty$, there are exactly two solutions of the equation. As $k\to\infty$, $F(t)\to 0$, so that $t\to 1$ or $t\to\infty$, and thus $x\to e$ or $x\to\infty$.

Joseph Silverman (Harvard University) obtained the following generalization:

Let $f: R^+\to R$ be a continuous, monotone increasing function, such that $f(1)=0$ and $\lim f(x)=\infty$ $(x\to\infty)$.
(a) For all sufficiently large k, the equation $x=f^k(x)$ has at least one solution greater than 1.
(b) If r_k is the smallest such solution, then $\lim r_k=f^{-1}(1)$ $(k\to\infty)$.

Also solved by H. L. Abbott, K. F. Andersen (Canada), D. M. Bloom, W. Boucher, P. Bracken, R. Breusch, D. M. Broline, B. W. Brunson, R. R. Burnside (Scotland), R. C. Carson, F. S. Cater, C. A. DeCarlucci, T. P. Dence, M. J. Dixon, W. F. Dwyer, M. Eggen, T. H. Foregger, L. L. Foster, A. Frank, M. R. Gopal, G. Gripenberg (Finland), D. Hackler, R. J. Hall, D. T. Hoffman, O. Jorsboe (Denmark), H. Kappus (W. Germany), H. Kestelman (England), L. Kuipers (Switzerland), J. Leech, B. Margolis, N. A. Martin, J. Milcetich, R. B. Nelsen, A. Nijenhuis, T. O'Neil, D. E. Orr, G. M. Ortner, B. Rice, St. Olaf College Problems Group, C. W. Schelin, D. R. Schmitz, H. J. Schultz, S. B. Seidman, R. E. Shafer, J. C. Smith, W. Snow, J. Theiler, B. Ware, W. V. Webb, J. Wiener, D. A. Wood, R. H. Wright, P. J. Zwier, and the proposer.

Dense Sequences in $[0,1]$

E 2788 [1979, 592]. *Proposed by Kwang-Nan Chow and David Protas, California State University, Northridge.*

Let $\{u_n\}$ be any sequence of real numbers such that $\{u_n\}\to\infty$ and $\{\cos u_n\}$ converges. Does there always exist a real number c such that $\{\cos cu_n\}$ diverges?

The answer is yes. R. M. Adin and L. E. Mattics (with Kevin Brown) located this problem in W. Rudin's textbook, *Principles of Mathematical Analysis*, p. 334, where it is given as an exercise, with generous hints for solution. The solvers who did not use the hints provided by Rudin generally approached the problem by considering the set of x's for which the sequence $u_n x \pmod{2\pi}$ has more than one cluster point, and showed that this set is large in some sense. All of these results are properly contained in the following theorem due to J. F. Koksma: If $u_n\to+\infty$ then for almost all x the sequence $u_n x \pmod{2\pi}$ is dense in $[0,2\pi]$. The proof can be found in L. Kuipers and H. Neiderreiter, *Uniform Distribution of Sequences*, New York, 1974, p. 35.

Also solved by R. M. Adin (Israel), F. S. Cater, M. W. Ecker, N. Glick, G. Gripenberg (Finland), J. Leech, L. E. Mattics, A. Meir (Canada), H. Noland, J. Theiler, and E. Triesch (West Germany).

Triangles with Vertices at Roots of Unity ζ^k

E 2789 [1979, 592]. *Proposed by Doug Hensley, Texas A & M University.*

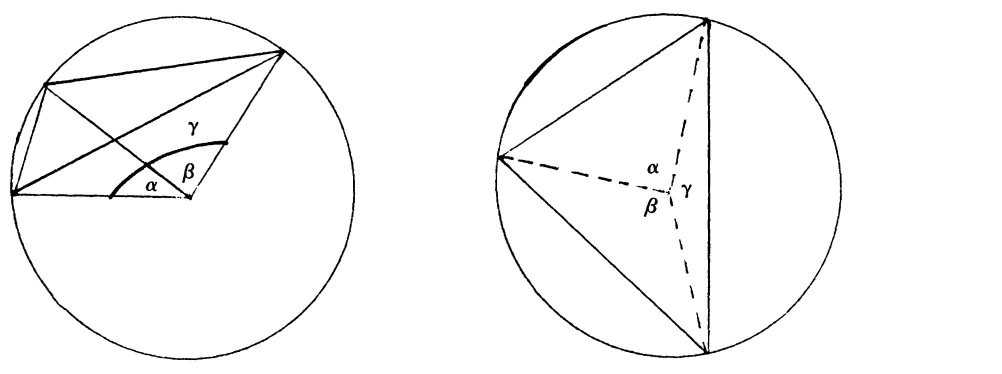
Suppose $\gcd(n,30)=1$ and $n\geqslant 13$. Let S_n be a set of n points equally spaced around a circle. Show that there are $(n^2-1)/12$ incongruent triangles with vertices in S_n. Show further that their areas are distinct when n is a prime.

Solution (*part* 1) *by Wayne Boucher, Montreal*; *D. M. Broline, University of Evansville*; *Michael Josephy, Universidad de Costa Rica*; *Jonathan Leech, Missouri Western State College*; *and L. E. Mattics, University of South Alabama* (*independently*). We show more generally that for any $n\geqslant 1$, the number $|T_n|$ of incongruent triangles is the integer nearest to $n^2/12$. This is clear for $0<n\leqslant 8$. We proceed to establish the relation (*) $|T_{n+6}|=|T_n|+n+3$. This will finish the proof, since $n^2/12+n+3=(n+6)^2/12$.

We take the n points of S_n to be $\exp 2\pi il/n$, $l=0,1,2,\ldots,n-1$. Let T_n be a maximal set of incongruent triangles with vertices in S_n, and let T_n' be the set of triples (k_1,k_2,k_3) of positive integers subject to the conditions $k_1+k_2+k_3=n$, $1\leqslant k_1\leqslant k_2\leqslant k_3$. There is a bijection between T_n and T_n'. Indeed, given a triangle with vertices $\exp 2\pi il_j/n$, $0\leqslant l_1<l_2\leqslant l_3\leqslant n$, set $k_1=l_2-l_1, k_2=l_3-l_2, k_3=n+l_1-l_3$; the triangle is included in T_n if $k_1\leqslant k_2\leqslant k_3$. Conversely, given a triple in T_n', take the triangle with vertices 1, $\exp 2\pi ik_1/n$, $\exp 2\pi i(k_1+k_2)/n$. Identify T_n, T_n' using this bijection.

To prove (*), count the triples $K=(k_1,k_2,k_3)$ in T_{n+6}'. If $k_1>2$, we make K correspond to (k_1-2,k_2-2,k_3-2) in T_n'. If $k_1=1$, we count directly the triples $(1,1,n+4)$, $(1,2,n+3)$, $(1,3,n+2),\ldots,(1,[\frac{1}{2}(n+5)],[\frac{1}{2}(n+6)])$: $[\frac{1}{2}(n+5)]$ triples. If $k_1=2$, there are (similarly) $[\frac{1}{2}(n+2)]$ triples. Since $[\frac{1}{2}(n+5)]+[\frac{1}{2}(n+2)]=n+3$, (*) is proved.

Solution (*part* 2) *by Jonathan Leech*. Let $n=p$, prime. Twice the area of a typical triangle is $\sin\alpha+\sin\beta\pm\sin\gamma$, where α, β, γ are among the $\frac{1}{2}(p-1)$ angles $\omega_k=2\pi k/p$, $k=1,2,\ldots,\frac{1}{2}(p-1)$. (See figure.)



It should be clear that, if the numbers $\sin\omega_k$ are linearly independent over the rationals, then no two different triangles can have the same area. Fact 1: Set $\zeta=\exp 2\pi i/p$. Then $\{\zeta^k\}_1^{p-1}$ are linearly independent over the rationals. (This amounts to the well-known fact that $1+\Sigma x^k$ is irreducible.) Fact 2: $2\pi i\sin\omega_k=\zeta^k-\zeta^{-k}$, and ζ^{-k} is a zero of the same irreducible polynomial. The assertion now follows immediately. Even more: if $n=p$, the areas of any three incongruent triangles are linearly independent over the rationals.

Essentially the same proof was given (part 2) by Duane Broline and by Michael Josephy. The latter noted that if $n=4p>8$, there exist $[\frac{1}{8}(n-4)]$ pairs of incongruent triangles with equal areas; if $n=24$ there is still another pair. If $4p\neq n<20$, there are no pairs.

Also solved by the proposer and (part 1) by Ron Adin (Haifa).

The Series Σa_n, Σa_n^3

E 2791 [1979,702]. *Proposed by John W. Vogel, Grinnell College.*

If the series of real numbers $\Sigma_{n=1}^{\infty} a_n$ converges, does $\Sigma_{n=1}^{\infty} a_n^3$ converge?

Solution by G. Pólya, Stanford University. The answer is: not necessarily. See my book Mathematical Discovery, vol. 2, pp. 49–50, Ex. 9.11. For analogous problems and a generalization, see G. Pólya and G. Szegö, Problems and Theorems in Analysis, vol. 1, p. 41, Ex. 185.1 and Ex. 185.2, also p. 78, Ex. 114.1.

Editorial notes. The problem already has appeared in this MONTHLY as 4142 [1944, 593; 1946, 283] in a considerably stronger form: If C is any subset of positive integers, there exists a sequence a_n so that Σa_n^{2l-1} converges for l in C and diverges for l not in C.

A result still more general than 4142 was found by V. Drobot: Given two points A and B in p-space, and odd integers $n_1 < n_2 < \cdots < n_p$, there exists a sequence a_k such that all the limit points of the partial sums of the series of vectors $\Sigma(a_k^{n_1},\ldots,a_k^{n_p})$ fill the line segment connecting A and B.

Leroy F. Meyers noted that if $|a_n|$ is eventually strictly monotonic, the answer is different. W. R. Emerson found necessary and sufficient conditions that $\Sigma f(a_n)$ converge whenever Σa_n converges: f is a constant times the identity function.

Also solved by 81 other readers, including the proposer.

ADVANCED PROBLEMS

Solutions of these Advanced Problems should be mailed in duplicate to Professor Roger C. Lyndon, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109 (USA), by February 28, 1981. The solver's full post-office address should be on each sheet.

6310. *Proposed by Seth Warner, Duke University.*

Is there a subfield K of the complex number field $\mathbb{C}$ such that $\mathbb{C}$ is two-dimensional over K and

(a) K is isomorphic to a proper subfield of the real number field $\mathbb{R}$?
(b) K is not isomorphic to any subfield of $\mathbb{R}$?

6311. *Proposed by Joseph Rotman, University of Illinois, Urbana.*

Let R be a commutative ring and let S be a multiplicatively closed subset of R. If E is an injective R-module, is the localization $S^{-1}E$ an injective $S^{-1}R$-module?

6312*. *Proposed by M. S. Klamkin, University of Alberta.*

Prove or disprove that the set of n equations in n unknowns

$$x_1^{l_i} + x_2^{l_i} + \cdots + x_n^{l_i} = 0 \quad (i = 1, 2, \ldots, n),$$

where the l_i's are relatively prime positive integers, has only the trivial solution $x_i = 0$ $(i = 1, 2, \ldots, n)$, if and only if each $m = 2, 3, \ldots, n$ divides at least one l_i.

6313. *Proposed by L. W. Tu, University of Michigan.*

Let $f(t) = \Sigma a_i t^i$ be a monic power series ($a_0 = 1$; $a_i = 0$ if $i < 0$). Let $\Delta_{pq}(f)$ be the $q \times q$ Hankel matrix with (i,j) element a_{p-i+j}. Prove that $\det \Delta_{pq}(1/f) = (-1)^{pq} \det \Delta_{qp}(f)$.

[Note that, with $q = 1$, this gives a determinantal formula for the coefficients of $1/f$. In the special case that $f(t)$ is the quotient of polynomials of respective degrees p, q, the result seems to be known.]

6314*. *Proposed by George Duncan and Joseph B. Kadane, Carnegie-Mellon University.*

Suppose X and Y are independent, nondegenerate symmetric random variables with respective centers of symmetry ξ,η. Prove or disprove the following: $W=XY$ is symmetrically distributed if and only if $\xi\eta=0$. (Under the additional hypothesis that X and Y have finite third moments, the statement is true.)

6315. *Proposed by Jan Mycielski, University of Colorado, and the editors.*

Let l be a prime power. Find the smallest integer $k=k(n,l)$ such that every element in the alternating group A_n can be expressed as a product of k lth powers.

SOLUTIONS OF ADVANCED PROBLEMS

Squares with Vertices in a Prescribed Set

6231 [1978, 686]. *Proposed by Terry R. McConnell, Stanford University.*

Let A be a subset of $\mathbf{R}^2$ with Lebesgue measure >0. Prove that A contains the vertices of a square.

Editor's note. T. A. Bick, Union College, observes that the problem is a special case of a result proved in his paper "Similar Configurations in Measurable Sets," this MONTHLY, 75 (1968) 31–34. He says: "After that note appeared, I was informed that its principal result had been obtained earlier by Professer H. Hadwiger, *Eine Erweiterung eines Theorems von Steinhaus-Rademacher*, Comm. Helv. 19 (1946) 236–239. Professor Hadwiger's proof is quite different from mine; so far as I know, he has prior claim to anyone else." Reference to Bick's paper was given also by Victor Pambuccian, student, University of Bucharest, Romania.

Also solved by John A. Baker, Charles L. Belna, Donald C. Benson, Ken Brown, F. S. Cater, Jim Essick, Gustaf Gripenberg (Finland), John M. Ingram, H. Kestelman (England), O. P. Lossers (Netherlands), Steven V. Noltie, N. T. Peck, Adam Riese, Arthur Rothstein, Rae Michael Shortt, Paul A. Vojta, Lawrence J. Wallen, and the proposer.

Composition of Functions

6244 [1978, 828]. *Proposed by John Myhill, State University of New York at Buffalo.*

Let f_i, $i=0,1,2,\ldots$, be a sequence of (everywhere defined) real functions. Prove that there exist two functions ϕ,ψ such that each of the f_i can be obtained from ϕ and ψ by composition.

Editor's Note. This problem has a rich history, but before turning to this we give one of the simplest solutions submitted.

Solution by Andrew Adler, University of British Columbia. Let J be a one-to-one map from $\mathbb{N}\times\mathbb{R}$ onto $\mathbb{R}$. Let $K: R\to R$ be defined by $Kx=J(0,x)$. Now every $u\in\mathbb{R}$ has the form $u=J(m,y)$ for unique $m\in\mathbb{N}$ and $y\in\mathbb{R}$. If $m=0$, then $u=J(0,y)=Ky=KJ(n,x)$ for unique $n\in\mathbb{N}$ and $x\in\mathbb{R}$. Define

$$Lu=\begin{cases} f_n(x) & \text{if } u=J(n+1,x)\\ J(n+1,x) & \text{if } u=J(0,y)=KJ(n,x).\end{cases}$$

It is easy to see that, for all $n\in\mathbb{N}$ and $x\in\mathbb{R}, L(LK)^{n+1}Kx=f_nx$.

Stanley Wagon points out that the result is contained in a theorem of Sierpiński [17], for which a simple proof was found by Banach [1]. For generalizations he refers to Sierpiński [18] and Łos [8].

Roy O. Davies (England) also gives the reference to Sierpiński [17], [16].

The proposer notes that, although the argument works for all infinite sets in place of $\mathbb{R}$, the semigroup of all mappings from a finite set E, of at least three elements, to itself cannot be generated by two elements. For it takes two permutations to generate the symmetric group on E, and another mapping to obtain the nonpermutations. Myhill also raises the question of what happens if the maps are required to be continuous, which is answered by the following discussion.

Comments by K. D. Magill, Jr., and S. Subbiah:

"The proposer did not require continuity. Indeed, if he had, the assertion would be false. We say more about this later. As it stands now, the proposer is asserting that any countable subsemigroup of the semigroup of all transformations on a set of cardinality c is contained in a subsemigroup with two generators. This is actually a corollary to considerably more general results already in the literature. Let X be a topological space and let $S(X)$ denote the semigroup, under composition, of all continuous functions mapping X into X. H. Cook and W. T. Ingram **[4]** and, independently, S. Subbiah **[19]** have shown that for a number of spaces X, every countable subsemigroup of $S(X)$ is contained in a subsemigroup generated by two elements. These spaces include all Euclidean N-cells, the space of rational numbers, the space of irrational numbers, the Cantor set, the Hilbert cube, and all *infinite discrete spaces*, whence the above problem follows.

"Some further remarks may be of interest. When X is locally compact and Hausdorff, $S(X)$ is a topological semigroup under the compact-open topology and if, in addition, X is second countable, then $S(X)$ is separable so that if X is any N-cell, the Cantor set, the Hilbert cube, or the countably infinite discrete space, the results of Cook, Ingram and Subbiah imply that $S(X)$ contains a dense (with respect to the compact-open topology) subsemigroup generated by two elements. We digress just a bit at this point to say something more about the separability of $S(X)$. It turns out that if X and Y are second countable and Y is regular, then the space of all continuous maps from X into Y with the compact-open topology is hereditarily separable. An extremely short and elegant proof of this fact has been given by E. A. Michael **[13]** and the proof deserves to be better known than it seems to be.

"Now we return to finitely generated dense subsemigroups of $S(X)$. Investigations along this line go back quite a few years. The first results of this type seem to be due to J. Schreier and S. Ulam **[20]** and W. Sierpiński **[15]** in 1934. Ulam apparently reported on such results in a Colloquium in 1936 which preceded his invitation to join the Harvard Society of Fellows **[20**, pp. 82, 83]. For a survey of some of these results one may consult Chapter 8 of **[9]**. Additional related results and problems are discussed in **[10]**."

Magill and Subbiah continue by showing that there exist four continuous maps from Euclidean n-space to itself which are not contained in any semigroup generated by three continuous functions. They conclude their discussion as follows.

"As we saw in the case of the topological semigroup $S(I^N)$, the fact that each countable subsemigroup is contained in one generated by two elements led immediately to the conclusion that $S(I^N)$ contains dense subsemigroups generated by two elements. In view of the proposition above, one cannot hope to prove in the same manner that $S(E^N)$ contains dense subsemigroups generated by two or even three elements. Nevertheless, it has been established (by necessarily different techniques) that $S(E^N)$ does contain a dense subsemigroup which is generated by three elements **[19]**. It is an open problem whether or not two elements will suffice. We conjecture that two elements will not suffice. We close with still another CONJECTURE: $S(E^N)$ contains countable subsemigroups which are not contained in any finitely generated subsemigroup."

George F. McNulty discusses connections and generalizations in a different direction, as follows.

"The problem posed by John Myhill is intimately related to several problems concerning universal terms (defined below) which were raised by Jan Mycielski about fifteen years ago. An explicit solution to problem 6244 can be found in **[12**, Corollary 2.6 and Example 2.8]. However, earlier solutions are implicit in the work of E. Post, M. Hall, J. Kalicki, and A. I. Mal'cev. **[12]** contains some remarks on these earlier papers.

"The language and notation of model theory is a convenient vehicle for the ideas that follow. We adhere to the usage of Chang and Keisler **[3]**. Fix L, a language whose only nonlogical symbols are g and h, two unary operation symbols. If σ is an L-term and $\mathfrak{A}$ is an L-structure, then $\sigma^{\mathfrak{A}}$ is the interpretation of σ in $\mathfrak{A}$ and it is a function from the universe of $\mathfrak{A}$ into itself. Let κ be a cardinal. A set Σ of L-terms is *jointly κ-universal* provided whenever

$$F:\Sigma\to{}^{\kappa}\kappa\,(\text{i.e., } F_\sigma:\kappa\to\kappa \text{ for all } \sigma\in\Sigma)$$

there exists an L-structure $\mathfrak{A}$ with universe κ such that $F_\sigma=\sigma^{\mathfrak{A}}$ for all $\sigma\in\Sigma$.

"To solve Myhill's problem it suffices to produce an infinite set of terms which is jointly 2^ω-universal. Actually, we will present three infinite sets of terms, each of which is jointly universal in every infinite cardinal. They are

$$\Gamma=\{\,ghgh^{i+3}gh^2x\colon i\in\omega\}$$

$$\Delta=\{\,ghg^{i+2}h^{i+2}x\colon i\in\omega\}$$

$$\Phi=\{\,g^2h^{i+2}ghx\colon i\in\omega\}.$$

Γ is gleaned essentially from Hall **[5]**, written in 1949; Δ comes from A. I. Mal'cev **[11]**, while Φ can be found in **[12]**. The verifications that these sets are jointly universal in every infinite cardinal are virtually the same. (All three

sets satisfy the subterm condition, cf. [12], of R. McKenzie, which generalizes a notion found in Isbell [7]).

"Let $\phi_0, \phi_1, \phi_2, \ldots$ be a one-to-one enumeration of any one of the sets above. Let κ be an infinite cardinal. For convenience we suppose that L is provided with exactly κ variables; then the set of L-terms has cardinality κ. Let A denote this set. Let f_i: $A \to A$ for all $i \in \omega$. We will define $\bar{g}, \bar{h} \in {}^A A$ and put $\mathfrak{A} = \langle A, \bar{g}, \bar{h} \rangle$ so that $f_i = \phi_i^{\mathfrak{A}}$ for all $i \in \omega$. Let $\sigma \in A$ define

$$\bar{h}(\sigma) = h\sigma$$

$$\bar{g}(\sigma) = \begin{cases} f_i(\tau) & \text{if } g\sigma = \phi_i(\tau) \\ g\sigma & \text{otherwise} \end{cases}.$$

The verification that the definition of $\bar{g}$ is sound presents no difficulty, but it does use the particular nature of Γ, Δ, or Φ. Let ψ be a proper subterm of ϕ_i. Then $\psi^{\mathfrak{A}}(\sigma) = \psi(\sigma)$ since the 'otherwise' case always prevails (again this is straightforward given the particular sets Γ, Δ, and Φ). But then $\phi_i^{\mathfrak{A}}(\sigma) = f_i(\sigma)$.

"The restriction that L involve only two operation symbols, both of which are unary, is not necessary for the development of these ideas. The concept of jointly κ-universal sets of terms can be quite naturally formulated for languages with any number of operation symbols of any finite rank. [12] includes an exposition of this notion."

W. Taylor presents a solution he attributes to McNulty, and states:

"Although this precise argument may not appear in the literature, one can easily locate a much more general result (i.e., McKenzie's subterm condition—see [12]). And the special case of considering only f_1 and f_2 (which really contains the general idea) appears explicitly in Burris [2])."

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Triods in the Plane

6250 [1979, 60]. *Proposed by Harold Shapiro, Kungl. Tekniska Hogskolan, Stockholm, Sweden.*

Let $w=f(z)$ be a continuous complex-valued function on the closed unit disc $|z|\leqslant 1$ which is one-to-one on the open disc $|z|<1$. Show that the set of boundary points of the image which have three or more distinct pre-images under the map f is at most countably infinite.

Solution by Charles L. Belna, Syracuse University. Let Ω denote the set of boundary points in question. Let $\omega_1,\omega_2,\omega_3$ be distinct pre-images of $\omega\in\Omega$, set $[\omega_j]=\{r\omega_j:\frac{1}{2}\leqslant r\leqslant 1\}$, and set $T_\omega=f(\cup_{j=1}^3[\omega_j])$. Since the restriction of f to the open disk $|Z|<1$ is necessarily a homeomorphism, we have $|\omega_j|=1$ $(j=1,2,3)$ and each T_ω forms a triod with $T_\omega\cap T_{\omega'}=\varnothing$ for distinct $\omega,\omega'\in\Omega$. The desired conclusion is now a consequence of R. L. Moore's theorem on triods in the plane ["Concerning Triods in the Plane and the Junction Points of Plane Continua," *Proc. Nat. Acad. Sci. U.S.A.*, 14 (1928), 85–88].

Also solved by James B. Essick, Gustaf Gripenberg (Finland), Martin Markl (Czechoslovakia), D. A. Overdijk (Netherlands), Jiří Veselý (Czechoslovakia), and the proposer.

An Inequality of Products

6254 [1979, 132]. *Proposed by Thomas E. Elsner, General Motors Institute, Flint, Michigan.*

For real numbers r_{ij} with $0\leqslant r_{ij}\leqslant 1$ for $i=1,2,\ldots,m$ and $j=1,2,\ldots,n$ prove that

$$1-\prod_{j=1}^{n}\left(1-\prod_{i=1}^{m}r_{ij}\right)\leqslant\prod_{i=1}^{m}\left[1-\prod_{j=1}^{n}(1-r_{ij})\right].$$

Editor's Note. Many solvers observed that the result follows directly from the set theoretic inclusion $\cup_j\cap_i A_{ij}\subseteq\cap_i\cup_j A_{ij}$. Hann Tzong Wang (Taiwan) and M. S. Klamkin noted that the inequality is stated by L. Carlitz in his solution (p. 404) of Problem 68-1, by J. C. Turner and V. Conway, *SIAM Review*, 11 (1969) 402–406. M. S. Klamkin also points out that further extensions are given by J. L. Brenner, "Some Inequalities from Switching Theory," *J. Combinatorial Theory*, 7 (1969) 197–205.

Also solved by Barry W. Brunson, Michael W. Chamberlain, Mihai Cipu (Rumania), L. E. Clarke (England), G. A. Heuer, Jay Hook, James Inglis, Robert B. Israel, John T. Kemper, Raymond Kennington (Australia), L. Kuipers (Switzerland), Albert W. Marshall, Edwin P. McCravy, Charles Riley, G. S. Rogers, James C. Smith, Doug Tyler, Peter Ungar, Stuart S-S. Wang, and the proposer.

Closed Graph Theorem

6255 [1979, 132]. *Proposed by Adam Riese, Wright State University, Dayton, Ohio.*

Let f: $R\to R$ be a function whose graph, considered as a subset of R^2, is both closed and connected. Prove f is continuous. What can be said when f: $R^m\to R^n$?

Solution by John Tripp, Southeast Missouri State University. We assume that the graph G of f is closed and show that discontinuity at a point a implies that G is disconnected.

We first note that if (x_n) is a sequence of real numbers such that $(x_n,f(x_n))\to(a,b)$ in the natural topology of the graph G, then we must have $b=f(a)$, since G is closed and $(a,f(a))$ is the only point in G with first coordinate equal to a. We next show that if (x_n) is a sequence such that $x_n\to a$, but such that the sequence $(f(x_n))$ does not converge to $f(a)$, then $(f(x_n))$ is unbounded. If $(f(x_n))$ does not converge to $f(a)$, then there is a number $\epsilon>0$ and a subsequence (x_{n_k}) such that $|f(x_{n_k})-f(a)|>\epsilon$ for all k. If the sequence $(f(x_{n_k}))$ were bounded, then it would have a convergent subsequence with limit not equal to $f(a)$. This is impossible by the above argument. So $(f(x_{n_k}))$ is unbounded, and so is $(f(x_n))$.

We assume that f has a discontinuity at a. Let $\delta>0$ be such that whenever $|x-a|\leqslant\delta$, we have either $|f(x)-f(a)|<1$ or $|f(x)-f(a)|>2|f(a)|+2$. If such a δ did not exist, then we could

find a sequence (x_n) with $x_n \to a$ such that $|f(x_n)-f(a)| \geqslant 1$ and $|f(x_n)| \leqslant 2|f(a)|+2$ for all n. Since the sequence $(f(x_n))$ would be bounded, it would have a subsequence, converging to a number other than $f(a)$. But this is impossible by the argument used in the previous paragraph.

We now define a real valued function F, defined on the set of points of G, the graph of f. We let $F(x,f(x)) = |f(x)|$ for $|x-a| \leqslant \delta$, where δ is as in the previous paragraph. We let $F(x,f(x)) = F(a-\delta, f(a-\delta))$ for $x \leqslant a-\delta$; and we let $F(x,f(x)) = F(a+\delta, f(a+\delta))$ for $x \geqslant a+\delta$. Clearly F is continuous. We consider the range of F. If $x \in [x-\delta, x+\delta]$ is such that $|f(x)-f(a)| < 1$, then $F(x,f(x)) = |f(x)| < |f(a)|+1$. If $x \in [x-\delta, x+\delta]$ is such that $|f(x)-f(a)| > 2|f(a)|+2$, then $F(x) = |f(x)| > |f(a)|+2$. So the range of F is contained in the set $(-\infty, |f(a)|+1) \cup (|f(a)|+2, \infty)$. The point $F(a,f(a)) = |f(a)|$ is in the interval $(-\infty, |f(a)|+1)$ and the unboundedness of f around a assures us that there are points of the image of F in the interval $(|f(a)|+2, \infty)$. So the image of G under F is disconnected. Since the image of a connected set under a continuous mapping is connected, it must follow that G is disconnected.

The previous proof can be easily adapted to the general case $f: R^m \to R^n$. We replace the absolute value by the usual Euclidean norms on R^m and R^n. We define the function F by $F(x,f(x)) = \|f(x)\|$ for $\|x-a\| \leqslant \delta$ and $F(x,f(x)) = F(y,f(y))$ where $y = a+\delta(x-a)/\|x-a\|$, for $\|x-a\| \geqslant \delta$.

Also solved by F. S. Cater, Boris Datskowsky, Frank Gibson, Man Kam Kwong, Martin Markl (Czechoslovakia), Charles Riley, Z. Z. Uoiea, Hann Tzong Wang (Taiwan), and the proposer.

Sets Formed by Iterated Closure, Interior, and Union

6260 [1979, 226]. *Proposed by Eric Langford, California Polytechnic State University, San Luis Obispo.*

If X is a subset of a topological space S, then it is known that there can be formed at most six new sets by repeated formations of closures and interiors iterated in any order. (This is related to the famous Kuratowski closure-and-complement problem; see E. Langford, *Characterization of Kuratowski 14-sets*, this MONTHLY 78 (1971) 362–367, for details.) It is also known that if we further allow the formation of unions then no more than six new sets can be generated for a maximum total of thirteen (Problem 5996 [1974, 1034], [1978, 283]). Given that we start with X and the additional six sets described in the first sentence, what is the *minimum* number of new sets that can occur when we further allow unions?

Solution by William Myers, Virginia Polytechnic Institute and State University. The minimum number of new sets is one. First I will give an example where there is only one new set; then I will show that this set can never be equal to one of the previous seven.

Example. Let $X = \cup_{n=1}^{\infty}(1/(n+1), 1/n) \cup \{r: r \text{ is rational and } (-1 \leqslant r \leqslant 0 \text{ or } 1 \leqslant r \leqslant 2)\} \cup \{3\}$. It is easy to check that the seven sets

$$X, X^i, \overline{X^i}, \overline{X^i}^{\,i}, \bar{X}, \bar{X}^i, \overline{\bar{X}^i} \tag{1}$$

together with the set $X \cup \overline{X^i}$ are distinct and that any union of two of these eight sets is again one of them.

THEOREM. *For any set X such that the seven sets* (1) *are distinct*, $X \cup X^i$ *is not equal to any of the previous seven.*

LEMMA 1. $X \cup \overline{X^i} \neq X$.

Proof. If $X \cup \overline{X^i} = X$, $\overline{X^i} \subseteq X$ and $\overline{X^i}^{\,i} \subseteq X^i \subseteq \overline{X^i}$. Since X^i is open, $X^i \subseteq \overline{X^i}^{\,i}$. Impossible, since $\overline{X^i}^{\,i} \neq X^i$.

LEMMA 2. $X \cup \overline{X^i} \neq X^i$.

Proof. If $X \cup \overline{X^i} = X^i$, $X \subseteq X^i \subseteq X$. Impossible, since $X \neq X^i$.

LEMMA 3. $X \cup \overline{X^i} \neq \overline{X^i}$.

Proof. If $X \cup \overline{X^i} = \overline{X^i}$, $X \subseteq \overline{X^i}$; thus, $\overline{X} \subseteq \overline{X^i}$. Since $X^i \subseteq X$, $\overline{X^i} \subseteq \overline{X}$. Impossible, since $\overline{X} \neq \overline{X^i}$.

LEMMA 4. $X \cup \overline{X^i} \neq \overline{X^i}^i$.

Proof. If $X \cup \overline{X^i} = \overline{X^i}^i$, $\overline{X^i} \subseteq \overline{X^i}^i \subseteq \overline{X^i}$. Impossible, since $\overline{X^i} \neq \overline{X^i}^i$.

LEMMA 5. $X \cup \overline{X^i} \neq \overline{X}$.

Proof. Suppose $X \cup \overline{X^i} = \overline{X}$. Let $A = \overline{X} \sim \overline{X^i}$. Since $\overline{X^i} \subseteq \overline{X}$ and $\overline{X^i} \neq \overline{X}$, $A \neq \varnothing$. Since $X \cup \overline{X^i} = \overline{X}$, $A = X \sim \overline{X^i}$ and $A \subseteq X \sim X^i$; thus, $A^i = \varnothing$. $\overline{X}^i \sim \overline{X^i}$ is open and $\overline{X}^i \sim \overline{X^i} \subseteq \overline{X} \sim \overline{X^i} = A$; thus, $\overline{X}^i \sim \overline{X^i} = \varnothing$ and $\overline{X}^i \subseteq \overline{X^i}$. Since X^i is open, $\overline{X}^i \subseteq \overline{X^i}^i$. Since $X^i \subseteq X, \overline{X^i}^i \subseteq \overline{X}^i$. Impossible, since $\overline{X^i} \neq \overline{X^i}^i$.

LEMMA 6. $X \cup \overline{X^i} \neq \overline{X}^i$.

Proof. If $X \cup \overline{X^i} = \overline{X}^i$, $X \subseteq \overline{X}^i$ and $\overline{X} \subseteq \overline{\overline{X}^i}$. Since $\overline{X}^i \subseteq \overline{X}$, $\overline{\overline{X}^i} \subseteq \overline{X}$. Impossible since $\overline{X} \neq \overline{\overline{X}^i}$.

LEMMA 7. $X \cup \overline{X^i} \neq \overline{\overline{X}^i}$.

Proof. If $X \cup \overline{X^i} = \overline{\overline{X}^i}, X \subseteq \overline{\overline{X}^i}$ and $\overline{X} \subseteq \overline{\overline{X}^i}$. Same contradiction as in Lemma 6.

Also solved by Leroy F. Meyers and the proposer.

REVIEWS

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with the assistance of the mathematics departments of St. Olaf, Carleton, and Macalester Colleges
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We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use and comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

Calculus with Analytic Geometry. By Harley Flanders and Justin J. Price. Academic Press, New York, 1978. xiv+1041 pp. $24.95. (TR, August/September 1978.)

This text covers the single-variable differential and integral calculus, vectors, the calculus of functions of two and three variables, and infinite series. There are also two chapters on analytic geometry and a nice chapter on approximation techniques. We used it during 1978-79 in several calculus courses at Case Western Reserve University, including an ordinary multi-section freshman calculus course and various accelerated and advanced placement courses. In one or another of these courses we have covered the entire book. The courses were populated by science and engineering students. On the whole, most of the teachers who taught from the book were favorably impressed, and we have adopted the book again for the year 1979-1980. In our opinion, anyone assigned to teach calculus to science and engineering students should seriously consider adopting this book.

The book is written in a lively, informal style that talks directly to the student—the sort of calculus book Isaac Asimov would write. There is advice on how to attack homework problems, warnings not to fall into certain well-known traps, and the like. The figures are plentiful and

remarkably well done. There are few typographical errors and most of them are obvious. There are surprisingly few errors in the answer section. On the other hand, many pages bristle unattractively with formulas, a result of typographical overcrowding and perhaps the desire of the authors to include many steps in computations so as not to make too many demands on the algebraic expertise of the student. The way important formulas are set off in blue boxes partly counteracts this overcrowding.

The level of exposition is uneven, with much more well done than badly done. One of the strongest points of the book is the exceptionally clear and conceptual development of vector algebra and vector calculus. The general development of differential and integral calculus is well planned and clearly written, with a few rough spots to be discussed below. There were a few pedagogical failings. For example, the concept of "parametric" in section 13.4 needs to be explained more extensively. More seriously, the authors ignore the distinction between integrating by "change of variable," as in the case of $\int \cos x e^{\sin x} dx$, and by "substitution," as when one determines $\int \sqrt{4-x^2}\, dx$ by setting $x=2\sin\theta$. This distinction has psychological reality for the students, though mathematically one can forcibly make the two techniques instances of the same phenomenon. There was some sloppy exposition: The paragraph before the definition of limit on page 94 (but not the definition itself) implies that a function cannot be continuous at an endpoint of its interval of definition. The "alternative definition" of limit on page 95 is really a theorem. Also 2^x is used on page 155, "defined" on page 164, then defined again on page 320.

The authors often introduce a topic in one place and return to it in more detail later. This spiral technique works very well in some instances. For example, in the section on tangent lines the authors show how to estimate the error in approximating the curve by the tangent line: further on there is a discussion of Taylor polynomial approximations of functions; then two chapters later Taylor series are treated. The error analysis in the earlier treatments reinforces the treatment of Taylor convergence in the later section. On the other hand, in section 5.6, change of variable and integration by parts are treated very briefly with few examples. With this inadequate preparation, the strong students feel insecure in their understanding of the topic and the weak students cannot use the technique at all. It is true that the students will return to these topics in Chapter 8, but meanwhile if a test intervenes (in our case, the first semester final exam) they are stuck.

The more theoretical aspects of a chapter are usually segregated into its last section. This practice is especially useful given our nontheoretical approach. On the other hand, the segregation of analytic geometry into Chapters 1 and 9 is inconvenient. Our students and our colleagues teaching physics and chemistry prefer us to dive right into the differential calculus at the beginning of the year, so integration of the analytic geometry would better serve our needs.

Perhaps the most awkward aspect of the book is the introduction of the natural logarithm a whole chapter later than the exponential, in order to be able to define the logarithm by an integral. Many of the problems on applications of exponentiation can be done more easily with the natural logarithm, which is not yet available. The authors assume too much familiarity with base-10 logarithms; a bit more review wouldn't hurt.

The treatment of the divergence theorem is inadequate, making the book unsuitable for our sophomore calculus course. The table of integrals is too skimpy, since many of the problems in the section on integration using tables cannot be solved without access to the CRC or other tables.

The quality and diversity of the exercises is one of the major strengths of the book. Refreshing new applications are stimulating to the teacher who is tired of fencing plots of land and watching the liquid level drop in inverted cones (but these are included too). There were some problems needing calculators, but we felt it necessary to hand out supplementary calculator problems, particularly early in the course. We also developed some of the topics in Chapter 10 into programming problems.

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TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print.*

GENERAL, S**, P. *Notes on Rubik's Magic Cube, Fifth Edition, Preliminary Version.* David Singmaster. (Polytechnic of the South Bank, London, SE1 0AA, UK), 1980, 75 pp, $4 (surface); $5.50 (air) (P). An introduction to the group theory structure of Rubik's Cube, the new, popular 3 x 3 x 3 puzzle cube with rotating colored faces. Twice as long as the *Fourth Edition,* this version includes a step by step solution, catalogues of useful processes and pretty patterns, analyses of several newly discovered subgroups (with many problems), and a bibliography. Contains a wealth of ideas for using the cube to demonstrate principles of group theory, from ordinary subgroups to wreath products. (The booklet may soon be marketed commercially in the U.S.) LAS

GENERAL, S, L. *Experiencing Science.* Jeremy Bernstein. Basic Books, 1978, xi + 275 pp, $11.95. [ISBN: 0-465-02185-9] Essays on physics (Kaplan and Rabi), biology (Lysenko, Rosalind Franklin, Lewis Thomas), logic (Gödel), and future science (Arthur Clarke, self-reproducing automata), most reprinted from *The New Yorker* or *The New York Times.* The "strange fictional love story" about Gödel's Theorem appears in this volume for the first time. LAS

GENERAL, S, L. *Mathematical Solitaires & Games.* Ed: Benjamin L. Schwartz. Excursions in Rec. Math. Ser., No. 1. Baywood Pub, 1980, viii + 152 pp, $6 (P). [ISBN: 0-89503-017-9] Reprints of 27 papers from the *Journal of Recreational Mathematics* and its predecessor *Recreational Mathematics Magazine* on such popular themes as instant insanity, SOMA cube, SIM, dots and squares, false coin problems, the four color problem. Because the selection of papers is from one source only, they do not always represent the best or definitive work on their topics; nevertheless, the volume is a handy compilation of interesting papers, each with references to other sources. LAS

PRECALCULUS, T(13: 1). *Trigonometry for College Students, Second Edition.* Karl J. Smith. Brooks/Cole, 1980, xiii + 320 pp, $16.95. [ISBN: 0-8185-0340-8] Better applications than most trigonometry books. Also includes problems called mind-bogglers and applications for further study. Good. (*First Edition,* TR, June-July 1977.) LLK

PRECALCULUS, T(13: 1). *College Algebra.* R. David Gustafson, Peter D. Frisk. Brooks/Cole, 1980, xv + 377 pp, $15.95. [ISBN: 0-8185-0325-4] This is a collection of precalculus topics rather than a review of intermediate algebra. Topics include conic sections, linear systems, logarithms and exponents, and series. LLK

EDUCATION, P. *Computers in Teaching: Mathematics and Statistics.* Herbert L. Dershem, David A. Smith. Conduit, 1979, 40 pp, $4 (P). A 1979 "State-of-the-Art" report on instructional computing: summary of survey results on courses using computing; examples of nontraditional uses of computing in mathematics; bibliographies of books, journals, and sources of information. A concise survey, already somewhat dated, that pulls together in one place much useful information. LAS

EDUCATION, P, L**. *Problem Solving in School Mathematics, 1980 Yearbook.* Stephen Krulik, Robert E. Reys. NCTM, 1980, xiv + 241 pp, $12. [ISBN: 0-87353-162-0] As "new math" was the slogan of mathematics education in the '60's, and "back to basics" in the '70's, so "problem solving" is the emerging theme of the '80's. Indeed, it is listed as the first priority in NCTM's recently released *Agenda for Action.* NCTM's 1980 *Yearbook* contains 21 essays and an annotated bibliography covering objectives, research, teaching strategies, heuristics, and difficulties of using problem solving as the focus of school mathematics. LAS

EDUCATION, P*, L*. *A Sourcebook of Applications of School Mathematics.* NCTM, 1980, xii + 361 pp, $15. [ISBN: 0-87353-164-7] Hundreds of realistic problems (with answers) organized by level (advanced arithmetic, algebra, geometry, trigonometry and logarithms, combinatorics and probability, odds and ends) and indexed by topic, supplemented by fine suggestions for extended projects (on, e.g., house heating, sports records), a fifty-page bibliography of additional sources, and two essays discussing strategies for using problems in the classroom. The result of a six year NSF supported joint effort of MAA and NCTM. Especially useful as a resource for mathematics teachers in junior and senior high school. LAS

HISTORY, P*, L*. *Philosophers at War: The Quarrel Between Newton and Leibniz.* A. Rupert Hall. Cambridge U Pr, 1980, xiii + 338 pp, $24.95. [ISBN: 0-521-22732-1] A thoroughly documented account of the "most celebrated controversy" in the history of science, an affair that scandalized scholarly society for decades. Hall sets the quarrel (not the mathematics on which it is based) in a cultural context in which "convergence" (i.e., simultaneous discovery) is a necessary consequence of vigorous research, and public challenge a common part of doing mathematics. LAS

COMBINATORICS, T*(14-17: 1, 2), S, L*. *Applied Combinatorics.* Alan Tucker. Wiley, 1980, ix + 385 pp, $20.95. [ISBN: 0-471-04766-X] Strategies for counting (generating functions, recurrence relations, inclusion-exclusion, Polya's enumeration formula) and analysis of graphs (circuits, trees, network algorithms, graph-based games), presented primarily by example rather than by formal proof.

Exercises promote proficiency in discrete problem solving, an essential skill for students in any of the mathematical sciences. By selecting topics and exercise difficulty, the book can be used for long or short courses, at almost any level. LAS

LINEAR ALGEBRA, T*(14-15: 1, 2), L. *Linear Algebra and its Applications, Second Edition.* Gilbert Strang. Acad Pr, 1980, xi + 414 pp, $17.95. [ISBN: 0-12-673600-X] Differs from the exciting *First Edition* (TR, June-July 1976; ER, January 1978) in numerous minor text improvements and many new exercises. A new *Instructor's Manual* contains rationale for various uncommon topics and suggestions for classroom approaches to many of the less well-known themes of the text. It remains a refreshing, forthright approach in an engaging personal style that blends lucid explanations (rather than formal proofs) with well-motivated links to diverse applications. LAS

FINITE MATHEMATICS, T(13: 1), S. *Finite Mathematics.* Ronald I. Rothenberg. Wiley, 1980, xii + 283 pp, $7.95 (P). [ISBN: 0-471-04320-6] Self-study format provides questions with answers immediately following. Self-tests at ends of chapters are the only exercises. LLK

CALCULUS, T(13: 1). *Calculus with Analytic Geometry for the Technologies.* Lawrence M. Clar, James A. Hart. P-H, 1980, xii + 352 pp, $15.95. [ISBN: 0-13-111856-0] A brief non-rigorous text for students who want calculus as a tool. First three chapters are analytic geometry, parametric equations and polar coordinates, and progressions, so there are fewer than 200 pages of calculus. LLK

DIFFERENTIAL EQUATIONS, T(17: 2). *Numerical Solution of Differential Equations.* M.K. Jain. Halsted Pr, 1979, xiii + 443 pp, $16.95. [ISBN: 0-470-26609-0] A comprehensive, sophisticated treatment which includes: singlestep and multistep methods; finite difference methods for boundary value problems in ordinary differential equations and for parabolic, hyperbolic and elliptic partial differential equations; and a short chapter on finite element methods. Exercises. TRS

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Mathematics-771: Approximation Methods for Navier-Stokes Problems.* Ed: R. Rautmann. Springer-Verlag, 1980, xvi + 581 pp, $34.30 (P). [ISBN: 0-387-09734-1] Proceedings of the Symposium held by the International Union of Theoretical and Applied Mechanics at the University of Paderborn, Germany on September 9-15, 1979. JAS

DIFFERENTIAL EQUATIONS, T(16-18: 1), S, L. *Numerical Solution of Differential Equations.* Isaac Fried. Comp. Sci. and Appl. Math. Acad Pr, 1979, xiii + 261 pp, $23.50. [ISBN: 0-12-267780-3] An excellent text which gives equal attention to initial value, boundary value and eigenvalue problems in ordinary and partial differential equations. Emphasizes the relationships among the analytical formulation of the physical event, the discretization techniques applied to it, the algebraic properties of the discrete systems created, and the properties of the digital computer. TRS

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Mathematics-799: Functional Differential Equations and Bifurcation.* Ed: A.F. Izé. Springer-Verlag, 1980, xxii + 409 pp, $24.50 (P). [ISBN: 0-387-09986-7] Papers which were presented at the conference held at the Instituto de Ciências Matemáticas de São Carlos, Universidade de São Paulo, São Carlos, Brasil, July 2-7, 1979. JAS

NUMERICAL ANALYSIS, P. *Numerical Methods of Approximation Theory.* Ed: L. Collatz, G. Meinardus, H. Werner. Int. Ser. Num. Math., V. 52. Birkhauser, 1980, 337 pp, $29 (P). [ISBN: 3-7643-1103-7] Excerpts from the conference held at Oberwolfach March 18-24, 1979. This is volume 5 of the subseries *Numerische Methoden bei der Approximationstheorie.* JAS

FUNCTIONAL ANALYSIS, P. *Lecture Notes in Mathematics-768: Approximate Identities and Factorization in Banach Modules.* Robert S. Doran, Josef Wichmann. Springer-Verlag, 1979, x + 305 pp, $19.40 (P). [ISBN: 0-387-09725-2] The authors attempt to present all the basic results on approximate identities in normed algebras with the aim of stimulating more research in this direction. The main tool in the study is the factorization theory of Banach modules, and the authors present much recent work in this area as well. TRS

OPTIMIZATION, S(16-18), L. *Theory of Games and Economic Behavior.* John von Neumann, Oskar Morgenstern. Princeton U Pr, 1980, xx + 641 pp, $9.95 (P). [ISBN: 0-691-00362-9] Paperback republication of the 1953 *Third Edition.* LAS

ANALYSIS, P. *Spectral Theory of Functions and Operators.* Ed: S.M. Nikol'skii. Proc. of Steklov Inst. of Math., No. 130. AMS, 1980, v + 233 pp, $40 (P). [ISBN: 0-8219-3030-9]

ANALYSIS, P. *The Logarithmic Potential and Other Monographs.* Griffith Conrad Evans, Gilbert Ames Bliss, Edward Kasner. Chelsea, 1980, viii + 117 pp, $17.95. [ISBN: 0-8284-0305-8] Republication of three monographs originally published by AMS: *The Logarithmic Potential* (1927) by Evans; *Fundamental Existence Theorems* (1913) by Bliss; and *Differential-Geometric Aspects of Dynamics* (1913) by Kasner. LAS

ANALYSIS, P. *Lecture Notes in Mathematics-794: Measure Theory, Oberwolfach 1979.* Ed: D. Kölzow. Springer-Verlag, 1980, xv + 573 pp, $34.60 (P). [ISBN: 0-387-09979-4] Proceedings of the conference held at Oberwolfach July 1-7, 1979. JAS

DIFFERENTIAL GEOMETRY, S(17-18), P. *Categories, Bundles and Spacetime Topology.* C.T.J. Dodson. Shiva Pub, 1980, xiii + 223 pp. [ISBN: 0-906812-01-1] From the introduction: "Once hooked on bundle theory, a theoretical physicist soon finds...vast families of maps going hither and thither among his spaces. ...The right way to view it all is through the language of categories... ." This development of geometric physics is more an expository monograph than a research report but it lacks the motivation and development of intuition necessary for a novice. JAS

DIFFERENTIAL GEOEMTRY, T(18: 1), S. *Lectures on Minimal Submanifolds, Volume 1.* H. Blaine Lawson, Jr. Publish or Perish, 1980, 178 pp, $10. [ISBN: 0-914098-18-7] An introductory exposition with some updating based on 1970 course notes originally published in Brazil. JAS

GEOMETRY, S*(15-17), L*. *Ideas of Space: Euclidean, Non-Euclidean, and Relativistic.* Jeremy Gray. Clarendon Pr, 1979, ix + 224 pp, $28.50. [ISBN: 0-19-853352-7] An admirable exposition for well educated laymen of the evolution of geometrical thought from before Euclid to black holes, featuring careful, concise exposition, sparing use of symbols, and frequent allusions to related scientific and philosophic issues. Its steep intellectual gradient from high school geometry to relativistic metrics and Lie groups requires and rewards careful reading. LAS

TOPOLOGY, P. *Lectures on Three-Manifold Topology.* William Jaco. CBMS Reg. Conf. in Math., No. 43. AMS, 1980, xii + 251 pp, $9.60 (P). [ISBN: 0-8218-1693-4] The lectures given by the author October 8-12, 1977 at Virginia Polytechnic Institute and State University, Blacksburg, Virginia. JAS

TOPOLOGY, T(15-18: 1, 2), S, L. *Topologie und Analysis.* Bernhelm Boob. Springer-Verlag, 1977, xiv + 352 pp, $17.50 (P). [ISBN: 0-387-08451-7] Part I concerns operators with index and includes Fredholm, finite order, compact, and Wiener-Hopf operators; homotopy invariance of the index; Kuiper theorem; the index bundle; Fourier series and integrals. Part II deals with analyses on manifolds and contains chapters on differential equations; differential and pseudodifferential operators; Sobolev spaces; elliptic operators; elliptic boundary value systems. Part III contains K-theory; the Index Formula in both the euclidean case and for closed manifolds; and closes with a survey of applications. Appendix. Bibliography. Indices. Exercises. RJA

PROBABILITY, S(13-18). *Take a Chance with Your Calculator: Probability Problems for Programmable Calculators.* Lennart Råde. Dilithium Pr, 1977, 163 pp, $8.95 (P). [ISBN: 0-918398-07-X] Part I contains 143 exercises dealing with simulations of discrete random experiments. It is expected that solutions will involve the use of programmable calculators. Part II contains commentaries on the exercises in the first part. Included are programming hints, references to the literature, historical anecdotes, and mathematical analysis of the experiment under consideration. Part III contains programs written for H-P25 and TI SR56 calculators for each of the exercises in the first part. RJA

STATISTICS, T*(14: 1, 2). *Statistics for Business and Economics.* Joseph G. van Matre, Glenn H. Gilbreath. Bus Pub, 1980, xiii + 589 pp, $18.50. [ISBN: 0-256-02276-3] Sophisticated introduction with optional calculus-based material. Unique features include detailed chapters on index numbers and time series analysis, a chapter on statistical decision theory, and a concluding chapter devoted to "the actual use of statistics in today's business world." Makes good use of real data for illustrations and exercises. Also includes short biographical sketches of "major and minor figures in statistics. RSK

STATISTICS, S(13-18). *Sondage, Compilation et Corrêlation.* Adolf Diegel. Pr U Quebec, 1977, xii + 162 pp, (P). Treats several aspects of compiling opinion polls: coding, computing, and constructing tables of information. Discusses how to interpret the correlation measure among sets or subsets. Presents several Fortran packages: TABCOD, TABSOM, TABCHI, TAB, TABORD. Index. RJA

STATISTICS, P*. *Fitting Equations to Data: Computer Analysis of Multifactor Data, Second Edition.* Cuthbert Daniel, Fred S. Wood. Wiley, 1980, xviii + 458 pp, $21.95. [ISBN: 0-471-05370-8] In the Wiley Series in Probability and Mathematical Statistics. Revision of the authors' well-known 1971 *First Edition* (TR, June-July 1974). Changes include extended use of "component and component-plus-residuals plots" as tools in the point by point study of the data. Also includes revisions of their two widely used computer programs for linear and nonlinear equations, and many new figures. RSK

STATISTICS, S*(16-17), L. *Lecture Notes in Statistics-1: R.A. Fisher: An Introduction.* Ed: S.E. Fienberg, D.V. Hinkley. Springer-Verlag, 1980, xi + 208 pp, $14 (P). [ISBN: 0-387-90476-X] First volume in a new series emphasizing applications of statistical methods. Contains edited versions of presentations made as part of a faculty-student seminar and during a Special Lecture Series at the University of Minnesota during the spring of 1978. Gives introductions to and reviews of the key ideas in some of Fisher's major papers, while also providing some insights into Fisher himself. RSK

STATISTICS, T*(17: 1, 2), P*. *Statistical Computing.* William J. Kennedy, Jr., James E. Gentle. Statistics, V. 33. Dekker, 1980, xi + 591 pp, $26.50. [ISBN: 0-8247-6898-1] Discusses numerical methods and algorithms, both general and statistical, and techniques for implementing them. Statistical areas treated include probability and percentage point approximations in selected distributions, random number generation (extensively), multiple linear regression, classification models, nonlinear regression, and model fitting based on criteria other than least squares. Extensive references. RSK

STATISTICS, P. *Jacob Wolfowitz, Selected Papers.* Ed: J. Kiefer, U. Augustin, L. Weiss. Springer-Verlag, 1980, xxiii + 642 pp, $35. [ISBN: 0-387-90463-8] Papers dating from 1939 to 1979 (over half later than 1960) covering a wide range of problems in statistics. A biographical sketch is followed by a thirteen-page discussion of his work. This volume, published to honor Jacob Wolfowitz on his 70th birthday on March 19, 1980, closes with a complete bibliography. JAS

COMPUTER PROGRAMMING. *Assembler I-III: Ein Lernprogramm.* R. Alletsee, H. Jung, G. Umhauer. Springer-Verlag, 1979. *I*, xi + 133 pp, $11 (P) [ISBN: 0-387-09204-8]; *II*, xi + 152 pp, $11.60 (P) [ISBN: 0-387-09205-6]; *III*, xii + 172 pp, $12.10 (P) [ISBN: 0-387-09206-4]. Set of three programmed texts that present the reader with all aspects of programming in assembly language for machines like the Siemens System 4004 and 7700, the IBM System 360/370, and the Univac System 9000. Each volume has an appendix and an index. RJA

COMPUTER PROGRAMMING, S. *Stimulating Simulations, Second Edition.* C.W. Engel. Hayden, 1979, 100 pp, $4.95 (P). [ISBN: 0-8104-5170-0] Twelve computer games written in Basic. Simulations include treasure hunts, space flights, a dungeon game and the like. Includes a brief introduction to the structure and design of programs like this and encourages the reader to write his own. CEC

COMPUTER PROGRAMMING, T(13-18: 1), S. *Structured Programming Using PL/1 and SP/k.* J.N.P. Hume, R.C. Holt, Reston Pub, 1975, xii + 340 pp, $10.95. [ISBN: 0-87909-792-2] Incorporates examples and problems from everyday life. Emphasis is on problem solving and structured programming. PL/1 is presented using a series of subsets SP/1,...,SP/8 that successively include more of the PL/1 language features until the entire language is included in SP/8. Final chapters place a special emphasis on scientific calculations and translation of a high-level language into machine language. Chapter summaries and exercises. Appendices. Bibliography. Index. RJA

COMPUTER PROGRAMMING, S(13-18). *Z-80 and 8080 Assembly Language Programming.* Kathe Spracklen. Hayden, 1979, 168 pp, $7.95 (P). [ISBN: 0-8104-5167-0] Written for the beginning assembly language programmer who knows at least one high-level language such as Basic. Emphasis is on design and coding. Many examples, diagrams, and exercises. Appendices on ASCII Code on 8080 Disassembler, a Z-80 Extension Disassembler, and answers to the exercises. Index. RJA

COMPUTER PROGRAMMING, S(13-18). *8080 Machine Language Programming for Beginners.* Ron Santore. Dilithium Pr, 1978, 104 pp, $6.95 (P). [ISBN: 0-918398-14-2] Each chapter introduces a small number of new instructions. This is accomplished by constructing the chapter around one useful long program that provides the reader with a context for the instructions introduced. Appendices. List of ASCII Codes. Answers to questions asked throughout the text. Index. RJA

COMPUTER PROGRAMMING, T(13: 1, 2), S, L. *Introduction to Computing, Structured Problem Solving Using WATFIV-S.* V.A. Dyck, J.D. Lawson, J.A. Smith. Reston Pub, 1979, xx + 604 pp, $15.95. [ISBN: 0-8359-3158-7] An introduction to structural programming at an elementary level. Programming is taught through the use of pseudo-code and conversion to WATFIV-S. Discusses algorithms, efficiency, and modular organization for dealing with large masses of data and simulation techniques. Includes an outstanding collection of exercises. CEC

COMPUTER PROGRAMMING, S(13-18). *Introduction to TRS-80 Graphics.* Don Inman. Dilithium Pr, 1979, x + 132 pp, $8.95 (P). [ISBN: 0-918398-18-5] Presents graphics programming using Level I Basic on a TRS-80. Begins by teaching how to turn individual points on a screen on and off and then how to print characters at any location on a display. Proceeds to manipulating points to draw rectangles, to using line drawings to form graphs and charts, to drawing vertical and horizontal lines, to forming lines at oblique angles, to composing triangles and curved lines. Chapter exercises and answers. Many suggestions in each chapter for improving upon presented programs and ideas. RJA

COMPUTER PROGRAMMING, S(13-18). *APL: An Introduction, A U-Program Worktext.* Howard A. Peelle. Hayden, 1978, 242 pp, $8.50. [ISBN: 0-8104-5122-0] Self-instruction workbook format. Chapters on command execution, program definition, evaluating expressions, branching, applying functions, interactive programs, arrays and array functions. Contains an interesting collection of APL expressions which are mind boggling. Summary of APL language features. Chapter exercises. Appendix of answers to the exercises. Index. RJA

COMPUTER SCIENCE, T, S*, L. *Understanding and Troubleshooting the Microprocessor.* James W. Coffron. P-H, 1980, xi + 338 pp, $17.95. [ISBN: 0-13-936625-3] For the computer addict who thinks the soldering pencil is mightier than the keyboard. This book provides a clear and thorough explanation of the insides of a microprocessor. The author avoids bogging down in solid state physics on the one hand and computer jargon on the other. For the reader who knows elementary electronics and a little programming (at the freshman or sophomore level), this is a good introduction to the interplay between ideas and machines. Also recommended for faculty whose students or children are starting to speak computerese and ask "why?". JAS

COMPUTER SCIENCE, S(16-18), P, L*. *Arithmetic Complexity of Computations.* Shmuel Winograd. CBMS Reg. Conf. in Appl. Math., No. 33. SIAM, 1980, 93 pp, $10.50 (P). [ISBN: 0-89871-163-0] A brief monograph based on lectures given at a CBMS-NSF Regional Conference at the University of Pittsburgh. Treats the problem of finding the minimum number of arithmetic operations needed to compute such things as the product of matrices, the product of polynomials, and the discrete Fourier transform. Unifying mathematical theme: computation of a bilinear form; major application: signal processing. LAS

APPLICATIONS, P. *Dynamics of Synergetic Systems.* Ed: H. Haken. Springer-Verlag, 1980, viii + 271 pp, $37.50. [ISBN: 0-387-09918-2] This is volume six of the Springer series in synergetics and presents the proceedings of the International Symposium on Synergetics held at Bielefeld, Federal Republic of Germany on September 24-29, 1979. JAS

APPLICATIONS, P. *Interdisciplinary Mathematics, V. XXI: Cartanian Geometry, Nonlinear Waves, and Control Theory, Part B.* Robert Hermann. Transl: Michael Ackerman. Math Sci Pr, 1980, xii + 585 pp, $60. [ISBN: 0-915-692-29-5] In addition to continuing his expositional development of geometric methods in mathematics and physics (the book contains more than the title promises), the author includes translations of two papers by Sophus Lie ("General Investigations of Differential Equations which Admit a Finite Continuous Group" and "Foundations of the Theory of Infinite Continuous Transformation Groups I"). There is also a section of "Reflections" containing a range of material from comments on computers (word processing and recent computer developments relevant to geometry and algebra) to self-admitted polemics concerning the author's relations with certain physicists. JAS

APPLICATIONS (BIOLOGY), P*. *Lecture Notes in Biomathematics-29: Kinetic Logic, A Boolean Approach to the Analysis of Complex Regulatory Systems.* Ed: René Thomas. Springer-Verlag, 1979, xiii + 507 pp, $26 (P). [ISBN: 0-387-09556-X] Proceedings of a course entitled "Formal Analysis of Genetic Regulation" held under the auspices of the European Molecular Biology Organization. Material by seven contributors demands good background in biology and emphasizes dynamic analysis of systems using logical methods. AWR

APPLICATIONS (ELECTRONICS), T(14-18: 1, 2), S, L. *Digital Circuits for Binary Arithmetic.* R.M.M. Oberman. Wiley, 1979, xii + 340 pp, $35. [ISBN: 0-470-26373-3] Chapters on codes, the four arithmetic operations on binary numbers, binary decimal arithmetic, floating point arithmetic circuits, and accumulative adding. Many examples and helpful diagrams. Chapter references. Index. RJA

APPLICATIONS (ENGINEERING), P. *Constructive Approaches to Mathematical Models.* Ed: C.V. Coffman, G.J. Fix. Acad Pr, 1979, xviii + 458 pp, $45. [ISBN: 0-12-178150-X] Contributions from 42 participants in a conference in honor of R.J. Duffin. Wide ranging subject matter is grouped under four headings: General Talks, Graphs and Networks, Mathematical Programming, Differential Equations. AWR

APPLICATIONS (ENGINEERING), S(13-18), P. *8080 Microcomputer Experiments.* Howard Boyet. Dilithium Pr, 1978, xviii + 354 pp, $13.95 (P). [ISBN: 0-918398-08-8] Begins with two chapters on digital logic and microcomputer architecture theory. Major part of the text devoted to more than 80 experiments divided among three categories: (1) software control; (2) single stepping; (3) interfacing with I/O devices. Appendices. Index. RJA

APPLICATIONS (HUMANITIES), T(13-16: 1), S, P, L. *Biblical Games: A Strategic Analysis of Stories in the Old Testament.* Steven J. Brams. MIT Pr, 1980, xi + 196 pp, $15. [ISBN: 0-262-02144-7] An unprecedented application of elementary nonquantitative game theory to Old Testament stories of conflict between man and God, concluding with various unorthodox suggestions about what this analysis reveals about the psychology of the Old Testament God ("tentative," "fundamentally distrusting," "unquestionably,...a superlative strategist"). Political scientist Brams uses mathematical models to provide not a magic key to theology ("the mystery remains impenetrable"), but a theory about the behavior of God that is "coherent, parismonious and rigorous." The book originated in an undergraduate humanities seminar at NYU, and could be used as a superb resource for any similar course. LAS

APPLICATIONS (MANAGEMENT), T(13-16). *Mathematics with Applications to Management and Economics, Fifth Edition.* Earl K. Bowen. Richard D. Irwin, 1980, xxiii + 962 pp, $18.95. [ISBN: 0-256-02349-2] Written to be accessible to students with no more than one year of secondary school algebra, the text covers enough linear algebra to introduce linear programming, enough differentiation for some simple optimization, enough integration to study probability in the continuous case. There is also a chapter on the mathematics of finance. (*Third Edition,* TR, May 1973; *Fourth Edition,* TR, December 1976.) AWR

APPLICATIONS (MANAGEMENT), T*(15-18: 1, 2), S, L. *Data Base Management Systems.* Alfonso F. Cardenas. Allyn, 1979, viii + 519 pp, $18.95. [ISBN: 0-205-16106-0] Objectives, architecture, and realms of data base management. Data organization techniques; discussion of generalized data base management systems; survey of the commercial market. Includes chapters on examples of the network and the hierarchic approaches, and describes the relational model approach. Concludes with the normalization and the design process of data bases. Appendices. Glossary. Index. Chapter references and exercises. RJA

APPLICATIONS (PHYSICS), T?(15-17: 1), S? *Group Theory Made Easy for Scientists and Engineers.* Nyayapathi V.V.J. Swamy, Mark A. Samuel. Wiley, 1979, viii + 174 pp, $14.50. [ISBN: 0-471-05128-4] Hopelessly incomplete presentation made easy only for those willing to live in a fog; full of imprecise definitions and unsubstantiated results. Group representation theory and continuous Lie groups, followed by applications to physics. The book may be a useful supplement for a student with a background in both group theory and quantum mechanics. No exercises. LCL

APPLICATIONS (PHYSICS), P. *Operator Algebras and Quantum Statistical Mechanics 1: C*- and W*-Algebras, Symmetry Groups, Decomposition of States.* Ola Bratteli, Derek W. Robinson. Springer-Verlag, 1979, xii + 500 pp, $36. [ISBN: 0-387-09187-4] This first volume presents the elementary theory of operator algebras and parts of the advanced theory which are relevant to subsequent applications in volume two to quantum statistical mechanics. The first chapter is a brief historical introduction. TRS

APPLICATIONS (SOCIAL SCIENCE), T(16-18: 1, 2), P, L. *The Genetics of Altruism.* Scott A. Boorman, Paul R. Levitt. Acad Pr, 1980, xx + 459 pp, $29.50. [ISBN: 0-12-115650-8] An extensive study, firmly set in mathematical language, of a basic question of social science: why are there so many social species, or--from another viewpoint--so few? The authors, both sociologists, have framed all comparative analyses in mathematical terms, "often at a high level of abstraction," "as a means of getting outside human--perhaps more generally primate and carnivore--frames of reference." In addition to advancing social science, this book could serve as an excellent source of mathematical models for advanced undergraduate seminars. Includes appendix on genetics background, glossary of terms, extensive reference list, and comprehensive author and subject indexes. LAS

APPLICATIONS (SOCIAL SCIENCE), T**(15-17: 1, 2), S***, P*, L***. *Mathematics for Social Scientists.* Ki Hang Kim, Fred William Roush. Elsevier North Holland, 1980, xiv + 277 pp, $14.95. [ISBN: 0-444-99066-6] Concise chapters on sets and relations, matrices, Boolean matrices and graphs, combinatorics, difference equations, differential equations, probability, and cluster analysis, followed by applications. As R. Duncan Luce says in a foreward, "This book should help many of us who wish to increase the mathematical skills of our social science majors." FLW

Reviewers Whose Initials Appear Above

Richard J. Allen, St. Olaf; Clifton E. Corzatt, St. Olaf; Lorraine L. Keller, St. Olaf; Richard S. Kleber, St. Olaf; Loren C. Larson, St. Olaf; A. Wayne Roberts, Macalester; Thomas R. Savage, St. Olaf; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY FRANK KOCHER, The Pennsylvania State University

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036

PERSONAL ITEMS

Assistant Professor *Lyndell M. Kerley*, East Tennessee State University, has been promoted to Associate Professor.

Professor *Harry Pollard* of Purdue University has been appointed Distinguished Visiting Professor at California State University, Chico for Fall semester, 1980.

Associate Professor *Thomas P. Wiggen* of Oklahoma City University has been appointed Assistant Professor of Computer Science at Northern Illinois University.

Louisiana State University: Associate Professor *Kenneth Reid* has been promoted to Professor. Dr. *R.D. Anderson*, Boyd Professor of Mathematics, and Professor *Dan R. Scholz* have retired. Dr. *Clark Lane* and Dr. *Charles N. Delzell* have been appointed Assistant Professors. Dr. *James K. Deveney* of Virginia Commonwealth University has been appointed Visiting Associate Professor.

University of North Carolina-Greensboro: Professor *Richard B. Sher* has been appointed Head of the Department of Mathematics. Professor *E.E. Posey* has returned to teaching and research after serving as Department Head for sixteen years.

University of Illinois at Urbana-Champaign: Associate Professor *William F. Stout* has been promoted to Professor. Associate Professor *Ross Lee Finney, Jr.* has accepted an appointment with the Massachusetts Institute of Technology and Education Development Center. Professor Emeritus *J.L. Doob* was awarded the President's National Medal of Sciene in January 1980.

Naval Postgraduate School: Professor Emeritus *Robert E. Gaskell* retired in December 1979. Professor *Craig Comstock* has retired. Dr. *Michael Humphries* has been appointed Adjunct Professor.

Rensselaer Polytechnic Institute: Associate Professor *Bobby F. Caviness* has been promoted to Professor. He will be on leave at the General Electric Research and Development Center in Schenectady, New York from September 1980 to August 1981. Professor *Norman S. Free* has retired with the rank of Professor Emeritus. Professor *Richard C. DiPrima* and Professor *Carlton E. Lemke* were jointly awarded the 1980 William H. Wiley Distinguished Faculty Award. This award is the highest recognition the faculty can bestow on its members. Professor *Edith H. Luchins* was appointed a Danforth Associate for a six-year term.

San Francisco State University: Associate Professors *David B. Meredith* and *Alfred S.-Y. Tang* have been promoted to Professor.

Pennsylvania State University: Professor *George E. Andrews* has been appointed Chairman of the Department of Mathematics. Professor *Donald C. Rung* has returned to teaching and research after serving as Department Chairman for five years.

Villanova University: Assistant Professor *William Miehle* has retired from the university. *John C. Quigg, Jr.* has been appointed assistant professor of mathematics.

Joel Alperin, M.A., of Scranton, Pennsylvania, died February 12, 1980. He was a member of the Association for nineteen years.

Dr. *Ruth Mason Ballard* of Wilmette, Illinois died June 7, 1980 at the age of 74. She was a member of the Association for forty-eight years.

Bruce A. Broemser, M.A., of El Sobrante, California, died November 16, 1979. He was a member of the Association for nine years.

Dr. *Francis P. Callahan* of Bluebell, Pennsylvania, died July 2, 1980 at the age of 55. He was a member of the Association for twenty three years.

Dr. *Hermon H. Conwell* of Beloit, Wisconsin, died in January 1980. A Dean and Professor Emeritus of Beloit College, he was a charter member of the Association.

Benjamin C. Glover of Columbus, Ohio, died in 1980. He was a member of the Association for fifty-eight years.

Professor Emeritus *Houston Karnes* of Louisiana State University died in May 1980. He was a member of the Association for forty-five years.

Dr. *Kenneth B. Leisenring* of Ann Arbor, Michigan, died in Math, 1980 at the age of 77. He was a member of the Association for thirty-one years.

Carl H. Rasmussen, M.S., of Livonia, Michigan, died August 15, 1979. He was a member of the Association for eight years.

Dr. *L. Vernon Robinson* of Argyle, Texas, died January 31, 1980 at the age of 82. He was a member of the Association for 57 years.

Dr. *Hubert H. Schneider* of Lincoln, Nebraska, died July 4, 1980. He was an Associate Professor of Mathematics at the University of Nebraska. He was a member of the Association for twenty-one years.

Bill Turner, M.S., of Hurst, Texas, died in the Fall of 1979 at the age of forty-nine. He was a member of the Association for one year.

Dr. *Franklin M. Turrell* of Riverside, California, died in May 1980 at the age of seventy-five. He was a member of the Association for twenty years.

Dr. *James W. Walker* of Kansas City, Missouri, died in June 1980 at the age of forty-two. He was a member of the Association for eight years.

Dr. *Robert E. Weber* of Sharpsville, Pennsylvania died March 12, 1980 at the age of thirty-six. He was a member of the Association for three years.

U.S.A. MATHEMATICAL OLYMPIAD

Eight U.S. and Canadian students earned Olympiad medals in a mathematics competition involving over 400,000 highschool mathletes. The final round in this competition was the Ninth USA Mathematical Olympiad (USAMO) in which 120 students competed in a challenging examination designed to test ingenuity as well as mathematical background.

The 120 USAMO competitors were the top performers in the Annual High School Mathematics Contest held in high schools throughout the United States and Canada on March 4, 1980.

The eight USAMO winners are:

Michael J. Larsen of Lexington, Massachusetts,
Eric D. Carlson of Munster, Indiana,
Michael V. Finn of Annandale, Virginia,
Bruce K. Smith of San Rafael, California,
Jeremy D. Primer of Maplewood, New Jersey,
Paul N. Feldman of Brooklyn, New York,
Daniel J. Scales of Westwood, Massachusetts, and
David W. Ash of Thunder Bay, Ontario, Canada.

The winners were honored on June 3 in Washington, D.C. at an awards ceremony in the National Academy of Sciences and the Diplomatic Reception Rooms of the U.S. Department of State.

DECLINE IN BACHELOR'S DEGREES, 1973-78

Data recently compiled by Educational Testing Service show that out of 20 disciplines, Mathematics suffered the largest percentage decline in number of bachelor's degrees between 1973 and 1978. Here are the figures.

Discipline	Percentage Change	Discipline	Percentage Change
Mathematics	-46%	Political Science	-14%
English	-45%	Geography	-12%
French	-45%	Psychology	-7%
History	-44%	Economics	+5%
Spanish	-38%	Engineering	+8%
Sociology	-37%	Chemisty	+11%
German	-35%	Biology	+21%
Philosophy	-34%	Music	+33%
Education	-30%	Geology	+47%
Physics	-22%	Computer Science	+67%

CALCULATOR INFORMATION CENTER

The Calculator Information Center at the Ohio State University has been funded by the National Institute of Education for another year. The Center, in operation since 1977, collects and disseminates information on calculator uses. The dissemination is accomplished primarily through two types of bulletins. *Reference* bulletins cite journal articles and other types of publications; *Information* bulletins contain suggestions on such topics as selecting calculators, using calculators at various instructional levels, and conducting workshops.

If you would like to contribute to the information exchange or if you would like a copy of the latest reference bulletin (and wish to put your name on the mailing list) you may call 614-422-8509 between 8 and 4 (Eastern time zone) or write:

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ARTICLES COMING SOON IN THIS MONTHLY

Simple analytic proof of the prime number theorem. Donald J. Newman
The geometry of root systems and signed graphs. Thomas Zaslavsky
Chains of circles with rational radii. M.D. Fox
An application of Poincaré's recurrence theorem to academic administration. Kenneth R. Meyer
Rigid and flexible frameworks. Ben G. Roth
What is a computer program? Michael C. Gemignani
Constructing Buffon curves from their distributions. Duane W. DeTemple
Increasing the participation of college women in mathematics-based fields. Lenore Blum and Steven Givant
Some historical remarks concerning degree theory. Hans-Willi Siegberg
Pólya's counting theorem via tensors. Russell Merris
Irrational sums and twin primes. Solomon W. Golomb

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

DIRECTORY OF WOMEN IN THE MATHEMATICAL SCIENCES

The AMS-MAA-NCTM-SIAM Committee on Women in Mathematics plans to publish a *Directory of Women in the Mathematical Sciences* with financial support from the American Mathematical Society, the Mathematical Association of America, the Society for Industrial and Applied Mathematics and, possibly, other organizations. It supersedes the original *Directory of Women Mathematicians* which the AMS issued in 1973-1974 with later supplements. The new *Directory* will include an updating of the original *Directory* and its supplements, as well as new listings.

The questionnaire on the following page will form the basis for the new *Directory*. Women who have earned a Ph.D. degree, or its equivalent, and those expecting to receive a Ph.D. in 1980 or 1981 should send the questionnaire to the address shown at the foot of the page. Information on women who expect to receive a Ph.D. at a later date will be kept on file.

The purpose of the *Directory* is to enable employers, individuals arranging colloquia, conferences or mathematics meetings, and officers of professional organizations seeking nominees for committees to find, in summary form, the curricula vitae of women in the mathematical sciences. It is planned to keep expenses at a minimum, which should permit frequent reissuance of the *Directory*.

Joint Committee on Women in Mathematics

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Alice T. Schafer (AMS), chairman
Joel E. Schneider (NCTM)
Barbara Searle (NCTM)

SPRING MEETING OF THE MISSOURI SECTION

The Missouri Section of the MAA met April 25 and 26, 1980, at Westminster College in Fulton, Missouri. Approximately 75 persons attended.

Invited addresses were given by Tim Wright of the University of Missouri-Rolla, *Thinkers Who Do Not Count and Counters Who Do Not Think*, and Dorothy L. Bernstein, MAA President, from Brown University, *Mathematical Models and Existence Theorems*.

Yudell L. Luke moderated a panel discussion on *Preparation for College Mathematics in High School and in College*. Panel members were Harry Oldweiler of Columbia Hickman High School, Elizabeth Berman of the University of Missouri at Kansas City, and NCTM President Shirley Hill of the University of Missouri at Kansas City.

The program also included a meeting for the MAA Department Representatives, a breakfast meeting for Mathematics Department Heads, and a breakfast meeting of Women in Mathematics. Also, those attending were guests at the Winston Churchill Memorial Concert Series presentation of *Music for an April Evening*.

Section Chairman Mike Z. Williams presided over the business meeting. Kevin Keating and Nathan Schroeder, students from Washington University in St. Louis, earned honorable mention on the Putman Competition and each was awarded the MAA publication *Mathematical Plums*.

Officers for academic year 1980-81 are: President, Merry McDonald, Northwest Missouri State University; Vice President, Glen Haddock, University of Missouri-Rolla; Past President, Mike Williams, Westminster College; Secretary-Treasurer, Jerry Wilkerson, Missouri Western State College; Governor, Troy Hicks, University of Missouri-Rolla; Chairman, High School Lecture Program, Leonard Palmer, Southeast Missouri State University; Chairman, MAA High School Mathematics Contest, Al Tinsley, Central Missouri State University.

The following were contributed: *What is a Statistical Metric Space*, Troy Hicks, University of Missouri-Rolla; *Polar Coordinates and Inversion in the Unit Circle*, Lyle Pursell, University of Missouri-Rolla; *Concerning Paraseparable Dendritic Spaces*, David John, Missouri Western State College; *Properties of Music Tables*, Curtis Cooper, Central Missouri State University; *Fibonacci Numbers and Permanents of Circulants*, Gerald Suchan, Missouri Southern State College; *Numerical Calculations of Cauchy Principal Values Arising From the Determination of Optical Properties of Cryofilms*, Kent Palmer, Westminster College; *An Investigation of a Generalization of a Divisibility Test*, Robert Kennedy, Central Missouri State University; *The Distance Set of a Generalized-Cantor Set in n-Space - Some Unsolved Problems*, Ken Lee, Missouri Western State College; *Prerequisite Math Knowledge for Learning Statistics*, Jeanne Sebaugh, University of Missouri-Columbia; *Secondary Students Solution for Algebra Word Problems*, Terry Goodman, Central Missouri State University.

John Kubicek, Secretary-Treasurer

SUGGESTION BOX

Members of the MAA are encouraged to send in suggestions, questions, etc., about the operations of the Association. Communications will be referred to the appropriate officer of the Association for answering; from time to time, those of general interest may also be answered in one or both of the official journals. Communications should be addressed to: Suggestion Box, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036

Directory of Women in the Mathematical Sciences

QUESTIONNAIRE FOR WOMEN PH.D'S AND PH.D. CANDIDATES

(*Please type or print*)

1. A. Name (last name first)____

   B. Cross listing of name (if desired) ______________________________

2. Mailing Address (Preferred)____________________________________

*Current Status: _______Employed _______Retired _______Graduate Student _______Unemployed
(*This information is for record keeping and will not be printed in the Directory.)

3. Name of employer or institution__________________________________

   Address of employer or institution________________________________

   Title of position____

4. Highest academic degree received________________________________

   Year degree conferred____________ Institution____________________________

5. Fields of mathematical interest: (Write **1** for primary field; **2** for other. Only two will be printed.)

   A. ______Algebra
   B. ______Analysis
   C. ______Applied mathematics
   D. ______Computer science
   E. ______Functional analysis
   F. ______Geometry
   G. ______Logic and foundations
   H. ______Number theory
   I. ______Operations research
   J. ______Statistics and probability
   K. ______Topology
   L. ______Other professional interest (Administration)

6. Publications: Bibliographic information for two most important publications. In addition to the title, give **MR number only** when available; otherwise for an article use MR abbreviation for journal and give volume number.

   i. ______________________________

   ii. ______________________________

Return this form as soon as possible to

DIRECTORY OF WOMEN IN THE MATHEMATICAL SCIENCES

c/o Professor Alice T. Schafer, Department of Mathematics, Wellesley College, Wellesley, MA 02181

CALENDAR OF FUTURE MEETINGS

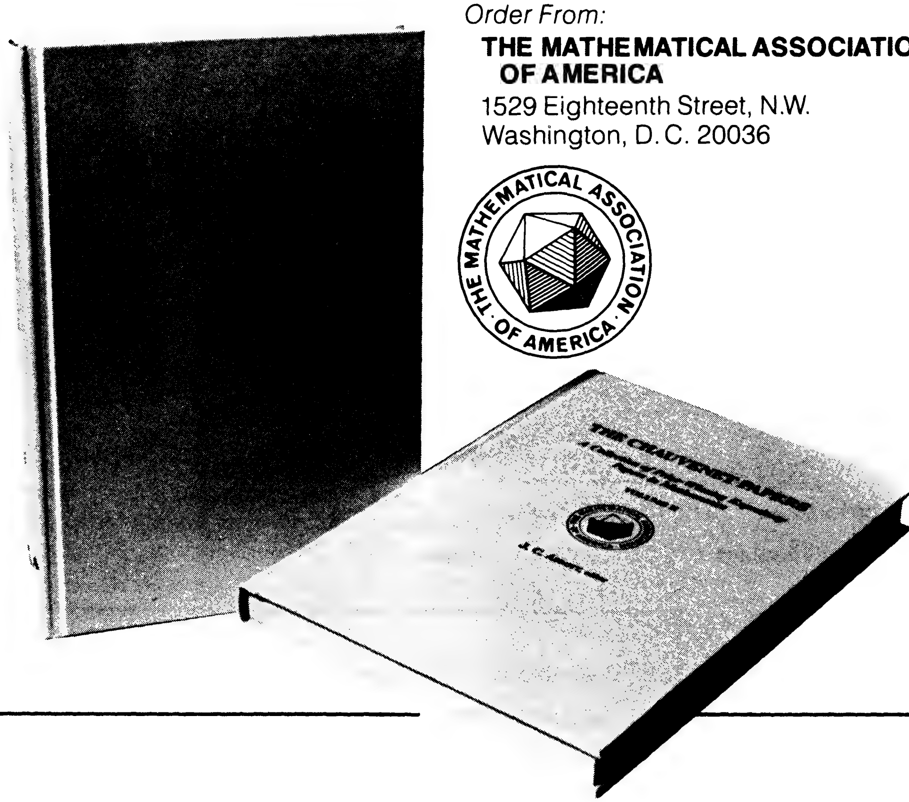
Sixty-fourth Annual Meeting, San Francisco, California, January 9–11, 1981.
Sixty-first Summer Meeting, Pittsburgh, Pennsylvania, August 17–19, 1981.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May 15–16, 1981.
EASTERN PENNSYLVANIA and DELAWARE, University of Delaware, Newark, November 22, 1980.
FLORIDA, Bethune Cookman College, Daytona Beach, March 6–7, 1981.
ILLINOIS, Illinois State University, Normal, May 1–2, 1981.
INDIANA
INTERMOUNTAIN
IOWA, third weekend in April. Deadline for papers February 1.
KANSAS, Benedictine College, Atchison, April 1981 (probably April 25).
KENTUCKY, Jefferson Community College, Louisville, April 3–4, 1981.
LOUISIANA–MISSISSIPPI, Mississippi State University, Mississippi State, February 13–14, 1981.
MARYLAND – DISTRICT OF COLUMBIA – VIRGINIA, Goucher College, Towson, Maryland, November 14–15, 1980.
METROPOLITAN NEW YORK, spring. Deadline for papers two weeks before meeting.
MICHIGAN, first Friday and Saturday in May. Deadline for papers six weeks before meeting.
MISSOURI, Northwest Missouri State University, Maryville, April 10–11, 1981.
NEBRASKA, University of South Dakota, Vermillion, South Dakota, April 11–12, 1981.
NEW JERSEY, Union College, Cranford, October 25, 1980.
NORTH CENTRAL, end of October and April. Deadline for papers October 1 and April 1.
NORTHEASTERN, Merrimack College, North Andover, Massachusetts, November 22, 1980.
NORTHERN CALIFORNIA, first or second Saturday in February.
OHIO, John Carroll University, University Heights, October 17–18, 1980.
OKLAHOMA–ARKANSAS, Oklahoma Christian College, Oklahoma City, March 27–28, 1981.
PACIFIC NORTHWEST, second Saturday in June. Deadline for papers six weeks before meeting.
ROCKY MOUNTAIN, Colorado College, Colorado Springs, May 1–2, 1981.
SEAWAY, Daemen College, Buffalo, New York, November 7–8, 1980.
SOUTHEASTERN, University of Alabama, Birmingham, April 10–11, 1981.
SOUTHERN CALIFORNIA, first or second Saturday in March.
SOUTHWESTERN, usually in April. Deadline for papers two weeks before meeting.
TEXAS, Friday and Saturday in early April. Deadline for papers March 1.
WISCONSIN, University of Wisconsin, La Crosse, late March–early April 1981.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Toronto, January 3–8, 1981.
AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES, Sheraton National Hotel, Arlington, Virginia, October 9–13, 1980.
AMERICAN MATHEMATICAL SOCIETY, San Francisco, California, January 7–10, 1981.
AMERICAN SOCIETY FOR ENGINEERING EDUCATION
ASSOCIATION FOR COMPUTING MACHINERY, Nashville, Tennessee, October 27–29, 1980.
ASSOCIATION FOR SYMBOLIC LOGIC, San Francisco, California, January 9–10, 1981.
ASSOCIATION FOR WOMEN IN MATHEMATICS, San Francisco, California, January 7–11, 1981.
CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHEMATIQUES
FIBONACCI ASSOCIATION
INSTITUTE OF MATHEMATICAL STATISTICS
MU ALPHA THETA
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, St. Louis, Missouri, April 22–25, 1981.
OPERATIONS RESEARCH SOCIETY OF AMERICA, Four Seasons Sheraton, Toronto, Canada, May 4–6, 1981.
PI MU EPSILON
SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Indianapolis, Indiana, October 30–November 1, 1980.
SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Stouffer's Greenway Plaza Hotel, Houston, Texas, November 6–8, 1980.



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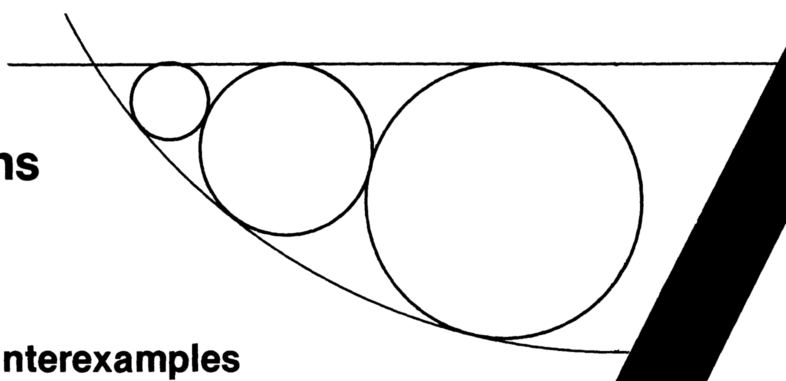

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1980

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MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

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THE MATHEMATICAL ASSOCIATION OF AMERICA

VOLUME 87 | NUMBER 9

(ISSN 0002-9890)

NOTICE TO AUTHORS

Research papers per se are unsuitable; see statement of policy (Vol. 85, p. 1).

Please follow the format in current issues of the MONTHLY. Manuscripts must be legibly typewritten or reproduced from typewritten copy, double spaced with wide margins, and on only one side of the paper. Three copies should be submitted to the appropriate editor and one kept by the author as protection against loss. The author's *full* mailing address must appear at the beginning of the paper, in a line following the author's name. (This is a change from the style previously used in this MONTHLY.) All authors are asked to supply subject classifications with their manuscripts, if possible from the "1980" classification that appears in the 1978 and later Mathematical Reviews Indexes.

Estimated median time from submission to publication: Main Articles 15 months, Progress Reports 6 months, Math. Notes 24 months, Research Problems 6 months, Classroom Notes 14 months, Math. Education 10 months.

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PRINTED IN THE UNITED STATES OF AMERICA

SIMPLE ANALYTIC PROOF OF THE PRIME NUMBER THEOREM

D. J. NEWMAN

Department of Mathematics, Temple University, Philadelphia, PA 19122

The magnificent prime number theorem has received much attention and many proofs throughout the past century. If we ignore the (beautiful) elementary proofs of Erdős [1] and Selberg [6] and focus on the analytical ones, we find that they all have some drawback. The original proofs [7] of Hadamard and de la Vallée Poussin were based, to be sure, on the nonvanishing of $\zeta(z)$ in $\operatorname{Re} z \geqslant 1$, but they also required annoying estimates of $\zeta(z)$ at ∞, the reason being that formulas for coefficients of Dirichlet series involve integrals over *infinite* contours (unlike the situation for power series) and so effective evaluation requires estimates at ∞.

The more modern proofs, due to Wiener [2] and Ikehara [8] (see also Heins's book [3]) do get around the necessity of estimating at ∞ and are indeed based only on the appropriate nonvanishing of $\zeta(z)$, but they are tied to certain results on Fourier transforms.

We propose to return to contour integral methods so as to avoid Fourier analysis, and also to use finite contours so as to avoid estimates at ∞. Of course certain errors are introduced thereby, but the point is that these can be effectively estimated away by elementary arguments.

So let us begin with the well-known fact [7] about the ζ-function:

$$(z-1)\zeta(z) \text{ is analytic and zero free throughout } \operatorname{Re} z \geqslant 1. \tag{1}$$

This will be assumed throughout and will allow us to give our proof of the prime number theorem.

In fact we give two proofs. The first one is the shorter and simpler of the two, but we pay a price in that we obtain one of Landau's equivalent forms of the theorem rather than the standard form, $\pi(N) \sim N/\log N$. Our second proof is a more direct assault on $\pi(N)$ but is somewhat more intricate than the first. Here we find some of Tchebychev's elementary ideas very useful.

Basically our novelty consists in using a modified contour integral,

$$\int_\Gamma f(z) N^z \left(\frac{1}{z} + \frac{z}{R^2}\right) dz,$$

rather than the classical one, $\int_C f(z) N^n z^{-1} dz$. The method is rather flexible, and we could use it to directly obtain $\pi(N)$ by choosing $f(z) = \log \zeta(z)$. We prefer, however, to derive both proofs from the following convergence theorem. Actually, this theorem dates back to Ingham [9], but his proof is à la Fourier analysis and is much more complicated than the contour integral method we now give.

THEOREM. *Suppose* $|a_n| \leqslant 1$ *and form the series* $\sum a_n n^{-z}$ *which clearly converges to an analytic function* $F(z)$ *for* $\operatorname{Re} z > 1$. *If, in fact,* $F(z)$ *is analytic throughout* $\operatorname{Re} z \geqslant 1$, *then* $\sum a_n n^{-z}$ *converges throughout* $\operatorname{Re} z \geqslant 1$.

Proof of the convergence theorem.. Fix a w in $\operatorname{Re} w \geqslant 1$. Thus $F(z+w)$ is analytic in $\operatorname{Re} z \geqslant 0$. We choose an $R \geqslant 1$ and determine $\delta = \delta(R) > 0$, $\delta \leqslant \frac{1}{2}$ and an $M = M(R)$ so that

$$F(z+w) \text{ is analytic and bounded by } M \text{ in } -\delta \leqslant \operatorname{Re} z, |z| \leqslant R. \tag{2}$$

Now form the counterclockwise contour Γ, bounded by the arc $|z| = R, \operatorname{Re} z > -\delta$, and the

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segment $\operatorname{Re} z=-\delta, |z|\leqslant R$. Also denote by A and B, respectively, the parts of Γ in the right and left half-planes.

By the residue theorem we have

$$2\pi i F(w)=\int_\Gamma F(z+w)N^z\left(\frac{1}{z}+\frac{z}{R^2}\right)dz. \tag{3}$$

Now on A, $F(z+w)$ is equal to its series, and we split this into its partial sum $S_N(z+w)$ and remainder $r_N(z+w)$. Again by the residue theorem we have

$$\int_A S_N(z+w)N^z\left(\frac{1}{z}+\frac{z}{R^2}\right)dz=2\pi i S_N(w)-\int_{-A} S_N(z+w)N^z\left(\frac{1}{z}+\frac{z}{R^2}\right)dz,$$

with $-A$ denoting as usual the reflection of A through the origin. Thus, changing z into $-z$, this can be written as

$$\int_A S_N(z+w)N^z\left(\frac{1}{z}+\frac{z}{R^2}\right)dz=2\pi i S_N(w)-\int_A S_N(w-z)N^{-z}\left(\frac{1}{z}+\frac{z}{R^2}\right)dz. \tag{4}$$

Combining (3) and (4) gives

$$\begin{aligned}2\pi i(F(w)-S_N(w))=&\int_A\left(r_N(z+w)N^z-\frac{S_N(w-z)}{N^z}\right)\left(\frac{1}{z}+\frac{z}{R^2}\right)dz\\&+\int_B F(z+w)N^z\left(\frac{1}{z}+\frac{z}{R^2}\right)dz,\end{aligned} \tag{5}$$

and to estimate these integrals we record the following (here as usual we write $\operatorname{Re} z=x$, and we use the notation $\alpha\ll\beta$ to mean simply that $|\alpha|\leqslant|\beta|$):

$$\frac{1}{z}+\frac{z}{R^2}=\frac{2x}{R^2} \text{ along } |z|=R \text{ (in particular on } A), \tag{6}$$

$$\frac{1}{z}+\frac{z}{R^2}\ll\frac{1}{\delta}\left(1+\frac{|z|^2}{R^2}\right)\leqslant\frac{2}{\delta}\text{on the line } \operatorname{Re} z=-\delta, |z|\leqslant R, \tag{7}$$

$$r_N(z+w)\ll\sum_{n=N+1}^{\infty}\frac{1}{n^{x+1}}\leqslant\int_N^\infty\frac{dn}{n^{x+1}}=\frac{1}{xN^x}, \tag{8}$$

$$S_N(w-z)\ll\sum_{n=1}^{N}n^{x+1}\leqslant N^{x-1}+\int_0^N n^{x-1}dn=N^x\left(\frac{1}{N}+\frac{1}{x}\right). \tag{9}$$

By (6), (8), (9) we have, on A,

$$\left(r_N(z+w)N^z-\frac{S_N(w-z)}{N^z}\right)\left(\frac{1}{z}+\frac{z}{R^2}\right)\ll\left(\frac{1}{x}+\frac{1}{x}+\frac{1}{N}\right)\frac{2x}{R^2}\leqslant\frac{4}{R^2}+\frac{2}{RN},$$

and so by the "maximum times length" estimate (M-L formula) for integrals we obtain

$$\int_A\left(r_N(z+w)N^z-\frac{S_N(w-z)}{N^z}\right)\left(\frac{1}{z}+\frac{z}{R^2}\right)dz\ll\frac{4\pi}{R}+\frac{2\pi}{N}: \tag{10}$$

Next by (2), (6), and (7) we obtain

$$\begin{aligned}\int_B F(z+w)N^z\left(\frac{1}{z}+\frac{z}{R^2}\right)dz&\ll\int_{-R}^{R}M\cdot N^{-\delta}\cdot\frac{2}{\delta}dy+2M\int_{-\delta}^{0}N^x\frac{2|x|}{R^2}\,\frac{3}{2}dx\\&\leqslant\frac{4MR}{\delta N^\delta}+\frac{6M}{R^2\log^2 N}.\end{aligned} \tag{11}$$

Inserting the estimates (10) and (11) into (5) gives

$$F(w)-S_N(w)\ll\frac{2}{R}+\frac{1}{N}+\frac{MR}{\delta N^{\delta}}+\frac{M}{R^2\log^2 N}$$

and if we fix $R=3/\epsilon$ we note that this right-hand side is $<\epsilon$ for all large N. We have verified the very definition of convergence!

First Proof of the Prime Number Theorem. Landau has pointed out that the convergence of $\sum\mu(n)/n$ is equivalent to the prime number theorem. Since $\sum\mu(n)/n^z=1/\zeta(z)$ for $\operatorname{Re} z>1$, however, (1) ensures that the hypotheses of our theorem hold, and Landau's form of the prime number theorem follows immediately.

Second Proof of the Prime Number Theorem. In this section we begin with Tchebychev's observation [5] that

$$\sum_{p<n}\frac{\log p}{p}-\log n \quad \text{is bounded,} \tag{12}$$

which he derives in a direct elementary way from the prime factorization of $n!$.

The point is that the prime number theorem is easily derived from

$$\sum_{p<n}\frac{\log p}{p}-\log n \quad \text{converges to a limit,} \tag{13}$$

by a simple summation by parts, which we leave to the reader. Nevertheless the transition from (12) to (13) is not a simple one and we turn to this now.

So form, for $\operatorname{Re} z>1$, the function

$$f(z)=\sum_{n=1}^{\infty}\frac{1}{n^z}\left(\sum_{p<n}\frac{\log p}{p}\right)=\sum_{p}\frac{\log p}{p}\left(\sum_{n>p}\frac{1}{n^z}\right).$$

Now

$$\sum_{n>p}\frac{1}{n^z}=\frac{1}{(z-1)p^{z-1}}+z\int_p^{\infty}\frac{1-\{t\}}{t^{z+1}}\,dt=\frac{p}{(z-1)}\left(\frac{1}{p^z-1}+A_p(z)\right)$$

where $A_p(z)$ is analytic for $\operatorname{Re} z>0$ and is bounded by

$$\frac{1}{p^x(p^x-1)}+\frac{|z(z-1)|}{xp^{x+1}}.$$

Hence

$$f(z)=\frac{1}{z-1}\left(\sum_{p}\frac{\log p}{p^z-1}+A(z)\right),$$

where $A(z)$ is analytic for $\operatorname{Re} z>\frac{1}{2}$ by the Weierstrass M-test.

By Euler's factorization formula, however, we recognize that

$$\sum_{p}\frac{\log p}{p^z-1}=\frac{-d}{dz}\log\zeta(z); \tag{14}$$

and so we deduce, by (1), that $f(z)$ is analytic in $\operatorname{Re} z\leqslant 1$ except for a double pole with principal part $1/(z-1)^2+c/(z-1)$, at $z=1$. Thus if we set

$$F(z)=f(z)+\zeta'(z)-c\zeta(z)=\sum\frac{a_n}{n^z},\qquad \text{where } a_n=\sum_{p<n}(\log p)/p-\log n-c,$$

we deduce that $F(z)$ is analytic in $\operatorname{Re} z\geqslant 1$.

From (12) and our convergence theorem, then, we conclude that

$$\sum \frac{a_n}{n} \text{ converges,}$$

and from this and the fact, from (14), that $a_n + \log n$ is nondecreasing we proceed to prove $a_n \to 0$.

By applying the Cauchy criterion we find that, for N large, we have both

$$\sum_{N}^{N(1+\epsilon)} \frac{a_n}{n} \leqslant \epsilon^2, \tag{15}$$

and

$$\sum_{N(1-\epsilon)}^{N} \frac{a_n}{n} \geqslant -\epsilon^2. \tag{16}$$

In the range N to $N(1+\epsilon)$ we have, by (14), that $a_n \geqslant a_N + \log(N/n) \geqslant a_N - \epsilon$ and so $\Sigma_N^{N(1+\epsilon)} a_n/n \geqslant (a_N - \epsilon)\Sigma_N^{N(1+\epsilon)} 1/n$ and (15) yields

$$a_N \leqslant \epsilon + \frac{\epsilon^2}{\sum_{N}^{N(1+\epsilon)} \frac{1}{n}} \leqslant \epsilon + \frac{\epsilon^2}{N\epsilon/N(1+\epsilon)} = 2\epsilon + \epsilon^2. \tag{17}$$

Similarly in $[N(1-\epsilon), N]$ we have $a_n \leqslant a_N + \log(N/n) \leqslant a_N + \epsilon/(1-\epsilon)$ so that

$$\sum_{N(1-\epsilon)}^{N} \frac{a_n}{n} \leqslant \left(a_N + \frac{\epsilon}{1-\epsilon}\right) \sum_{N(1-\epsilon)}^{N} \frac{1}{n}$$

and (16) gives

$$a_N \geqslant \frac{-\epsilon}{1-\epsilon} - \frac{\epsilon^2}{\sum_{N(1-\epsilon)}^{N} \frac{1}{n}} \geqslant -\frac{\epsilon}{1-\epsilon} - \frac{\epsilon^2}{N\epsilon/N} = \frac{\epsilon^2 - 2\epsilon}{1-\epsilon}. \tag{18}$$

Taken together (17) and (18) establish that $a_N \to 0$ and so (13) is proved.

The research for this paper was supported in part by NSF Grant No. MCSF8-02171.

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MISCELLANEA

43. If you ask mathematicians what they do, you always get the same answer; they think. They are trying to solve difficult and novel problems. (They never think about ordinary problems—they just write down the answers.)

—M. Evgrafov, *Literaturnaya Gazeta*, no. 49 (1979) 12.

THE PHILOSOPHICAL IMPLICATIONS OF THE FOUR-COLOR PROBLEM

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In a recent article, Thomas Tymoczko has suggested that the four-color theorem (4CT) [1] either is not a mathematical theorem at all or is an entirely new kind of theorem. He goes on to draw the conclusion that the theorem is not an a priori truth, in the classical mathematical tradition, but an a posteriori truth, and this conclusion forms the main thesis of his article.

Tymoczko is at pains to point out that he is not suggesting that the 4CT is untrue but that it is not "truth" as mathematicians have used the word heretofore and that its different nature springs from the fact that its proof relies on the use of a computer.

Although the issues raised by Tymoczko are undoubtedly important to the world of mathematics, it is difficult to see how he can justify his main thesis; and I will endeavor to show that it is more reasonable to continue to regard all mathematical truths as a priori—no matter how they are arrived at—although it may indeed be necessary to introduce a new mathematical entity intermediate between a conjecture and a theorem.

Tymoczko correctly points out that one of the main attractions of the four-color problem lies in the fact that it is "so simple to state that a child can understand it" (p. 57). Yet he goes on to assert that "the four-color problem is not a formal question" (p. 79).

In point of fact it is difficult to conceive of a more clearly formulated problem. It is every bit as well formulated as Fermat's last theorem or Goldbach's conjecture; and whatever the nature and status of the existing proofs of the 4CT may be, they can hardly serve to convert the problem from a formal question into an empirical one.

There is one unfortunate error of fact in Tymoczko's otherwise remarkably lucid thumbnail exposition of the existing method of proof. He says (p. 60) that in order to establish reducibility it must be proved that "every four-coloring of the ring around a given configuration can either be extended to the configuration, or modified first by one or more Kempe chain interchanges and then extended or modified by suitable identification of distinct vertices and then extended." The type of modification mentioned creates what is referred to in four-color terminology as a "reducer" for the original configuration.

Yet even in the case of classical hand reductions by Birkhoff [2], Winn [3], and others, reducers were never restricted to those obtained by identifying vertices and have always involved other constraints, such as requiring vertices to be adjacent. In point of fact any configuration that is smaller than the original configuration is a potential reducer, provided only that it has fewer vertices and/or edges and is not capable of creating a loop in the modified graph. And, without the use of a wide variety of different kinds of reducers, there is no hope of establishing the truth of the four-color conjecture within the framework of the existing methods of proof.

Most surprisingly, Tymoczko fails to recognize the fact that in so far as there is any weakness in the Haken/Appel proof of the 4CT it lies not so much in the reducibility testing—which is

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almost certainly correct and has been independently corroborated to a large extent—but in the discharging procedure, which gives rise to the unavoidable set of configurations.

It is by no means an easy matter to check that the final working list of 1482 unavoidable configurations embraces all the unavoidable configurations arising from their discharging procedure. Thus the Haken and Appel proof *is* subject to some degree of uncertainty, and Tymoczko is definitely incorrect when he says that no mathematician "has argued against it." At least one mathematician has openly stated that, in his opinion, they "failed to establish a crucial proposition" [4]. And those mathematicians who do not reject the proof in such a summary manner are not necessarily persuaded that the Haken/Appel discharging procedure is indubitably without flaws.

It should perhaps be pointed out that there are now two proofs of the 4CT: the original one by Haken, Appel, and Koch and the more recent one by F. Allaire [5]. Allaire's proof also involves a discharging/reducibility approach but only requires some 50 hours of computer time. It is, moreover, based on an entirely different discharging procedure and a completely independently developed reducibility testing program. At the very least Allaire's proof must rank as an independent corroboration of the truth of the four-color conjecture, and there can be little doubt that even if the Haken/Appel proof is flawed the theorem is nevertheless true.

Tymoczko chooses to compare the use of computers in mathematical theorem proving on earth to the use by Martians of a mathematical genius called Simon (pp. 71–72), who justifies his lemmas and even some theorems (particularly those of a combinatorial nature) by appeals to personal verification, moral rectitude, or political stature. He further compares the use of computers to the practice of some of Simon's followers of using the phrase "Simon says" for establishing the truth of difficult theorems—without actually obtaining Simon's seal of approval.

Even a cursory examination shows that this is a most unconvincing analogy. No one claims that computer results are correct because computers are morally upright or politically sound or enjoy great status, and mathematicians are certainly not in the habit of reporting computer-assisted results that they have not actually run on a computer (quite apart from anything else, someone else may carry out the necessary computer investigation and establish the erroneous nature of their claims).

The very reason those of us who have worked on reducibility testing are happy about Haken, Appel, and Koch's reducibility results is that they have to a large extent been independently checked by the use of different programs on different computers.

One of the problems is that Tymoczko fails to do the reader the service of supplying a coherent definition of the term "a priori truth," which is commonly taken to mean a truth that perforce possesses universal and necessary validity or a truth whose validity is independent of the impressions of our senses. Tymoczko appears to favor the latter definition (p. 77) but gives no reason for his rejection of the former, although he alludes to it in several places (e.g., p. 78 and p. 80); nor does he appear to realize that the latter definition is not really viable and, at the very least, is extremely nonutilitarian in the absence of some qualification.

These two definitions are not a priori (I use the term in an informal sense) the same; and, as I shall endeavor to show, it is really only the first that is a viable utilitarian definition, although the latter can be modified so as to be compatible with it.

It is certainly immediately obvious that there are many a priori truths that can be verified by recourse to our senses. The example cited by Tymoczko himself (p. 59) is a case in point. The truth concerned is the assertion that the sum of the first one hundred strictly positive integers is 5050. Using Gauss's procedure we can verify this truth by multiplying 101 by 50 in our heads. However, we can also verify it by guiding our pencils across a piece of paper—using our sense of touch—and literally writing down the first 100 integers; and then—by the use of our sense of sight and a few extra pencil marks—we can actually carry out the addition of the numbers in question, once again obtaining the result 5050.

In this case the use of actual pencil and paper and the use of sight to verify the truth in question is not essential, since it can be verified using the Gaussian approach in our heads [6];

but, as I shall show, there are many mathematical truths that cannot be verified "in our heads" and can only be accessed by recourse to our physical senses—to the actual carrying out of a type of experiment.

At this stage it is convenient to consider the four-color theorem in the context of other well-established graph-theoretic theorems that do not make use of computers and which Tymoczko would therefore regard as a priori truths [7].

1. The Four-Color Theorem in Context. The proof of many graph-theoretic theorems (as well as theorems from other branches of mathematics) falls naturally into three parts.

(i) Establishing the fact that the theorem is true provided a certain finite set of graphs, configurations, or—in general—cases possess (or do not possess, as the case may be) a stated property.

(ii) Obtaining an exhaustive listing of these cases.

(iii) Confirming that all the members of this set do possess the required property.

The finite set of cases concerned may, at one extreme, be so small and so simple that the case testing can be done in our heads, or it may, at the other extreme, be so large and/or so complicated that it is impossible to carry out without the help of a computer.

I shall use as an example a theorem due to O. Ore and J. Stemple [8] concerning the four-color conjecture, which may be stated as follows:

Every triangulated planar graph which has less than 40 *vertices is* 4*-colorable.*

Ore and Stemple obtained their result by reducibility testing coupled with a type of discharging that differs both from that used by Haken and Appel and from that used by Allaire. All of Ore's reducibility results were obtained by hand, and the discharging section of the proof involved the testing of 42 cases to ensure that they all discharged to an appropriate extent.

The figures and calculations involved covered a total of 118 pages and were too bulky to be published in a journal. They were thus written out in incredibly neat longhand and lodged in the Mathematics Department's library at Yale University (Haken, Appel, and Koch ran into a similar problem as regards the listing of their unavoidable configurations and had to resort to the use of microfiche). The referees no doubt checked the set of 42 cases carefully, but I do not think that many other people did so and I believe I can count myself as one of the few people who actually went to the trouble of obtaining a photocopy and checking it thoroughly by hand.

The point I wish to make is that carrying out the case testing necessary to establish the truth of Ore's theorem was, and is, an extremely time-consuming and tedious affair and there is literally no possibility of carrying out the case testing in our heads. Yet no one, to my knowledge, has ever suggested that Ore's theorem is not an a priori truth.

In practice, theorems (or potential theorems) involving case testing may fall into four categories:

(i) Those theorems in which the case testing can be done in our heads.

(ii) Those theorems in which the case testing is impossible to carry out without the help of pencil and paper.

(iii) Those theorems in which the testing can be carried out with immense effort by means of pencil and paper—requiring, say, several thousand man hours of effort.

(iv) Those theorems which are entirely beyond the reach of hand calculation and for which the case testing *has* to be carried out by computer [9].

The divisions between these categories are not clear-cut and they tend rather to shade into each other. Moreover, a given theorem has no necessary permanent location in one particular category, and the method we use for its proof is not entirely forced upon us by its category.

Many theorems in category (i) will not, in practice, be handled in our heads, since, except in the simplest cases, a resort to pencil and paper is more convenient and reliable. Although theorems in category (iii) can be proved by hand, recourse to a computer would often (though

not always) be a natural route to follow. And even in the case of theorems in category (ii) it may well be more appropriate to resort to a computer when there are many elaborate cases to check —once again for the sake of convenience and reliability.

As of now Ore's theorem is in category (ii) and the 4CT itself is in category (iv). But no one has proved that this is a permanent categorization, for these two theorems and the development of more powerful mathematical techniques may, in the course of time, change the existing categorization of theorems out of all recognition. Who can say what might not happen in a hundred or even fifty years' time?

It thus seems rather farfetched to classify theorems as a priori or a posteriori on the basis of their present categorization. Are we to assume that the 4CT is, as of now, an a posteriori theorem but that it may in time become an a priori one? This is surely a most odd use of the term a priori.

Tymoczko concedes (p. 71) that a simple proof of the 4CT cannot be ruled out entirely but contends that it can nevertheless be conveniently regarded as a paradigm for theorems whose proofs must perforce be permanently computer assisted. But even this class of theorems, if it exists, does not drive "a wedge" (p. 75) between mathematical theorems in this class and the remaining theorems in mathematics—neither in terms of reliability nor in a deeper philosophical sense.

The primality of some primes can be verified by hand—by sieving or other techniques—whereas the primality of certain large primes can only be verified by doing the necessary checks on a computer. Are we to assume that the former are a priori primes and the latter a posteriori ones?

Tymoczko seems to have an almost naïve faith in the reliability of human beings as opposed to computers and makes this very rash statement:

> The reliability of the 4CT, however, is not of the same degree as that guaranteed by traditional proofs, for this reliability rests on the assessment of a complex set of empirical factors.

This statement is clearly indefensible, as can be readily seen by considering the theorems in category (iii). If people choose to prove such theorems using traditional methods of pencil and paper, Tymoczko would be forced, on his own cognizances, to regard such theorems as being a priori truths in the traditional mold and, in terms of the above statement, of greater reliability than proofs of the same theorems in which the case testing is done on a computer.

In point of fact the complete reverse is the case, and for such theorems computer-assisted proofs would often be more reliable than traditional proofs.

Human beings get tired, and their attention wanders, and they are all too prone to slips of various kinds; a hand-checked proof may justifiably be said to involve a "complex set of empirical factors." Computers do not get tired and almost never introduce errors into a valid implementation of a logically impeccable algorithm.

I would go so far as to say that any lack of reliability of the present proofs of the 4CT resides less in the use of a computer for the reducibility testing and more in the fact that a computer was not used to create the unavoidable set of configurations arising from the discharging procedure. I will justify this statement in somewhat more detail in the next section, but before doing so it is appropriate, at this stage, to explore the meaning of the term "a priori" in some detail.

The term a priori as applied to any truth may mean:

(i) a truth that possesses universal and necessary validity; that is to say, a truth that is true in all possible worlds, *or*

(ii) a truth whose validity can be established without recourse to sense experience of the physical world.

Tymoczko concedes (p. 78) that, when using the first definition, the 4CT is no different from any other theorem and bases his contention that the 4CT is not an a priori truth on the second definition (p. 77). We can only assume that he regards the first definition as invalid (or as having

reference to some other kind of truth), and in focusing attention on the second definition he appears to confuse the nature of truth and the manner in which we come to know its truth.

He says (p. 60) that "mathematical truths are known a priori," which is tantamount to saying that mathematical truths are known to be true before we have proved them to be true. This might be a happy state to be in, but it is very far from the realities of mathematical research.

If we accept the fact that mathematical theorems *are* a priori truths, then in terms of definition (i) they are certainly true *prior* to and *independently of* the uncovering of an actual proof by an actual human being—but it says nothing about the manner in which we uncover their truth. If, in order to uncover their truth, it is necessary to use pencil and paper or a calculator or even a computer, this can make no difference to the nature of their truth.

What then are we to make of definition (ii)? Surely the only reasonable course of action is to qualify it in some way or other so as to bring it into accord with definition (i). This could be done, for example, by changing it to read:

(ii)′ a truth whose validity can in principle be established without recourse to sense experience of the physical world.

This brings definition (ii) into line with definition (i) and implies that an a priori truth is one that can be uncovered by a sentient being, with a brain sufficiently large for the purpose, without recourse to any experiment. It does not imply that the particular type of sentient beings that happen to have arisen on earth, which we choose to call Homo sapiens, may not need to resort to an experiment to *know* the truth of a specific a priori truth **[10]**. Such experiments do not convert a priori truths into a posteriori ones, and I will consider the true distinction between a priori and a posteriori truths in somewhat more detail in the final section.

If we do not take definition (i) as a valid definition of a priori and stick to definition (ii) in its unmodified form, then any philosophical problems raised by the 4CT arose a long time ago when men took to writing symbols in the sand. Computers are really just a highly sophisticated and highly efficient form of automated pencil and paper, and if there is any philosophical distinction of significance between the various categories of mathematical theorems it surely lies between category (i) and all the rest.

Those theorems in category (i) are indubitably a priori truths (whatever definition we use), since a particular type of sentient being actually has verified (or can verify) them by pure unaided reason. The rest do not enjoy the privilege of such purely abstract verification, and in terms of Tymoczko's concept of a priori they are all a posteriori.

Tymoczko (pp. 79–80) spends some time discussing Haken and Appel's heuristic argument, which led them to believe that it was probably possible to prove the 4CT by investigating configurations with a ring size no greater then 14, and he suggests (p. 80) that:

> This probability cannot be accounted for in ontological terms according to which any statement is true or false in all possible worlds.

A much clearer example of a probability statement concerns the search for large primes on a computer. In some cases, while it has not yet proved possible to establish the primality of a given large number, it is possible using Monte Carlo methods to assess the probability that the number in question is a prime. Such probability assignments are well founded and unambiguous, but they hardly raise ontological questions of much moment.

A statement to the effect that a particular integer has a particular probability of being prime does not purport to be an ontological statement, and it is thus not necessary to account for it in ontological terms. The only ontological reality is the a priori primality or nonprimality of the integer in question.

Mathematicians who work in this area distinguish such Monte Carlo methods from what they call "rigorous algorithms for primality testing" **[11]**. And although they may use such numbers for practical purposes (e.g., in cryptography) as if they were primes, their primality remains an openly unproven conjecture.

We should not let the word *probability* beguile us into drawing false analogies between

probabilistic statements in the physical sciences and probabilistic statements concerning as yet opaque truths in mathematics. When physicists assess the probability of an electron's presence in a particular location at a particular time, they mean either that the electron spends an appropriate fraction of its time in the location in question or that an electron has no specific location in space. They are in fact caught in the problem of wave/particle duality and the uncertainty that this brings.

When mathematicians say that a particular number has probability x of being prime, they do not mean that it spends a fraction x of its time as a prime and a fraction $1-x$ of its time being factorizable. Nor do they mean that it is some new kind of number that is neither prime nor factorizable (or both prime and factorizable). All they are doing is assessing the likelihood of its primality being worth pursuing or worth counting on (for cryptography or some other purpose).

There is therefore no need even to consider modifying "the concept of proof" (p. 80) to include probabilistic arguments, since they neither prove nor purport to prove anything and are merely potentially useful in directing our efforts into appropriate channels. Haken and Appel's proof did not depend on their probability arguments, which merely encouraged them to pursue the matter to completion, and they have no bearing on the nature of the truth that they uncovered.

Tymoczko recognizes this point (p. 80) when he suggests that probabilistic methods might with profit be added to our repertoire of tools for tackling theorem proving, since they can serve to avoid wasted effort or encourage perseverance in the case of difficult theorems. The only thing that can be added is that this is their sole role and their sole import as far as theorem proving is concerned.

2. Algorithms and the Four-Color Theorem. Tymoczko singles out three characteristics of mathematical proofs (p. 59), namely:

(a) proofs are convincing,
(b) proofs are surveyable,
(c) proofs are formalizable,

and in addition to contending that the 4CT is an a posteriori truth rather than an a priori one he contends that although it is formalizable it is not surveyable.

This view seems to rest on an unnecessarily narrow conception of what surveyability involves and an insufficient appreciation of the manner in which the formalization and surveying of a proof interact with each other.

There is one point on which there can be little doubt and that is that the four-color theorem (as opposed to the four-color problem) has not as yet been adequately formalized, and I intend to outline the manner in which this might be done and to show that once this has been done it will become much more readily surveyable. In order to appreciate the lack of formalization in the present proofs and the extent to which they can, as of now, be surveyed, it is first of all necessary to consider the subject of algorithms and their implementation on computers.

Just as many proofs in graph theory involve case testing, so also many proofs are concerned with establishing the efficacy of algorithms. As an example of a graph-theoretic algorithm we may take the so-called "greedy" algorithm, which finds a spanning tree of maximum weight in a graph on n vertices in which each edge has been assigned a specific weight (we may assume for the purpose of the present discussion that each edge has an integer weight or that the set of weights has been scaled so as to make them integer). The greedy algorithm proceeds as follows:

1. Choose an edge of maximum possible weight that
   (a) has not yet been chosen
   (b) does not form a circuit with those edges that have already been chosen.
2. If the number of edges chosen is $n-1$ then stop, else return to step 1. □

The reason for the appellation "greedy" is self-evident, and it is not at all difficult to prove that

this patently naïve algorithm does achieve what it purports to achieve. Moreover, the algorithm is so simple to implement that it can easily be implemented by hand for small and medium-sized graphs. And each time we apply it we uncover a new mathematical truth, namely, that the weighted graph in question has the resulting spanning tree of maximum weight. Moreover, on his own cognizances, Tymoczko would have to accept it as an a priori truth.

But what if the graph is so large that we are forced to resort to a computer? We then need the assurance not merely that the algorithm is sound but that its computer implementation (usually in terms of some high-level computer language) is likewise sound. Tymoczko touches on the subject of evaluating computer programs (p. 74) but does not capture the flavor of the current state of the art. The branch of computer science that deals with ensuring that the computer implementation of a particular algorithm in a particular high-level language achieves what it purports to achieve is now highly sophisticated and well understood.

Some of the more recent high-level languages, such as Pascal, that use top down parsing and avoid "go to" statements, practically force the programmer to implement algorithms in a logically impeccable manner.

It is true that computer programs do sometimes have "bugs," but so do attempts to prove theorems by means of pencil and paper. It is also true, as Tymoczko says, that the "flaws in programs may sometimes go unnoticed for a long time" (p. 74). But so do flaws in proofs that have nothing to do with computers. The first published "proof" of the 4CT, by Kempe, *was* flawed, and it was not until 10 years later that Heawood uncovered the flaw.

The point that Tymoczko seems to miss entirely is that flaws in the computer implementation of algorithms are nothing other than *errors of logic*, no different in essence from errors that crop up in proofs that have nothing to do with computers.

So if the greedy algorithm is implemented on a computer by means of a program in one or another high-level language, which conforms to the requirements of a valid implementation, then any spanning tree of maximum weight found by means of such a computer-implemented algorithm would surely also be a new a priori truth. The fact that it was found by computer has no effect on its nature whatsoever.

It is true that in some cases algorithms with logically sound implementations that are themselves logically sound do not always produce the results they should when run on a computer. This is because computers are finite and we are thus forced either to truncate or to round off irrational numbers or numbers with a recurring binary representation. For example, many algorithms for inverting matrices do not work very well on ill-conditioned matrices, and much of computer-oriented numerical analysis is concerned with overcoming such problems.

But such an unhappy situation does not apply to the greedy algorithm (for a graph with integer weights) or to the algorithm used for reducibility testing, both of which are restricted to operations involving integers or Boolean truth values.

What then can we say about the determination of reducibility on a computer? The algorithm itself is so simple **[12]** that its correctness is not at issue, and any doubts must reside in its computer implementation. Unfortunately, although the algorithm itself is very simple, its computer inplementation is anything but trivial; and the number of logical checks required to confirm the reducibility of a 13 or 14 ring configuration is so large that the actual programs for the proofs were written in assembler language in order to make them as efficient as possible.

This raises problems as regards any formal check on the correctness of the programs, since the formal checks used at present apply to high-level language programs rather than assembler language programs. Nevertheless the reducibility results in the Haken, Appel, and Koch paper have been indirectly surveyed in a manner very little different from the hand surveying of other proofs involving a large amount of case testing—namely, by means of independently written computer programs run on different computers.

In fact, the case testing for the reducibility results in the 4CT has probably been more thoroughly surveyed than the case testing in some other theorems that are not dependent on the

computer. Moreover, as pointed out above, it is rather the discharging procedure, which has been carried out by hand, that has conceivably not been adequately surveyed.

But even in this regard the theorem has been indirectly surveyed by virtue of Frank Allaire's proof, which uses a different discharging procedure and nevertheless arrives at the same conclusion. When hand-checking a proof, referees sometimes adopt precisely this procedure. They check a particular facet of the proof by means of a closely related but not necessarily identical argument.

So, all in all, it is fair to say that although the surveyability of the 4CT leaves much to be desired it has been fairly effectively surveyed and is no worse off in this regard than theorems concerning other equally intractable problems.

It is not surprising that the initial proofs of important and difficult theorems such as the 4CT sometimes appear in a manner that is anything but formalized, and the interaction between formalization and surveyability may well proceed through several stages before such proofs become truly satisfactory ones. It seldom happens, moreover, that mathematicians insist on formalizing a proof to the extent of literally presenting it in terms of, say, Zermelo-Fraenkel set theory. It is usually regarded as adequate to formalize it to a sufficient extent to satisfy the cognoscenti that it could be so formalized if one wished to do so. And there is surely no reason for the 4CT to be formalized to a greater extent than any other theorem.

It is, however, perfectly legitimate to contend that the proof has not as yet been adequately formalized, if for no other reason than that this makes it very difficult to survey it properly. But contrary to Tymoczko's view, this difficulty has very little to do with the reliance of the proofs on the use of a computer; and it is at least arguable that it arises partly because inadequate use has been made of the computer.

There are, moreover, certain steps that could be taken in the direction of formalizing the proof that would make it fairly readily surveyable, and it is appropriate to give some indication of what these steps might be—namely:

(1) The discharging procedure and the resulting creation of the unavoidable set of configurations could be computerized. (This might involve a judicious choice between various possible discharging schemes.)

(2) The progams for both the discharging and the reducibility testing could be written in a high-level language and adequately checked to ensure that they do implement the algorithms they purport to implement.

At this stage there would be no great difficulty attached to surveying the proof. And if surveyors so wished they could even rerun at least some of the computer programs on an entirely different computer to assure themselves that their surveys were sufficiently thorough. As pointed out above, this has already been done to quite a large extent as far as the reducibility testing is concerned.

It is perhaps appropriate to begin to draw this section to a close with some discussion of the obvious first requirement of mathematical proofs—namely, that they should be convincing. At this juncture in history the 4CT has not been properly integrated into graph theory as a whole and stands to some extent as a monument on its own, but there is little doubt that this is not its permanent lot.

Indeed, it already has strong connections with at least some branches of graph theory that have no direct reliance whatsoever on computer programs. Several mathematicians, such as Walter Stromquist, Frank Bernhart, and Frank Allaire, who did research on the question of reducibility also developed a coherent theory of irreducibility that is in complete agreement with the reducibility results that have been obtained thus far on the computer. Moreover, in the light of such irreducibility theory, it became possible to determine anti-configurations for all planar configurations that are not freely reducible. And Frank Allaire was able to make excellent use of such anti-configurations in finding reducers for intractable reducible configurations. Such developments can surely only serve to strengthen the confidence that mathematicians have in the truth of the 4CT.

And in the years to come, when the theorem is even more inextricably intertwined with graph theory as a whole, it will seem not a little quaint to even suggest that it is not an a priori theorem with a surveyable proof. The four-color conjecture served as an excellent stimulus to graph-theoretic research in general and the 4CT may continue to exert a benign influence on graph theory until such time as it has been brought into "the body of the kirk."

Having said all this there may well be a justifiable nagging doubt in the mind of the reader that an issue of some importance has been evaded by not facing up to it directly. Even though there are convincing reasons for regarding all mathematical theorems as a priori truths and even though there is very little reason for regarding the use of computers for theorem proving as the driving of a wedge between one kind of theorem and another, we are still left with lengthy proofs (whether achieved by hand or on a computer) that have been neither adequately formalized nor adequately surveyed and are suggestive rather than definitive—simply because mathematicians have only a finite amount of time and energy at their disposal.

Such "proofs" do indeed have every appearance of being mathematical experiments of less than even the bare minimum of reliability, and since the number of such "proofs" is likely to grow it would do mathematicians no harm to start thinking about them in advance.

Mathematicians already work with lemmas, propositions, theorems, corollaries, and conjectures; perhaps they need a new kind of entity that lies somewhere between a theorem and a conjecture. Perhaps these additional entities could be called agnograms, meaning thereby theoremlike statements that we have verified as best we can but whose truth is not known with the kind of assurance we attach to theorems and about which we must thus remain, to some extent, agnostic.

Such agnograms would have a status somewhat different from that of conjectures but their elevation to the status of theorems would have to await the arrival of a more adequately formalized and surveyable proof. I do not myself feel that the 4CT falls into this category, but it might well be regarded as a forerunner of the many genuine agnograms to come.

In addition to invoking the concept of an agnogram we may also need to wean ourselves away from the natural inclination to think that in order to qualify as a theorem an agnogram must not only be made surveyable but be made surveyable by a single mathematician on his own and be literally surveyed by at least two or three mathematicians working completely independently. Computer scientists have found that when writing lengthy programs it is essential to break them down into self-contained modules, each of which can be checked on its own. If the interconnections between the modules are likewise checked, this serves to ensure that the program as a whole is logically sound.

And the factors that come into play in a lengthy program are much the same as the factors that come into play in lengthy proofs of mathematical theorems. If such lengthy proofs are broken down into modules and each module is surveyed on its own and the interconnection between the modules is likewise surveyed in detail, there is no reason that the proofs should not be regarded as proofs of actual mathematical theorems. (Mathematicians, of course, already do this to some extent by using lemmas and corollaries.) And, if the surveying of the proof has to be carried out by a team of mathematicians each examining a different module rather than by a single individual, this should not, of itself, be regarded as invalidating the proof.

3. A Priori and a Posteriori Truths. It has been suggested above that:

> An a priori truth is one with universal and necessary validity that can in principle be established without recourse to sense experience of the physical world.

We have, however, not as yet defined what we mean by an a posteriori truth. Traditionally, a posteriori truths have been regarded as truths that can only be known by means of experiments. But for our purpose this is a little too strong.

We have already argued that many a priori truths are known by means of a form of experiment and may, in at least some cases, be inaccessible to us human beings in any other

way. In just the same way it seems pointless to insist that all a posteriori truths must be the result of actual experiments. As a de facto phenomenon we certainly do come to know many truths about the physical world without carrying out actual experiments, and we make excellent use of such knowledge for practical purposes. A more appropriate definition would thus seem to be:

> An a posteriori truth is one whose truth is contingent upon the nature of the universe to which it applies and cannot in principle be known without carrying out at least some experiments.

There is a significant difference in the use of the phrase "in principle" in these two definitions. In the first it means "in principle though not in practice." But in the second it implies both "of necessity" and "by assumption," the assumption being that more than one kind of world is possible and that we must therefore of necessity carry out experiments to uncover the truth about the nature of our particular world.

What then *is* the difference between a priori truths and a posteriori ones if the touchstone of an a posteriori truth is an actual experiment and, at the same time, some a priori truths are inaccessible without an experiment? The difference lies surely in their antecedents. The basic a priori truths can be arrived at without recourse to experiments since they are truths that must appertain in all possible worlds, whereas the basic a posteriori truths can only be known by recourse to experiments since they are not necessarily true for all possible worlds. We may note, moreover, that the experiments involved in the two cases are very different in nature. The experiments we carry out to try to uncover a priori truths involve the manipulation of numbers and logical symbols. But the experiments used to uncover a posteriori truths involve the measurement of length, pressure, volume, mass, temperature, time, and many other physical quantities.

It should thus be clear that agnograms are neither a priori truths nor a posteriori truths, but conjectures to which we can attach a high degree of credence. And even if the 4CT is regarded as an agnogram it cannot be so classified simply because its proof depends on the use of a computer, and it certainly cannot be regarded as an a posteriori truth. Physicists do not accept into their ensemble of a posteriori truths the results of inadequately verified experiments; if mathematicians likewise feel unable to accept into their ensemble of a priori truths the results of inadequately surveyable proofs, it would be somewhat invidious to call such results a posteriori truths.

Making a distinction between a priori and a posteriori truths does, of course, raise many philosophical problems, and not everyone would agree that the distinction is a watertight one. Eddington's epistemological theory was based on the contention that the physical world as we know it is in a very deep sense the only possible world that can exist and attempted in his "Fundamental Theory" [**13**] to determine the basic characteristics of the universe from first principles.

His work has never appealed to a very wide audience and may well be hopelessly flawed in numerous respects. But the philosophy behind it may ultimately be more appropriate than any other alternative. If it ever became possible to determine a dimensionless quantity, such as the electron/proton mass ratio de novo, in a manner acceptable to physicists as a whole, this would certainly make the distinction between a priori and a posteriori truths very difficult to maintain.

And if the four-color problem has inadvertently caused philosophers to reflect anew on these issues, then it has served a purpose in a wider field than we graph theoreticians ever dared to hope.

I would like to thank my colleagues F. Allaire, R. C. Read, W. T. Tutte, and D. H. Younger for helpful comments on the first draft of this paper.

Notes and References

1. The momentous first proof of the four-color theorem by Haken, Appel, and Koch was announced in the *Bulletin of the American Mathematical Society*, 82 (5) (September 1976), was presented by Haken to a packed

audience at a meeting of the American Mathematical Society in Toronto in September 1976, and is published under the title "Every Planar Map is Four-Colorable" in the *Illinois Journal of Mathematics*, 21 (84) (September 1977) 429 – 567. Tymoczko's article "The Four-Color Problem and Its Philosophical Significance," appeared in the *Journal of Philosophy*, 76 (2) (February 1979) 57 – 83.

2. "The Reducibility of Maps," *American Journal of Mathematics*, 35 (1913) 114 – 128.

3. "On Certain Reductions in the Four-Color Problem," *Journal of Mathematics and Physics*, 16 (1938) 159 – 171.

4. G. Spencer-Brown, *The Laws of Form*, E. P. Dutton, New York, 1979, p. xii. Unfortunately Spencer-Brown does not say what the proposition is, so it is impossible to assess the validity of this contention.

5. "Another Proof of the Four-Color Theorem—Part I," *Proceedings of the Seventh Manitoba Conference on Numerical Mathematics and Computing*, 1977, pp. 3 – 72. Part II of this paper was also presented at the conference but has not yet been published.

6. By the time we are able to do such problems in our heads we have, of course, made extensive use of our senses in learning a wide range of a priori truths.

7. Not all mathematicians would, of course, accept all the current graph-theoretic proofs as valid, and I speak merely in terms of the general consensus among mathematicians. In particular, the intuitionist approach of L. E. J. Brouwer and the constructionist approach of E. Bishop rule out many forms of proof that are widely used. It would, however, take us outside the scope of this article to consider these approaches, which are not really germane to Tymoczko's arguments anyway. For a recent discussion, see A. Calder, "Constructive Mathematics," *Scientific American*, 141 (4) (October 1979) 146 – 171.

8. "Numerical Calculations on the Four-Color Problem," *J. Combinatorial Theory*, 8 (1970) 65 – 78.

9. Haken himself has suggested, in "An Attempt to Understand the Four-Color Problem," *Journal of Graph Theory*, 1 (1977) 193 – 206, that in the light of the 4CT proof this category may well contain a significantly large class of theorems and that it is at least worth investigating whether or not other well-known intractable conjectures can also be solved by computer. There may be, of course, some a priori truths that are inaccessible to any form of proof, but these are not even potential theorems.

10. The reference to a hypothetical being with an adequately large brain may perhaps be regarded as philosophically suspect, but there is another way of looking at the problem.

The nature of the interface between a human being and the external world is a somewhat hazy one. We might argue that the surface of our skin delineates this interface—but what about the air in our lungs and the food in our digestive tract, and what if we are wearing clothes, as is normally the case. Do they alter the interface or not? They certainly make us *look* very different to an outside observer.

We might alternatively argue that it is the collection of living cells from which we are made that distinguishes us from the world outside—but what about our hair (which is dead) or the water and enzymes in our body (which are as essential to our existence as the living cells themselves), and what about any cancer or pre-cancer cells that may be present, or the bacteria in our digestive tract?

It seems at least a permissible point of view to argue instead that the interface between us and the external world is ever changing and somewhat indefinite. In terms of this view, carpenters' hitting nails into pieces of wood do indeed extend beyond their skin, are something more than just their living cells, and are, at any rate as an operational concept, people-with-hammers.

So, also, in the case of a scientist looking through a microscope at some bacteria, it is reasonable to regard the interface between the scientist and the external world as lying between the microscope and the specimen and not between the scientist's eye and the microscope. Such people are, in fact, people-with-microscopes. Likewise, a good musician playing a violin is often "at one" with the instrument and is very much a person-with-a-violin.

Such examples can be multiplied indefinitely, and it seems clear that by means of our tools and instruments we can extend the interface between ourselves and the outside world out into the world, both in terms of what our senses experience and in terms of our ability to alter the world around us.

And, since human beings in the act of checking a long mathematical proof by hand are, for the most part, acting as rather inefficient computers, the resort to a computer simply converts them into people-with-(more efficient) computers, which can certainly act as stand-ins for sentient beings with an adequately large brain. It may even be possible one day to wear a micro-computer directly on our heads.

The reason that the 4CT was not proved before the advent of computers may thus be simply because there were no sentient beings with adequately large brains around before that time. To maintain that a computer-assisted proof of a mathematical theorem does not lead to an a priori truth is thus suspiciously close to maintaining that bacteria do not exist because scientists can only see them with the help of a microscope.

11. See, for example, H. C. Williams, "Primality Testing on a Computer," *Ars Combinatoria*, 5 (June 1978) 127–186.

12. F. Allaire and E. R. Swart, "A Systematic Approach to the Determination of Reducible Configurations in the Four-Color Conjecture," *J. of Combinatorial Theory Ser. B*, 25 (3) (December 1978) 339–361.

13. *Fundamental Theory*, Cambridge University Press, 1949. See also A. S. Eddington, *The Philosophy of Science*, Cambridge University Press, 1939.

FORMULAE FOR THE CURVATURES OF CIRCLES IN CHAINS

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In this paper a *chain* is a sequence of circles each touching its two neighboring circles and two boundary curves which are either circles or straight lines. Formulae are given for the bends (i.e., curvatures) of the circles in any chain, and parametric expressions which give rational chains, that is, chains in which all the circles have rational bends.

The main results can be obtained by inversive methods [2], [4], [6], [8], the following lemmas being particularly useful. With respect to a circle center O and radius k, (i) *a line at a distance d from O inverts into a circle of bend $2d/k^2$*; (ii) *a circle radius r whose center is at a distance d from O inverts into a circle of bend $|(d^2-r^2)/k^2r|$*. (Wilker [9] ascribes these results to Morley [7], but they are to be found in older text books, e.g., [8].)

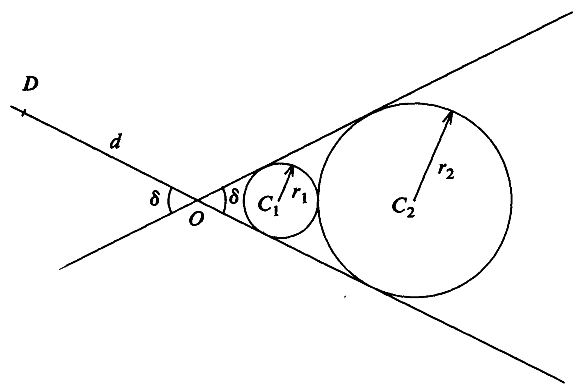

FIG. 1

1. Chains with Intersecting Bounds. (a) *Chains bounded by two straight lines*. If the bounds intersect at an angle δ (Fig. 1), and r_1, r_2 are radii of consecutive circles in the chain, then

$$r_2 = \frac{1+\sin\frac{1}{2}\delta}{1-\sin\frac{1}{2}\delta}\, r_1,$$

and, in general,

$$r_n = \left(\frac{1+\sin\frac{1}{2}\delta}{1-\sin\frac{1}{2}\delta}\right)^n r_0.$$

Our later work is made easier if we introduce a variable θ such that

$$\sinh\tfrac{1}{2}\theta = \tan\tfrac{1}{2}\delta, \tag{1}$$

which gives

$$r_n = r_0 e^{n\theta} \tag{2}$$

(b) *Chains bounded by a circle and an intersecting straight line*. If, in case (a), C_n is the center

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of the circle radius r_n, and D is a point on one of the bounds such that $DO=d$ (Fig. 1), we have

$$DC_n^2=d^2+r_n^2\operatorname{cosec}^2\tfrac{1}{2}\delta+2r_nd\cot\tfrac{1}{2}\delta.$$

Using lemma (ii) and equation (2) we find that the circle center C_n inverts with respect to a unit circle center D into a circle with bend

$$u_n=\frac{d^2}{r_0}e^{-n\theta}+r_0e^{n\theta}\cot^2\tfrac{1}{2}\delta+2d\cot\tfrac{1}{2}\delta.$$

Taking A and ε such that $A=2d\cot\frac{1}{2}\delta$, and $r_0e^{\varepsilon}=d\tan\frac{1}{2}\delta$, we obtain

$$u_n=A[1+\cosh(n\theta+\varepsilon)]. \tag{3}$$

The bounds invert into a circle and an intersecting line (Fig. 2). We take the radius of the circle

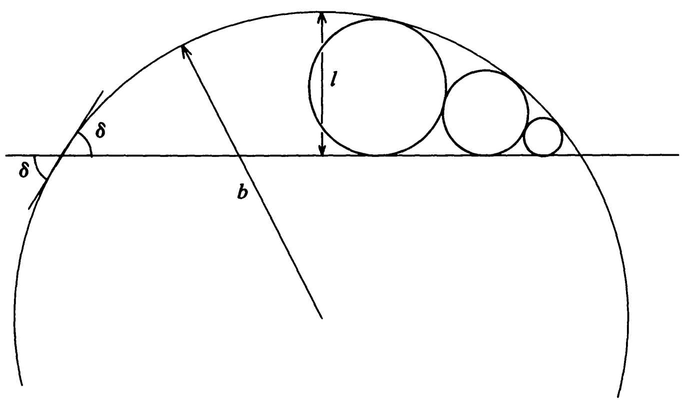

FIG. 2

as b and, if the height of the segment containing the chain is l, the largest possible circle in the chain has radius $\frac{1}{2}l$. Thus $A=1/l$. Clearly $\cos\delta=(b-l)/b$, so (1) gives

$$\cosh\theta=\frac{2b+l}{2b-l}. \tag{4}$$

So for any chain in that segment,

$$u_r=\frac{1}{l}[1+\cosh(r\theta+\varepsilon)]=\frac{2}{l}\cosh^2\tfrac{1}{2}(r\theta+\varepsilon). \tag{5a}$$

Outside the circle the region whose bounds intersect at an angle δ contains chains for which

$$u_r=\frac{1}{l}[-1+\cosh(r\theta+\varepsilon)]=\frac{2}{l}\sinh^2\tfrac{1}{2}(r\theta+\varepsilon). \tag{5b}$$

In either of these regions

$$u_{r+1}-2u_r\cosh\theta+u_{r-1}=\pm\frac{2}{l}(\cosh\theta-1),$$

i.e.,

$$(2b-l)u_{r+1}-2(2b+l)u_r+(2b-l)u_{r-1}=\pm 4, \tag{6}$$

with the positive sign for the outer region, and the negative for the finite segment. From (6) we see that if b and l are rational, any chain containing two adjacent rational circles has all its circles rational. For this to occur, it is sufficient to take ε such that $\cosh\varepsilon=(2b+\lambda^2l)/(2b-\lambda^2l)$, with rational λ such that $\lambda^2\leqslant 2b/l$.

If a chain is to contain a given circle, we may take its bend as u_0 and obtain $\cosh\varepsilon$ from (5a)

or (5b). Coshθ comes from (4), and when u_1 has been found from (5a) or (5b) the relation (6) will give the sequence of values of u_r. For example, in a semicircle of unit radius, $b=l=1$; so $\cosh\theta=3$. For the chain with $u_0=2$ we find $\cosh\varepsilon=1$, whence $u_1=4$. Using

$$u_{r+1}-6u_r+u_{r-1}=-4 \tag{7}$$

we find $u_2=18$, $u_3=100$, $u_4=578$, $u_5=3364, \ldots$.

It is possible to obtain an algebraic expression for u_r by solving (7), namely,

$$u_r=\tfrac{1}{2}\left[(3+2\sqrt{2})^r+(3-2\sqrt{2})^r+2\right].$$

(c) *Chains bounded by intersecting circles.* If we invert case (b) with respect to a unit circle whose center is on the axis of symmetry of the segment, we obtain chains bounded by circles S_1, S_2, say, whose radii may be taken as a,b, respectively, and whose centers are a distance c apart. If the arcs bounding the region $S_1\cap S_2$ cut at an angle δ, then

$$\cos\delta=-\frac{a^2+b^2-c^2}{2ab}=1-\frac{s}{2ab}, \tag{8}$$

where $s=(a+b)^2-c^2$.

It is possible to show that chains in this region have bends u_r of the form

$$u_r=A+B\cosh(r\theta+\varepsilon);$$

and, since δ is invariant under inversion, we still have

$$\sinh\tfrac{1}{2}\theta=\tan\tfrac{1}{2}\delta.$$

Thus

$$\cosh\theta=\frac{3-\cos\delta}{1+\cos\delta}=\frac{4ab+s}{4ab-s}. \tag{9}$$

The values of A and B can be shown to be $2(a+b)/s$ and $2c/s$, respectively; so

$$u_r=\frac{2}{s}[a+b+c\cosh(r\theta+\varepsilon)]. \tag{10}$$

This applies to any of the regions bounded by arcs of S_1 and S_2 if the following sign convention is used.

Region	a	b	c
$S_1\cap S_2$	+	+	+
$S_1'\cap S_2$	−	+	−
$S_1\cap S_2'$	+	−	−
$S_1'\cap S_2'$	−	−	+

(In the case $S_1'\cap S_2'$, negative values of u_r may be obtained: these correspond to circles which surround the bounds.) In all cases successive bends are related by

$$u_{r+1}-2u_r\cosh\theta+u_{r-1}=\frac{4(a+b)}{s}(1-\cosh\theta). \tag{11}$$

To obtain rational chains it is sufficient to take a,b,c rational and ε satisfying $\cosh\varepsilon=(4ab+\lambda^2 s)/(4ab-\lambda^2 s)$ for rational λ with $\lambda^2\leqslant 4ab/s$. For example, given S_1,S_2 with radii 5,6, respectively, and centers 9 units apart, for chains in $S_1\cap S_2'$ we set $a=5$, $b=-6$, $c=-9$; so $s=-80$ and $\cosh\theta=5$. Thus (10) becomes $u_r=[1+9\cosh(r\theta+\varepsilon)]/40$, and (11) gives $u_{r+1}=10u_r-u_{r-1}-\frac{1}{5}$. If $\lambda=\frac{1}{2}$, then $\cosh\varepsilon=7/5$ and we find the sequence

$$\ldots, u_{-3}=\tfrac{2294}{50}, u_{-2}=\tfrac{233}{50}, u_{-1}=\tfrac{26}{50}, u_0=\tfrac{17}{50}, u_1=\tfrac{134}{50}, u_2=\tfrac{1313}{50}, \ldots .$$

Clearly by varying λ we can find any number of different rational chains with the given bounds.

2. Chains with Nonintersecting Bounds. We shall adopt a sign convention that, although perhaps not the most obvious, brings out the similarity between the results in 1 and 2.

(d) *Chains bounded by nonintersecting circles.* If the bounds are external to one another, both radii will be taken as negative; if one bound surrounds the other, the larger will be taken as positive, the smaller as negative. Analogous to the angle of intersection of the bounds in 1 is the *inversive distance* δ between nonintersecting circles [3]. This is invariant under inversion. With our sign convention we have

$$\cosh\delta = -\frac{a^2+b^2-c^2}{2ab} = 1-\frac{s}{2ab}, \tag{12}$$

where a,b are the radii of the circles, c is the distance between the centers, and $s=(a+b)^2-c^2$. Note that, if $c=0$, then

$$\delta=\ln\left(-\frac{a}{b}\right) \quad \text{and} \quad \frac{a}{b}=-e^{\delta}. \tag{13}$$

In Fig. 3, the two circles with center O, radii α,β, bound a chain of equal circles with centers

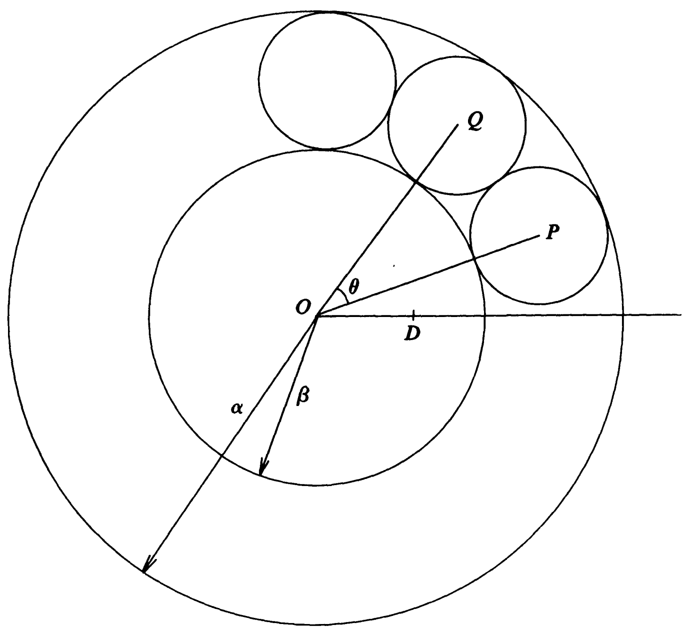

FIG. 3

P,Q, etc., and radii p. With respect to a unit circle with center D, the circle with center Q inverts into one with bend

$$u=\left|\frac{OD^2+OQ^2-2OD\cdot OQ\cos\phi-p^2}{p}\right|$$

which is of the form $A+B\cos\phi$, where $\phi=$ angle DOQ, and A,B are constants. The bounds invert into circles whose radii may be taken as a,b (with $b<0$), and with centers c apart. With our sign convention the extreme possible bends for circles in the inverse chain are $2(a+b-c)^{-1}$ and $2(a+b+c)^{-1}$. Also ϕ is of the form $r\theta+\varepsilon$, where r is an integer, θ is angle POQ and ε is

constant for a given chain. Clearly $\sin\frac{1}{2}\theta=(\alpha+\beta)/(\alpha-\beta)$, whence, using (13),

$$\sin\tfrac{1}{2}\theta=\frac{e^{\delta}-1}{e^{\delta}+1}=\tanh\tfrac{1}{2}\delta. \tag{14}$$

Since δ is invariant under inversion, we have for the inverse configuration, using (14) and (12), that

$$\cos\theta=\frac{3-\cosh\delta}{1+\cosh\delta}=\frac{4ab+s}{4ab-s}. \tag{15}$$

Thus

$$u_r=\frac{2}{s}[a+b+c\cos(r\theta+\varepsilon)], \tag{16}$$

and

$$u_{r+1}-2u_r\cos\theta+u_{r-1}=\frac{4(a+b)}{s}(1-\cos\theta). \tag{17}$$

The analogy between equations (9), (10), (11) and (15), (16), (17) is obvious.

For rational chains we may take a,b,c rational and

$$\cos\varepsilon=\frac{(1+\cos\theta)-\lambda^2(1-\cos\theta)}{(1+\cos\theta)+\lambda^2(1-\cos\theta)}=\frac{4ab+\lambda^2 s}{4ab-\lambda^2 s} \tag{18}$$

with $\lambda abs<0$ and λ rational.

To construct a chain with a given value of θ we may take

$$\begin{aligned} a&=b(\mu-1)[\mu(\cosh\delta-1)-(\cosh\delta+1)]/2\\ &=b(\mu-1)\left(\mu\tan^2\tfrac{1}{2}\theta-\sec^2\tfrac{1}{2}\theta\right)/\mu, \end{aligned}$$

and

$$\begin{aligned} c&=b\left[\mu^2(\cosh\delta-1)-(\cosh\delta+1)\right]/2\mu\\ &=b\left(\mu^2\tan^2\tfrac{1}{2}\theta-\sec^2\tfrac{1}{2}\theta\right)/\mu, \end{aligned} \tag{19}$$

with $b<0, \mu\neq 0$. All possibilities are obtained if $\mu>-1$; c may be negative, but this has no significance in the geometrical configuration.

If $\theta=2m\pi/n$, where m and n are co-prime positive integers, we obtain a *ring* of circles (or Steiner chain [2], [3], [5], [6]), that is, a closed chain that surrounds one of the bounds m times and comprises n distinct circles. It is well known that bounds that contain one such ring contain an infinity of rings. Rational rings exist for $m=1$ with $n=3$, 4, or 6. They can be obtained from (18) or from the table below given rational values for λ, μ, and any one of a, b, or c.

Number of circles in ring	$\cos\varepsilon$	$\frac{a\mu}{b(\mu-1)}$	$\frac{c\mu}{b}$	$\frac{s}{ab}$
3	$\frac{1-3\lambda^2}{1+3\lambda^2}$	$3\mu-4$	$3\mu^2-4$	-12
4	$\frac{1-\lambda^2}{1+\lambda^2}$	$\mu-2$	μ^2-2	-4
6	$\frac{1-3\lambda^2}{1+3\lambda^2}$	$\mu-4$	μ^2-4	$-4/3$

For example, the bounds for a 3-ring with $\mu=-2$ and $b=-2$ have $a=30$, $c=8$ and $s=720$. For

the circles in the ring $u_r=[7+2\cos(2r\pi/3+\varepsilon)]/90$. If we take $\lambda=\frac{1}{2}$, then $\cos\varepsilon=1/7$, and $\cos(2r\pi/3+\varepsilon)$ takes values $1/7,-13/14,11/14$ for $r=0,1,2$. Thus $u_0=17/210, u_1=2/35, u_2=2/21$. Any such ring can be scaled to give integral radii. In this example, with a scale factor of 34 we find that circles radii 1020 and 68 and centers 272 apart contain a ring of circles of radii 420, 595, 357.

It is easy to see that bounds which contain one rational ring contain an infinity of such rings.

We now generalize a result given by Coxeter [1]. The expression $(A+B\cos\phi)^k$ can be written in the form $\Sigma_{j=0}^{k}a_j\cos j\phi$, where the a_j are independent of ϕ. Now for any ring ϕ takes all values of the form $2mr\pi/n+\varepsilon$ for $0\leqslant r<n$, with m and n co-prime positive integers. But it is easy to prove that

$$\sum_{r=0}^{n-1}\cos j\left(\frac{2mr\pi}{n}+\varepsilon\right)=0,\quad \text{for}\quad 0<j<n,\quad j \text{ an integer.}$$

Thus, if

$$S_k=\sum_{r=0}^{n-1}u_r^k, \text{ where } k \text{ is an integer,}\quad 0\leqslant k<n,$$

we have

$$S_k=\sum_{r=0}^{n-1}\left(\sum_{j=0}^{k}a_j\cos j\phi\right)=\sum_{j=0}^{k}\left(\sum_{r=0}^{n-1}a_j\cos j\phi\right)=na_0,$$

where a_0 depends only on the value of a,b,c. We see, therefore, that if given bounds contain n-rings, the sum of the kth powers of the bends (for any k such that $0\leqslant k<n$) is constant for all rings with these bounds. Coxeter proves the constancy of S_0, S_1, and S_2 only.

The first few values of S_k are

$$\begin{aligned}
&S_0=n,\\
&S_1=\frac{2n}{s}(a+b),\\
&S_2=\frac{2n}{s^2}(2s+3c^2), && (n\geqslant 3),\\
&S_3=\frac{4n}{s^3}(a+b)(2s+5c^2), && (n\geqslant 4),\\
&S_4=\frac{2n}{s^4}(8s^2+40sc^2+35c^4), && (n\geqslant 5).
\end{aligned}$$

In general

$$S_k=\frac{n}{s^k}\sum_j\binom{k}{2j}\binom{2j}{j}[2(a+b)]^{k-2j}c^{2j},$$

with the summation over all integer values of j with $0\leqslant 2j\leqslant k$, and with k an integer such that $0\leqslant k<n$.

(e) *Chains bounded by a circle and a nonintersecting straight line.* This case arises by taking D on one of the bounds (Fig. 3), or as a limiting case of (d) where $a\to\infty, c\to\infty, |a/c|\to 1$. We take the circle to have negative radius b and the line to be at a distance l from the nearest point of the circle. It may be shown that $\cosh\delta=(b-l)/b$ ([3]), whence

$$\cos\theta=\frac{2b+l}{2b-l}. \tag{20}$$

We find

$$u_r = \frac{1}{l}[1+\cos(r\theta+\varepsilon)] = \frac{2}{l}\cos^2\tfrac{1}{2}(r\theta+\varepsilon), \tag{21}$$

and

$$u_{r+1} - 2u_r\cos\theta + u_{r-1} = \frac{2}{l}(1-\cos\theta),$$

that is,

$$(2b-l)u_{r+1} - 2(2b+l)u_r + (2b-l)u_{r-1} = -4. \tag{22}$$

Equations (20), (21), (22) are analogous to (4), (5a), (6).

For b and l rational we obtain rational chains if $\cos\varepsilon = (2b+\lambda^2 l)/(2b-\lambda^2 l)$ with λ rational. Rings are obtained if $\theta = 2m\pi/n$, with m and n co-prime integers; and rational rings with 3, 4, or 6 circles if $l=-6b, \cos\varepsilon = (1-3\lambda^2)/(1+3\lambda^2)$; $l=-2b, \cos\varepsilon=(1-\lambda^2)/(1+\lambda^2)$; or $l=-2b/3, \cos\varepsilon=(3-\lambda^2)/(3+\lambda^2)$, respectively. For example, taking $b=-1$ and $\lambda=\frac{1}{2}$, we obtain a 4-ring if $l=2$, the curvatures being $4/5, 1/10, 1/5, 9/10$. This is equivalent to a ring with radii 45, 360, 180, 40 bounded by a circle radius 36 and a line 108 from its center.

The bends of n-rings with given bounds have constant sums of kth powers for $0 \leqslant k < n$. Here

$$\begin{aligned} &S_0 = n, \\ &S_1 = n/l, \\ &S_2 = 3n/2l^2, \qquad (n \geqslant 3), \\ &S_3 = 5n/2l^3, \qquad (n \geqslant 4), \end{aligned}$$

and, in general,

$$S_k = \binom{2k}{k} n/(2l)^k \quad \text{for} \quad 0 \leqslant k < n.$$

(f) *Chains bounded by parallel lines*. These trivial chains are inversely equivalent to the following.

3. Chains with Bounds That Touch. Wilker gives a detailed discussion of this case in [**9**]. With the sign convention of 2 it can be shown that if the bounds have *bends* α, β, then

$$u_r = \frac{\alpha\beta}{\alpha+\beta} - (\alpha+\beta)(n+\mu)^2$$

for arbitrary μ, and

$$u_{r+1} - 2u_r + u_{r-1} = -2(\alpha+\beta).$$

For rational chains we take α, β, μ rational.

The following supplementary results are offered without proof. Any three mutually touching circles can be regarded as a 3-ring. If the bends are x,y,z, where $z=(\lambda^2-xy)/(x+y)$, then the bounds have curvatures $-x-y-z\pm 2\lambda$, and we have a rational configuration for rational x,y,λ. In general, the distance between the centers of the bounds is irrational, being $(\alpha^2+\beta^2+14\alpha\beta)^{1/2}/\alpha\beta$, where α, β are the curvatures of the bounds.

A 3-ring with its bounds contains six "triangular" regions each bounded by arcs of three of the circles. If the ring and its bounds are rational, then the circles inscribed in each of the six triangles will be rational. These in turn create new triangular regions, each of which can contain a rational inscribed circle. This process can be continued indefinitely to give a configuration bounded by a circle and containing an infinity of rational circles each touching an infinite number of the others. The simplest way of beginning the construction of such a net of circles is

to take two rational circles which touch externally, together with the smallest circle which surrounds them.

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FOURSOMES, FIVESOMES, AND ORGIES
or
UPON READING POEMS BY FREDERICK SODDY

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Four mutually touching circles
have radial reciprocals such that
the square of their sum
    is two times the sum of their squares.

Five mutually touching spheres
have radial reciprocals such that
the square of their sum
    is three times the sum of their squares.

Six cyclically touching spheres
    which each touch each of
        three mutually touching spheres
have radial reciprocals such that
the sum of the cyclical six
    is two times the sum of three cyclical alternates
        and three times the sum of two cyclical opposites
            and six times the sum of the mutual three.

Author's Note. Frederick Soddy (1877–1956) won the 1921 Nobel Prize for Chemistry for his conceptualization of isotopes. His mathematical verses appear in volumes 137 and 138 of *Nature* and are discussed by Soddy and others in volume 139. By the way, the radial reciprocals of $n+2$ mutually touching n-spheres have the property that the square of their sum is n times the sum of their squares. The result on four circles was known to Descartes in 1643.

ROBBINS'S THEOREM FOR MIXED MULTIGRAPHS

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1. Introduction. We consider the question, When is it possible to find an assignment of one-way directions for all the streets in a town while preserving the property that it is possible to reach any point in town from any other point? In a classic paper Robbins [7] answered this question for a town that has no one-way streets.

We resolve the more general case for a town in which some, but not all, of the streets have already been made one-way. We show that if reachability has been preserved in arriving at this initial condition, then there is some assignment of directions to convert the remaining two-way streets to one-way if and only if it was originally possible to find an assignment for converting all the streets to one-way. Specifically it is shown that the decisions regarding the assignment of directions to the individual streets can be made one at a time, in any order, if a solution is possible at all. The original Robbins Theorem is not used in the proof. Since the Robbins Theorem is a special case, we also have a new proof of that result.

We shall comment on another generalization of the question that arises from introducing a penalty function for making the conversion. Finally we include a short discussion of algorithms for these problems.

We follow Harary [6] for all basic notation and terminology. For the convenience of the reader, however, we review a few basic concepts here. A *graph or undirected graph* (V,X) has a finite *point set* V and a *line set* X which consists of two-element subsets of V. A *digraph* or *directed graph* (V,Y) has a finite point set V and an *arc set* Y containing ordered pairs from V.

A *multigraph* allows for more than one line between a pair of points, and a *multidigraph* allows more than one arc in the same direction between a pair of points. A multigraph is said to be *connected* if there is a path between every pair of distinct points. A multidigraph is called *strongly connected* if there is a dipath (directed path) between every ordered pair of distinct points. A *bridge* of a connected multigraph G is a line whose removal disconnects G; if G has no bridges it is called *bridgeless*.

Robbins's problem as a graph-theoretic question amounts to choosing a direction for each line of a multigraph G so that the resulting multidigraph $D(G)$ is strongly connected. If this is possible, then the original multigraph G is called *strongly orientable*. Robbins's theorem verifies that a multigraph is strongly orientable if and only if it is connected and bridgeless.

2. Preliminaries. In order to proceed with the case of both one-way and two-way streets, we define an appropriate graph-theoretic model. A *mixed multigraph* G is an ordered triple (V,E,η) where V is a finite set called *points*, E is a finite set called *edges*, and η is a function assigning to each element $e \in E$ either an ordered pair from V or a two-element subset of V. If $\eta(e)$ is the

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ordered pair (u,v), we say e is a *directed edge from u to v*; otherwise e is an *undirected edge*. Clearly, the mixed multigraph includes a multigraph and a multidigraph as special cases. We say that *directing an undirected edge e* of G means choosing one of the two possible orderings of its endpoints and changing the value of $\eta(e)$ accordingly. An *orientation* of a mixed multigraph G is any multidigraph obtained by directing each undirected edge of G. Similarly, we say that *undirecting a directed edge e* of G means changing the value of $\eta(e)$ from the ordered pair (u,v) to the unordered pair $\{u,v\}$. The *underlying multigraph* of a mixed multigraph is the multigraph obtained by undirecting all the directed edges.

An appropriate generalization of Robbins's problem is to consider when there exists a strongly connected orientation of a mixed multigraph. To answer this, we define a few more terms. A *walk* from u to v in a mixed multigraph G is an alternating sequence of points and edges $u=v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n=v$ such that each e_i has $\eta(e_i)=\{v_{i-1},v_i\}$ or $\eta(e_i)=(v_{i-1},v_i)$. When such a walk exists, we say that v is *accessible* from u in G; we also stipulate that any point is accessible from itself. Finally, we define a mixed multigraph as *connected* if every pair of its points is mutually accessible from each other.

3. The General Robbins Theorem. Clearly, a mixed multigraph can be viewed as the result of a partial attempt to strongly orient a multigraph. Certainly it seems possible that such an attempt to assign directions edge by edge may lead to a mixed multigraph G representing a cul-de-sac, in which no further progress toward a strongly connected multidigraph can be made, despite the fact that G may be connected. That this does not occur is the main result of this paper.

THEOREM 1. *Let e be an undirected edge of a connected mixed multigraph G. Then e may be directed to produce another connected mixed multigraph if and only if e is not a bridge of the underlying multigraph of G.*

Proof. Since the "only if" portion of the theorem is obvious, we proceed by showing that if neither orientation of e produces a connected mixed multigraph, then e is a bridge of the underlying multigraph of G.

Let u and w denote the endpoints of e. We show first that there are no walks between u and w in $G-e$. For suppose there is a walk W from u to w in the mixed multigraph $G-e$. Then setting $\eta(e)=(w,u)$ results in a connected mixed multigraph, since any occurrence of $\ldots\ u,\ e,\ w\ \ldots$ in a walk can be replaced by W so that e need not be traversed in the direction from u to w. The hypothesis that neither orientation of e produces a connected mixed multigraph is contradicted; thus the walk W cannot exist. Likewise there is no walk W' from w to u in $G-e$.

Now let U be the set of points in G that are accessible in $G-e$ from u, and let v be some point in U. We claim that u is accessible from v in $G-e$. This follows from the fact that u is accessible from v in G; and if this accessibility required e, then w would be accessible in $G-e$ from u.

Define $Y=V-U$, where V is the set of all points, and note that Y is not empty as $w\in Y$. Furthermore, we claim that all points in Y are accessible in $G-e$ from w. To verify this we merely note that $t\in Y$ must be accessible from w in G; and if this accessibility required e, then it would be accessible in $G-e$ from u.

The proof can now be completed by showing that e is the only line of the underlying multigraph that has one endpoint in Y and the other in U. Clearly the definition of U precludes the existence of an undirected edge between U and Y as well as precluding a directed edge from U to Y. Finally, if there were a directed edge from Y to U, then there would be a walk W' in $G-e$ from w to u because u is accessible in $G-e$ from each point of U and each point of Y is accessible in $G-e$ from w. But we showed above that W' cannot exist.

An obvious inductive proof now produces Robbins's theorem for mixed multigraphs.

THEOREM 2. *A mixed multigraph G has a strongly connected orientation if and only if G is connected and the underlying multigraph of G is bridgeless.*

It should be noted that the original form of the Robbins Theorem, which is a special case of Theorem 2, was not used here in the proof of Theorem 2. We thus have a new proof of the original Robbins Theorem.

4. Another Extension. Another generalization of the Robbins Theorem follows from defining a penalty function for directing the lines of a multigraph. One obvious measure involves the distances between points. For a multigraph (multidigraph), the number of lines (arcs) in a path (dipath) is called the *length* of that path (dipath). The distance $d(u,v)$ from point u to point v in a multigraph (multidigraph) is the length of the shortest path (dipath) from u to v. The diameter $d(G)$ of a multigraph or multidigraph is the largest of the distances between all pairs of points. We could then measure the cost of some orientation D_i by $d(D_i(G))-d(G)$, and we would then wish to minimize this difference over all possible orientations. Thus we define

$$\rho(G)=\min_i \{d(D_i(G))-d(G)\}$$

It is interesting to calculate $\rho(G)$ for some special classes of multigraphs. We give the result here for two cases: The *complete graph K_p* has exactly one line between each pair of the p points of V. If the point set V is partitioned into two subsets V_1 and V_2 such that each point of V_1 is joined to each point of V_2 by one line and there are no other lines, then G is called the *complete bigraph $K_{m,n}$* where $|V_1|=m$ and $|V_2|=n$.

THEOREM 3.

$$\begin{aligned} \rho(K_p)&=1 \quad \text{for } p\geqslant 3 \text{ but } p\neq 4\\ \rho(K_4)&=2\\ \rho(K_{n,n})&=1 \quad \text{for } n\geqslant 2. \end{aligned}$$

Proof. We first observe that $d(K_p)=1$. Now, since K_p has no multiple lines between any pair of points, it follows that either $d(u,v)\geqslant 2$ or $d(v,u)\geqslant 2$ in $D_i(K_p)$. Hence, $\rho(K_p)\geqslant 1$. To show that $\rho(K_p)=1$, we produce an orientation of K_p with diameter two for $p\neq 4$. Now it is easily verified directly that diameter-two orientations exist for K_3 and K_6 but not K_4. (One finds in the process that $\rho(K_4)=2$.) The general result is then obtained by induction applied separately to the cases of p odd and p even; the details are as follows: Define the join of two graphs $G_1=(V_1,X_1)$ and $G_2=(V_2,X_2)$ as a graph G_1+G_2 whose point set is $V_1\cup V_2$ and whose line set consists of $X_1\cup X_2$ together with all possible lines joining V_1 with V_2. We then represent K_k as $K_{k-2}+K_2$, where K_2 is formed from any two points u and v of K_k. Now orient $\{u,v\}$ from u to v; orient into u all other lines connected to u, and orient away from v all other lines connected to v. By the induction hypothesis K_{k-2} can be oriented to produce diameter two if $k-2=3$ or $k-2\geqslant 5$. It is a simple matter to verify that the resulting orientation of $K_{k-2}+K_2$ has diameter two. Notice that the starting case of $k=3$ yields all the odd values; the starting case of $k=6$ is required to verify the even values, as K_4 cannot be oriented to produce diameter two.

Turning now to $K_{n,n}$, it is easy to see that $\rho(K_{n,n})\geqslant 1$ since $d(K_{n,n})=2$ and in $D_i(K_{n,n})$ the length of any path connecting $u\in V_1$ with $v\in V_2$ must be odd. The desired result is therefore obtained by producing a diameter-three orientation of $K_{n,n}$. This may be accomplished by choosing any n mutually disjoint lines and orienting them all from V_1 to V_2, with all remaining lines oriented from V_2 to V_1. It is easy to see that any pair of points in the resulting digraph is connected by a dipath of length 1, 2, or 3.

Unfortunately it does not appear likely that one can produce a reasonable procedure for determining ρ for a general graph because Chvátal and Thomassen **[4]** have shown that the problem of deciding whether an arbitrary undirected graph admits an orientation of diameter two belongs to the class of problems called NP-hard **[1]**.

Concluding Remarks. There are several proofs of the Robbins Theorem in the literature [2], [3], [7], [8]. The proof given by Roberts [8] is particularly interesting because it yields an efficient algorithm that produces a strongly connected orientation of a multigraph. The algorithm is based on the "depth-first search procedure," which is a term used in computer science to denote a certain process of searching a graph [1]. Although the search process is identical to the Trémaux maze search [5], the modern interpretation is important because it has been used to derive a number of valuable properties of the algorithm. The depth-first search process can also be used when the initial input is a mixed multigraph. We shall not prove this here but merely comment that a proof can be constructed by using the results given in [1] and [7].

As noted above, the diameter generalization poses a difficult problem for which there is no known efficient algorithm. It is interesting, in this regard, that the depth-first search algorithm can produce very large diameters; e.g., depth-first search can yield diameter $p-1$ for its orientation of K_p.

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ITERATED BINOMIAL COEFFICIENTS

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1. Introduction. It is traditional in mathematics to explore iteration of operations, and when these are noncommutative and nonassociative there are usually quite interesting properties to be uncovered. Thus, there is certainly nothing bizarre about considering such expressions as c^{b^a}, the result of iterating the exponential operation. Far less common, however, is discussion of the expression

$$\binom{\binom{c}{b}}{a},$$

the result of iterating the "binomial coefficient operation." Yet this expression even has a natural combinatorial interpretation: "the number of subsets of size a which can be formed from the collection of subsets of size b from a class of size c"; that is, the number of a-tuples of b-tuples in a set of size c. Perhaps it is because the notation is ungainly, especially when we try to

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consider further iterates, such as

$$\left[\begin{pmatrix}\binom{d}{c}\\ b\end{pmatrix}\atop a\right],$$

that this subject has been neglected. (The older notation was no better:

$$C_a{}^{C_b^c} \quad\text{and}\quad C_a^{C_b{}^{C_c^d}},$$

etc!)

Let us therefore establish a notation more amenable to typesetting. Define $(a_1;a_2;\ldots;a_k)$ inductively by

$$(a_1)=a_1,\quad (a_1;a_2)=\binom{a_1}{a_2},\quad\text{and}\quad (a_1;a_2;\ldots;a_{k-1};a_k)=((a_1;a_2;\ldots;a_{k-1});a_k).$$

Thus

$$(a_1;a_2;a_3)=\left[\binom{\binom{a_1}{a_2}}{a_3}\right].$$

For $k\leqslant 3$, we will frequently continue to use the "vertical notation" as well as the "horizontal notation," because of its greater familiarity and visual impact.

In this paper, we will explore identities, inequalities, equations, divisibility properties, extremal problems, and reduction formulas involving iterated binomial coefficients. This is not intended as a definitive treatment, but as an initial exploration into rich new territory.

2. The First Identity.

THEOREM 1.

$$\binom{\binom{n}{2}}{2}=3\binom{n+1}{4}.$$

Proof I (Algebraic).

$$(n;2;2)=\left(\frac{n^2-n}{2};2\right)=\frac{1}{2}\left(\frac{n^2-n}{2}\right)\left(\frac{n^2-n-2}{2}\right)=\frac{1}{8}(n+1)(n)(n-1)(n-2)=3\binom{n+1}{4}.\quad\blacksquare$$

Proof II (Combinatorial). $(n;2;2)$ is the number of pairs of 2-subsets from a set of n objects. If we pick 4 elements (say a,b,c,d) from the set of n objects, there are 3 ways to arrange them into two pairs (ab vs. cd, ac vs. bd, ad vs. bc). We do *not* get $(n;2;2)=3(n;4)$, because in a pair of 2-subsets there may be an element in common. Let us enlarge our deck of n cards by the addition of one joker, and now take a sample of size 4. If we find four ordinary cards, there are 3 ways to arrange them into two pairs. If we find three ordinary cards (say a,b,c) and the joker, there are 3 ways (again!) to form two pairs (ab vs. ac, ab vs. bc, ac vs. bc). Thus, $(n;2;2)=3(n+1;4)$. ■

Theorem 1 is simultaneously a reduction formula (from a three-tiered to a two-tiered binomial coefficient), an inequality (that $(n;2;2)>(n;4)$ for all $n\geqslant 3$), and a divisibility result (that $(n+1;4)$ divides $(n;2;2)$, where both expressions are regarded as polynomials in n over the rational field). These are all among the directions which we will explore further in the rest of this paper.

The content of Theorem 1 first appeared in **[1]**.

3. The Divisibility Result.

THEOREM 2. *For all $k\geqslant 1$,*

$$\binom{n+k-1}{2k} \text{ divides } \left[\begin{matrix}\binom{n}{2}\\ \binom{k}{2}+1\end{matrix}\right],$$

where both expressions are considered as polynomials in n over the field of rational numbers.

Proof. We observe that

$$\left(\begin{matrix}\binom{n}{2}\\ a\end{matrix}\right)=\frac{(n^2-n)(n^2-n-2)(n^2-n-4)\cdots(n^2-n-2a+2)}{2^a\cdot a!}.$$

The quadratics corresponding to $a=1,2,4,7,\ldots,\binom{k}{2}+1,\ldots$ factor as follows:

$$n^2-n=n(n-1),n^2-n-2=(n+1)(n-2),n^2-n-6=(n+2)(n-3),$$
$$n^2-n-12=(n+3)(n-4),\ldots,n^2-n-2\binom{k}{2}=(n+k)(n-k-1),\ldots.$$

Thus

$$\left[\begin{matrix}\binom{n}{2}\\ \binom{n}{k}+1\end{matrix}\right]$$

includes, among its polynomial factors,

$$\prod_{j=-k-1}^{k}(n+j)=(2k)!\binom{n+k-1}{2k}.$$

Since constant factors, such as $(2k)!$, do not affect divisibility of polynomials over the field of rationals, the result follows. ■

Note. Only for $k=1$ and $k=2$ do the two polynomial expressions in Theorem 2 have the same degree. For $k>2,(n;2;(k;2)+1)$ contains irreducible quadratic factors not found in $(n+k-1;2k)$.

4. Comparison of $(c;b;a)$ **with** $(c;(b;a))$. For positive integers a, b, and c, we seek all solutions to the "associative" equation

$$\left(\begin{matrix}\binom{c}{b}\\ a\end{matrix}\right)=\left(\begin{matrix}c\\ \binom{b}{a}\end{matrix}\right).$$

THEOREM 3. *Except for the situation where* $1<a<b<c$, *all solutions of*

$$\left(\begin{matrix}\binom{c}{b}\\ a\end{matrix}\right)=\left(\begin{matrix}c\\ \binom{b}{a}\end{matrix}\right)$$

are of the following five types:

(i) $a=1$, *for all* b *and* c.

(ii) *if* $a>b$, *solution occurs iff* $a=\binom{c}{b}$.

(iii) *if* $b>c$, *solution occurs iff* $a<b$.

All other solutions involve $1<a\leqslant b\leqslant c$, *where*

(iv) *if* $a=b$, *solution occurs iff* $c=a+1$.

(v) *if* $b=c$, *solution occurs iff* $a<c-1$.

Proof.

(i) With $a=1$, $\binom{\binom{c}{b}}{1}=\binom{c}{b}=\binom{c}{\binom{b}{1}}$.

(ii) If $a>b$, then $\binom{b}{a}=0$, and $\binom{c}{\binom{b}{a}}=\binom{c}{0}=1$.

Then $\binom{\binom{c}{b}}{a}=1$ if and only if $a=\binom{c}{b}$.

(iii) If $b>c$, then $\binom{\binom{c}{b}}{a}=\binom{0}{a}=0$. Then $\binom{c}{\binom{b}{a}}=0$ if and only if $\binom{b}{a}>c$. Since $c\geqslant 1$, and $b\geqslant c$, this excludes $a\geqslant b$, but allows all $a<b$.

(iv) If $a=b$, $\binom{c}{\binom{b}{a}}=\binom{c}{1}=c$. Then $\binom{\binom{c}{b}}{a}=\binom{\binom{c}{a}}{a}=c$ occurs iff $a=1$ or $a=c-1$. Since $a=1$ is covered under Case (i), the only "new" solution is $a=b=c-1$.

(v) If $b=c$, and $a>1$, then $\binom{\binom{c}{b}}{a}=\binom{1}{a}=0$. Then $\binom{c}{\binom{b}{a}}=\binom{c}{\binom{c}{a}}=0$ if and only if $\binom{c}{a}>c$; that is, iff $1<a<c-1$.

As noted, there are no other solutions unless $1<a<b<c$. However, in this case, empirical evidence appears to indicate

$$\binom{\binom{c}{b}}{a}>\binom{c}{\binom{b}{a}}$$

for "all" choices of a, b, and c. Appearances are in fact deceptive, for while it is true that there are no counterexamples with $(c;b;a)<10^{18}$, there is a sense in which the counterexamples outnumber the supporting instances. The situation is as follows:

THEOREM 4. *For fixed a and b with $\binom{b}{a}>ab$, there is a constant $c_0=c_0(a,b)$ such that*

$$\binom{\binom{c}{b}}{a}<\binom{c}{\binom{b}{a}}$$

for all $c\geqslant c_0$.

Proof. Regarding n as variable and k as fixed,

$$\binom{n}{k}\sim\frac{n^k}{k!}\quad\text{as } n\to\infty.$$

Thus,

$$\binom{\binom{c}{b}}{a}\sim\binom{\frac{c^b}{b!}}{a}\sim\frac{c^{ab}}{a!(b!)^a}\quad\text{as } c\to\infty$$

while

$$\binom{c}{\binom{b}{a}}\sim\frac{c^{\binom{b}{a}}}{\binom{b}{a}!}\quad\text{as } c\to\infty.$$

Since, by assumption, $\binom{b}{a}>ab$, we have $c^{\binom{b}{a}}$ growing faster than c^{ab}; and even though $\binom{b}{a}!>a!(b!)^a$, these are mere constants whose effect is overcome for sufficiently large c. ■

Notes. 1. The "smallest" cases where $\binom{b}{a}>ab$ include $15=\binom{6}{2}>2\cdot 6=12$, and $21=\binom{7}{2}>$

$2\cdot 7=14$. Extensive numerical investigation has determined that the smallest $c_0=c_0(a,b)$ is $c_0(2,7)=75$, which also minimizes $(c_0;b,a)$ and $(c_0;(b;a))$, which have the values

$$\left[\begin{pmatrix}\binom{75}{7}\\ 2\end{pmatrix}\right]\approx 1.97\times 10^{18}<\left[\begin{pmatrix}75\\ \binom{7}{2}\end{pmatrix}\right]\approx 2.10\times 10^{18}.$$

2. For $a=2$, $b=6$, we find $c_0(2,6)=132$, with

$$\left[\begin{pmatrix}\binom{132}{6}\\ 2\end{pmatrix}\right]\approx 2.143\times 10^{19}<\left[\begin{pmatrix}132\\ \binom{6}{2}\end{pmatrix}\right]\approx 2.154\times 10^{19}.$$

3. The actual value of c_0 is somewhat larger than the value c_1 of c which solves

$$\frac{c^{\binom{b}{a}}}{\binom{b}{a}!}=\frac{c^{ab}}{a!(b!)^a}.$$

The asymptotic relation $\binom{n}{k}\sim n^k/k!$ as $n\to\infty$ may be replaced by the identity $(n/k)=\bar{n}^k/k!$, where $\bar{n}=\text{g.m.}\ (n,n-1,\dots,n-k+1)\approx n-\frac{1}{2}(k-1)$, where "g.m." refers to the geometric mean, which is less than the arithmetic mean $n-\frac{1}{2}(k-1)$. In setting

$$\begin{pmatrix}c\\ \binom{b}{a}\end{pmatrix}=\frac{\bar{c}^{\binom{b}{a}}}{\binom{b}{a}!}\quad\text{and}\quad\begin{pmatrix}\binom{c}{b}\\ a\end{pmatrix}=\frac{\hat{c}^{ab}}{a!(b!)^a},$$

it is easy to show that $\bar{c}>\hat{c}$, which explains why $c_0>c_1$.

4. Whether there are solutions of

$$\begin{pmatrix}\binom{c}{b}\\ a\end{pmatrix}=\begin{pmatrix}c\\ \binom{b}{a}\end{pmatrix}$$

with $1<a<b<c$ remains an open question. It seems quite unlikely that such solutions exist, but it appears very difficult to prove this.

5. A Simple Inequality. For any set of n objects large enough to form triples, there are more triples of pairs than pairs of triples. That is,

THEOREM 5.

$$\begin{pmatrix}\binom{n}{2}\\ 3\end{pmatrix}>\begin{pmatrix}\binom{n}{3}\\ 2\end{pmatrix}\quad \textit{for all } n\geqslant 3.$$

Proof.

$$\begin{pmatrix}\binom{n}{2}\\ 3\end{pmatrix}=\frac{\dot{n}^6}{48}>\begin{pmatrix}\binom{n}{3}\\ 2\end{pmatrix}=\frac{\tilde{n}^6}{72},$$

since $\dot{n}>\tilde{n}$ as well as $72>48$. ∎

Notes. 1. The proof shows that in fact

$$\begin{pmatrix}\binom{n}{2}\\ 3\end{pmatrix}>\frac{3}{2}\begin{pmatrix}\binom{n}{3}\\ 2\end{pmatrix}$$

holds, for all $n\geqslant 3$. The inequality of Theorem 5 will be applied in the next section. No obvious purely combinatorial reason for this inequality has yet been offered.

2. For the analogous problem, $a^{(b^c)}=(a^b)^c$, see [2]. For this problem, with integers a,b,c

which satisfy $1<a<b<c$, the strict inequality $a^{(b^c)}>(a^b)^c$ holds.

3. A generalization of Theorem 5 is that if $1<a<b<c$, then

$$\binom{\binom{c}{a}}{b}>\binom{\binom{c}{b}}{a}.$$

The proof of Theorem 5 can be paralleled closely. In particular, if $1<a<b$, then $a!(b!)^a> b!(a!)^b$.

6. Maximization Over Partitions. We are interested in the partition of n into positive integers, $n=a_1+a_2+\cdots+a_k$, which maximizes the iterated binomial coefficient $(a_1;a_2;\ldots;a_k)$. To motivate the methodology and anticipate the form of the solution, we will first consider two similar but simpler problems, both of which have appeared in the literature. (For Theorem 7, see [3].)

THEOREM 6. *Let $g(n)=\max a_1\cdot a_2\cdots a_k$, where the product is maximized over all partitions of n into positive integers, $n=a_1+a_2+\cdots+a_k$. Then for $n>1$,*

$$g(n)=\begin{cases} 3\cdot 3\cdots 3=3^{n/3} & \text{if } n\equiv 0 \pmod 3\\ 3\cdot 3\cdots 3\cdot 4=4\cdot 3^{(n-4)/3} & \text{if } n\equiv 1 \pmod 3\\ 3\cdot 3\cdots 3\cdot 2=2\cdot 3^{(n-2)/3} & \text{if } n\equiv 2 \pmod 3\end{cases}.$$

Proof. By exhaustive verification, $g(n)=n$ for $n\leqslant 4$. For $n\geqslant 5$, it is clear that the "optimality principle" applies, i.e.,

$$g(n)=\max_{1<a<n} a\cdot g(n-a).$$

Moreover, $a<5$ because $5g(n-5)<2\cdot 3g(n-5)$ shows that n has a better partition than any containing 5 as a part; and a fortiori for $a>5$. For large n, the value of a which maximizes $a\cdot g(n-a)$ is found by comparing the three candidates: $2g(n-2), 3g(n-3), 4g(n-4)$. If we iterate each of these enough times to make them comparable, we find $2^6g(n-12)<3^4g(n-12)> 4^3g(n-4)$. Thus, for large n, the best value of a to use is $a=3$. In fact, for $n>4, g(n)=3g(n-3)$, which defines the general solution, which follows three lines of descent (since the "standard" value of a is 3), based on $g(2)=2$, $g(3)=3$, and $g(4)=4$.

THEOREM 7. *Let $h(n)=\max a_1^{a_2^{\cdot^{\cdot^{a_k}}}}$, where the maximum is taken over all partitions $n=a_1+a_2+\cdots+a_k$ of n into positive integers. Then for $n>4$,*

$$h(n)=\begin{cases} 2^{2^{\cdot^{\cdot^{2^{3^2}}}}} & \text{for } n \text{ odd}\\ 2^{2^{\cdot^{\cdot^{2^{3^3}}}}} & \text{for } n \text{ even}\end{cases}.$$

Proof. By exhaustive verification, $h(n)=n$ for $n\leqslant 4$. For $n\geqslant 5$, it is clear that the "optimality principle" applies, i.e.,

$$h(n)=\max_{1<a<n} a^{h(n-a)}.$$

Moreover, $a<4$ because $4^{h(n-4)}<2^{2^{h(n-4)}}$ as soon as $h(n-4)>2$. It remains only to compare $2^{h(n-2)}$ with $3^{h(n-3)}$, or $h(n-2)\log 2$ with $h(n-3)\log 3$. We have $h(n-2)\log 2>h(n-3)\log 3$ as soon as $h(n-2)/h(n-3)>\log 3/\log 2\approx 1.585$. This is easily seen to occur for all $n>6$. Thus the critical cases are $h(5)=3^2=9$ and $h(6)=3^3=27$, beyond which $h(n)=2^{h(n-2)}$. ∎

THEOREM 8. *Let $f(n)=\max(a_1;a_2;\ldots;a_k)$, where the maximum is taken over all partitions of n into positive integers. Then for $n>10$,*

$$f(n)=\begin{cases} (5;2;2;3;3;\ldots;3) & \text{if } n\equiv 0 \pmod 3\\ (5;2;2;2;2;3;3;\ldots;3) & \text{if } n\equiv 1 \pmod 3\\ (5;2;2;2;3;3;\ldots;3) & \text{if } n\equiv 2 \pmod 3\end{cases}.$$

For smaller n, $f(1)=1, f(2)=2, f(3)=3, f(4)=4, f(5)=5, f(6)=\binom{4}{2}=6, f(7)=\binom{5}{2}=10, f(8)=\binom{6}{2}$ $=15, f(9)=\left[\binom{\binom{5}{2}}{2}\right]=45, f(10)=\left[\binom{\binom{5}{2}}{3}\right]=120.$

Proof. The values of $f(n)$ for $n \leqslant 10$ can be verified by exhaustive search. For $n>10$, the "principle of optimality" holds:

$$f(n)=\max_{1<a<n}\binom{f(n-a)}{a}.$$

In fact, $a<4$, because

$$\binom{f(n-4)}{4}<\left[\binom{\binom{f(n-4)}{2}}{2}\right]$$

by Theorem 1, and a fortiori for $a>4$; so only $a=2$ and $a=3$ are possible choices. By Theorem 5, since

$$\binom{\binom{n}{2}}{3}>\binom{\binom{n}{3}}{2},$$

a "2" can never occur below a "3" in the expression for $f(n)$. Thus, for all $n>5$, $f(n)$ is either

$$\binom{f(n-2)}{2} \text{ or } \binom{f(n-3)}{3}.$$

We also note that

$$\binom{\binom{m}{3}}{3}\sim\frac{m^9}{1296} \text{ while } \left[\binom{\binom{\binom{m}{2}}{2}}{2}\right]\sim\frac{m^8}{128}$$

as $m\to\infty$, which implies that for all $m \geqslant m_0$,

$$\binom{\binom{m}{3}}{3}>\left[\binom{\binom{\binom{m}{2}}{2}}{2}\right],$$

and hence, for all $n \geqslant n_0$, $f(n)=\binom{f(n-3)}{3}$. Careful examination reveals that $n_0=14$, with the result that for $n \geqslant 14$, depending on the value of n modulo 3, $f(n)$ equals one of $(f(11);3;3;\ldots;3)$ or $(f(12);3;3;\ldots;3)$ or $(f(13);3;3;\ldots;3)$.

Notes. 1. The problems relating to the functions $g(n)$ and $h(n)$ in Theorems 6 and 7 readily generalize to maximizing over all partitions of the positive real number x into positive parts. (In both these problems, the real number $e=2.718\ldots$ is specially favored as a "part.") A similar generalization can be made for the $f(n)$ in Theorem 8 if we define $\binom{x}{a}$ as

$$\Gamma(x+1)/\Gamma(a+1)\Gamma(x-a+1).$$

2. If $j(n)=\max \text{L.C.M.}(a_1,a_2,\ldots,a_k)$, where again the maximum is taken over all partitions of n into positive integers, the solution has a quite different flavor, for the "principle of optimality," i.e.,

$$j(n)=\max_{1<a<n}\text{L.C.M.}(a,j(n-a))$$

does not hold. (The problem is that the value of a which maximizes $a \cdot j(n-a)$ may not be relatively prime to $j(n-1)$.) This maximization problem also has a literature [**4**].

7. Reduction Formulas. The result of Theorem 1 can be generalized as follows:

$$\binom{\binom{n}{1}}{2} = 1 \cdot \binom{n}{2}$$

$$\binom{\binom{n}{2}}{2} = 3\binom{n+1}{4}$$

$$\binom{\binom{n}{3}}{2} = 6\binom{n+2}{6} + 3\binom{n+1}{6} + \binom{n}{6}$$

$$\binom{\binom{n}{4}}{2} = 10\binom{n+3}{8} + 15\binom{n+2}{8} + 10\binom{n+1}{8}$$

$$\binom{\binom{n}{5}}{2} = 15\binom{n+4}{10} + 45\binom{n+3}{10} + 55\binom{n+2}{10} + 10\binom{n+1}{10} + \binom{n}{10}.$$

The general form is

$$\binom{\binom{n}{b}}{2} = \sum_{j=1}^{b} \begin{bmatrix} \binom{b}{j} + \varepsilon_j \\ 2 \end{bmatrix} \binom{n+b-j}{2b}, \quad \text{where } \varepsilon_j = \begin{Bmatrix} 1 & \text{if } j \text{ odd} \\ 0 & \text{if } j \text{ even} \end{Bmatrix}.$$

This formula may be interpreted as follows: Since $\binom{\binom{n}{b}}{a}$ is the number of pairs of b-tuples from a set of n objects, we basically must select $2b$ objects, which need not all be distinct. The expression $\binom{n+b-j}{2b}$ refers to the selection of $2b$ objects from the original set of n, augmented by the adjunction of $b-j$ "jokers," where $0 \leqslant b-j \leqslant b-1$, to allow for the fact that two distinct b-tuples may overlap in as few as 0 elements or as many as $b-1$ elements. The appropriate coefficient for this expression turns out to be

$$\begin{bmatrix} \binom{b}{j} \\ 2 \end{bmatrix} \quad \text{for even } j, \qquad \text{and} \quad \begin{bmatrix} \binom{b}{j}+1 \\ 2 \end{bmatrix} \quad \text{for odd } j.$$

The skeptical reader may wonder what the advantage is of a "reduction formula" which expresses

$$\binom{\binom{n}{b}}{2}$$

in terms of

$$\begin{bmatrix} \binom{b}{j} \\ 2 \end{bmatrix},$$

an expression "of the same type." The answer is that for fixed b, even with arbitrarily large (or variable) n, the coefficients

$$\begin{bmatrix} \binom{b}{j} \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \binom{b}{j}+1 \\ 2 \end{bmatrix}$$

are fixed and comparatively small numbers. This reduction formula may also be written

$$\left(\begin{pmatrix} n \\ b \end{pmatrix} \atop 2\right) = \sum_{k=0}^{b-1} \left[\left(\begin{pmatrix} b \\ k \end{pmatrix} + \varepsilon_k^b \atop 2\right) \right] \binom{n+k}{2b},$$

where

$$\varepsilon_k^b = \left\{ \begin{matrix} 0 & \text{if } b \equiv k \pmod 2 \\ 1 & \text{if } b \not\equiv k \pmod 2 \end{matrix} \right\}.$$

Another reduction formula is

$$\left(\begin{pmatrix} n \\ b \end{pmatrix} \atop 2\right) = \sum_{j=1}^{b} \binom{2j-1}{j} \binom{b+1}{2j} \binom{n}{b+j},$$

where the terms now have three factors, but no three-tiered binomial coefficients appear. Here, instead of adding "jokers" to the deck, only $b+j$ objects are selected from the n-card deck, and $b-j$ of them are used in both b-tuples.

Depending on which approach one uses, $(n;2;3)$ may be given either an "increasing" or a "decreasing" reduction:

$$\left(\begin{pmatrix} n \\ 2 \end{pmatrix} \atop 3\right) = \binom{n+1}{6} + 13\binom{n+2}{6} + \binom{n+3}{6} = \binom{n}{3} + 16\binom{n}{4} + 30\binom{n}{5} + 15\binom{n}{6}.$$

Similarly,

$$\left(\begin{pmatrix} n \\ 3 \end{pmatrix} \atop 3\right) = 4\binom{n+5}{9} + 80\binom{n+4}{9} + 120\binom{n+3}{9} + 65\binom{n+2}{9} + 10\binom{n+1}{9} + \binom{n}{9}$$

$$= 4\binom{n}{4} + 100\binom{n}{5} + 480\binom{n}{6} + 945\binom{n}{7} + 840\binom{n}{8} + 280\binom{n}{9}.$$

No simple reduction formulas have yet been found for the most general case of $\left(\begin{pmatrix} n \\ b \end{pmatrix} \atop a\right)$.

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44. Nature does not consist entirely, or even largely, of problems designed by a Grand Examiner to come out neatly in finite terms, and whatever subject we tackle the first need is to overcome timidity about approximating.

—H. and B. S. Jeffreys, *Methods of Mathematical Physics*, 2nd ed., Cambridge University Press, 1950, p. 8.

AN ELEMENTARY APPROACH TO JORDAN THEORY

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The usual proofs of the existence and uniqueness of the Jordan canonical form employ a certain amount of machinery: module theory, quotient spaces, or at least a fairly sophisticated manipulation of polynomial matrices. We offer an alternative and, to the best of our knowledge, new development that uses only the elementary concepts of linear algebra. The distinctive feature of this approach is that it works with what we call "λ-Jordan matrices," rather than with nilpotent operators.

As usual, we define the $k\times k$ λ-Jordan block (or λ-block) to be the $k\times k$ matrix

$$J(k,\lambda):=\begin{bmatrix}\lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1\\ & & & \lambda\end{bmatrix},\quad (J(1,\lambda):=[\lambda], J(0,\lambda):=\varnothing).$$

We will call a matrix λ-Jordan if it has the block diagonal form

$$\begin{bmatrix}A & & & \\ & J(k_1,\lambda) & & \\ & & \ddots & \\ & & & J(k_m,\lambda)\end{bmatrix},\tag{1}$$

where A is a square matrix and λ is not an eigenvalue of A.

Let V_0 be an n-dimensional complex linear space and let $\mathcal{T}_0$ be a linear transformation over V_0. Let T_0 be the matrix of $\mathcal{T}_0$ with respect to some basis $b_0=\{v_1,\dots,v_n\}$ for V_0. We call T_0 the Jordan canonical form for $\mathcal{T}_0$ if it has the block diagonal form, every block being a λ-block for some or other value of λ. In this case b_0 is called a Jordan canonical basis.

We first consider the uniqueness of the Jordan canonical form. Given a Jordan canonical form for a linear transformation $\mathcal{T}_0$, observe that if we single out all the λ-blocks of some λ and group all the others together into A, we obtain a λ-Jordan matrix. Such a conversion can be achieved by renumbering the basis vectors. Thus, to show uniqueness of the Jordan form it is enough to show that each transformation uniquely determines the sizes of the λ-blocks in its λ-Jordan matrix. In fact, it suffices to show this for $\lambda=0$, since if T' and T'' are two different λ-Jordan matrices for $\mathcal{T}_0$, then $T'-\lambda I$ and $T''-\lambda I$ are two different 0-Jordan matrices for the transformation $\mathcal{T}_0-\lambda\mathcal{I}$. (Here I is the identity $n\times n$ matrix and $\mathcal{I}$ is the identity transformation over V_0).

Suppose then that $\mathcal{T}_0$ has a 0-Jordan matrix

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$$T_0:=\begin{bmatrix} A & & & \\ & J(k_1,0) & & \\ & & \ddots & \\ & & & J(k_m,0) \end{bmatrix} \tag{2}$$

with respect to some basis b_0 for V_0. Notice that A must be nonsingular, since 0 is not an eigenvalue of A. It is helpful first to consider a specific example, say when $n=10$ and $\mathfrak{T}_0$ has the matrix

$$T_0=\begin{array}{c} \begin{matrix} V_1 & V_2 & V_3 & V_4 & V_5 & V_6 & V_7 & V_8 & V_9 & V_{10} \end{matrix} \\ \begin{bmatrix} A & & & & & & & & & \\ & & 0 & 1 & 0 & 0 & & & & \\ & & 0 & 0 & 1 & 0 & & & & \\ & & 0 & 0 & 0 & 1 & & & & \\ & & 0 & 0 & 0 & 0 & & & & \\ & & & & & & 0 & 1 & 0 & \\ & & & & & & 0 & 0 & 1 & \\ & & & & & & 0 & 0 & 0 & \\ & & & & & & & & & 0 \end{bmatrix} \end{array} \tag{3}$$

with respect to the basis $b_0=\{v_1,\dots,v_{10}\}$, where A is some nonsingular 2×2 matrix. Here each column is labeled by the symbol of the corresponding basis vector.

Let V_1 be the image of the transformation $\mathfrak{T}_0$, i.e., $V_1:=\operatorname{im}\mathfrak{T}_0:=\mathfrak{T}_0 V_0$. It is easy to read off from (3) a basis for $V_1: b_1=\{v_1,v_2,v_3,v_4,v_5,v_7,v_8\}$. In particular, $\dim V_1=7$. It can be seen also that $\dim V_0-\dim V_1$ is equal to the number of 0-blocks in T_0. Inasmuch as V_1 is an invariant subspace of $\mathfrak{T}_0$, we can speak about the restriction $\mathfrak{T}_1$ of $\mathfrak{T}_0$ to V_1. This restriction has the matrix

$$T_1=\begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_7 & v_8 \end{matrix} \\ \begin{bmatrix} A & & & & & & \\ & & 0 & 1 & 0 & & \\ & & 0 & 0 & 1 & & \\ & & 0 & 0 & 0 & & \\ & & & & & 0 & 1 \\ & & & & & 0 & 0 \end{bmatrix} \end{array}$$

with respect to the basis b_1.

Now we can repeat the reasoning. It will be the same because T_1 is 0-Jordan again. So we obtain the basis $b_2=\{v_1,v_2,v_3,v_4,v_7\}$ for the image $V_2:=\mathfrak{T}_1 V_1\ (=\mathfrak{T}_0 V_1=\mathfrak{T}^2 V)$ of the transformation $\mathfrak{T}_1$. Also one can see that $\dim V_2=5$ and $\dim V_1-\dim V_2$ is again the number of 0-blocks in T_1. Proceeding in this way we obtain the following sequence of transformations: $\mathfrak{T}_2$ over V_2 with the basis b_2 and the block-diagonal matrix T_2 with blocks A, $J(2,0)$, $J(1,0)$; $\mathfrak{T}_3$ over V_3 with the basis $b_3=\{v_1,v_2,v_3\}$ and the matrix T_3 with blocks A, $J(1,0)$; $\mathfrak{T}_4$ over V_4 with the basis $b_4=\{v_1,v_2\}$ and the matrix $T_4=A$. Here the sequence terminates, since $V_5=V_4$. Notice that the index of the last member of the sequence coincides with the maximal size of T_0's 0-blocks.

Applying the same analysis to our general linear transformation $\mathfrak{T}_0$ with 0-Jordan matrix (2), we obtain the sequence

$$\mathfrak{T}_0,\mathfrak{T}_1,\mathfrak{T}_2,\dots,\mathfrak{T}_h=\mathfrak{T}_{h+1} \tag{4}$$

of successive restrictions of the transformation $\mathfrak{T}_0$ to the respective spaces,

$$V_0,V_1,V_2,\dots,V_h=V_{h+1} \qquad (V_i:=\operatorname{im}\mathfrak{T}_{i-1}). \tag{5}$$

In addition, we have the following procedure for forming a basis b_i for V_i, $i=1,2,\dots$: simply delete from b_0 the last i vectors corresponding to each 0-Jordan block in T_0. (Delete all vectors corresponding to a block of size less than i.) Consequently, we see that if s_i, $i=1,\dots,h$, is

the number of $i \times i$ 0-blocks in T_0, then

$$\begin{aligned}
\dim V - \dim V_1 &= s_1 + s_2 + \cdots + s_h \\
\dim V_1 - \dim V_2 &= s_2 + \cdots + s_h \\
&\cdots\cdots\cdots\cdots\cdots \\
\dim V_{h-1} - \dim V_h &= s_h \\
\dim V_h - \dim V_{h+1} &= 0.
\end{aligned}$$

Solving these equations for $s_1, \ldots, s_h$, we see that the sizes of the 0-blocks in any 0-Jordan matrix for $\mathcal{T}_0$ are expressed in terms of the invariants $\dim V_0$, $\dim V_1, \ldots$, of the transformation $\mathcal{T}_0$ and so do not depend on the choice of a basis. This completes the proof of the uniqueness of the Jordan form.

To establish the existence of the Jordan canonical form it is enough to establish the existence of a 0-Jordan matrix for $\mathcal{T}_0$. For, by considering $\mathcal{T}_0 - \lambda \mathcal{I}$, this shows the existence of a λ-Jordan matrix for $\mathcal{T}_0$ for any eigenvalue λ of $\mathcal{T}_0$. Then apply induction to the restriction of $\mathcal{T}_0$ to the subspace corresponding to the matrix A in (1).

For motivation, we consider once more the process described above for constructing a basis b_i for V_i, given 0-Jordan matrix (2) for $\mathcal{T}_0$ and the corresponding basis b_0. (The reader may wish to follow the example given above while reading this paragraph.) Our process stops when we have exhausted all 0-blocks, i.e., when we are left with only the portion of the original matrix corresponding to the submatrix A. Suppose this happens at stage h. Then A is the matrix of the restriction $\mathcal{T}_h$ of $\mathcal{T}_0$ to V_h with respect to the basis b_h. Looking at the previous stage of our process we see that our basis for V_{h-1} consists of our basis for V_h together with certain vectors of the basis b_0 which lie in $\ker \mathcal{T}_{h-1}$. This step is atypical, however. At each earlier step i we may have discarded from b_{i-1} some basis vectors which lie in $\ker \mathcal{T}_{i-1}$, but we have also eliminated some others. These others, however, are mapped by $\mathcal{T}_{i-1}$ in a one-to-one fashion onto the basis vectors which we will discard at the next stage. Thus we have decompositions $V_{h-1} = V_h \oplus \tilde{N}_h$ and $V_{i-1} = V_i \oplus (\tilde{N}_i \oplus L_i)$ for $i = 1, \ldots, h-1$. Combining these gives a decomposition

$$V_0 = V_h \oplus \tilde{N}_h \oplus (\tilde{N}_{h-1} \oplus L_{h-1}) \oplus \cdots \oplus (\tilde{N}_1 \oplus L_1), \tag{6}$$

where

(i) $\mathcal{T}_0$ is nonsingular on V_h,
(ii) each $\tilde{N}_i \subset \ker \mathcal{T}_{i-1}$,
(iii) $\mathcal{T}_{i-1}$ maps L_i isomorphically onto $\tilde{N}_{i+1} \oplus L_{i+1}$ for $i = 1, \ldots, h-2$,
(iv) $\mathcal{T}_{h-2}$ maps L_{h-1} isomorphically onto $\tilde{N}_h$.

In fact, (iv) is (iii) for $i = h-1$ taking into account that $L_h = \{0\}$. Notice also that the partial sum $V_h \oplus \tilde{N}_h \oplus \cdots \oplus (\tilde{N}_{i+1} \oplus L_{i+1})$ is equal to V_i.

Conditions (i) through (iv) may be illustrated schematically as follows (the arrows mean isomorphic mapping by $\mathcal{T}_0$):

$$\begin{array}{lll}
V_0 = V_h \oplus & & (V_h \to V_h) \\
 & \tilde{N}_h \oplus & (\tilde{N}_h \text{ perishes}) \\
 & \tilde{N}_{h-1} \oplus L_{h-1} \oplus & (\tilde{N}_{h-1} \text{ perishes}) \\
 & \tilde{N}_{h-2} \oplus L_{h-2} \oplus & (\tilde{N}_{h-2} \text{ perishes}) \\
 & \cdots\cdots\cdots\cdots & \\
 & \tilde{N}_2 \oplus L_2 \oplus & (\tilde{N}_2 \text{ perishes}) \\
 & \tilde{N}_1 \oplus L_1 & (\tilde{N}_1 \text{ perishes})
\end{array} \tag{7}$$

Our existence proof consists in constructing such a decomposition for an arbitrary transformation $\mathfrak{T}_0$.

So, given a transformation $\mathfrak{T}_0$ over V_0, consider sequences (4) and (5). The discussion above suggests how to find h so that (i) is satisfied: choose h so that $V_h = V_{h+1}$. This must happen eventually since $V_0 \supset V_1 \supset V_2 \supset \cdots$. We let h be the least such integer.

Let N_i be the kernel of $\mathfrak{T}_{i-1}$. Choose $\tilde{N}_i$ to be any subspace of N_i such that $V_i \oplus \tilde{N}_i = V_i + N_i$ (i.e., extract from N_i a complement for V_i in $V_i + N_i$, for example, by adding elements from N_i to complete a basis for V_i to one for $V_i + N_i$). To choose the L_i we need the following fact.

PROPOSITION. *If M is a subspace of V_i $(i>0)$ such that*

$$V_i = V_{i+1} \oplus M, \tag{8}$$

then there exists a subspace L of V_{i-1} such that

$$V_{i-1} = (V_i + N_i) \oplus L \tag{9}$$

and

$$\mathfrak{T}_0 L = \mathfrak{T}_{i-1} L = M, \qquad \dim L = \dim M \tag{10}$$

(i.e., $\mathfrak{T}_0$ maps L isomorphically onto M).

Proof. Let M satisfy (8). Choose a basis $\{y_1,\ldots,y_s\}$ for M. Since $M \subset V_i = \mathfrak{T}_{i-1} V_{i-1}$, there exist $z_1,\ldots, z_s \in V_{i-1}$ such that $\mathfrak{T}_{i-1} z_j = y_j$. These z_j's must also be linearly independent. Then for the span L of $z_1,\ldots, z_s$, we have $L \subset V_{i-1}$, $\mathfrak{T}_0 L = \mathfrak{T}_{i-1} L = M$ and $\dim L = \dim M$ (i.e., (10) holds).

To prove (9) we first verify that $(V_i + N_i) \cap L = \{0\}$. In fact, $x \in (V_i + N_i) \cap L$ implies $\mathfrak{T}_0 x \in \mathfrak{T}_0 L = M$ and $x = v + n$ for some $v \in V_i$, $n \in N_i$. Hence $\mathfrak{T}_0 x = \mathfrak{T}_0 v \in V_{i+1}$, so $\mathfrak{T}_0 x \in M \cap V_{i+1} = \{0\}$. But $\mathfrak{T}_0$ is one-to-one on L, so this forces $x = 0$.

It remains to show that $(V_i + N_i) + L = V_{i-1}$. Let $w \in V_{i-1}$. Then $\mathfrak{T}_{i-1} w \in V_i = V_{i+1} \oplus M$ and so $\mathfrak{T}_{i-1} w = v + m$ for some $v \in V_{i+1}$, $m \in M$. Hence there exist $u \in V_i$ and $l \in L$ such that $\mathfrak{T}_{i-1} u = v$ and $\mathfrak{T}_{i-1} l = m$. Therefore $\mathfrak{T}_{i-1} w = \mathfrak{T}_{i-1}(u + l)$ and $\mathfrak{T}_{i-1}(w - u - l) = 0$, i.e., $w - u - l \in N_i$. This means that $w \in (V_i + N_i) + L$.◀

For $i = h$ (8) takes the form $V_h = V_{h+1}$ (i.e., $M = \{0\}$), so by the preceding proposition $L = \{0\}$ and $V_{h-1} = V_h + N_h$. Obviously, $V_{h-1} = V_h \oplus N_h$, i.e., $\tilde{N}_h = N_h$. For $i = h-1$ and $M = \tilde{N}_h$ the proposition guarantees the existence of an L_{h-1} that complements $V_{h-1} + N_{h-1}$ to V_{h-2} and is mapped by $\mathfrak{T}_0$ isomorphically onto $\tilde{N}_h$. Choosing a complement $\tilde{N}_{h-1}$ of V_{h-1} to $V_{h-1} + N_{h-1}$ we will have $V_{h-2} = V_{h-1} \oplus (\tilde{N}_{h-1} \oplus L_{n-1})$. Applying the proposition again for $i = h-2$ and $M = \tilde{N}_{h-1} \oplus L_{h-1}$ we find L_{h-2} mapped by $\mathfrak{T}_0$ isomorphically onto $\tilde{N}_{h-1} \oplus L_{h-1}$ and such that $V_{h-3} = (V_{h-2} + N_{h-2}) \oplus L_{h-2}$ or, after a proper choice of $\tilde{N}_{h-2}$, $V_{h-3} = V_{h-2} \oplus (\tilde{N}_{h-2} \oplus L_{h-2})$. Continuing in this way, we obtain decomposition (6) with properties (i)-(iv).

We now describe the process for choosing the desired basis (it is convenient to follow it on the diagram (7)). Let a and n_h be any bases for V_h and $\tilde{N}_h$, respectively. Since $\mathfrak{T}_0$ maps L_{h-1} isomorphically onto $\tilde{N}_h$, there is a basis l_{h-1} for L_{h-1} which $\mathfrak{T}_0$ maps onto n_h (vector by vector). Choose any basis n_{h-1} for $\tilde{N}_{h-1}$. Then we can choose a basis l_{h-2} for L_{h-2} which $\mathfrak{T}_0$ maps (vector by vector) onto $n_{h-1} \cup l_{h-1}$. Continuing in this fashion, we obtain a basis $a \cup n_h \cup n_{h-1} \cup l_{h-1} \cup \cdots \cup n_1 \cup l_1$ for V_0 with the property: $\mathfrak{T}_0$ maps l_i onto $n_{i+1} \cup l_{i+1}$ for $i = 1,\ldots, h-2$, and onto n_h for $i = h-1$.

Each vector of $n_1 \cup l_1$ determines a "string" of basis vectors. In fact, let $v_1 \in n_1 \cup l_1$. If $v_1 \in n_1$, the string terminates. If $v_1 \in l_1$, join $v_2 := \mathfrak{T}_0 v_1$ to v_1. If $v_2 \in n_2$, the string terminates. If $v_2 \in l_2$, join $v_3 := \mathfrak{T}_0 v_2$ to it, and so on. Each string terminates with a vector of $\cup n_i$ and the number m of strings coincides with the number of vectors in $\cup n_i$. Thus, by regrouping vectors one can represent the basis in the form $a \cup s_1 \cup \cdots \cup s_m$, where s_j are strings. The matrix of $\mathfrak{T}_0$ with respect to the basis ordered in this fashion is easily seen to be 0-Jordan. Then a gives a nonsingular block and s_j gives a 0-Jordan block. Notice that the maximal size of 0-blocks is h

and the number of such blocks is equal to $\dim\tilde{N}_h$. The number of blocks of size $h-1$ is $\dim\tilde{N}_{h-1}$ and so on.

REMARKS.

1. In terms of the construction above, one can easily see the extent of arbitrariness in the choice of basis $n_h \cup n_{h-1} \cup l_{h-1} \cup \cdots \cup n_1 \cup l_1$. Each n_h is chosen arbitrarily as a basis of N_h. Each vector of the basis n_{h-1} is chosen to within an arbitrary summand from N_h. Each vector of l_{h-1} is chosen to within an arbitrary summand from $N_h + \tilde{N}_{h-1}$, and so on.
2. Our approach is "weakly sensitive" to the assumption of finite dimension, and so it allows one to extend some results to the infinite dimensional case.
3. The title of this paper refers not only to the main theorem but also to other applicable theorems of the Jordan theory: for example, the Cayley-Hamilton theorem, the chain of theorems clearing up the structure of invariant subspaces of a linear transformation, and many others. In other expositions, these theorems are preparatory to a proof of the main theorem and use rather complicated techniques. But, once available, the main theorem converts all of them into simple corollaries.

FIFTY YEARS AGO

Some of the articles in Volume 37 of this MONTHLY were: Applications of groups to geometry; The rectangular hexagon; Polygenic functions; Zolotarev's ideal theory (by Chebotarev); Mathematics and the problem of ore location; Present tendencies in projective geometry; Transmission of impulses through a muscle; and Integral equations in theoretical physics.

One can sense the changing currents in Mathematics from a sampling of the books reviewed: Kellogg's *Foundations of Potential Theory*; Dickson's *Introduction to the Theory of Numbers*; Hobson's *Theory of Functions of a Real Variable*; Fowler's *Statistical Mechanics*; Landau's *Grundlagen der Analysis*; Nevanlinna's *Le Théorème de Picard-Borel et la théorie des fonctions méromorphes*; Granville's *Calculus* (revised by Smith and Longley, priced at $3.20); and *Poetry and Mathematics* by Scott Buchanan, who (the reviewer said) deplored "the scarcity of clear and distinct ideas today." (Buchanan's name is little known fifty years later, and his book seems almost unreadable now; but Clifton Fadiman dedicated *Fantasia Mathematica* to him.)

The *Annals of Mathematical Statistics* was founded in 1930.

The 1929–30 Annual Meeting of AMS and MAA was held December 29 and January 1 at Des Moines, and was attended by 140 (104 of them from MAA). The Summer Meeting at Brown attracted 305 (170 from MAA). Sixty-three Ph.D.'s in Mathematics were awarded in the United States and Canada in 1929.

This MONTHLY stated (somewhat confusedly) that the Rockefeller Institute [actually the International Education Board] had endowed an Institute of Mathematics at Göttingen, to be directed by Richard Courant. [In point of fact, the money had been promised at the end of 1926 and the Institute had been dedicated late in 1929.]

The Harvard Mathematical Colloquium was established. This was, apparently, such a novel idea that this MONTHLY described it as being a new mathematical society.

R.P.B.

EDITED BY DEBORAH TEPPER HAIMO AND FRANKLIN TEPPER HAIMO

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ON LEGENDRE'S PRIME NUMBER FORMULA

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1. The first conjecture for $\pi(x)$, the number of primes less than x, was published by Legendre in 1798 in his book [**4**(a), p. 19]. With the notation $\pi(a)=b$ he wrote: "Finally, it is plausible that the rigorous formula that gives the value of b, if a is very large, is of the form $b=a/(A\log a+B)$, A and B being constant coefficients and $\log a$ denoting the natural logarithm. The exact determination of the coefficients will be a singular task, worthy of exercising the acumen of the Analysts."

In the second edition of this book [**4**(b), p. 394], in the year 1808, he made a quantitative conjecture also for the values of the constants A and B, based on tabulations of primes: "Though the sequence of the prime numbers is extremely irregular, one can find with a very satisfactory precision how many of them lie between 1 and a given limit x. The formula that resolves the question is

$$y=\frac{x}{\log x-1.08366}$$

where $\log x$ is the natural logarithm."

Legendre's conjecture can be written in modern terminology as

$$\pi(x)=\frac{Bx}{\log x-A+o(1)} \tag{1.1}$$

where he conjectured

$$A=1.08366\ldots \quad \text{and } B=1. \tag{1.2}$$

It is not clear from Legendre that he intended an asymptotic formula, but it is true that, for large x, the best value of A, even for numerical work, is 1. But, within the range of the tables that Legendre had, his formula is remarkably accurate.

Gauss, also, investigated the distribution of primes empirically, in the years 1792–1793, when he was only 15–16 years old, but he didn't prove anything and didn't publish any conjecture. In a letter to Encke in the year 1849, he wrote the following (see [**2**, Bd. 2, pp. 444–447]): "I soon observed that for all oscillations this frequency is on the average near in inverse ratio to the logarithm, so that the number of all primes under a given limit n is determinable approximately by the integral

$$\int\frac{dn}{\log n},$$

where we mean the natural logarithm."

Thus Gauss conjectured:

$$\pi(x)\approx \operatorname{li} x \overset{\text{definition}}{=} \int_2^x \frac{dt}{\log t}\left(\sim\frac{x}{\log x}\right). \tag{1.3}$$

He commented on Legendre's conjecture (1.1)–(1.2) as follows: "It seems that with increasing n the (average) value of A decreases, but whether the limit as n increases to infinity is 1 or not, I don't dare to conjecture." (Gauss's n is now x.)

Somewhat later, however, he wrote in his letter: "I cannot say that there is any reason to expect a simple limit; on the other hand, the excess of A over 1 can very well be a quantity of the order $1/\log n$."

Thus (as expressed in modern notation) he held it to be possible that

$$\pi(x) = \frac{x}{\log x - 1 + \dfrac{C + o(1)}{\log x}}. \tag{1.4}$$

As (1.4) is certainly true for $\operatorname{li} x$ instead of $\pi(x)$, (1.4) could be motivated by his other conjecture (1.3).

The problem concerning the possible value of A in (1.1) was decided in 1848 by Chebyshev [1], only forty years after Legendre announced (1.2). In his paper he disproved Legendre's conjecture. That is, he showed that if (1.1) was true with certain constants A and B then necessarily

$$B = 1 \quad \text{and} \quad A = 1, \tag{1.5}$$

as Gauss conjectured (see (1.4)). So the interesting phenomenon occurred that, at that time, it was impossible to prove the truth of (1.1) with any constants A and B; but, supposing (1.1), it was possible to determine the values of A and B. Only 50 years later, in 1899, could de la Vallée Poussin [5] prove that (1.1) is really satisfied (so naturally with $A = B = 1$). Chebyshev's proof is analytic and relatively complicated. He used the properties of $\zeta^{(m)}(s)$ for real $s \to 1+0$. However, he proved more than (1.5); that is, he showed that for any $q > 0$

$$\limsup_{x \to \infty} \frac{\pi(x) - \operatorname{li} x}{\dfrac{x}{\log^q x}} \geqslant 0 \tag{1.6}$$

and

$$\liminf_{x \to \infty} \frac{\pi(x) - \operatorname{li} x}{\dfrac{x}{\log^q x}} \leqslant 0. \tag{1.7}$$

(The fact that the limit exists and is equal to 0 was proved by de la Vallée Poussin [5] in 1899.)

From the case $q = 2$, one gets easily (1.5). Landau simplified the proof for $q = 2$; but this proof uses the zeta function and its derivative, and it is again not simple enough (needing more than 3 pages in Chapter 10 of his *Handbuch* [3]).

2. In the present note we shall show that Legendre's conjecture (1.2) can be disproved; more precisely, (1.5) can be proved in a very simple, elementary way. So it is curious that Legendre's conjecture (1.2) had remained open for 40 years.

With the notation

$$\Psi(x) \overset{\text{definition}}{=} \sum_{n \leqslant x} \Lambda(n) \overset{\text{definition}}{=} \sum_{p^m \leqslant x} \log p, \tag{2.1}$$

we shall prove the

THEOREM. *If with certain constants C and D*

$$\Psi(x) = Cx + \frac{(D + o(1))x}{\log x} \tag{2.2}$$

then $C = 1$ and $D = 0$.

From this by partial summation one immediately gets the

COROLLARY. *If with certain constants A and B*

$$\pi(x)=\frac{Bx}{\log x-A+o(1)} \tag{2.3}$$

then $B=1$ and $A=1$.

3. Using Stirling's formula (in a weaker form) and Legendre's formula for $n!$ we get

$$\begin{aligned}\log x+O(1)&=\frac{1}{x}\log[x]!=\frac{1}{x}\sum_{n\leqslant x}\Lambda(n)\left[\frac{x}{n}\right]\\&=\sum_{n\leqslant x}\frac{\Lambda(n)}{n}+\frac{1}{x}\sum_{n\leqslant x}\Lambda(n)\cdot O(1)=\int_2^x\frac{\Psi(t)}{t^2}\,dt+\frac{\Psi(x)}{x}+O\left(\frac{\Psi(x)}{x}\right)\\&=\int_2^x\frac{C+\dfrac{D+o(1)}{\log t}}{t}\,dt+O(1)=C\log x+(D+o(1))\log\log x.\end{aligned} \tag{3.1}$$

This gives $C=1$, $D=0$ and thus proves the theorem.

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POLYNOMIAL GROUP LAWS

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In [**P**] R. Palais defined a map $m:K\times K\to K$ (where K is an infinite field) to be separately polynomial if for each $x\in K$ the two partial maps $y\to m(x,y)$ and $y\to m(y,x)$ of K to K are polynomial; and he showed that when K is uncountable this implies $m(x,y)$ is a polynomial in x and y [**P**, p. 392, Section 5], although this conclusion can fail when K is countable [**P**, p. 390, Example 4].

The purpose of this note is to establish that even when K is countable the conclusion is valid if m is a group law, and in fact we can exhibit an explicit formula for $m(x,y)$:

THEOREM. *If K is an infinite field and $m:K\times K\to K$ is a separately polynomial group law, then m is of the form $m(x,y)=x+y-c$ for some $c\in K$; so c is the identity for m, and the group inversion is given by $x^{-1}=2c-x$.*

The theorem shows that a separately polynomial group law is strong enough to furnish K with the only polynomial group structure up to a polynomial isomorphism, although there are in general many abstract group structures that are close to being polynomial. For example, let

$K=\mathbb{R}$ (the real number field) and define a group law that is not separately polynomial by $m(x,y)=(x^3+y^3)^{(1/3)}$ for $x,y \in K: m(0,y)$ and $m(x,0)$ are the only polynomial maps of K into K.

The proof of the following lemma is routine [**B**, p. 58, Theorem 4] and so omitted. The lemma is the one-variable Jacobian problem, which is unsolved for several variables [**V**]:

LEMMA. *Every polynomial bijection (possessing a polynomial inverse) $P: K \to K$, where K is an infinite field, is of the form $P(x)=ax+b$ for some $a \in K^* = K - \{0\}$ and $b \in K$; and conversely.*

Now we will demonstrate the theorem:

Proof. Write $m(x,y)=x^*y$ and define $L_x(y)=x^*y=R_y(x)$. Then by assumption L_x and R_x are polynomial maps $K \to K$ for each $x \in K$. Now if P and Q are polynomial bijections $K \to K$ such that $P(Q(z))=z$ for all $z \in K$, then clearly P and Q have degree one by the lemma. (Here is where we need K infinite, to guarantee that a polynomial is determined by its values at points of K.) Applying this to L_x and $L_{x^{-1}}$ (and to R_x and $R_{x^{-1}}$), we see that $L_x(y)=B(x)y+A(x)=R_y(x)$ $=D(y)x+E(y)$ where A, B, D and E are polynomials of degree at most one, and B and D are nonzero everywhere. It follows of course that B and D are nonzero constants b and d, while $A(x)=ax-c$ and $E(y)=ey-f$. Clearly, $a=d$, $e=b$, and $f=c$; so $x^*y=ax+by-c$ for $a, b \in K^*$ and $c \in K$. Finally, the associativity of the group law $(x^*y)^*z=x^*(y^*z)$ for all x, y, and $z \in K$ implies that $a^2=a$ and $b^2=b$, so $a=b=1$. Hence $m(x,y)=x^*y=x+y-c$. Evidently c is the identity for m and the inverse of x under m is $2c-x$. This completes the proof of the theorem.

While our method does not extend directly to separately polynomial group laws $K^n \times K^n \to K^n (n \geqslant 2)$, some generalizations can be found in [**M**], [**N**].

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A CHANGE-OF-VARIABLES FORMULA WITHOUT CONTINUITY

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There exist in the literature a good number of change-of-variables formulas for integrals in R^n and a good number of different proofs for them. In the textbooks on measure and integration one usually finds the classical change-of-variables formula, which requires the differentiability of the transformation of variables. In more specialized treatises, like the one by Radó and Reichelderfer [3], one can find formulas that relax this condition, requiring instead the continuity of the transformation and involving the notion of the generalized Jacobian (cf. [3, p. 273]).

In this note it will be shown that it is still possible, by very elementary means, to obtain a satisfactory formula without even requiring the continuity of the transformation. From this general formula one easily obtains some other formulas, classical and less classical.

The (Lebesgue) measure of a measurable subset A of R^n will be denoted by $|A|$.

GENERATING THE k-SUBSETS OF AN n-SET

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While discussing with colleagues certain questions about generalized measures on finite measure spaces, the author came upon a combinatorial lemma—on generating all of the k-subsets of an n-set. In the context of these discussions, one is given an n-set, say $X=\{x_1,x_2,\ldots,x_n\}$, together with a collection $\mathcal{K}$ of its subsets that is closed under disjoint union and under complementation. A *generalized measure* μ is a nonnegative real-valued function on these "measurable" subsets, one however that is additive over disjoint unions. Of course, there are any number of such generalized measure spaces (g.m.s.), some interesting and some not, depending on one's point of view. Two useful but quite different introductions to the subject are those of Rado [1] and Gudder [2].

In the present context, we would like to be assured, first of all, that there are generalized measures that are not induced by an ordinary signed measure on the points of X. For this purpose, we let $X=\{1,2,3,4,5,6\}$, and as the measurable sets we take

$$\mathcal{K}=\mathcal{K}(6,3)=\{A\subseteq X:|A|=0,\ 3,\text{ or }6\}$$

with measures

$$\begin{array}{ll}\mu(\varnothing)=0 & \mu(\{1,3,5\})=\frac{1}{4}\\ \mu(X)=1 & \mu(\{2,4,6\})=\frac{3}{4}\end{array}$$

and the measures of all other 3-subsets are taken to be $\frac{1}{2}$. It will be a consequence of the matrix argument following our lemma that, if μ is induced by a signed measure ρ on the points of X, we must have $\rho=\frac{1}{6}$ for every point. The given measures of $\{1,3,5\}$ and $\{2,4,6\}$ then provide a contradiction.

In the opposite direction altogether, it is of interest to establish whole classes of g.m.s. for which any generalized measure is necessarily induced by an ordinary (though perhaps signed) point measure. One reason for our interest is the fact that additivity of the integral over measurable functions is a necessary consequence. In the general case, we note that this has proved to be a rather difficult question [3]. At any rate, this is the context in which our combinatorial lemma appears. For $X=\{x_1,x_2,\ldots,x_n\}$ as before, we introduce the class of g.m.s. $\mathcal{K}(n,k)$ of *elementary cardinality* $k\geqslant 2$, in which a set is measurable iff its cardinality is a multiple of k. It is then essential that we assume $n=mk$, and we further suppose that $m>2$ to avoid certain trivialities. With all of the k-subsets (combinations) of X being measurable, the situation is somewhat reminiscent of that found in a number of puzzles: "By always weighing exactly k balls at a time, determine such and such." Moreover, using disjoint unions, it is clear that $\mathcal{K}$ is generated by these k-subsets. But our lemma goes considerably further:

LEMMA. *Let $n=mk$ with $k\geqslant 2$ and $m>2$. The $\binom{n}{k}$ k-subsets of $\{1,2,\ldots,n\}$ are generated (by complementation of disjoint unions) from the $n-1$ k-subsets*

$$\{1,2,\ldots,k\},\ldots,\{n-k,n-k+1,\ldots,n-1\},\ \{n-k+1,\ldots,n-1,1\},\ldots,\{n-1,1,\ldots,k-1\}.$$

Proof. In the interest of notational clarity, we first agree to refer to any k-subset as a *simplex*, and we also rename the index n as 0, thus permitting the use of modulo n addition. With this understanding, we quite naturally refer to each simplex of the form $\{j,j+1,\ldots,j+k-1\}$ as a *sequential* simplex. At the same time, we introduce, for any index i, the idea of an *i-interior* simplex $\{j,\ldots,\hat{i},\ldots,j+k\}$ and the complementary notion of an *i-exterior* simplex $\{i,j,\ldots,j+k-2\}$, refusing, however, to allow the limiting case of being sequential. For any i,

there are thus $k-1$ interior and $n-k-1$ exterior simplexes. We remark that a collection $\mathcal{K}$ of simplexes that contains all n sequential simplexes and that is closed under complementation of disjoint unions will contain all i-interior simplexes (for some i) iff it contains all i-exterior simplexes. Parenthetically, we note also that the same remark extends quite easily to the case of I-interior (respectively, I-exterior) simplexes for any set of sequential indices I having a cardinality $|I| \leqslant k-1$.

With the new terminology, our lemma claims that if we begin with all of the 0-interior simplexes, together with the $n-k$ sequential simplexes that do not contain 0, then we generate a collection $\mathcal{K}$ (by taking complements of disjoint unions), which in fact consists of all the k-subsets. We first observe that the k remaining sequential simplexes are easily generated, and, by the remark above, we have all of the 0-exterior simplexes as well. The first phase of our proof then consists in showing that, together with all of the sequential simplexes, the 0-interior (and 0-exterior) simplexes will generate all of the 1-interior (and, hence, also the 1-exterior) simplexes.

We obtain one of these 1-interior simplexes immediately, namely, $\{0,2,\ldots,k\}$, as the complement of $\{n-k+1,\ldots,n-1,1\}$ and disjoint sequential simplexes. We now show, in effect, that each successive simplex $\{n-j,\ldots,0,2,\ldots,k-j\}$ generates $\{n-(j+1),\ldots,0,2,\ldots,k-(j+1)\}$ in turn. Consulting Fig. 1, we see that the 0-interior and 0-exterior simplexes at the left will generate the $(n-(j+1))$-exterior simplex shown. Together with $\{n-j,\ldots,0,2,\ldots,k-j\}$, this will generate the 1-exterior simplex $\{1,n-(j+k),\ldots,n-(j+2)\}$, and the latter gives rise to the desired $\{n-(j+1),\ldots,0,2,\ldots,k-(j+1)\}$.

By finite induction, it follows that we obtain all i-interior and i-exterior simplexes at each

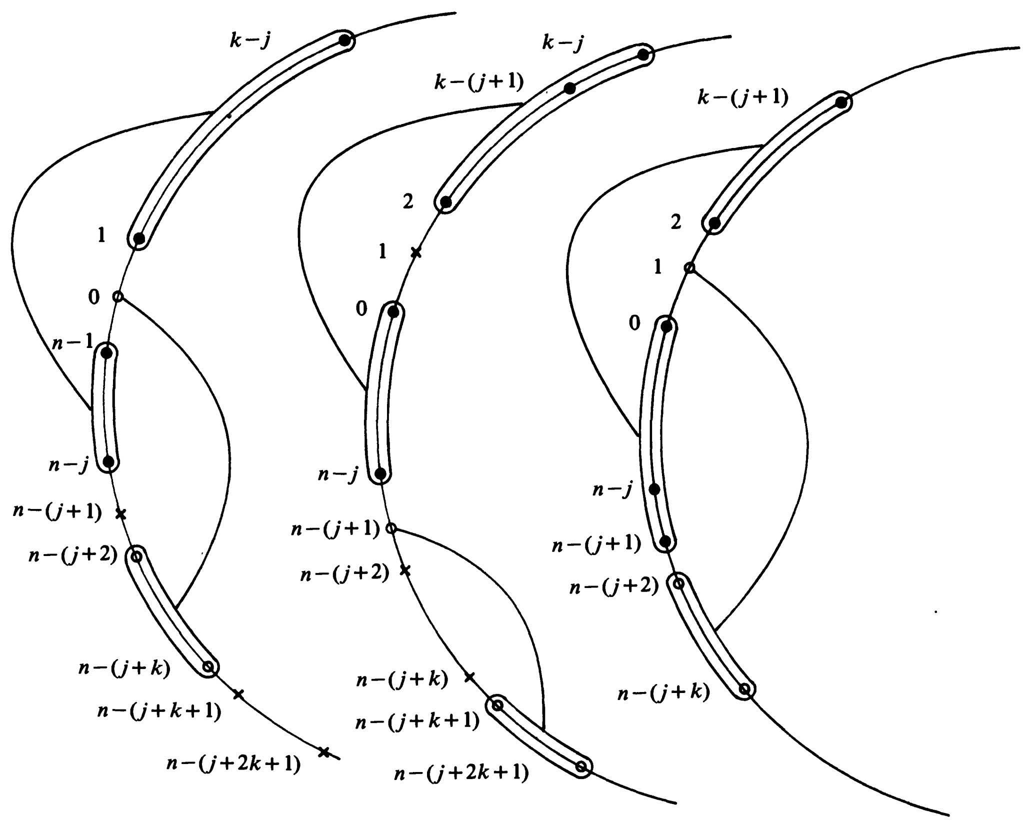

FIG. 1

point $i=0,1,\ldots,n-1$. As a result, the seemingly unsymmetric situation regarding the given $n-1$ simplexes is resolved to one of complete circular symmetry. We are thus permitted to speak now of the question of generating various circular "patterns" as simplexes, without regard to the actual position of the points.

From this point of view, we shall refer to the number of components of patterns in regard to their circular connectivity. To show that all patterns (hence, all simplexes) are generated, we proceed by induction on p, the number of components. For $p=1$, we have the sequential simplexes already known to be present. We now treat the case $p=2$ before providing a general inductive step.

When $p=2$, we are dealing with the I-interior (and I-exterior) simplexes mentioned earlier, and a little reflection shows that it is sufficient to consider $|I|=r$, with r in the range $1\leqslant r\leqslant k-1$. We then argue by induction on r. If $r=1$, we have the case of the i-interior and i-exterior simplexes, as has already been treated. So we suppose that the I-interior and I-exterior simplexes have been generated in case $|I|\leqslant r-1$, and we try to generate some I-interior simplex with $|I|=r$. We split the set I into two sequential subsets, each having $r-1$ or fewer points. Using the inductive hypothesis, we take two exterior simplexes, accordingly, to surround the simplex in question, thus obtaining the latter as a complement of disjoint simplexes already generated.

For the inductive step, we assume that any simplex with p components can be generated ($p\geqslant 2$), and we then consider a simplex L with $p+1$ components. We show that we can find a constructible set M in the complement of L such that the number of components of $L\cup M$ is p or less. This means that M fills (at least) one of the "gaps" between the components of L. This will complete the proof, for we can then use the inductive hypothesis to generate, k points at a time, the complement C of $L\cup M$. The disjoint union $M\cup C$ is thereby constructible, thus allowing us to generate its complement L, completing the argument.

It remains only to show how to construct M so as to fill a gap. If any one gap G contains $k-1$ or more points, then any other gap H can be filled by the disjoint union M of as many sequential simplexes as will fit, together with a simplex formed using points of G to complement any remaining points in H. This latter simplex can be chosen to have connectivity $p=2$ and hence can already be generated. If no gap has as many as $k-1$ points, then we use the fact that the total number of points in all $p+1$ gaps is at least $2k$ in concluding that the union of any p of the gaps contains at least k points. We may then form a simplex M by adding to all of the points in any one gap H only points from $p-1$ or fewer gaps. Such an M has at most p components, so that according to the inductive hypothesis it can already be generated. $\square$

If we have a signed point measure ρ defined on the points of $X=\{x_1,x_2,\ldots,x_n\}$, such that

$$\sum_{j=1}^{k}\rho_j\geqslant 0$$

after we have reordered the points according to increasing (or at least, nondecreasing) measure, then ρ induces a generalized measure on $\mathfrak{X}(n,k)$ upon our writing

$$\mu(A)=\sum_{x_j\in A}\rho_j \qquad (A\in\mathfrak{X}(n,k)).$$

Conversely, the lemma can be used to show that every generalized measure μ on a space $\mathfrak{X}(n,k)$ of elementary cardinality is induced by a unique signed point measure ρ on $x_1, x_2,\ldots,x_n$. For we have that μ is completely determined by its values $\mu_1,\mu_2,\ldots,\mu_{n-1}$ on the $n-1$ k-subsets listed in the statement of the lemma. Recalling that k divides n, one shows that the $(n-1)\times(n-1)$ system:

$$
\begin{bmatrix}
\overbrace{1\;1\cdots 1}^{k} & & & & 0\\
& 1\;1\cdots 1 & & & \\
& & \ddots & & \\
& 0 & & 1\;1\cdots 1 & \\
1 & & & 1\cdots 1 & \\
\ddots & & & \ddots & \\
\underbrace{1\cdots 1}_{k-1} & & & & 1
\end{bmatrix}
\begin{bmatrix}
\rho_1\\ \rho_2\\ \vdots\\ \\ \rho_{n-1}
\end{bmatrix}
=
\begin{bmatrix}
\mu_1\\ \mu_2\\ \vdots\\ \\ \mu_{n-1}
\end{bmatrix}
$$

has the determinant $k \neq 0$. This is accomplished by employing row operations to shift the bottom triangular submatrix to the right, k columns at a time, successively, until we finally obtain a lower right-hand $(k-1)\times(k-1)$ array:

$$
\begin{bmatrix}
2 & & 1\\
& \ddots & \\
1 & & 2
\end{bmatrix}
$$

whose determinant is that of the original system. Upon checking that

$$
\frac{1}{k}\begin{bmatrix}
k-1 & & -1\\
& \ddots & \\
-1 & & k-1
\end{bmatrix}
$$

serves as an inverse, our claim is verified, and the unique solution ρ provides a signed point measure on $x_1, x_2, \ldots, x_n$ that we extend to x_n in the obvious way. Clearly, the given μ is then only the restriction of the ordinary signed measure ρ to the sets in $\mathcal{X}(n,k)$. This point of view was first brought to the author's attention by J.-P. Marchand, who also suggested a conjecture close to the statement of our lemma.

Acknowledgment. The author wishes to acknowledge most helpful discussions with two of his colleagues, Professors Stanley P. Gudder and Jean-Paul Marchand.

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UNSOLVED PROBLEMS

EDITED BY RICHARD GUY

In this department the MONTHLY *presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada T2N 1N4.*

ARE ALL PRIMES $32k+17$ $(k>0)$ SQUARE SEPARABLE?

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In an investigation of unitary perfect polynomials over $GF(p^d)$, M. S. Harbin and the author [1] were led to a condition which generalizes to the concept of **square-separability** (s-s) in the following way. Let h be a primitive root of an odd prime $p=2^et+1$ where t is odd, and let s be a divisor of t. The set S of the $2^{e-1}t/s$ least positive residues of the *odd* powers of h^s may be considered as a subset of the integer interval $[1,p-1]$ and, as such, the set S is independent of the choice of h. "Order" the elements of S according to their natural order as integers. We say that p is s-**square-separable** (s-s-s) if every closed integer interval $[h^{us},h^{vs}]$ determined by elements h^{us},h^{vs} of S contains a quadratic residue of p (an *even* power of h). If d is the least s for which p is s-s-s, then p is s-s **of index** d.

For example, $p=3$ is the only prime which is 1-s-s. Indeed, no other (Fermat) prime 2^e+1 (5, 17, 257, ...?) is s-s. The primes 7 and 11 are s-s, but $13=2^23+1$ is not since the interval $[5, 8]=[2^9, 2^3]$ contains none of the quadratic residues ±1, ±3, ±4 of 13. It can be shown [2] that, *apart from the primes $p=5$ and $p=13$ in the case $e=2$, all primes $p=2^et+1$ are* s-s *for $e=1$, 2 and* 3.

What About $e\geqslant 4$? A computer study made at Emory University with Doyle and Mandelberg [2] for the cases (i) $4\leqslant e\leqslant 7$, $1<t<10^6$, and (ii) $e=8$ and 9, $1<t<10^5$, has given the following results. The second column of the table gives the number of **admissible** values of t, i.e., odd $t>1$ for which 2^et+1 is prime.

e	total # of primes	# of s-s primes	exceptional but admissible values of t
4	64335	64335	—
5	61657	61653	3, 11, 29, 3989
6	58981	58966	3, 75, 229, 247, 337, 423, 429, ..., 93009, 115353, 888547
7	56769	56742	5, 9, 11, 21, 35, 75, 89, 105, ..., 99035, 301361, 511377
8	6218	6180	3, 13, 31, 37, 55, 105, 141, 147, ..., 19095, 69673, 76531
9	6000	5936	15, 21, 23, 35, 45, 51, 63, 89, 131, ..., 71481, 77169, 85191

It is clear that if $u|v$ and p are u-s-s, then p is v-s-s, but what more can be said? The example $p=2521=2^3315+1$ which is s-s-s for $s=35$, 45, 105, 315 but not for $s=3$, 5, 7, 9, 15, 21, 63, shows that the index d doesn't necessarily divide s whenever p is s-s-s of index d. Nor is it necessary for p to be s-s-s just because $s>d$. Other primes showing both of these "bad" behaviors are 5881 and 7481.

Can one characterize those s-s primes whose index d divides s whenever p is s-s-s? Or those which are s-s-s for all $s>d$? Or those with index $d=t$? For the last question, the condition that t

is prime is sufficient, but not necessary: $73=2^3 9+1$ is s-s of index 9 and $241=2^4 15+1$ is s-s of index 15.

It has been conjectured [1] that $p=2^e t+1$ is s-s for all sufficiently large admissible t. If this is so for $t>t_e$, what can be said about the least t_e as a function of e?

In contrast, the Editor of this Section has posed questions of finiteness. Is 7 the only 3-s-s prime? Are 11 and 41 the only 5-s-s primes? (Recall that 3 is the only prime which is 1-s-s.) Is there only a finite number N_s of s-s-s primes for each odd s? If so, can N_s be calculated explicitly?

Is the following question more tractable than the notorious twin-prime conjecture? Are there infinitely many odd t for which $2^e t+1$ and $2^e(t+2)+1$ are both prime?

The author gratefully acknowledges conversations with L. Carlitz and Trevor Evans, the enthusiasm of Kevin Doyle and Kenneth Mandelberg, and the suggestions of Richard K. Guy.

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CLASSROOM NOTES

EDITED BY DEBORAH TEPPER HAIMO AND FRANKLIN TEPPER HAIMO

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DIFFERENTIABILITY AND PERMUTATIONS OF QUANTIFIERS

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1. Introduction. This note is the result of a one-hour junior-level individual study project. The problem considered is of a type suggested to us by Professor B. Jónsson. We feel that this kind of problem provides an excellent exercise for undergraduate students in either a logic or a real variables course. A much easier exercise is obtained by considering continuity rather than differentiability. The proof of the main result uses elementary facts about the convergence of Cauchy sequences and the most basic facts about the order of quantifiers.

All functions considered in this paper are real-valued functions defined on the set $\mathbb{R}$ of all real numbers. A *differentiable function* is one that satisfies the formula F:

$$\forall a \exists b \forall \epsilon \exists \delta \forall x\Big((0<|x-a|<\delta)\to\Big(\Big|\frac{f(x)-f(a)}{x-a}-b\Big|<\epsilon\Big)\Big),$$

where the domain of a, b, and x is $\mathbb{R}$, and the domain of ϵ and δ is the set of positive real numbers. By a *quantifier* we mean, as usual, the quantifier symbol together with the associated variable. We are interested in formulas obtained from F by permuting the five quantifiers, leaving the rest of the formula as is. Each such formula defines a class of functions, i.e., all those

functions that satisfy the given formula. For notational convenience, we identify such a formula with the corresponding permutation of the quantifiers. Also, we let G, P, Q, and DQ denote the expressions

$$(0<|x-a|<\delta)\to\left(\left|\frac{f(x)-f(a)}{x-a}-b\right|<\epsilon\right),\quad 0<|x-a|<\delta,$$

$$\left|\frac{f(x)-f(a)}{x-a}-b\right|<\epsilon,\quad \text{and}\quad \frac{f(x)-f(a)}{x-a},$$

respectively.

The goal of this study is to determine the classes of functions defined by formulas obtained from F by permuting the quantifiers. The conclusion is as follows:

Result. Any formula obtained from F by permuting the quantifiers defines one of the following classes:

(i) the class of all functions,
(ii) the class of differentiable functions,
(iii) the class of differentiable functions with uniformly continuous derivatives,
(iv) the class of linear functions.

The discovery that class (ii) admits another definition ((18) below) and that class (iii) appears was a complete surprise. We expected to find many more classes of functions, not necessarily linearly ordered by inclusion.

2. The First Reduction. The following two facts greatly reduce the number of formulas we must consider. The first fact is a basic result of logic (see [**1**, p. 57, 2.19, parts c and h]), and the second is due to the symmetry of x and a in the formulas under consideration.

Fact 1. Adjacent, like quantifiers are interchangeable; i.e., any two permutations differing only in the order of adjacent, like quantifiers are equivalent.

Fact 2. Permutations differing only in the relative position of the universal quantifiers on x and a are equivalent.

By extensive application of these two facts, we find that each of the 120 permutations of the five quantifiers is equivalent to one of the following 30 permutations (of which (17) is the original version):

(1) $\exists b\forall a\forall x\exists\delta\forall\epsilon G$	(17) $\forall a\exists b\forall\epsilon\exists\delta\forall x G$
(2) $\exists\delta\forall a\forall x\exists b\forall\epsilon G$	(18) $\forall a\forall\epsilon\exists b\exists\delta\forall x G$
(3) $\exists b\forall\epsilon\forall a\forall x\exists\delta G$	
(4) $\exists\delta\forall\epsilon\forall a\forall x\exists b G$	(19) $\forall\epsilon\exists\delta\forall a\exists b\forall x G$
(5) $\forall a\exists b\forall x\exists\delta\forall\epsilon G$	
(6) $\forall a\exists\delta\forall x\exists b\forall\epsilon G$	(20) $\exists b\forall\epsilon\forall a\exists\delta\forall x G$
(7) $\forall a\exists b\forall\epsilon\forall x\exists\delta G$	(21) $\exists b\forall a\exists\delta\forall\epsilon\forall x G$
(8) $\forall a\exists\delta\forall\epsilon\forall x\exists b G$	(22) $\exists b\forall\epsilon\exists\delta\forall a\forall x G$
(9) $\forall\epsilon\exists b\forall a\forall x\exists\delta G$	(23) $\exists b\exists\delta\forall\epsilon\forall a\forall x G$
(10) $\forall\epsilon\exists\delta\forall a\forall x\exists b G$	(24) $\forall\epsilon\exists b\forall a\exists\delta\forall x G$
(11) $\forall a\forall x\exists b\exists\delta\forall\epsilon G$	(25) $\exists\delta\forall\epsilon\exists b\forall a\forall x G$
(12) $\forall a\forall x\exists b\forall\epsilon\exists\delta G$	(26) $\forall\epsilon\exists b\exists\delta\forall a\forall x G$
(13) $\forall a\forall x\exists\delta\forall\epsilon\exists b G$	(27) $\forall a\exists b\exists\delta\forall\epsilon\forall x G$
(14) $\forall\epsilon\forall a\exists b\forall x\exists\delta G$	(28) $\forall a\exists\delta\forall\epsilon\exists b\forall x G$
(15) $\forall\epsilon\forall a\exists\delta\forall x\exists b G$	(29) $\exists\delta\forall a\exists b\forall\epsilon\forall x G$
(16) $\forall\epsilon\forall a\forall x\exists b\exists\delta G$	(30) $\exists\delta\forall\epsilon\forall a\exists b\forall x G$.

3. The Class of All Functions. It follows from the following fact that any of the permutations numbered (1)–(16) defines the class of all functions.

Fact 3. If the universal quantifiers on x and a both precede one of the existential quantifiers in a given permutation, then that permutation defines the class of all functions.

To see this, suppose K is such a permutation and let f be any function. If $x=a$, then P is false; so G is true. Thus we may assume that $x\neq a$. If the existential quantifier is on δ, then we choose δ so that $0<\delta<|x-a|$. Then again P is false; so G is true. If the existential quantifier is on b, then we choose $b=(f(x)-f(a))/(x-a)$. Then Q is true; so G is true. In any case f satisfies K.

4. The Class of Differentiable Functions. We now show that permutations (17) and (18) are equivalent. The fact that the class of functions defined by (17) is a subclass of the class defined by (18) ensues from the following result of logic (see [2, p. 57, 2.19, part i]).

Fact 4. If permutation K' is obtained from permutation K by shifting a universal quantifier from the left to the right of an existential quantifier, then the class of functions defined by K' is a subclass of the class of functions defined by K.

To see that the classes of functions defined by (17) and (18) are in fact equal, suppose f satisfies (18). Let a be given. For each positive integer n, let b_n and δ_n be such that, whenever $0<|x-a|<\delta_n$, we have $|DQ-b_n|<1/n$. A standard argument shows that $\{b_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Now let $b=\lim_{n\to\infty}b_n$. For $\epsilon>0$, choose N such that (i) $N\geqslant 2/\epsilon$ and (ii) for $n\geqslant N$, $|b_n-b|<\epsilon/2$. Then if x is such that $0<|x-a|<\delta_N$, we have

$$|DQ-b|=|DQ-b_N+b_N-b|\leqslant|DQ-b_N|+|b_N-b|<1/N+\epsilon/2\leqslant\epsilon.$$

Thus f satisfies (17).

5. The Class of Functions with Uniformly Continuous Derivatives. We now show that (19) defines the class of functions with uniformly continuous derivatives.

Suppose f satisfies (19). Using one application of Fact 4 and two applications of Fact 1, we see that f is differentiable. For given $\epsilon>0$, we let δ_ϵ denote the assured δ, and, for given $\epsilon>0$ and given a, we let $b_{\epsilon,a}$ be the guaranteed b. Now given $n>0$ and a, we get $\delta_{1/n}>0$ such that $|DQ-f'(a)|<1/n$ if $0<|x-a|<\delta_{1/n}$. Then for x such that $0<|x-a|<\min\{\delta_\epsilon,\delta_{1/n}\}$ we have

$$|f'(a)-b_{\epsilon,a}|\leqslant|f'(a)-DQ|+|DQ-b_{\epsilon,a}|<1/n+\epsilon.$$

Thus we must have $|f'(a)-b_{\epsilon,a}|\leqslant\epsilon$. Now given $\epsilon>0$ we take $\delta=\delta_{\epsilon/2}$. Thus if $0<|x-a|<\delta$ we have

$$|DQ-f'(a)|\leqslant|DQ-b_{\epsilon/2,a}|+|b_{\epsilon/2,a}-f'(a)|<\epsilon.$$

Thus f satisfies the formula

$$\forall\epsilon\exists\delta\forall a\forall x((0<|x-a|<\delta)\to(|DQ-f'(a)|<\epsilon)).$$

Given $\epsilon>0$ choose δ such that $|DQ-f'(a)|<\epsilon/2$ whenever $0<|x-a|<\delta$. Then for such x, a we have

$$|f'(a)-f'(x)|\leqslant|f'(a)-DQ|+|DQ-f'(x)|<\epsilon,$$

and f' is uniformly continuous.

Suppose now that f is a function with uniformly continuous derivative. Given $\epsilon>0$, choose $\delta>0$ such that $|f'(x)-f'(a)|<\epsilon$ whenever $0<|x-a|<\delta$. Now given a, let $b=f'(a)$. Then if $0<|x-a|<\delta$, the Mean-Value Theorem gives c between x and a such that

$$f'(c)=\frac{f(x)-f(a)}{x-a}.$$

Since $0<|c-a|<\delta$, we have $|DQ-b|=|f'(c)-f'(a)|<\epsilon$, and f satisfies (19).

6. The Class of Linear Functions. The remaining permutations, i.e., (20)–(30), define the class

of linear functions. By repeated applications of Fact 1 and Fact 4, we see that for any permutation K, numbered (20)–(30), we have

(i) the class defined by permutation (23) is a subclass of the class defined by K;

(ii) the class defined by K is a subclass of either the class defined by permutation (24) or the class defined by permutation (28).

Thus it is enough to show that any function satisfying (24) or (28) is linear and that any linear function satisfies (23).

If f is linear, we can take $b=f'(x)$ and $\delta=1$ to satisfy (23).

In the case where f satisfies (24) a Cauchy sequence argument can be used to show that $b=f'(a)$; so f is linear.

Similarly, if f satisfies (28) one can show by a Cauchy sequence argument that we can take $b=f'(a)$ (b now depending on a). From this it can be shown that, for any a, f is linear on a neighborhood of a, and thus f is linear.

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WHEN DOES $T^{n+1}x - T^n x \to 0$ IMPLY CONVERGENCE?

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Hillam [1] proved:

Theorem. *Let $f:[0,1]\to[0,1]$ be a continuous function. Let x_1 be a point in $[0,1]$ and let x_n denote the resulting sequence of successive approximations. Then the sequence x_n converges to a fixed point of f if and only if $\lim(x_{n+1}-x_n)=0$.*

The main object of this note is to show that Hillam's result does not extend beyond the one-dimensional case. We give a counterexample for the closed disc. On the other hand, Hillam's proof uses the ordering of the unit interval, which is irrelevant. We give a proof applying to any one-dimensional finite simplicial complex.

We consider only the convergence of the sequence $T^n x$; if it converges, the limit must be a fixed point.

Example. *There is a continuous mapping T of the closed unit disc in the plane such that the origin and points on the unit circle are fixed points and every other point p satisfies: $T^n p - T^{n+1}p \to 0$ but $T^n p$ is not convergent.*

To define the mapping (using polar coordinates (r,θ)) we put $T0=0$ and:

If $0<r\leqslant\frac{1}{2}$ we call (r,θ) an "inner point" and put

$$T(r,\theta)=\left(\tfrac{4}{3}r,\theta+\tfrac{1}{2}\right).$$

If $\frac{1}{2}\leqslant r\leqslant 1$ we call (r,θ) an "outer point" and put

$$T(r,\theta)=\left((2-r)^{-1},\theta+(1-r)\right).$$

The geometric effects of this definition are:

(A) If p is an inner point then $T^n p$ is an outer point for n sufficiently large.

(B) If p is an outer point at distance k^{-1} from the unit circle then Tp is an outer point at

distance $(k+1)^{-1}$ from the unit circle, and

$$\arg(Tp)=k^{-1}+\arg p \text{ (see Fig. 1).}$$

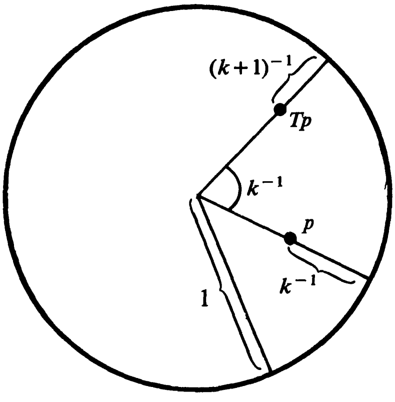

FIG. 1. Mapping of an "outer point" in the example.

It is clear from (A) and (B) that $T^np-T^{n+1}p\to 0$ as $n\to\infty$. Also if p is an outer point (at distance k^{-1} from the unit circle),

$$\arg(T^Np)=\arg p+\sum_{n=0}^{N-1}(k+n)^{-1}.$$

Since $\Sigma(k+n)^{-1}$ is divergent we deduce that T^Np cannot converge. If p is an inner point it now follows from (A) that T^np cannot converge.

We now give some positive results. Without restricting the dimension, the best result seems to be:

THEOREM 1. *Let M be a metric space with metric $d(.,.)$. If T is a continuous map of M into M, if $x_1\in M$, and if $d(T^nx_1,T^{n+1}x_1)\to 0$, then any limit point y of the set $\{T^nx_1\}$ is a fixed point of T.*

Proof. For some $n(i)\to\infty$,

$$y=\lim T^{n(i)}x_1=\lim T^{n(i)+1}x_1=T\lim T^{n(i)}x_1=Ty.$$

We now give our extension of Hillam's theorem.

THEOREM 2. *Let M be a one-dimensional finite simplicial complex, $T: M\to M$ continuous, and $x\in M$. Then T^nx converges to a fixed point of T if and only if $d(T^nx,T^{n+1}x)\to 0$.*

Proof. Assume that $d(T^nx,T^{n+1}x)\to 0$. Assume that $\{T^nx\}$ has two limit points y_1,y_2. Then for each positive ε we can connect y_1 to y_2 by an ε-chain of the form $T^mx,T^{m+1}x,\dots,T^kx$ (taking m sufficiently large, T^mx close to y_1 and T^kx close to y_2). There are a finite number of minimal arcs from y_1 to y_2; one of these arcs, A, must contain an infinite number of our ε-chains. Thus each point in A is a limit point of the set $\{T^nx\}$ and thus is a fixed point of T. Choose a point T^mx in A; then $T^sx=T^mx$ for $s>m$. Thus $\{T^nx\}$ can have only one limit point, so that T^nx is convergent.

The reverse implication is trivial, as is the fact that the limit of T^nx must be a fixed point of T.

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AN ELEMENTARY PROOF OF INCLUSION-EXCLUSION FORMULAS

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Many probability problems involve finding the probability that exactly m (or at least m) among N events occur simultaneously. In this author's judgment every upper-division course in Probability should cover the mathematical formulas needed for such problems. Although most texts treat this subject, it is surprising how many present only the special formula for computing the probability of at least one out of N events. Perhaps one reason for this is that the standard proof of the general result, using Loève's method of indicators, is tedious and requires properties of mathematical expectation (cf. Neuts [3] and Parzen [4]). Organizational problems arise for those texts that present the subject matter involving inclusion-exclusion formulas before introducing expectation. Feller [1] avoids this difficulty by presenting a combinatorial proof valid *only* for discrete sample spaces.

The purpose of this note is to provide a rigorous proof of the theorem below that is valid for arbitrary sample spaces and does not require properties of expectation. In the author's opinion, the argument is easier to follow than the approach using the method of indicators.

THEOREM. *For $1 \leqslant m \leqslant N$, let B_m denote the event that exactly m among the events $A_1, A_2, \ldots, A_N$ occur simultaneously. Then*

$$P(B_m) = \sum_{k=m}^{N} (-1)^{k-m} \binom{k}{m} S_k,$$

where

$$S_k = \sum_{1 \leqslant j_1 < j_2 < \cdots < j_k \leqslant N} P(A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_k}).$$

Proof. Let $\{E_1, E_2, \ldots, E_{2^N}\}$ be the partition of the sample space determined by the 2^N possible events of the form $A_1^{\delta_1} \cap A_2^{\delta_2} \cap \cdots \cap A_N^{\delta_N}$, where each $A_j^{\delta_j}$ is either A_j or A_j'. For $1 \leqslant i \leqslant 2^N$, let

$$(l_i(1), l_i(2), \ldots, l_i(r_i), l_i(r_i+1), \ldots, l_i(N))$$

be a permutation of $(1,2,\ldots,N)$ such that $l_i(1), l_i(2), \ldots, l_i(r_i)$ are the subscripts of those A-events that occur if and only if E_i occurs. Then

$$r_i = \text{Number of } A\text{-events that occur when } E_i \text{ occurs},$$

and

$$E_i = A_{l_i(1)} \cap A_{l_i(2)} \cap \cdots \cap A_{l_i(r_i)} \cap A'_{l_i(r_i+1)} \cap \cdots \cap A'_{l_i(N)}.$$

Clearly, B_m is the disjoint union of the $\binom{N}{m}$ E-events that consist of N-fold intersections where exactly $N-m$ of the A-events are complemented. Therefore,

$$P(B_m) = \sum_{i \in \Lambda} P(E_i), \quad \text{where } \Lambda = \{i : r_i = m\}.$$

Since each event of the form $A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_k}$ is the disjoint union of 2^{N-k} E-events, consider the expression

$$P(A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_k}) = \text{Sum of } 2^{N-k}\ E\text{-event probabilities.} \tag{1}$$

The right-hand side of (1) contains the term $P(E_i)$ if and only if

$$\{j_1, j_2, \ldots, j_k\} \subset \{l_i(1), l_i(2), \ldots, l_i(r_i)\}.$$

Thus, exactly $\binom{r_i}{k}$ of the summands on the right-hand side of

$$S_k = \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq N} P(A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_k})$$

contain the term $P(E_i)$ in their formula-(1) decompositions. Writing S_k, using (1), and collecting terms according to distinct E-events, it follows that

$$S_k = \sum_{i=1}^{2^N} \binom{r_i}{k} P(E_i).$$

Therefore,

$$\sum_{k=m}^{N} (-1)^{k-m} \binom{k}{m} S_k = \sum_{i=1}^{2^N} \left\{ \sum_{k=m}^{N} (-1)^{k-m} \binom{k}{m} \binom{r_i}{k} \right\} P(E_i)$$
$$= \sum_{i=1}^{2^N} c_i P(E_i).$$

Observe that $c_i = 0$ if $r_i < m$; $c_i = 1$ if $r_i = m$ (i.e., if $i \in \Lambda$); and

$$c_i = \sum_{k=m}^{r_i} (-1)^{k-m} \binom{k}{m} \binom{r_i}{k} = \binom{r_i}{m} \sum_{s=0}^{r_i - m} (-1)^s \binom{r_i - m}{s}$$
$$= \binom{r_i}{m} (1-1)^{r_i - m} = 0 \quad \text{if } r_i > m.$$

Hence,

$$\sum_{k=m}^{N} (-1)^{k-m} \binom{k}{m} S_k = \sum_{i \in \Lambda} P(E_i) = P(B_m). \quad \square$$

COROLLARY. *If C_m denotes the event that at least m among the events $A_1, A_2, \ldots, A_N$ occur simultaneously, then*

$$P(C_m) = \sum_{k=m}^{N} (-1)^{k-m} \binom{k-1}{m-1} S_k, \quad \text{for } 1 \leq m \leq N.$$

This corollary follows from the theorem by induction (cf. Feller [1] or Neuts [3]).

References

1. W. Feller, An Introduction to Probability Theory and Its Applications, vol. 1, 3rd ed., Wiley, New York, 1968.
2. M. Loève, Sur les systèmes d'événements, Ann. Univ. Lyon, Sec. A(3), 5 (1942) 55–74.
3. M. Neuts, Probability, Allyn and Bacon, Boston, 1973.
4. E. Parzen, Modern Probability Theory and Its Applications, Wiley, New York, 1960.

MISCELLANEA

45. Some of the most important results (e.g., Cauchy's theorem) are so surprising at first sight that nothing short of a proof can make them credible.

—H. and B. S. Jeffreys, *Methods of Mathematical Physics*, 2nd ed., Cambridge University Press, 1950, p. v.

MATHEMATICAL EDUCATION

EDITED BY W. E. MASTROCOLA

Material for this department should be sent to Robert F. Wardrop, Department of Mathematics, Central Michigan University, Mount Pleasant, MI 48859.

THE PREDICTABILITY OF COUNTEREXAMPLES

DESMOND MACHALE

Department of Mathematics, University College, Cork, Ireland

When teaching mathematics, one is often faced with the task of convincing a student that the converse of a given theorem is false or that the theorem is as strong as possible. The usual method involves the production of a counterexample. However, even a quick glance at a wide range of textbooks shows a singular lack of variety in the counterexamples exhibited, arising no doubt from a lack of imagination on the part of the authors. Indeed, an inexperienced student might be led to conjecture that the following statements are theorems:

1. The function $f(x)=|x|$ is the only real function that is continuous but not differentiable.
2. The real interval $[0,1]$ is the only uncountable set.
3. The function defined on $[0,1]$ by
$$f(x)=\begin{cases} 0, & x \text{ rational} \\ 1, & x \text{ irrational} \end{cases}$$
is the only function that is not Riemann integrable.
4. The only noncommutative operations in algebra are multiplication of 2×2 matrices, subtraction of integers, and composition of permutations on three objects.
5. $\Sigma_{r=1}^{\infty}(1/r)$ is the only divergent series.
6. $\Sigma_{r=1}^{\infty}(-1)^{r+1}/r$ is the only series that is convergent but not absolutely convergent.
7. The numbers π and e are transcendental, but the proof is beyond the scope of any textbook.
8. Cubic and quartic polynomials are soluble by radicals, but nobody knows the details.
9. $\sqrt{2}$ is the only irrational number.
10. The alternating group on four symbols is the only finite group that does not satisfy the converse of Lagrange's Theorem.
11. Any result about the natural numbers can be proved by induction, but the details can always be omitted.
12. The only Pythagorean triple is $(3,4,5)$.

PROBLEMS AND SOLUTIONS

EDITED BY VLADIMIR DROBOT

Send all ***proposed problems****, in duplicate if possible, to Professor Vladimir Drobot, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053 (USA). Please include solutions, relevant references, etc.*

An asterisk (*) indicates that neither the proposer nor the editors supplied a solution.

Solutions *should be sent to the addresses at the head of each problem set.*

A publishable solution must, above all, be correct. Given correctness, elegance and conciseness are preferred. The answer to the problem should appear right at the beginning. If your method yields a more general result, so much the better. If you discover that a MONTHLY *problem has already been solved in the literature, you should of course tell the editors; include a copy of the solution if you can.*

SOLUTIONS OF PROBLEMS DEDICATED TO EMORY P. STARKE

Tetrahedron Temperatures

S 11 [1979, 392]. *Proposed by R. C. Buck and E. F. Buck, University of Wisconsin, Madison.*

A solid tetrahedron carries a continuous temperature distribution. What is the maximum number of points having the same temperature one can be sure of finding on the edges of the tetrahedron?

Solution by James W. Walker, Massachusetts Institute of Technology. First, note that it suffices to consider temperature distribution on the edges of the tetrahedron. Any continuous function on the tetrahedron restricts to one on the edges; and by the Tietze theorem, any continuous real function on the edges can be extended to the whole tetrahedron.

Let K_n denote the 1-skeleton of an $n-1$ simplex. (This is the standard notation for a complete graph on n vertices.) Let $T(n)$ denote the function $T(1)=1, T(2m)=m^2, T(2m+1)=m(m+1)$, where m is a positive integer. Then $T(n)$ is the maximum number of points having the same temperature that one can be sure of finding on K_n. In particular, $T(4)=4$ is the answer to the stated problem.

To show that there need be no more than $T(n)$ points with the same temperature, we consider a specific temperature distribution. If $v_1, v_2, \ldots, v_n$ are the vertices of K_n, let v_i have temperature i, and use linear interpolation along the edges. There are $i-1$ vertices colder than v_i, and $n-i$ vertices hotter than v_i. Any edge from a colder vertex to a hotter one contains exactly one point at temperature i. Therefore temperature i occurs at exactly $(i-1)(n-i)+1$ places. By similar reasoning, if $i<t<i+1$, then temperature t occurs at exactly $i(n-i)$ places. It is straightforward to show that $T(n)$ is the smallest number which is greater than or equal to both $(i-1)(n-i)+1$ and $i(n-i)$, for all $i=1,2,\ldots,n$. So for this particular distribution, $T(n)$ is the exact maximum size of an isotherm.

Now I will show that there are always at least $T(n)$ points of the same temperature. The important properties of the above distribution were that

(1) the vertices could be listed in order of strictly increasing temperature, and

(2) if v is a vertex hotter than t and w is a vertex colder than t, then the edge from v to w contains at least one point of temperature t.

Property (2) holds for any continuous temperature distribution, by the intermediate value theorem. Therefore, it only remains to be shown that it is no loss of generality to assume that all vertices have distinct temperatures.

Leaving the vertex temperatures fixed, we can use the intermediate value theorem to see what kind of temperature distribution will minimize the maximum size of an isotherm. We may therefore assume that

(a) if vertices v and w are not isothermal, then temperature is strictly monotone along the edge from v to w,

(b) if vertices v and w are isothermal, then there is a point z on the edge from v to w such that temperature is strictly monotone from v to z and from z to w, and

(c) in the situation of (b), z has a temperature close enough to that of v and w so that no vertex is isothermal to an interior point of the edge from v to w.

Given several isothermal vertices, the idea is to perturb their temperatures (and the temperatures within a small neighborhood of them, for the sake of continuity). We will perturb them one by one, and we have to be careful about which vertices we heat and which we cool. After perturbing each vertex, we have to adjust the edge temperatures so that properties (a), (b), and (c) will still hold.

Given a vertex v, let $h(v)$ denote the number of edges incident to v which heat up as you move away from v, and let $c(v)$ denote the number of edges which cool down. Thus $h(v)+c(v)=n-1$. Now, if $h(v)<c(v)$, we can cool v; if $h(v)>c(v)$, we can heat v; and if $h(v)=c(v)$, we can perturb the temperature of v in either direction. This procedure will not increase the maximum size of an isotherm.

Also solved by F. S. Cater, Nick Franceshine III, P. F. Gibson, Michael Josephy (Costa Rica), V. Kannon (India), Man Kam Kwong, O. P. Lossers (Netherlands), Mark Merriman, Mark D. Meyerson, Brian M. O'Connor, Robert H. Overton, Victor Pambuccian (Romania), L. C. Singhal (India), F. B. Strauss, and the proposers.

Closed Complex Additive Subsemigroups

S 16 [1979, 591]. *Proposed by I. J. Schoenberg, University of Wisconsin, Madison.*

Characterize the closed sets S of the complex plane such that $d(z+w)\leqslant d(z)+d(w)$ for all complex numbers z and w, where $d(z)$ denotes the euclidean distance from z to S.

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, Netherlands. Clearly S must be a closed subsemigroup for addition; conversely, if S is a closed subsemigroup for addition, then S has the required property. For $z,w\in C$, there is $z',w'\in S$ such that $d(z)=|z-z'|$, $d(w)=|w-w'|$. Then $d(z+w)\leqslant|z+w-(z'+w')|\leqslant d(z)+d(w)$.

Note. Cater and Meyerson (independently) observed that if one drops the condition that S is topologically closed then the desired sets would be those whose topological closures are additive semigroups.

Also solved by Ron M. Adin (Israel), Duane M. Broline, F. S. Cater, N. J. Fine, Gustaf Gripenberg (Finland), Michael Josephy (Costa Rica), Mark D. Meyerson, Thomas L. Moore, J. E. Nymann, Colorado College Problem Solving Group, John H. Riley, Jr., V. Šverák (Czechoslovakia), James Theiler, and the proposer.

ELEMENTARY PROBLEMS

Solutions of these Elementary Problems should be mailed in duplicate to Dr. J. L. Brenner, 10 Phillips Road, Palo Alto, CA (USA) 94303, by March 31, 1981. Please place the solver's name and mailing address on each (double-spaced) sheet. Include a self-addressed card or label (for acknowledgment).

E 2853. *Proposed by John Barnden and David E. Daykin, Reading University, England.*

Consider the set C of all partially ordered sets (posets) constructible as follows. Start with the one-element poset. Then at each stage, in the obvious way, replace some element by one of the two two-element posets, using the inherited order relation with preference (if preference is needed). Show that the poset $P\in C$ if and only if every four-element subset of P (with inherited order) is in C.

E 2854. *Proposed by James Chew, North Carolina Agricultural and Technical University.*

Let $f(x,y)$ be real valued and continuous on the unit square $0\leqslant x,y\leqslant 1$. Set $y^*(a)=\inf\{y|y \text{ maximizes } f(a,y)\}$. Is $y^*(a)$ continuous? What if, for each a, $f(a,y)$ attains its maximum *uniquely* at $y^*(a)$?

E 2855. *Proposed by W. W. Meyer, General Motors Research Laboratories.*

For real nonnegative x, y, solve $\Gamma(x+y+1)\geqslant(1+xy)\Gamma(x+1)\Gamma(y+1)$.

E 2856. *Proposed by F. W. Luttmann, Sonoma State University.*

Does there exist a simple closed curve, other than the circle, such that two consecutive Steiner symmetrizations with respect to two orthogonal lines always produce a circle?

E 2857. *Proposed by John Leggett, Atascadero, California.*

If x, y, z, w are real numbers, $xy=4$, $z^2+4w^2=4$, prove that $(x-z)^2+(y-w)^2 \geqslant 1.6$.

E 2858. *Proposed by Oto Strauch, University of Bratislava, Czechoslovakia.*

Let I be the open interval $(0,1)$, let Y be a countable subset of I, and let a_n $(n \in N)$ be positive real numbers such that $\sum_{n=1}^{\infty} a_n \leqslant 1$. Prove that there is a sequence I_n $(n \in N)$ of pairwise disjoint closed subintervals of I such that length $(I_n)=a_n$ for all n, and every point of Y is in the interior of some I_n.

SOLUTIONS OF ELEMENTARY PROBLEMS

Filling an Open Set with Squares of Specified Areas

E 2790 [1979, 593]. *Proposed by Mark D. Meyerson, United States Naval Academy.*

Suppose we have a collection of squares, one each of area $1/n$ for $n=1,2,3,\ldots$, and any open set G in the plane. Show that we can cover all of G except a set of area 0 by placing some of the squares inside G without overlap. (The edges of the squares are allowed to touch.)

Solution by Jonathan Leech, Missouri Western State College, St. Joseph, Missouri. Let H denote the sequence $1/n$ for $n=1,2,3,\ldots$. Let S denote a finite subset of H, or a subsequence of H whose terms form a convergent series, and let $\bar{S}$ denote its complement. For any subset T of H, by a set of T squares we shall mean a set of squares in one-to-one correspondence with T such that the square corresponding to $1/n$ has area $1/n$. Finally, let U_i be any enumerated collection of open unit squares which cover (tile) the plane, and let $G_i = U_i \cap G$. Then all of G except for a set of area zero is covered by $\cup G_i$.

We shall show that for any nonempty G_i and any subsequence $\bar{S}$ of H there is a subsequence T of $\bar{S}$ such that G_i is covered by T squares which lie inside G_i. Once this assertion is established, we then cover the first nonempty G_i with T' squares for some $T' \subset H$. Next we cover the second nonempty G_i with T'' squares for some $T'' \subset H - T'$ and so on until (in perhaps an infinite number of steps using the diagonal process) G is exhausted. G then contains a disjoint union of T squares for some subset T of H and the complement of this union in G is a set of area zero.

So now let us consider a nonempty G_i and a subsequence $\bar{S}$ of H. First subdivide U_i into 4 squares each of area $1/4$, subdivide it again into 16 squares each of area $1/16$ and so on until there are 4^n squares each of area $1/4^n$ such that $3/4$ of the area of G_i is accounted for by the squares which lie in G_i. By increasing n if necessary, we may assume that $\sum\{m^{-1} | m \geqslant 4^n, m^{-1} \in S\} \leqslant 1/4$.

Let $\sigma_n = 1/(4^n+1) + 1/(4^n+2) + \cdots + 1/4^{n+1}$. Note that (1) σ_n has $3\cdot 4^n$ terms, from which it follows that (2) $3 > \sigma_n > 3/4$; hence (3) if $\tau_p = 1/a_1 + 1/a_2 + \cdots + 1/a_p$ is a subsum of σ_n then $p < 4^{n+1}\cdot\tau_p$ and finally (4) if $\tau_p < 1/4$ then $p < 4^n$. Thus by our assumption on n there are at most 4^n elements of S among the terms of σ_n, and there are at least $2\cdot 4^n$ terms of σ_n not in S. Furthermore, since the $2\cdot 4^n$ largest terms of σ_n are $1/(4^n+1)$ through $1/(3\cdot 4^n)$, each of these largest terms is greater than or equal to $1/(3\cdot 4^n)$.

Now we place squares of descending areas, where the areas come from $\{1/(4^n+1),\ldots,1/4^{n+1}\}$, in the interiors of the squares of area $1/4^n$ which lie in G_i. The squares which we insert which are not S squares will have area at least $[(1/3\cdot 4^n)/(1/4^n)](3/4)$ (area G_i) $=(1/4)$ area G_i. Thus by removing a finite disjoint union of a subset of $\bar{S}$ squares which has area at least

$(1/4)$ area G_i, we obtain a new open set G_i' which has area $<(3/4)$ area G_i. We add to S the corresponding finite number of areas to obtain a new set S' which still consists of a finite subset of H or a subsequence of H whose terms form a convergent series. We repeat the process with G_i' and $\overline{S'}$ and continue to obtain a descending chain of open sets $G_i \supset G_i' \supset G_i'' \supset \cdots \supset G_i^m \supset \cdots$ such that area $G_i^{m+1} \leqslant (3/4)$ area G_i^m; and hence their areas tend to 0. Moreover, the sets $G_i - G_i^m$ form an ascending chain of finite disjoint unions of squares whose areas come from $\overline{S}$ and are distinct. Finally, $\cup(G_i - G_i^m)$ is a disjoint union of squares whose areas come from $\overline{S}$, are distinct, and sum to the area of G_i. Thus the complement of $\cup(G_i - G_i^m)$ in G_i has area zero.

Also solved by F. S. Cater, N. J. Fine, H. Kestelman (England), L. E. Mattics, T. L. Moore, the University of New Orleans Problems Group, and the proposer. Cater, Moore, and the proposer generalize the problem to any collection of squares one each of area a_i with $\{a_i\}$ a decreasing null sequence whose sum diverges. Leech mentions that the problem can also be generalized to higher dimensions.

Odd Intersections of Point Sets

E 2792 [1979, 702]. *Proposed by Robert Patenaude, California State College, Bakersfield.*

Let U be a finite set. Characterize those collections C of subsets of U with the following property: there is a unique subset R of U such that the number of sets in C which R intersects is odd.

Solution by Arnold Adelberg, Grinnell College. We will show that, apart from $\varnothing$, whose presence in C is irrelevant, C is characterized as the set of supersets of a proper subset of U, the subset being $U-R$.

If $P(U)$ is the set of subsets of U, $C \subseteq P(U)$ and $S \subseteq U$, let $n(S)$ be the number of sets in C meeting S and $m(S)$ be the number of sets in C contained in S. It follows that $n(S)+m(U-S) = |C|$; so if $|C|$ is even, $n(S)$ and $m(U-S)$ have the same parity. Note that if S is a minimal set in C, then $m(S)=1$.

Assume first that C is the set of supersets of a proper subset A of U. Letting $R=U-A$, we have $|C|=|P(R)|=2^{|R|}$; so $n(R)=|C|-1$, which is odd. Let B be a subset of U other than R. If B meets A, then $n(B)=|C|$, whereas if $B \subset R$, then $n(B)$ is divisible by $|P(R-B)|$, so that in either case, $n(B)$ is even.

Conversely suppose C has the property of the problem. Let C' be the set of supersets of $U-R$. Taking $\varnothing \in C$ if and only if $R=U$, we have $|C|$ is even. If A is a minimal set in C, then $m(A)=1$; so $n(U-A)$ is odd, which implies that $A=U-R$. It follows that $C \subseteq C'$.

Next, we apply the initial remarks with the collection $C'-C$ replacing C. It follows from the direction already proved that each subset of U meets an even number of sets in $C'-C$ and $|C'-C|$ is even; so $C'-C$ has no minimal sets. Hence $C=C'$.

(0,1)-matrices with Prescribed Row- and Column-Sums

E 2794*[1979, 703]. *Proposed by Robert A. Leslie, Agnes Scott College, Decatur, Georgia.*

Let m, n, r, and c be positive integers with $rm=cn$. How many $m \times n$ matrices are there with each entry either 0 or 1 and where every row sum is r and every column sum is c?

Comments by Richard Stanley, Massachusetts Institute of Technology. This problem has been the subject of considerable study, and it is unlikely that a simple formula exists. The case $r=c=2$ is solved by Anand, Dumir, and Gupta in Duke Math J., 33 (1966) 757–769. A formula for the case $r=c=3$ appears on page 236 of L. Comtet, Advanced Combinatorics, Reidel, 1974. The numbers in question can be related in various ways to the representation theory of the symmetric group or of the complex general linear group, but this does not make their computation any easier.

E. Triesch called attention to the paper of R. C. Read, *The enumeration of locally restricted*

graphs (II), J. London Math. Soc., 35 (1960) 344–351, where the problem is rephrased. Richard Stanley has kindly furnished a solution for the case $r=2$, $c=3$ based on this reformulation. The number of $r\times s$ (0, 1)-matrices with row-sum vector $(a_1,\dots,a_r)$ and column-sum vector $(b_1,\dots,b_s)$ is the coefficient of $x_1^{a_1}\cdots x_r^{a_r}y_1^{b_1}\cdots y_s^{b_s}$ in

$$\prod_{i=1}^{r}\prod_{j=1}^{s}(1+x_iy_j)=\exp\log\prod_{i,j}(1+x_iy_j)$$

$$=\exp\left[\sum_{i,j}\sum_{n\geq 1}\frac{(-1)^{n-1}}{n}(x_iy_j)^n\right]$$

$$=\exp\left[\sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\Big(\sum_i x_i^n\Big)\Big(\sum_j y_j^n\Big)\right]$$

$$=\prod_{n\geq 1}\exp\left[\frac{(-1)^{n-1}}{n}\Big(\sum_i x_i^n\Big)\Big(\sum_j y_j^n\Big)\right]$$

$$=\prod_{n\geq 1}\sum_{k\geq 0}\frac{(-1)^{(n-1)k}}{n^kk!}\Big(\sum_i x_i^n\Big)^k\Big(\sum_j y_j^n\Big)^k \quad (*).$$

One can now extract the desired coefficient in the case $r=2$, $c=3$. We seek the coefficient of $x_1^2\cdots x_{3l}^2y_1^3\cdots y_{2l}^3$ in (*). We need to consider only $n\leq 2$, so we can replace (*) by

$$\Big(\sum_{k\geq 0}(k!)^{-1}\Big(\sum x_i\Big)^k\Big(\sum y_j\Big)^k\Big)\Big(\sum_{k\geq 0}(-1)^k2^{-k}(k!)^{-1}\Big(\sum x_i^2\Big)^k\Big(\sum y_j^2\Big)^k\Big).$$

The coefficient of $x_{i_1}^2\cdots x_{i_k}^2y_{j_1}^2\cdots y_{j_k}^2$ in the second factor is $(-1)^kk!2^{-k}$, and there are $\binom{3l}{k}\binom{2l}{k}$ choices of $x_{i_1},\dots,x_{i_k}, y_{j_1},\dots,y_{j_k}$. The coefficient of

$$x_1^2\cdots x_{3l}^2y_1^3\cdots y_{2l}^3/x_{i_1}^2\cdots x_{i_k}^2y_{j_1}^2\cdots y_{j_k}^2$$

in the first factor is $(6l-2k)!/2!^{3l-k}3!^{2l-k}$. It follows that the desired coefficient is

$$C_l=\sum_{k=0}^{2l}(-1)^kk!\binom{3l}{k}\binom{2l}{k}(6l-2k)!2^{-3l}6^{-2l+k}$$

$$=288^{-l}\sum_{0}^{2l}(-1)^kk!\binom{3l}{k}\binom{2l}{k}(6l-2k)!6^k.$$

Thus $C_1=1$, $C_2=1860$. Enormous difficulties arise in trying to extend this method to arbitrary r, c.

Lee Erlebach rediscovered the formula for $r=c=2$.

Properties of Regular Bipartite Graphs

E 2795. *Proposed by Doug Wiedemann, Institute for Defense Analyses, Princeton, N.J.*

Let S be a nonempty subset of $\{0,1\}^n=\{0,1\}\times\cdots\times\{0,1\}$ such that each member of S is adjacent to exactly k other members of S, where "adjacent" means differing in one coordinate position. Show that the size of S is even and at least 2^k. Furthermore, if the graph of the adjacency relation of S is connected, show that it will still be connected after removal of any point.

The following two statements generalize the results called for in the problem. Statement I was proved by *O. P. Lossers, Department of Mathematics, Eindhoven University of Technology, Eindhoven, the Netherlands*. Statement II was proved by *Allen J. Schwenk, U.S. Naval Academy, Annapolis, Maryland.*

I. Given that each $x \in S$ is adjacent to at least k other members of S, we have $|S| \geqslant 2^k$.

Proof. If $(0,\ldots,0) \notin S$, then for $y \in S$ consider $S' = S + y = \{s + y | s \in S\}$, where addition is carried out modulo 2. Note that (i) S' satisfies the hypothesis of the statement if and only if S does and (ii) $|S'| = |S|$. Thus without loss of generality assume $(0,\ldots,0) \in S$. For $x \in S$ set $w(x) = i$ if x has exactly i coordinates equal to 1. Further set $S_i = \{x \in S | w(x) = i\}$.

Clearly $|S_0| = 1$ and $|S_1| \geqslant k$. Now an $x \in S_1$ has at most i neighbors y with $w(y) = i - 1$ so that x must have at least $k - i$ neighbors of weight $i + 1$. On the other hand, each $x \in S_{i+1}$ is a neighbor of at most $i + 1$ members of S_1. Thus considering the number of adjacencies between S_{i+1} and S_i it follows that $(i+1)|S_{i+1}| \geqslant (k-i)\,|S_i|$. Thus by induction,

$$|S_i| \geqslant \binom{k}{i} \quad \text{for } 0 \leqslant i \leqslant k,$$

so that

$$|S| \geqslant 2^k.$$

II. If G is a regular bipartite graph, then G is even and has no cutpoints. (Recall that a graph is regular if every vertex is adjacent to the same number of vertices; and bipartite in case the vertex set can be partitioned into two sets such that two vertices can be adjacent only if they are not in the same set of the partition. Finally, a cutpoint of a graph is one whose removal will create more "components".)

Proof. Let A and B be the sets which partition the vertex set with $|A| = a$, $|B| = b$. Then, considering A, it is clear that G has ka edges if each vertex is adjacent to k vertices. Similarly G has kb edges. Thus $a = b$, and the number of vertices is $|A| + |B| = 2a$, even. Assume G has a cutpoint $v \in A$ and that $G - v$ has "new" component B, where v is adjacent to i vertices of B, $1 \leqslant i \leqslant k - 1$. Being a subgraph of a bipartite graph, B must be bipartite with, say, c vertices in one set and D in the other. Counting the edges incident to each set and noting that v is adjacent to vertices in only one of these sets we get the equation $kc = kd - i$. This is impossible because of the bounds on i.

Finally note that statement II applies to the graph in the problem, as can be seen by taking $A = \{x \in S | w(x) \text{ even}\}$ and $B = \{x \in S | w(x) \text{ odd}\}$.

Also solved by S. F. Barger, F. S. Cater, Paul Eitner, E. Triesch, student (West Germany), and the proposer.

ADVANCED PROBLEMS

Solutions of these Advanced Problems should be mailed in duplicate to Professor Roger C. Lyndon, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109 (USA), by March 31, 1981. The solver's full post-office address should be on each sheet.

6316. *Proposed by David Winter, University of Michigan.*

Let S be a set of $3n$ points in $\mathbb{R}^3$, no four of which are coplanar. Suppose that $S = R \cup Y \cup G$, where each of R, Y, G has n points. Is it possible to partition S into n triples $\{r_i, y_i, g_i\}$, $1 \leqslant i \leqslant n$, where each r_i is in R, each y_i is in Y, and each g_i is in G, in such a way that the n triangles $T_i = \operatorname{conv}\{r_i, y_i, g_i\}$ are pairwise disjoint?

6317*. *Proposed by Michael R. Anderson, University of Michigan.*

Let A be a finite set of cardinality m, and B a set of cardinality n, where $m > n > 0$. Find a small family F of functions $f: A \to B$ such that, for each subset A' of A of cardinality n, there exists some f in F whose restriction to A' is one-to-one.

6318. *Proposed by Cole A. Giller, University of California, Berkeley.*

Let M_n be the $n! \times n!$ matrix obtained from the multiplication table of the full symmetric

group S_n by replacing each entry s of the table by 1 if s is an n-cycle, and by 0 otherwise. When is $\det(M_n)\neq 0$?

6319. *Proposed by Charles Ryavec, University of California at Santa Barbara.*

Given two complex l_1 sequences $(a_n)_{-\infty}^{\infty}$ and $(b_n)_{-\infty}^{\infty}$ such that $\Sigma a_n=\Sigma b_n$, show that there exists $f\in L^1(R)$ such that $f(n)=a_n$ and $\hat{f}(n)=\int_{-\infty}^{\infty}e^{2\pi inx}f(x)\,dx=b_n$ for all n. Is the function f unique?

6320. *Proposed by Edgar A. Cohen, Jr., Naval Surface Weapons Center,, Silver Spring, Maryland.*

The principal value integral

$$P=\lim_{T\to\infty}\frac{1}{\pi^2}\int_{-T}^{T}\int_{-T}^{T}\frac{\sin ht}{t}\,\frac{\sin ku}{u}\,\frac{\sin(t+u)}{t+u}\,dt\,du,\qquad 0\leqslant h\leqslant 1,\quad 0\leqslant k\leqslant 1,\tag{1}$$

arises in probability theory,where it furnishes a representation for the cumulative distribution function whose mass is concentrated on the line $y=x$ [**1**, p. 123 ($a=0,b=0$)]:

$$F(x,y)=\begin{cases} x, & x\leqslant y\\ y, & y\leqslant x\end{cases}\tag{2}$$

where $(x,y)\in[0,1]\times[0,1]$. Therefore, one sees that $P=\min(h,k)$. Is it true that the iterated integral

$$\frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin ht}{t}\,\frac{\sin ku}{u}\,\frac{\sin(t+u)}{t+u}\,dt\,du\tag{3}$$

likewise exists? Can one evaluate (1) directly? If (3) exists, can it be obtained from *ab initio* considerations?

1. Fisz, Marek, *Probability Theory and Mathematical Statistics*, Wiley, New York, 1963, Third Edition.

6321. *Proposed by N. P. Erugin, Minsk, USSR.*

Give a qualitative description of the integral curves of the system of differential equations

$$\dot{x}=y-x+x^3,\qquad \dot{y}=-x-y+y^3.$$

Find a periodic solution.

SOLUTIONS OF ADVANCED PROBLEMS

Inequality of L_p Norms of a Derivative of a Function

6185 [1977, 829]. *Proposed by John Milcetich, University of the District of Columbia.*

Let $f(z,\theta)=(1+e^{i\theta}z)^{\beta}(1-z)^{-\alpha}$ where $|z|<1$, $\theta\in R$ and $\alpha\geqslant\beta\geqslant 1$. Show that for $p>0$ and $0<r<1$

$$\int_{-\pi}^{\pi}|f'(re^{i\phi},\theta)|^p\,d\phi\leqslant\int_{-\pi}^{\pi}|f'(re^{i\phi},0)|^p\,d\phi.$$

Solution by the Proposer. We have

$$f'(z,\theta)=(1+e^{i\theta}z)^{\beta-1}(1-z)^{-\alpha-1}\big[(\alpha-\beta)ze^{i\theta}+\alpha+\beta e^{i\theta}\big].$$

Hence

$$\begin{aligned}I&=\int_{-\pi}^{\pi}|f'(re^{i\phi},\theta)|^p\,d\phi\\ &=\int_{-\pi}^{\pi}|1+re^{i(\theta+\phi)}|^{p(\beta-1)}\,|1-re^{i\phi}|^{p(-\alpha-1)}\,|(\alpha-\beta)re^{i\phi}+\beta+\alpha e^{-i\phi}|^p\,d\phi.\end{aligned}$$

We use the notion of the symmetrically decreasing rearrangement of a nonnegative function found in [2]. Since $p(-\alpha-1)<0$, the symmetrically decreasing rearrangement of $|1-re^{i\phi}|^{p(-\alpha-1)}$ on $[-\pi,\pi]$ is itself. That of $|1-re^{i(\theta+\phi)}|^{p(\beta-1)}$ is $|1+re^{i\phi}|^{p(\beta-1)}$ because $p(\beta-1)\geqslant 0$. Write $\beta+\alpha e^{-i\theta}$ as $a(\theta)e^{i\gamma(\theta)}$ with $0\leqslant\alpha-\beta\leqslant a(\theta)\leqslant\beta+\alpha$. Then the symmetrically decreasing rearrangement of $|a(\theta)e^{i\gamma(\theta)}+(a-\beta)re^{i\phi}|^p$ is $|a(\theta)+(a-\beta)re^{i\phi}|^p$, since $a(\theta)\geqslant 0$, $a-\beta\geqslant 0$. We now use the inequality in [1] which states that the integral of a product of three nonnegative functions is not larger than the integral of the product of the symmetrically decreasing rearrangements of the functions over the interval of integration. Hence

$$I\leqslant\int_{-\pi}^{\pi}|1+re^{i\phi}|^{p(\beta-1)}\,|1-re^{i\phi}|^{p(-a-1)}\,|a(\theta)+(a-\beta)re^{i\phi}|^p\,d\phi.$$

Because $a-\beta\leqslant a(\theta)\leqslant\alpha+\beta$, we have $|a(\theta)+(\alpha-\beta)re^{i\phi}|^p\leqslant|\alpha+\beta+(\alpha-\beta)re^{i\phi}|^p$. Replacing the third factor in the integrand by this larger quantity we have the desired inequality.

References

1. J. Clunie and P. L. Duren, Addendum: An arclength problem for close-to-convex functions, J. London Math. Soc., 41 (1966) 181–182.
2. G. H. Hardy, J. E. Littlewood, and G. Pólya, Inequalities, Cambridge University Press, 1952.

Relations Between a Matrix and Its Adjoint

6222 [1978, 599]. *Proposed by Emilie V. Haynsworth, Auburn University, Alabama.*

Let A be an $n\times n$ matrix over the complex field. Let $\operatorname{Adj}A$ denote the standard adjoint matrix for A, that is $\operatorname{Adj}A=(C_{ji})$ where C_{ij} is the cofactor of a_{ij} in A. Prove that if $A+\operatorname{Adj}A=kI$, then

(1) A has at most two distinct eigenvalues, λ_1, and λ_2.

(2) The Jordan form, J, for A has blocks no larger than 2×2, and if $\lambda_1\neq\lambda_2$, A is diagonalizable.

(3) If $\lambda_1\lambda_2\neq 0$, and λ_1 has multiplicity m, then $\lambda_1^{m-1}\lambda_2^{n-m-1}=1$.

(4) If $\lambda_1=0$, $A\neq 0$, $n>2$, then λ_1 is a simple root and $\lambda_2^{n-2}=1$.

(5) Let $S=A+J-kI$. Then S^2 commutes with both A and J and, if S is nonsingular, $S^{-1}AS=J$.

(6) If A is nonnegative and λ_1 and λ_2 are both positive then A^{-1} is an M-matrix.

Conversely, if properties (1), (2) and (3) hold, then $A+\operatorname{Adj}A=kI$.

Composite of solutions by Jeffrey M. Cohen, University of Pittsburgh; Lorraine L. Foster, California State University, Northridge; Eli L. Isaacson, New York University; Michael Josephy, Universidad de Costa Rica; N. Miku, Catholic University, Nijmegen, Netherlands; D. W. Robinson, Brigham Young University; Gary L. Walls & Wallace C. Pye, University of Mississippi; Steven A. Wegmann, student, Iowa State University; and the proposer. (i) If A is the $n\times n$ scalar matrix lI, then $A+\operatorname{Adj}A=(l+l^{n-1})I$, so that A satisfies the hypothesis, but if $l^{n-2}\neq 1$, (3) does not hold. Insert: "$\lambda_1\neq\lambda_2$" in (3).

(ii) The last sentence should read: Conversely, if properties (1), (2), (3), (4) hold, then $A+\operatorname{Adj}A=kI$. (If A is $\operatorname{diag}[0,0,1,1]$, then (1), (2) hold, and (3) holds vacuously, but $A+\operatorname{Adj}A\neq kI$. If A is

$$\begin{bmatrix}2&1&0\\0&2&0\\0&0&2\end{bmatrix},$$

then (1), (2), (4) hold, but $A+\operatorname{Adj}A$ is not diagonal.)

(iii) An M-matrix is understood to mean a real matrix in which the diagonal elements are nonnegative, the nondiagonal elements are nonpositive, and every principal minor has nonnegative determinant.

First, note that (†) $T(\operatorname{Adj}A)T^{-1}=\operatorname{Adj}(TAT^{-1})$. For, this is clearly true if A is invertible, and

$A^{-1}=(\mathrm{Adj}A)/\det A$, so (†) is true even if A is singular. Next, if (*) $A+\mathrm{Adj}A=kI$, then (*T) $TAT^{-1}+T(\mathrm{Adj}A)T^{-1}=kI$. Thus A can be taken in Jordan normal form, $TAT^{-1}=J$. But then, the assertions (1)–(4), and the converse, follow with a little computation.

Alternatively, from (*), it follows that (**) $A^2+(\det A)I=kA$, so that the minimal polynomial $m(\lambda)$ of A has degree 1 or 2, and must be one of $\lambda-\lambda_1,(\lambda-\lambda_1)^2,(\lambda-\lambda_1)(\lambda-\lambda_2)$. This establishes (1) and (2). If $m(\lambda)=(\lambda-\lambda_1)(\lambda-\lambda_2)$, $\lambda_1\neq\lambda_2$, then (**) shows that $\det A=\lambda_1\lambda_2$; this is (3). In (4), if 0 is not a simple root, the alternative $n>2$, $\mathrm{Adj}A=0$, leads to the contradiction $A=kI$, $k=0$. The relation $\lambda_2^{n-2}=1$ then follows from (*).

(5) follows from the relations

$$AS=A^2+AJ-kA=AJ+J^2-kJ=SJ;\ SA=JS,$$

so that $S^2A=SJS=AS^2$, $S^2J=SAS=JS^2$. (Only the relations (*),(*T) are needed in this argument—the form of J plays no role.)

As for (6), if $\mathrm{Adj}A=(\lambda_1+\lambda_2)I-A$, then (since A has proper values λ_1,λ_2) $\mathrm{Adj}A$, and thus

$$A^{-1}=(\det A)^{-1}\mathrm{Adj}A=(\lambda_1\lambda_2)^{-1}\mathrm{Adj}A,$$

is an M-matrix.

The (corrected) converse is immediate.

Also solved (in another corrected version) by J. C. Mauldon.

Sum of Sums of the Möbius Function

6235 [1978, 770]. *Proposed by Robert J. Anderson and M. Ram Murty, Massachusetts Institute of Technology.*

Let $M(x)=\sum_{n\leqslant x}\mu(n)$, where μ is the Möbius function. H. Gupta conjectured that $\sum_{n\leqslant x}M(n)=O(x\log x)$. (See Journal Indian Math. Soc., 1949.) He also gave numerical evidence to support this conjecture. Settle this conjecture.

Solution by the proposers. Suppose that the conjecture is true. We will derive a contradiction. We have $A(x)=\sum_{n\leqslant x}(x-n)\mu(n)=\sum_{n\leqslant x}M(n)+O(x)=O(x\log x)$. For $\mathrm{Re}\,s>2$, $\int_1^\infty M(x)x^{-s}dx=((s-1)\zeta(s-1))^{-1}$ and $\int_1^\infty(\sum_{n\leqslant x}n\mu(n))x^{-s-1}dx=(s\zeta(s-1))^{-1}$, so by subtracting we get $\int_1^\infty A(x)x^{-s-1}dx=(s(s-1)\zeta(s-1))^{-1}$. This integral converges to an analytic function for $\mathrm{Re}\,s>1$, since $A(x)=O(x\log x)$. Thus $1/\zeta(s-1)$ is analytic for $\mathrm{Re}\,s>1$, which contradicts the fact that $\zeta(s)$ has zeros on the line $\mathrm{Re}\,s=\frac{1}{2}$. In fact our argument shows that $\sum_{n\leqslant x}M(n)$ is not $O(x^\delta)$ for any $\delta<\frac{3}{2}$.

Also solved by Harold G. Diamond, L. E. Mattics, R. W. K. Odoni (England), and Lajos Takács.

Sum of the Form $\Sigma\alpha^k[\sqrt[m]{k}\,]$

6247 [1979, 59]. *Proposed by Bencze Miha'ly, University of Babes-Bolyai, Cluj-Napoca, Romania.*

Let $\alpha>1$, $m>1$, and $n>1$ with m and n integers. Also let $[x]$ denote the greatest integer in x. Prove that:

$$\sum_{k=1}^{n^m-1}\alpha^k[\sqrt[m]{k}\,]\leqslant(n-1)\frac{\alpha^{n^m}-\alpha^{(n/2)^m}}{\alpha-1}.$$

Editor's Note. The following is the shortest of several similar simple solutions.

Solution by South Alabama Problem Group, University of South Alabama.

$$\sum_{k=1}^{n^m-1}\alpha^k[k^{1/m}]=\sum_{j=1}^{n-1}j\sum_{k=j^m}^{(j+1)^m-1}\alpha^k$$

$$= \sum_{j=1}^{n-1} j(\alpha-1)^{-1}(\alpha^{(j+1)^m} - \alpha^{j^m})$$

$$= \left((n-1)\alpha^{n^m} - \sum_{k=1}^{n-1} \alpha^{k^m}\right)(\alpha-1)^{-1}$$

It is a simple calculus exercise to show that if $1 \leqslant x \leqslant n-1$, then $f(x) = \alpha^{(n-x)^m} + \alpha^{x^m} \geqslant 2\alpha^{(n/2)^m}$ (because $f'(n/2) = 0$ and $f''(x) > 0$). So whether or not $n-1$ is odd,

$$\sum_{k=1}^{n-1} \alpha^{k^m} \geqslant (n-1)\alpha^{(n/2)^m}$$

Also solved by K. F. Anderson, L. Carlitz, L. E. Clarke (England), Chuck DeCarlucci, L. Kuipers (Switzerland), B. L. R. Shawyer, and Michael Skalsky.

Linear Independence of Functions $\exp[i\cos(\theta-\theta_j)]$

6253 [1979, 132]. *Proposed by Maurice Machover, St. John's University, Jamaica, N.Y.*

If $0 \leqslant \theta_1 < \theta_2 < \theta_3 < \cdots < \theta_n < 2\pi$, are the functions

$$\exp[i\cos(\theta-\theta_j)] \quad (j = 1,2,\ldots,n)$$

linearly independent over the complex numbers?

I. *Solution by L. E. Clarke, University of East Anglia, Norwich, England.* Let $a_1, a_2, \ldots, a_n$ be complex numbers, and let

$$f(z) = \sum_{j=1}^{n} a_j \exp\{i\cos(z-\theta_j)\} = \sum_{j=1}^{n} a_j g_j(z),$$

say. We are to show that if $f(z) = 0$ for all real z then all $a_j = 0$. But if $f(z) = 0$ for all real z then $f(z) = 0$ for all complex z (by analytic continuation).

Now consider a particular j, and let $z = \theta_j + \frac{1}{2}\pi + iy$, where y is real. Then, for $k \neq j$,

$$|g_k(z)|/|g_j(z)| = \exp[\{\cos(\theta_k - \theta_j) - 1\}\sinh y] \to 0 \quad \text{as } y \to \infty$$

because $\cos(\theta_k - \theta_j) < 1$. Since $a_j = -\sum_{k\neq j} a_k g_k(z)/g_j(z)$, it follows that $a_j = 0$.

Almost the same argument shows that no nontrivial linear combination of $g_1, g_2, \ldots, g_n$ can be of finite order.

II. *Solution by R. B. Israel, University of British Columbia.* More generally, for any fixed nonzero complex number r, the functions $\exp[ir\cos(\theta-\theta_j)]$ are linearly independent. A linear combination $f(\theta) = \sum_j a_j \exp[ir\cos(\theta-\theta_j)]$ can be written as $f = \mu * g$, where μ is the measure $\sum_j a_j \delta_{\theta_j}$ (here δ_{θ_j} is the Dirac measure concentrated at θ_j), g is the function $\exp(ir\cos\theta)$, and $*$ denotes convolution. Therefore the Fourier coefficients of f are $\hat{f}(k) = 2\pi\hat{\mu}(k)\hat{g}(k)$, and $f \equiv 0$ if and only if $\hat{g}(k) = 0$ whenever $\hat{\mu}(k) \neq 0$. Now unless all the a_j are zero, $\hat{\mu}$ has infinitely many nonzero terms (otherwise μ would be an L^2 function). So it is sufficient to prove that $\hat{g}$ has only finitely many zeros.

Now g is an even function and so

$$\hat{g}(k) = \frac{1}{\pi}\int_0^{\pi} e^{ir\cos\theta}\cos k\theta \, d\theta = i^k J_k(r),$$

where J_k is the Bessel function of order k (Erdélyi et al., *Higher Transcendental Functions*, vol. 2, McGraw-Hill, 1953, p. 81 (2)). Consequently, by the power series for the Bessel function,

$$|\hat{g}(k)| = \left| \sum_{j=0}^{\infty} \frac{(-1)^j(\frac{1}{2}r)^{2j+k}}{j!(k+j)!} \right|,$$

and $\hat{g}(k)\neq 0$ because the first term of the series dominates: in fact, for $k>|r^2|$,

$$|\hat{g}(k)|>\frac{|r|^k 2^{-k}}{k!}\left(1-\sum_{j=1}^{\infty}\frac{|r^2/4k|^j}{j!}\right)>\frac{|r|^k 2^{-k}}{k!}(2-e^{1/4}).$$

Actually all $\hat{g}(k)\neq 0$ if r is not real or $|r|<2.4048$.

Also solved by Boris Datskovsky, Rafael de la Llave (Spain), L. F. Moyers, and Roger Nussbaum.

The solution by de la Llave was similar to Solution II. He observes that the same general argument establishes the linear independence of the functions $\psi(\theta-\theta_j)$ provided that the Fourier transform of ψ (in the distributional sense) has a wide enough support so that the exponentials $\exp(it\theta_j)$ are linearly independent over it.

Mean, Median, and Mode

6265 [1979, 311]. *Proposed by John H. Cook & David Sanders, Metropolitan Life, New York City.*

Prove or disprove the following assertion: If $x=s_n$ is the solution to the equation

$$e^{-x}\left(1+x+\frac{1}{2}x^2+\cdots+\frac{1}{n!}x^n\right)=1/2$$

then s_n-n approaches $2/3$ as n goes to ∞.

Editor's Note. Leroy F. Meyers (Ohio State University), L. Takács (Case Western Reserve University), and Gaston Gonnett (Canada) identified the problem as part of problem 211, part 2, vol. I, Problems and Theorems in Analysis, by G. Pólya and G. Szegö. O. P. Lossers (Eindhoven Institute of Technology, Netherlands), G. Bach (Stuttgart), and Leon Gerber (St. John's University) gave proofs *de novo*. M. J. Laird (Kings College, London) found a treatment by R. Furch in Z. für Physik, 112 (1939) 92–95, and by D. E. Knuth, in The Art of Computer Programming, vol. 1, Addison-Wesley, 1969, pp. 113–116.

The first discussion of 6265 of which we are aware was given by Arthur T. Doodson, *Relation of the mode, median and mean in frequency curves*, Biometrika, 11 (1915–17) 425–429. A rule of thumb of Edwardian statistics was that, for slightly skewed data, (median-mode) = 2(mean-mode)/3. Problem 6265 is a precise statement of this rule of thumb for the gamma distribution.

Problem V of A. DeMoivre's in The Doctrine of Chances is: "To find how many trials are necessary to make it equally probable that an event will happen three, four, five, &c. times; supposing that a is the number of chances for its happening in any one trial, and b the number of chances for its failing." DeMoivre discussed the equation $e^{-x}(1+x+\cdots+x^n/n!)=1/2$ as part of the "&c."

Also solved by David Borwein, Paul Bracken (undergraduate), Robert Breusch, Paul F. Byrd, Thad Dankel, Jr., & Kenneth R. Gurganus, Harvey Diamond, Leo F. Epstein, Richard A. Groeneveld, Robert B. Israel, L. E. Mattics, Karl K. Norton, Mark A. Pinsky, N. M. Timme (Netherlands), Richard S. Varga, Buck Ware, and John S. White.

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR., AND LYNN A. STEEN
with the assistance of the mathematics departments of St. Olaf, Carleton, and Macalester Colleges
COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

Applied Mathematical Demography. By Nathan Keyfitz. Wiley, New York 1977, xxiv+388 pp. $19.95. (Telegraphic Review, December 1977.)

Mathematical social science is a young and growing field, and this book is a valuable contribution to its study. In my junior-senior level introductory probability and statistics course (with calculus prerequisite) this semester, two class periods were devoted to some of the material in Chapter 11 (Microdemography). The geometric distribution came to life. The problem of family size (under certain simplified pregnancy and birth assumptions), which replaced the artificial mental exercises on coin tossing, excited the students and added incentive for learning some of the summation tricks.

Although written primarily for demographers, this text also contains straightforward solutions to many problems of more general interest. Let me cite just a few. How many people have lived on earth? If we use published assumptions on the number of births in various years and assume uniform increase between them, the answer can be found by simple use of exponentials and logarithms, integration, and algebraic manipulation. What annual rate of increased growth in the doctoral faculty would result if each doctoral faculty member produced, over his lifetime, just three students who became doctoral faculty members? Making certain assumptions about the age of faculty members and students, the author obtains a difference equation. To solve this, he assumes an exponential solution and uses iteration. How would a change in population growth-rates affect the premiums necessary to sustain our Social Security system? Certain simplifying assumptions are made so that the premium can be expressed as a ratio of the integral representing the pensioners to the integral representing the workers. Then logarithmic differentiation is used, and we obtain an approximate solution to our problem.

Since demography is a social science, there are certain underlying difficulties. Even in modern countries, census data and vital statistics are far from perfect. This information is poorer still in the less developed countries. Since advances in science and medicine, as well as social, economic, and political changes, affect birth, marriage, and death rates, even the best current data do not provide a clear picture of the future. Nonetheless, the text gives us experience in model-building. The author makes certain reasonable assumptions and shows, mathematically, how these lead to various conclusions. While we cannot take the conclusions as laws in the sense of physical laws, we do have a clear sense of direction for the results from current, or anticipated, trends.

The mathematical exposition itself is, at times, a bit uneven. For example, at the bottom of page 120, we have a function $\phi(r,\sigma^2)=$ constant and wish to obtain $dr/d\sigma^2$. Why does the author have to go through a derivation of this well-known result?

There are no organized problem sets. However, from time to time, the reader is asked to derive a result. Written in the spirit of some texts' "it is easy to show that," these seem designed to keep text size reasonable and are not of uniform quality.

Should this book be used in a course in mathematical model-building? Yes, with reservations. The mathematical background assumed is elementary calculus (including series, partial differentiation, and multiple integration), a semester of linear algebra, and a semester of probability and statistics. I would hope that all mathematics majors have this as part of their education. However, only students that also have some course work and interest in the social sciences should undertake to read this book. Students who are not interested in the ideas and problems of demography may not be able to immerse themselves in its mathematical underpinnings.

On the other hand, this book would certainly be a welcome addition to a mathematical library. It is excellent for browsing. An undergraduate mathematics teacher would enjoy using some of its material for examples.

Wolfe Snow, Brooklyn College (CUNY)

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, S, L*, *Mathematics and Humor*. John Allen Paulos. U of Chicago Pr, 1980, 116 pp, $12.95. [ISBN: 0-226-65024-3] A brief but serious inquiry into common elements in mathematics and humor: both are forms of intellectual play that depend on distinct rules; both require a willingness to engage in reasoning from hypotheses contrary to fact; both rely heavily on *reductio ad absurdum* arguments; and both climax in "getting it"--either the joke or the proof. Paulos uses self-reference paradoxes and catastrophe theory as mathematical paradigms of various types of humor. LAS

GENERAL, P, L, *Science Education Databook*. Alphonse Buccino, et al. NSF, ix + 154 pp, (P). Highlights in graphical and tabular form of recent data (1977-78 and earlier) concerning resources, participation, attitudes, tests, degrees and employment germane to science education. A most useful reference for anyone concerned about national policy in science education. LAS

GENERAL, L*, *Lecture Notes in Mathematics-601-700: An Index and Other Useful Information*. Ed: A. Dold, B. Eckmann. Springer-Verlag, 1979, 26 pp, free (P). Complete contents of these 100 *Lecture Notes*, together with rationale for contents of the series and instructions for preparation of manuscripts. LAS

GENERAL, P, *The Typing of Mathematics*. Roger Schenkman. Repro Handbook, 1978, vii + 247 pp, $6.95. Intended for typists, this volume covers matters of style, spacing and layout of both mathematical, chemical and computer science text. Strangely, it deals exclusively with IBM typewriters and composers, totally ignoring computer composition or text editing. LAS

GENERAL, P, *Lecture Notes in Mathematics-770: Séminaire Bourbaki, Vol. 1978/79, Exposés 525-542*. Springer-Verlag, 1980, iv + 341 pp, $19.40 (P). [ISBN: 0-387-09733-3]

GENERAL, P, *Women and the Mathematical Mystique*. Ed: Lynn H. Fox, Linda Brody, Dianne Tobin. Johns Hopkins U Pr, 1980, xi + 211 pp, $16.50. [ISBN: 0-8018-2341-2] 12 revised and updated papers from a 1976 AAAS and Hyman Blumberg Symposium on sex differences in mathematical talent and achievement. Includes papers portraying female mathematicians, research on sex differences in mathematics achievement and enrollment patterns, and descriptions of programs designed to facilitate women's entry into mathematics. LAS

GENERAL, P, *Analytic Number Theory, Mathematical Analysis and their Applications*. Ed: S.M. Nikol'skii. Proc. of Steklov Inst. of Math., No. 143. AMS, 1980, vii + 220 pp, $56 (P). [ISBN: 0-8218-3044-9]

PRECALCULUS, T(13: 1), *Algebra and Trigonometry*. Walter Fleming, Dale Varberg. P-H, 1980, xvi + 528 pp, $17.95 [ISBN: 0-13-021824-3]; *College Algebra*, xvi + 495 pp, $15.95 [ISBN: 0-13-141606-5]; *Plane Trigonometry*, xiii + 235 pp, $15.95 [ISBN: 0-13-679043-7]. Good text in three versions. The authors have made an effort to "spark the readers' curiosity, to draw them into the sections." They do this by beginning each section with a "challenging problem, historical anecdote, a famous quotation, or an appropriate cartoon." *College Algebra* contains nine chapters of *Algebra and Trigonometry*, plus chapters on probability and mathematics of finance. *Plane Trigonometry* contains four chapters from *Algebra and Trigonometry*, plus a chapter on conic sections. LLK

EDUCATION, P, *Research in Mathematics Education*. Ed: Richard J. Shumway. NCTM, 1980, vi + 487 pp, $27. [ISBN: 0-87353-163-9] A comprehensive reference on the process (e.g., comparative experiments, clinical research, survey research) and problems (e.g., cognitive development, skill learning, problem solving, curriculum and instruction) in mathematics education, intended for doctoral candidates and others engaged in research in mathematics education. Contains 14 coordinated chapters by many different authors. LAS

EDUCATION, P, *Source Book of Projects, Science Education Development and Research: Fiscal Year 1979, With References to Earlier Years*. NSF, xi + 155 pp, (P). Abstracts of 1979 NSF DISE and RISE projects; lists of previous projects; various keyword and project director indexes. An interesting source of current innovation in mathematics and science education research. LAS

HISTORY, P, *Lie Groups: History, Frontiers and Applications, V. IX: Development of Mathematics in the 19th Century*. Felix Klein. Trans: M. Ackerman. Math Sci Pr, 1979, ix + 629 pp, $50. [ISBN: 0-915692-28-7] The first half is Klein's book in translation. The style is highly subjective, though always interesting and illuminating. Many times the reader will wish that the mathematics would be more explicit. But that can be found elsewhere while the topics here cannot. The second half is Robert Hermann's "Kleinian Mathematics From an Advanced Standpoint" which attempts to bring back much of the 19th century mathematics to bear on problems of Riemannian geometry. TLS

HISTORY, P, *Studies in the Scientific and Mathematical Philosophy of Charles S. Peirce: Essays by Carolyn Eisele*. Ed: R.M. Martin. Mouton Pub, 1979, xii + 386 pp, DM 96. [ISBN: 90-279-7808-5] 30 essays (23 reprinted from various sources, 7 appearing here for the first time) by Carolyn Eisele, the

pre-eminent scholar of Peirce and editor of his four-volume *The New Elements of Mathematics*. Eisele's studies compose the major body of scholarly work on the relation of Peirce's philosophy to his scientific and mathematical work. LAS

HISTORY, L, *Black Mathematicians and Their Works*. Ed: Virginia K. Newell, et al. Dorrance & Co, 1980, xvi + 327 pp, $12.50 (P); $18. [ISBN: 0-8059-2677-1; 0-8059-2556-2] Reprints of 25 research papers in mathematics and mathematics education by black authors; professional biographies of over 60 American black mathematicians and mathematics educators; and, perhaps of most interest to those under 40, copies of letters from the mid-fifties concerning the efforts of various individuals (especially Lee Lorch) to change the stance of the AMS and MAA regarding discrimination against black members at regional and national meetings. LAS

HISTORY, P, L, *Early British Computers, The Story of Vintage Computers and the People Who Built Them*. Simon Lavington. Digital Pr, 1980, 139 pp, $8 (P). [ISBN: 0-932376-08-8] A history of British computers from 1935 to 1955, embedded in appropriate economic, political and technological contexts. Includes brief discussion of concurrent developments in the U.S., and appendices on computer vocabulary and on technical specifications of early computers. LAS

NUMBER THEORY, T(18: 1), S, P, *Transcendence Methods*. Michel Waldschmidt. Pure and Appl. Math., No. 52. Queen's U, 1979, 122 pp, (P). A series of ten lectures presented at Queen's University in July 1979. An introduction to the subject of transcendental numbers along with a selection of some of the recent developments. No exercises, and a limited list of references. CEC

LINEAR ALGEBRA, T(13: 1), *Elementary Linear Algebra with Applications*. Adil Yaqub, Hal G. Moore. A-W, 1980, xiii + 369 pp, $16.95. [ISBN: 0-201-08825-8] Good basic text with little to distinguish it from many others. LLK

COMPLEX ANALYSIS, P, *Lecture Notes in Mathematics-798: Analytic Functions, Kozubnik 1979*. Ed: J. Lawrynowicz. Springer-Verlag, 1980, x + 476 pp, $27 (P). [ISBN: 0-387-09985-9] Proceedings of the conference held in Kozubnik, Poland on April 19-25, 1979. Contains a list of lectures not included in this volume. JAS

NUMERICAL ANALYSIS, P, *The Mathematics of Finite Elements and Applications II, MAFELAP 1975*. Ed: J.R. Whiteman. Acad Pr, 1976, xiii + 573 pp, $48.50. [ISBN: 0-12-747252-5] 43 papers from a conference held at Brunel University on finite element methods, theory and applications. RWN

NUMERICAL ANALYSIS, S(16-17), P, *Algorithmes d'Accélération de la Convergence, Étude Numérique*. Claude Brezinski. Editions Technip, 1978, xi + 392 pp, 195F (P). [ISBN: 2-7108-0341-0] A guide to the use of various acceleration algorithms, e.g., Richardsen and Overholt, in their scalar, vector and continuous forms. Presents the algorithms, theoretical results, numerical examples and production codes. RWN

NUMERICAL ANALYSIS, P, *Proceedings of the 1980 Army Numerical Analysis and Computers Conference*. US Army Research Office (P.O. Box 12211, Research Triangle Park, NC), 1980, xiv + 480 pp, (P). 28 papers from a February 1980 Army Mathematics Steering Committee conference at the NASA Ames Research Center on the general theme of computation of fluid flows involving shocks and discontinuities. LAS

NUMERICAL ANALYSIS, P, *Lecture Notes in Mathematics-773: Numerical Analysis*. Ed: G.A. Watson. Springer-Verlag, 1980, x + 184 pp, $11.80 (P). [ISBN: 0-387-09740-6] Proceedings of the 8th Biennial Conference held at Dundee, Scotland, June 26-29, 1979. JAS

NUMERICAL ANALYSIS, P, *Polynomial and Spline Approximation: Theory and Applications*. Ed: Badri N. Sahney. Reidel, 1979, vii + 321 pp, $37. [ISBN: 90-277-0984-X] 16 invited papers from the NATO Advanced Study Institute held in Calgary in 1978. About half of the papers are expository. Coverage includes polynomial, least squares, multivariate, periodic and non-linear splines and applications to initial value problems, fracture mechanics, surface generation, optimal recovery and Fourier analysis. RWN

NUMERICAL ANALYSIS, T(18: 1), P, *The General Problem of Approximation and Spline Functions*. A.S.B. Holland, B.N. Sahney. Krieger, 1979, viii + 344 pp, $22.50. [ISBN: 0-88275-598-6] Discusses the general problem of approximation in normed linear spaces. Major examples include Chebyshev polynomials and B-, L-, L_g- and cardinal splines. RWN

FUNCTIONAL ANALYSIS, P, *C*-Algebras and Their Automorphism Groups*. Gert K. Pedersen. London Math. Soc. Mono., No. 14. Acad Pr, 1979, ix + 416 pp, $60. [ISBN: 0-12-549450-5] First half of monograph presents the general theory of C*-algebras and includes extensive use of sequentially closed algebras in the decomposition theory, while the second half develops the theory of locally compact groups of automorphisms of C*-algebras. TRS

FUNCTIONAL ANALYSIS, P, *Fourier Analysis*. Ed: Miguel de Guzmán, Ireneo Peral. (Impreso en Graficas Moscat, Margaritas 20, Madrid 29) 1980, 200 pp, (P). [ISBN: 84-300-2172-8] Six papers (five in English, one in French) from a June 1979 seminar held in El Escorial, Spain, the first international meeting organized by the newly founded Spanish Mathematical Association. LAS

OPTIMIZATION, T(16-17: 1), P, L, *Mathematical Programming: Structures and Algorithms*. Jeremy F. Shapiro. Wiley, 1979, xvi + 388 pp, $23.95. [ISBN: 0-471-77886-9] Major theorems and algorithms in linear, integer, dynamic and nonlinear programming and network and combinatorial optimization. Well-organized development with unity provided by duality and convexity theory. Examples. RWN

ANALYSIS, P, *Lecture Notes in Mathematics-781: Harmonic Analysis, Iraklion 1978*. Ed: N. Petridis, S.K. Pichorides, N. Varopoulos. Springer-Verlag, 1980, 213 pp, $14 (P). [ISBN: 0-387-09756-2] Proceedings (all in English) of the conference held at the University of Crete, Iraklion, Greece in July 1978. JAS

ANALYSIS, P. *Lecture Notes in Mathematics-779: Euclidean Harmonic Analysis.* Ed: J.J. Benedetto. Springer-Verlag, 1980, 177 pp, $11.80 (P). [ISBN: 0-387-09748-1] Proceedings of the seminars held at the University of Maryland in the spring of 1979. JAS

ANALYSIS, P. *Tauberian Theory and its Applications.* A.G. Postnikov. Proc. of Steklov Inst. of Math., No. 144. AMS, 1979, v + 138 pp, $26 (P). [ISBN: 0-8218-3048-1] An inviting exposition of classical Tauberian theory in 29 brief sections. LAS

GEOMETRY, S(16-17), P. *Hilbert's Fourth Problem.* Aleksei Vasil'evich Pogorelov. Trans: Richard A. Silverman. V.H. Winston, 1979, vi + 97 pp, $16. [ISBN: 0-470-26735-6] By defining a completely additive set of functions on the space RP^3 and using Crofton's definition of the length of curves, the author is able to characterize all continuous Desarguesean metrics of RP^3. He then is able to solve Hilbert's Fourth Problem. It is written at the general level and contains much enjoyable mathematics. TLS

GEOMETRY, P. *Contributions to Geometry.* Ed: Jürgen Tölke, Jörg M. Wills. Birkhäuser Verlag, 1979, 406 pp, $49. [ISBN: 3-7643-1048-0] A collection of papers dedicated to H. Hadwiger that constitutes the Proceedings of the Geometry Symposium in Siegen, 1978. The volume is in three parts: convexity, differential geometry, and foundations. The part on convexity contains 9 broad survey articles and one research paper together with a list of 105 open problems. The second part contains 10 papers more evenly split between surveys and research. Finally, there are four research papers on the foundations of geometry. SS

TOPOLOGY, P. *Topological Entropy and Equivalence of Dynamical Systems.* Roy L. Adler, Brian Marcus. Memoirs No. 219. AMS, 1979, iv + 84 pp, $6.40 (P). [ISBN: 0-8218-2219-5] This monograph proves a theorem for topological entropy analogous to Ornstein's result. The main objects of study are topological Markov shifts, and many interesting properties are explored using this discrete system. TLS

PROBABILITY, P. *Lecture Notes in Mathematics-774: Ecole d'Eté de Probabilités de Saint-Flour VIII-1978.* R. Azencott, Y. Guivarc'h, R.F. Gundy. Springer-Verlag, 1980, xiii + 334 pp, $20.40 (P). [ISBN: 0-387-09741-4] Three lecture series from July 1978: *Grandes Déviations et Applications* by R. Azencott; *Quelques Propriétés Asymptotiques des Produits de Matrices Aléatoires* by Y. Guivarc'h; and *Inégalités pour Martingales à un et deux Indices: L'espace H_p* by R.F. Gundy. JAS

PROBABILITY, S(16-18), P. *Géométrie Différentielle Stochastique.* Paul Malliavin. Pr U Montreal, 1978, 144 pp, $8 (P). [ISBN: 0-8405-0425-X] Notes from a course given by the author at the University of Montreal from June 20-July 15, 1977. Topics include the structure theorem and differential calculus of Ito for stochastic integrals, comparison theorems, a principle for transferring from differential geometry to stochastic geometry, annulation of harmonic forms and ergodic properties of diffusions, and the method of optimal stochastic control from complex analyses in C^n. Bibliography. RJA

PROBABILITY, T(17), P. *Reversibility and Stochastic Networks.* F.P. Kelly. Wiley, 1979, viii + 230 pp, $39.95. [ISBN: 0-471-27601-4] A stochastic process $\{x(t)\}$ is reversible if $\{x(t_1), x(t_2)\ldots x(t_n)\}$ has the same distribution as $\{x(t_1-), x(t_2-)\ldots x(t_n-)\}$ for any choices of $t_1,t_2,\ldots,t_n$. After finding conditions for reversibility, the author shows the importance of this idea to a variety of applications, notably queues, migration, and genetic models. Some exercises, many examples. Assumes a probability background at the level of Feller's *Volume I.* TAV

STATISTICS, T(14-15; 1, 2). *Statistical Concepts with Applications to Business and Economics.* Richard W. Madsen, Melvin L. Moeschberger. P-H, 1980, xv + 653 pp, $19.95. [ISBN: 0-13-844878-7] Mature treatment, assuming only college algebra but containing some optional discussions which require calculus. Advanced topics include regression and correlation, analysis of variance, nonparametric tests, Bayesian inference, and decision theory. Contains none of the unique topics usually found in business statistics texts, such as index numbers. RSK

STATISTICS, P. *Asymptotic Theory of Statistical Tests and Estimation in Honor of Wassily Hoeffding.* Ed: I.M. Chakravarti. Acad Pr, 1980, xiv + 350 pp, $25. [ISBN: 0-12-166650-6] Proceedings of the Advanced International Symposium on Asymptotic Theory of Statistical Tests and Estimation, held at the University of North Carolina at Chapel Hill, April 16-18, 1979. Papers treat "large deviations theory, distributions--exact and asymptotic, extreme multivariate distributions, sequential analysis, probability inequalities, estimation of density, support and contours of support of probability laws, statistical inference for the compound Poisson and Wiener-Lévy, processes, properties of Wiener processes, and applications." RSK

STATISTICS, P. *Contributions to Statistics: Jaroslav Hájek Memorial Volume.* Ed: Jana Jurečková. Reidel, 1979, 317 pp, $50. [ISBN: 90-277-0883-5] A collection of 24 original papers related to Hájek's work, together with a personal recollection of Hájek. JAS

COMPUTER PROGRAMMING, T*(13-14: 1). *Problems for Computer Solutions Using BASIC.* Henry M. Walker. Winthrop Pub, 1980, x + 189 pp, $12.95 (P). [ISBN: 0-87626-717-7] Designed to teach problem solving using Basic, this book emphasizes the development and programming of algorithms for a wide variety of mathematical and scientific problems. Optional material demonstrates much of the power of RSTS on a PDP 11/70 computer, though most of the material is appropriate for any substantial version of Basic. Only a very few sections require calculus. JAS

COMPUTER PROGRAMMING, L. *Basic Computer Programs in Science and Engineering.* Jules H. Gilder. Hayden, 1980, 247 pp, $8.95 (P). [ISBN: 0-8104-0761-2] A collection of over 100 simple programs in Basic for doing elementary mathematical and (electrical) engineering calculations. The range of programs includes: combinations, complex arithmetic, matrix computations, resistor color code interpretation, power supply design, and attenuator pad design. JAS

COMPUTER PROGRAMMING, T(13: 1). *Fundamentals of Fortran Programming, Second Edition.* Robert C. Nickerson. Winthrop Pub, 1980, xi + 450 pp, $11.95 (P). [ISBN: 0-87626-301-5] This *Second Edition* contains an appendix listing differences in Fortran versions including ANSI Fortran 77. Requires minimal mathematics background. Examples and explanations very clearly presented. Good problem sets. (*First Edition,* TR, May 1975.) LLK

COMPUTER PROGRAMMING, T(13: 1). *Top-down, Modular Programming in FORTRAN with WATFIV.* Rahul Chattergy, Udo W. Pooch. Winthrop Pub, 1980, xvi + 217 pp, $11.95 (P). [ISBN: 0-87626-879-3] A text designed for amateur programmers. Demonstrates basic programming techniques with its stated purpose to show how to write programs of quality without first developing habits that are hard to break. LLK

COMPUTER PROGRAMMING, T?(13: 1). *Business Basic.* Robert J. Bent, George C. Sethares. Brooks/Cole, 1980, xi + 223 pp, $11.95 (P). [ISBN: 0-8185-0359-9] An introduction to minimal Basic via business-oriented examples, according to the standards proposed in 1976 by the American National Standards Institute. This book is, from a practical point of view, flawed by the following omissions: no mention of random files, no mention of lower case ASCII, and minimal treatment of string manipulation. It does, however, treat matrix operations. JAS

COMPUTER PROGRAMMING, S. *More Basic Computer Games.* Ed: David H. Ahl. Creative Computing Pr, 1979, xi + 185 pp, $7.50 (P). [ISBN: 0-916688-09-7] Listings of eighty-four games and sample runs. Many of the programs have appeared previously in *Creative Computing.* CEC

COMPUTER PROGRAMMING, T(13-14: 1), S, L. *Foundations of Programming Through Basic.* Peter Moulton. Wiley, 1979, xii + 271 pp, $12.95 (P). [ISBN: 0-471-03311-1] Basic programming (including matrix statements) presented through problem solutions. More emphasis on programming and less on algorithms than many competing books. JAS

COMPUTER PROGRAMMING, T(13: 1). *Applied BASIC Programming.* Roy Ageloff, Richard Mojena. Wadsworth, 1980, xvi + 441 pp, $13.95 (P). [ISBN: 0-534-00808-9] Good format. Lots of examples, exercises, and debugging procedures in each programming chapter. LLK

COMPUTER PROGRAMMING, T??(13: 1). *BASIC Made Easy, A Guide to Programming Microcomputers and Minicomputers.* Don Cassel, Richard Swanson. Reston Pub, 1980, 251 pp, $12.95. [ISBN: 0-8359-0399-0] An introduction to Basic which is made easy more by oversimplification than by sticking to basics. For example, matrix operations are introduced but string manipulation is treated very quickly ("the majority of processing in Basic deals with the manipulation of numbers") and ASCII is presented devoid of punctuation and lower case letters. JAS

COMPUTER SCIENCE, P. *Performance of Computer Systems.* Ed: M. Arató, A. Butrimenko, E. Gelenbe. North-Holland, 1979, ix + 565 pp, $70.75. [ISBN: 0-444-85332-4] 36 papers from the Fourth International Symposium on Modelling and Performance Evaluation of Computer Systems. Concerns probabilistic models (based on queueing theory), deterministic models, and measurements of their effectiveness. Applications include communication networks, multiprocessor systems, data bases, and virtual memories. Each paper is in English. RWN

COMPUTER SCIENCE, S*(16-17), P. *Research Directions in Software Technology.* Ed: Peter Wegner. MIT Pr, 1979, xiii + 869 pp, $24.95. [ISBN: 0-262-23096-8] Survey papers on 20 research areas such as software engineering, program verification, concurrent programming, data base management, and artificial intelligence. Discussions and critiques by other authorities are included. A good source for obtaining recent, expert overviews of software research. RWN

COMPUTER SCIENCE, T(15-16: 1), L. *Modeling and Analysis: An Introduction to System Performance, Evaluation, Methodology.* Hisashi Kobayashi. A-W, 1978, xvii + 446 pp, $18.95. [ISBN: 0-201-14457-3] An overview of the evaluation of the performance of systems, especially computer systems, and the techniques for solving the resulting models; includes Markov chains, queueing analysis, simulation and experimental design. RWN

COMPUTER SCIENCE, S, P, L. *More Chess and Computers: The Microcomputer Revolution, The Challenge Match.* David Levy, Monroe Newborn. Computer Sci Pr, 1980, 117 pp, $11.95 (P). [ISBN: 0-914894-07-2] State-of-the-art survey of computer chess, including a blow-by-blow account (annotated by both Levy and by his computer opponent) of Levy's victory in the 1978 challenge match for a $2,500 prize. Levy's winning strategy was to "do nothing but do it well." The only game he lost was the only one he tried actively to win. LAS

APPLICATIONS, P. *Lecture Notes in Mathematics-782: Bifurcation and Nonlinear Eigenvalue Problems.* Ed: C. Bardos, J.M. Lasry, M. Schatzman. Springer-Verlag, 1980, viii + 296 pp, $19.50 (P). [ISBN: 0-387-09758-9] Proceedings of a conference held October 2-4, 1978 at the University of Paris 13, Centre Scientifique et Polytechnique, Villetaneuse, France. A wide range of scientific applications is represented. JAS

APPLICATIONS (ENGINEERING), T(18: 1), P. *Lecture Notes in Mathematics-749: Finite Element Approximation of the Navier-Stokes Equations.* V. Girault, P.-A. Raviart. Springer-Verlag, 1979, vii + 200 pp, $12.50 (P). [ISBN: 0-387-09557-8] Reviews background from elliptic boundary value problems and functional analysis. Gives the existence and uniqueness for solutions to Navier-Stokes problems. Analyzes the mixed finite element method including convergence and error bounds. RWN

APPLICATIONS (OPERATIONS RESEARCH), P. *Research Games: An Approach to the Study of Decision Processes.* K.C. Bowen. Halsted Pr, 1978, xi + 126 pp, $15.95 (P). [ISBN: 0-470-26535-3] Expensive, introductory discussion of research games used to study decision processes. Conclusions general and consistent with common sense. Useful for persons desiring initial background. WC

APPLICATIONS (PHYSICS), T(14-16: 1, 2), L, *Mechanics*. W. Chester. Allen & Unwin, 1979, xv + 432 pp, $18.95 (P); $34. [ISBN: 0-04-510059-4; 0-04-510058-6] A presentation for the post-calculus student of a minimal amount of vector calculus (no tensors) and a lot of classical mechanics. JAS

APPLICATIONS (PHYSICS), P, *General Relativity and Gravitation*. Ed: A. Held. Plenum Pr, 1980. *Volume 1*, xix + 598 pp [ISBN: 0-306-40265-3]; *Volume 2*, xviii + 540 pp, $57.50 each. [ISBN: 0-306-40266-1] Two volumes of review articles surveying the state-of-the-art in gravitation and general relativity gathered by the International Society for General Relativity and Gravitation to celebrate the 100th anniversary of Albert Einstein's birth. JAS

APPLICATIONS (PHYSICS), T*(15-16: 1), S*, L*, *Vector and Tensor Analysis with Applications*. A.I. Borisenko, I.E. Tarapov. Trans: Richard A. Silverman. Dover, 1979, x + 257 pp, $4.50 (P). [ISBN: 0-486-63833-2] Unabridged, corrected republication of 1968 original edition (TR, August-September 1968). Self-contained, readable, but not wordy. Introduction to the algebra and calculus of vectors and tensors, with applications in fluid dynamics and electromagnetic theory. Worked out problems and exercises (some with answers) at chapter ends. Good index and bibliography (even though brief and out-of-date). A "best" buy. JK

APPLICATIONS (PHYSICS), P, *Lecture Notes in Physics-122: New Developments in Semiconductor Physics*. Ed: F. Beleznay, G. Ferenczi, J. Giber. Springer-Verlag, 1980, 276 pp, $19 (P). [ISBN: 0-387-09988-3] Lecture notes and selected contributed papers from the International Summer School on New Developments in Semiconductor Physics held at the University of Szeged, July 1-6, 1979. JAS

APPLICATIONS (PHYSICS), P, *Lecture Notes in Physics-124: Gravitational Radiation, Collapsed Objects and Exact Solutions*. Ed: C. Edwards. Springer-Verlag, 1980, vii + 487 pp, $30.70 (P). [ISBN: 0-387-09992-1] Lectures from the Einstein Centenary Summer School held in Perth, Western Australia, during January 1979. Experimental and theoretical aspects are given approximately equal time. JAS

APPLICATIONS (PHYSICS), P, *Lecture Notes in Physics-123: The Ruelle-Araki Transfer Operator in Classical Statistical Mechanics*. Dieter H. Mayer. Springer-Verlag, 1980, viii + 154 pp, $12 (P). [ISBN: 0-387-09990-5] An extension of work by Ruelle and Araki which began a mathematical description of one-dimensional lattice systems with long range interactions. JAS

APPLICATIONS (PHYSICS), S(15-16), P, *Solitons*. Ed: R.K. Bullough, P.J. Caudrey. Springer-Verlag, 1980, xviii + 389 pp, $44.90. [ISBN: 0-387-09962-X] Volume 17 of Topics in Current Physics. Provides both an historical survey and a source book for a number of significant recent achievements in the theory of solitary waves. JAS

APPLICATIONS (PHYSICS), P, *Lecture Notes in Mathematics-775: Geometric Methods in Mathematical Physics*. Ed: G. Kaiser, J.E. Marsden. Springer-Verlag, 1980, vii + 257 pp, $16.80 (P). [ISBN: 0-387-09742-2] Papers presented at the CBMS Regional Conference held at the University of Lowell, Lowell, MA on March 19-23, 1979. JAS

APPLICATIONS (PHYSICS), P, *Quantum Theory and Gravitation*. Ed: A.R. Marlow. Acad Pr, 1980, x + 267 pp, $18.50. [ISBN: 0-12-473260-7] Proceedings of a series of meetings held at Loyola University, New Orleans, Louisiana in May 1979. This volume both exhibits and celebrates the current active interplay between mathematics (not just geometry) and theoretical physics in the search for a unification of quantum and relativistic theories. JAS

APPLICATIONS (RELATIVITY), T*(16-17: 1, 2), L, *Essential Relativity: Special, General, and Cosmological, Revised Second Edition*. Wolfgang Rindler. Springer-Verlag, 1977, xv + 284 pp, $19.80 (P). [ISBN: 0-387-10090-3] A paperback version of the 1977 *Second Edition* with corrections and a rewritten section on the metric of static fields. (*First Edition*, TR, February 1970; *Second Edition*, TR, April 1978.) JAS

APPLICATIONS (RELATIVITY), T(17-18: 1), S, L, *Elementary General Relativity*. C. Clarke. Wiley, 1979, ix + 131 pp, $18.95 (P). [ISBN: 0-470-26930-8] A brief but sophisticated treatment. The mathematics is not developed in depth (for the sake of brevity) but tensors and groups do appear informally. Some nice intuition is presented; the book reads like the sort of preface which makes great reading after studying the book. JAS

APPLICATIONS (SOCIAL SCIENCE), T**(15-17: 1, 2), S***, P*, L***, *Mathematics for Social Scientists*. Ki Hang Kim, Fred William Roush. Elsevier North Holland, 1980, xiv + 277 pp, $14.95. [ISBN: 0-444-99066-6] Concise chapters on sets and relations, matrices, Boolean matrices and graphs, combinatorics, difference equations, differential equations, probability, and cluster analysis, followed by applications. As R. Duncan Luce says in a foreward, "This book should help many of us who wish to increase the mathematical skills of our social science majors." FLW

APPLICATIONS (SOCIAL SCIENCE), P, *Data Bases in the Humanities and Social Sciences*. Ed: Joseph Raben, Gregory Marks. North Holland, 1980, xii + 329 pp, $48.75. [ISBN: 0-444-85499-1] 58 papers from an August 1979 IFIP conference at Dartmouth College discussing the role of computer data bases in such fields as linguistics, economics, music, sociology and history. LAS

Reviewers Whose Initials Appear Above

Richard J. Allen, St. Olaf; William Carlson, St. Olaf; Clifton E. Corzatt, St. Olaf; Lorraine L. Keller, St. Olaf; Richard S. Kleber, St. Olaf; Joseph Konhauser, Macalester; R.W. Nau, Carleton; Thomas R. Savage, St. Olaf; Seymour Schuster, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; Timothy L. Strotman, St. Olaf; T.A. Vessey, St. Olaf; Frank L. Wolf, Carleton.

MATHEMATICAL ASSOCIATION OF AMERICA
NEWS AND NOTICES

EDITED BY FRANK KOCHER, The Pennsylvania State University

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C., 20036.

ASSOCIATE DIRECTOR JOINS MAA STAFF

The Mathematical Association of America announces the appointment on September 1, 1980, of Dr. Marcia P. Sward to the newly created position of Associate Director.

Until September 1, 1980, Dr. Sward was Associate Professor of Mathematics and Chairman of the Department of Mathematics at Trinity College, Washington, D.C. She has served since 1978 as a member of the MAA Committee on Placement Examinations.

Dr. Sward received a Bachelor of Arts degree from Vassar College in 1961 and a Ph.D. in mathematics from the University of Illinois in 1967. She has been a member of the faculty of Trinity since 1968. During 1979 she served as a University Fellow at the National Highway Traffic Safety Administration of the Department of Transportation. She lives in Chevy Chase, Md., with her husband, Gilbert, who is a Professor of Mathematics at Montgomery College, Rockville, Md., and their two children.

Dr. Sward will assist the MAA Executive Director in the general administration of the Washington Headquarters and have specific responsibility for the Association's publication program. She will serve as editor of the newsletter which the Association plans to begin publishing in 1981.

PERSONAL ITEMS

Associate Professor Raymond Cannon of Stetson University has been appointed Associate Professor at Baylor University.

E.L. Perry, Jr., formerly of the Department of Mathematics at Baylor University, has accepted a position with the Thompson-Ramo-Wooldridge Corporation of Houston, Texas.

F. Eugene Tidmore of Baylor University has been promoted to Professor.

The National Science Foundation has recently named M. Ray Perryman, Associate Professor of Economics at Baylor University and a member of the MAA, one of "the top ten social scientists in the United States."

M. Gweneth Humphreys has retired at Randolph-Macon Woman's College with the title of Gillie A. Larew and Charles A. Dana Professor, Emeritus.

Paul L. Irwin of Randolph-Macon Woman's College has been promoted to Associate Professor and is serving a three-year term as Department Head.

Professor Mary Diegert, formerly Chairman of the Department of Mathematics at Broome Community College, is now chairing the newly-formed Department of Computer Science at that institution.

Three recently appointed Assistant Professors at San Jose State University are Dr. Veril Phillips, Santa Clara University; Dr. Marilyn Roth, Johns Hopkins University, and Dr. Edward Schmeichel, University of Southern California.

Dr. Leslie Foster, University of Pittsburgh, has been appointed Assistant Professor at San Jose State University effective January 1981.

Professor Rodney Anderson of San Jose State University retired in June 1980 with the title of Professor Emeritus.

Assistant Professor Max Agoston of San Jose State University has been promoted to Associate Professor.

Frederic Gooding, Jr., Assistant Professor at the York Campus of the Pennsylvania State University, has been appointed to the faculty of Trinity College, Washington, D.C.

Frank Kocher has retired from the Pennsylvania State University and is now a Visiting Lecturer at Baylor University.

Haskell B. Curry, Professor Emeritus at the Pennsylvania State University, was honored on the occasion of his eightieth birthday by a reception September 14, 1980, and the presentation of a *Festschrift*.

Professor Verner Hoggatt of San Jose State University died on August 11, 1980, at the age of 59. He was a member of the Association for thirty-two years.

Arthur Sard, Professor Emeritus of Queens College (CUNY) died on August 30, 1980, in Basle, Switzerland. He was a member of the Association for forty-three years.

SHORT COURSE AT TUCSON

The College of Engineering of the University of Arizona will present a course entitled *Probabilistic Methods in Mechanical and Structural Design* January 5-9, 1981, at the Ramada Inn, 404 North Freeway, Tucson, Arizona. The objective of this short course and workshop is to provide practical information on engineering applications of probabilistic and statistical methods, and design under random vibration environments. Modern methods of structural and mechanical reliability analysis will be presented. Special emphasis will be given to fatigue and fracture reliability.

Further information may be obtained from

Dr. Paul H. Wirsching
Associate Professor of Aerospace and Mechanical Engineering
The University of Arizona
College of Engineering
Tucson, Arizona 85721

COMPUTER SCIENCE EMPLOYMENT REGISTER

In connection with the ACM Computer Science Conference in St. Louis, Missouri, February 23-26, 1981, the Ninth Annual Computer Science Employment Register will be held at Stouffer's Riverfront Towers Hotel. This unique register aids in matching computer scientists and data processing specialists with employer opportunities.

Registrations must be submitted on official forms, which may be obtained from

Orrin E. Taulbee
ACM Computer Science Employment Register
Department of Computer Science
University of Pittsburgh
Pittsburgh, Pennsylvania 15260

The request should state whether it is desired for an applicant, an academic employer, or an employer from business, industry or government. Employers should submit different forms for different positions.

Completed applications forms will be compiled into four books of listings: student applicant, experienced applicant, academic employer, and business, industry and government employer. Multiple copies of these books will be available at the conference. The closing date for acceptance of forms is January 26, 1981.

COMING SOON IN THIS MONTHLY

Thomas Zaslavsky, *The Geometry of Root Systems and Signed Graphs*

Kenneth R. Meyer, *An Application of Poincaré's Recurrence Theorem to Academic Administration*

Ben G. Roth, *Rigid and Flexible Frameworks*

Michael C. Gemignani, *What Is A Computer Program?*

Duane W. DeTemple and Jack M. Robertson, *Constructing Buffon Curves from Their Distributions*

Lenore Blum & Steven Givant, *Increasing the Participation of College Women in Mathematics-based Fields*

William Abikoff, *The Uniformization Theorem*

Alan H. Schoenfeld, *Teaching Problem Solving Skills*

James G. Kennedy, *Arithmetic in Roman Numerals and Nondecimal Number Systems*

Bernie Grofman, *Partisan Preferences, Fair Apportionment, and the Banzhaf Index*

Chester J. Salwach, *Planes, Biplanes and Their Codes*

Paul Schaefer, *Sum-Preserving Rearrangements of Infinite Series*

Robert B. Burckel, *Iterating Holomorphic Self-maps of Discs*

Ezra Brown, *The First Proof of the Quadratic Reciprocity Law, Revisited*

Hans-Willi Siegberg, *Historical Remarks Concerning Degree Theory*

Russell Merris, *Pólya's Counting Theorem via Tensors*

George M. Phillips, *Archimedes the Numerical Analyst*

Ray Bobo, *Foursomes, Fivesomes, and Orgies*

R. Arthur Knoebel, *Exponentials Reiterated*

Louis J. Ratliff, Jr., *A Brief History and Survey of the Catenary Chain Conjectures*

Yakar Kannai, *An Elementary Proof of the No-Retraction Theorem*

Saunders MacLane, *Mathematical Models, a Sketch for the Philosophy of Mathematics*

Philip J. Davis, *Are There Coincidences in Mathematics?*

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

APRIL MEETING OF THE INDIANA SECTION

The Indiana Section of the MAA met at Valparaiso University on Saturday, April 26, 1980, with fifty members present. It followed a symposium on *The Use of History in the Teaching of Mathematics* held the previous day in commemoration of Professor Arthur E. Hallerberg. Also on Saturday fifteen teams participated in the Indiana Small College Mathematics Competition.

The following addresses were presented in the History Symposium:

History as a Tool in the Teaching of Mathematics, Harry Pollard, Purdue University
Some Ideas About the History and Teaching of Mathematics, Karl Menger, Illinois Institute of Technology
Mathematics: Enough of Its History to Improve Its Teaching, Philip S. Jones, University of Michigan

The following papers were presented at the Saturday meeting:

Total Torsion, Michael Penna, Indiana-Purdue University, Indianapolis
Mathematics in the Aircraft Industry, Douglas McCarthy, Indiana-Purdue University, Fort Wayne
Achievement and Avoidance Games of Geometric Configurations, Frank Harary, University of Michigan
Angle Trisectors I have Known, Underwood Dudley, DePauw University

Dan Johnson, Merrillville Junior High School, was recognized for having solved the greatest number of problems during the year from the *Indiana School Mathematics Journal.* He and his teacher Elizabeth Banzen were presented books.

At the business meeting, Chairman Underwood Dudley announced the receipt of a grant of $250 from the MAA for the publication of the symposium lectures. Officers elected for 1980-81 are: Chairman D. E. Deal, Ball State University; Vice-chairman, M. Mundt, Valparaiso University; Secretary-Treasurer, R. Patterson, Indiana-Purdue University at Indianapolis; Governor, G. Sherman, Rose-Hulman Institute of Technology. Memberships in the MAA were awarded to Randy Ekl, Rose-Hulman Institute of Technology; and Daniel Brunner, Purdue University, in recognition of their performance on the Putnam examination.

R.R. Patterson, Secretary-Treasurer

MAY 1980 MEETING OF THE MICHIGAN SECTION

The spring meeting of the Michigan Section was hosted by Hope College on May 1 and 2 on the Hope Campus. Section chairperson, Delia Koo, presided at the business meeting. At this meeting H. T. Slaby of Wayne State University was elected as chairperson for the coming year. David James of the University of Michigan-Dearborn, and Philip Mahler, Henry Ford Community College, were elected as vice-chairpersons; and Paul Garlich, University of Michigan-Flint, as secretary-treasurer. It was announced that Delia Koo had been elected to succeed Yousef Alavi as Governor of the Michigan Section.

Major addresses were:

Excellence in Theory and Practice, George Piranian, University of Michigan
Achievement and Avoidance Games, Frank Harary, University of Michigan
Examples of Involutions of Closed Surfaces and 3-Manifolds, Kyung W. Kwun, Michigan State University
M.C. Escher, Plane Symmetry Groups, and the Tektronic 4051, Elliot A. Tanis, Hope College
Using the Microcomputer as a Visual Aid in the Classroom, Thomas E. Elsner, General Motors Institute
Weather, Elephants, and the Harmonic Series, Ralph P. Boas, Northwestern University

Shorter contributed papers were:

How to Deposit Checks in the Bank, Jane I. Robertson, Ann Arbor
Egyptian Fractions, Then and Now, Frank Sherburne, Jr., Hope College

A panel discussion, *What Next for the Michigan Section?* held on Friday afternoon, featured Phillip Tuchinsky of Ford Motor Company, Brian Winkel of Albion College, Helen Hoke of Lansing Community College, and David Johnson of Eastern Michigan University.

On Saturday morning another panel discussed *The Job Market for Mathematicians.* It was moderated by Zane Motteler of Michigan Technical University and included Marc Konvisser of Wayne State University, Thomas Miles of Central Michigan University, and M.S. Ramanujan of the University of Michigan as panelists.

Three student papers presented were:

Relativity in Perspectivity, Bill Terkeurst, Hope College
Square Limit: An Approach to Infinity, Powell Quiring, Hope College
Simulating Transformations in the Plane, Symmetry Patterns Using the Tektronic 4051, James McElheny, Hope College

The Friday evening banquet was held at the Point West at the juncture of Lakes Michigan and Macatawa. Richard Schwing of General Motors Research Laboratories gave the banquet address: *What Is the Cost of Living--a Longer Life?*

The Saturday luncheon was held at Marigold Lodge, a former private estate on Lake Macatawa. Approximately 140 persons attended the meeting. Local arrangements were made by Frank Sherburne and Elliot Tanis of Hope College.

MAY MEETING OF THE SEAWAY SECTION

The Seaway Section of the MAA held its spring meeting at Herkimer Community College, Herkimer, New York, May 2 and 3, 1980. Approximately seventy people were in attendance. Section chairperson, Howard Bell of Brock University presided.

On Friday the section's executive committee met to consider future activities of the section. A banquet followed at historic Beardslee Manor. Nicolas Goodman of SUNY Center at Buffalo spoke on *Mathematics as an Objective Science.*

The Saturday sessions included a panel discussion, moderated by Jack Graver, Syracuse University on *The New Three-Year Sequence for High School Mathematics in New York State* with panel members Janet Burt, Stephen Cavior, Howard Johnson, and Fred Paul. The *Thirteenth Annual Harry M. Gehman Lecture* was given by H.S.M. Coxeter of the University of Toronto titled *My Graph.*

The following contributed papers were also presented:

Application of Bayesian Techniques to Reliability Demonstration: Estimation and Up-dating of the Distribution, T.S. Bolis, SUC, Oneonta
A Generating Property of Pythagorean Triples, P.J. Arpaia, St. John Fisher College
There are No Correct Answers in the Real World: An Illustration of Modeling as a Process, R.H. Wright LeMoyne College
When Is a Rational Quaternion Algebra a Division Algebra? C.W. Kohls, Syracuse University
A Simple Probability Model for the Reliability of a Jury Trial, R.C. Williams, Alfred University
Mathematics and Art, J.R. Kolod, College of St. Rose

At the business session of the section, the following officers were elected: Chairperson-elect, Kenneth D. Magill, Jr., SUNY Center at Buffalo; First vice-chairperson, Jack E. Graver, Syracuse University. Continuing in office: Chairperson, Howard Bell, Brock University; Second vice-chairperson, Harriette Stephens, SUNY College at Canton; Secretary-Treasurer and Newsletter Editor, Donald Trasher, SUC at Geneseo; Section Governor, Mabel Montgomery, SUC at Buffalo; High School Contest Chairman, Dennis Martin, SUC at Brockport.

The section recognized Michael H. Albert of the University of Waterloo as the winner of the section's prize for highest score in the Putnam Competition.

D.W. Trasher, Secretary-Treasurer

JUNE MEETING OF THE NORTHEASTERN SECTION

The Northeastern Section met at Williams College in Williamstown, Massachusetts, June 20-21, 1980. Invited addresses were given by:

Prof. Raymond M. Smullyan, *Fixed-Points, Mockingbirds, and Self-Reference*
Prof. Stephen L. Snover, *Integrating Programmable Calculators from 1970-1990*
Prof. Albert Nijenhuis, *Khachian's Catchy Algorithm: What's Behind the New York Times Story*
Prof. Ross L. Finney, *The Prisoner's Dilemma, Cardiac Output, and President Ford's War on Red Tape: Introducing Applications in the Classroom*

Workshops were presented by:

Prof. Laurie Snell, *Classroom Use of UMAP Materials*
Prof. Sinan Koont, *Classroom Use of UMAP Materials*
Prof. Stephen L. Snover, *Classroom Use of Calculators*

At the business meeting it was announced that the fall meeting of the section will be held at Merrimac College in North Andover, Massachusetts, on Saturday, November 22. The section will sponsor a short course especially directed at faculty members of two-year colleges in June 1981.

The meeting concluded with a harpsichord concert by Prof. Victor E. Hill of Williams College.

Roger L. Cooke, Chairman

ADULT EDUCATION

The Committee on Adult Education in the Mathematical Sciences has noted the near dearth of articles on adult education in both the MONTHLY and the TWO-YEAR COLLEGE MATHEMATICS JOURNAL and has asked that members be encouraged to submit such articles. The same committee has noted the apparent nonexistence of a bibliography of such articles. If any readers have compiled such a bibliography and are willing to share it with this committee, please so notify David P. Roselle, 100 Sandy Hall, Virginia Tech, Blacksburg, Virginia 24061.

ENDOWMENTS FOR INSTITUTIONAL MEMBERSHIPS

Funds from a recent gift to St. Peter's College have been utilized to endow the College's institutional membership in the Association and thus to endow student memberships in the Association. Members of the Association are encouraged to investigate similar use of funds available to their institutions. Also, members may want to consider making gifts to their institutions for such use.

60th SUMMER MEETING
THE MATHEMATICAL ASSOCIATION OF AMERICA
AUGUST 18-20, 1980

The Sixtieth Summer Meeting was held at the University of Michigan in Ann Arbor, MI during the period 18-20 August. There were 1333 registrants including 778 members of the Association. The Meeting was in conjunction with meetings of the American Mathematical Society, the Association for Women in Mathematics, the Institute for Mathematical Statistics, the Mathematicians Action Group, and Pi Mu Epsilon.

Sessions of the Association were held in Rackham Lecture Hall (RLH) and the Modern Languages Building (MLB). The Program Committee consisted of Joseph E. Adney, Yousef Alavi, A. Bharucha-Reid, Peter L. Duren (Co-Chairman), Wilfred L. Kaplan (Co-Chairman), John O. Kiltinen, Delia Koo, John S. Kostoff, Phillip H. Mahler, and Elliot A. Tanis. The program was comprised of the following presentations:

Monday, 9:10-10:00 A.M.; RLH: THE EARLE RAYMOND HEDRICK LECTURES: "Partitions I: Elementary Elegance," George E. Andrews, The Pennsylvania State University.

The first Lecture considered the Gaussian polynomials and their relation to the partitions of numbers. The fact that these polynomials are generalizations of binomial coefficients was utilized to describe and discover facts about partitions. Subsequently, it was shown how the study of partitions leads directly to one of Gauss's original formulas for these polynomials. This formula allowed Gauss to evaluate the Gaussian sum.

Monday, 10:10-11:00 A.M.; RLH: "Mathematicians, Cryptography, and Computers in the Second World War," Peter J. Hilton, Case Western Reserve University.

The speaker quoted extensively from his own experience as a cryptographer in World War II and derived heuristic conclusions as to the appropriate training of mathematicians to enable them to deal with unexpected applications. There was discussion of means employed for encipherment and decipherment (as limited by the Official Secrets Act) and the role of machines in the process. A study was offered of the nature of the special genius of Alan Turing.

Monday, 11:10-Noon; RLH: "Some Ideas in Nonlinear Analysis," Ivar Stakgold, University of Delaware.

Many problems of analysis can be recast in the fixed-point form $u = Tu$, where u is an unknown element of a suitable space and T is a given transformation (possibly nonlinear) on the space. Various analytical methods for characterizing and calculating fixed points were compared. Monotone methods were emphasized and their application to bifurcation problems was discussed.

Monday, 11:10-Noon; MLB: "A Course Designed to Reduce Math Anxiety," Shirley Emerson and Barbara Riehl, Schoolcraft College.

Studies have shown that much of the difficulty of students who have problems learning mathematics is related to anxiety. Because of this fear many students avoid mathematics and this limits career options. This prompted the development of a course at Schoolcraft College, a suburban community college, particularly for such students. The course combines individualized mathematics instruction with supportive counseling techniques to build skills and competence while dealing with attitudes and feelings about mathematics. The sessions are led by a counselor as well as a mathematics instructor.

Monday, 1:20-2:10 P.M.; RLH: EARLE RAYMOND HEDRICK LECTURES: "Partitions II: Applications," George E. Andrews.

In this Lecture, some of the diverse areas of mathematics and science in which partitions of numbers have played a useful role were discussed. Areas considered included nonparametric statistics, atomic physics, and the Richmond-Szekeres application of partition asymptotics to special values of the dilogarithm.

Monday, 3:20-4:10; RLH: "Pensively Penetrating Penrose's Pentapieces," John H. Conway, University of Cambridge.

Monday 4:15-5:30 P.M.; RLH: "Employment of the Non-Ph.D Mathematician," a panel discussion with Donald W. Bushaw, Washington State University, Arthur Coxford, University of Michigan, and Orrin Taulbee, University of Pittsburgh. Moderator: David W. Ballew, South Dakota School of Mines.

The discussion focused on the employment opportunities available for the non-Ph.D. mathematician. Professor Bushaw called attention to a survey of industrial employers. Professor Coxford discussed the emerging crisis and shortage of secondary mathematics teachers. Professor Taulbee reported on the opportunities for mathematicians in computer science.

Monday, 4:15-5:30 P.M.; RLH: "Machine Grading of College Mathematics Courses," Lisl Gaal, University of Minnesota and E. O. Milton, University of California at Davis.

Multiple-choice, machine-graded tests are not ideally suited for mathematics, but the experience with this format in large classes has been good. To minimize the effect of a trivial numerical error, all but the shortest problems are subdivided into a sequence of questions. This allows for partial credit. The

tests are entirely subject-matter and are achievement-, not ability-, tests. There are never any trick questions and guessing is not penalized. This has minimized student opposition and many, especially minorities, consider it fairer than usual testing. Statistics indicate that the tests measure achievement accurately and alleviate the problem caused by large classes.

Tuesday, 9:00-9:50 A.M.; RLH: EARLE RAYMOND HEDRICK LECTURES: "Partitions III: Ramanujan's 'Lost' Notebook," George E. Andrews.

These lectures were concluded by a look at the implications for partitions of some of the formulas from Ramanujan's "Lost" Notebook (cf. "An Introduction to Ramanujan's 'Lost' Notebook," American Mathematical Monthly, 86 (1979), 89-108). Section 5 of the article just alluded to provides just one example of the partition-theoretic consequences arising from Ramanujan's marvelous discoveries.

Tuesday, 10:00 - 10:50 A.M.; RLH -- Business Meeting of the Association. Presentation of Carl B. Allendoerfer, Lester R. Ford, and George Polya Awards for mathematical exposition.

Tuesday, 11:00 - 11:50 A.M.; RLH: "Building Algebraic Models," Georgia M. Benkart, University of Wisconsin.

A common technique used by physicists to describe a physical system possessing certain symmetries is to attach an algebra to the system in such a way that the symmetries become incorporated into the model as automorphisms of the algebra. The resulting problem then can be restated as: given a little information about the algebra desired, and given a subgroup of its group of automorphisms, how does one find the algebra? One method, of course, is to make a judicious guess. The other more practical and more widely used method is to employ some ideas about representations.

In situations arising from physics, the group of automorphisms is what is called a Lie group, and attached to every Lie group is a Lie algebra. The Lie algebra in this case consists of derivations--which are linear transformations satisfying the usual product rule from calculus for derivatives. The derivations carry roughly the same information as the automorphisms but are generally easier to compute, so it is representations of the Lie algebra of derivations which are of prime importance. This approach to building models using derivations perhaps is best illustrated by the models which have been constructed to describe the interactions between quarks, antiquarks, and gluons.

Tuesday, 11:00 - 11:50 A.M.; MLB: "Microcomputers in a Two-Year College Mathematics Class," John Kostoff, Delta College.

"Microcomputers in a Two-Year Mathematics Class" was something similiar to a "show-and-tell" session in which original computer programs were demonstrated. These may be used in Freshman or Sophomore mathematics classes, either as a teaching aid (electronic blackboard) or as a learning tool for students. Some of the areas featured were precalculus, calculus, statistics, and differential equations, a few of which use programs having color graphics. These programs have been written by members of the Delta College mathematics faculty (with student assistance in some cases) on the APPLE II microcomputer.

Wednesday, 1:20 - 2:10 P.M.; RLH: "Some Personal Experiences for Popularization of Mathematics in China," L. K. Hua, Institute of Mathematics, Academia Sinica, Peking.

The author discussed a point of view of some phenomena in the study of linear partial differential equations of second order in higher dimensions. This talk was divided into three parts: (1) geometric treatment of partial differential equations of mixed type; (2) the proper boundary value problem for an elliptic partial differential equation which is degenerate on the boundary; (3) a linear partial differential equation equivalent to a system of partial differential equations.

Wednesday, 3:20 - 4:10 P.M.; RLH: "The Ellipsoidal Algorithm for Linear Programming," Vasik Chvatal, McGill Univ.

Wednesday, 3:20 - 4:10 P.M.; MLB: "Mathematical Modeling in the Biological Sciences," John A. Jacquez, University of Michigan.

The rapid growth of the biological sciences in the last 50 years has nourished the growth of mathematical biology just as the growth of physics in the last century led to a flowering of mathematical physics. Mathematical modeling of biological systems is made difficult by the complexity and inherent variability of such systems. Both of these characteristics were illustrated by the modeling of exchange of materials between blood in the capillaries and the surrounding tissue cells. The inherent variability of biological systems suggests a criterion of modeling for some types of biological systems. The inherent variability means that by a criterion of biological function there is a class of real systems whose members are equivalent. The principle of modeling this suggests is that one should not have to specify exactly those features of the real systems which show considerable variability in the biological prototypes; one must seek models which stand for the equivalence class of prototypes.

Wednesday, 4:20 - 5:10 P.M.; RLH: "An Introduction to the Finite Element Method," Richard S. Falk, Rutgers University.

In recent years the finite element method has become an increasingly important tool for numerical solution of partial differential equations. In this talk the essential features of the method are intro-

duced, with emphasis given to the role of a variational principle in the derivation of the approximate equations and the properties of piecewise polynomials used to represent the approximate solution.

Wednesday, 4:20 - 5:30 P.M.; MLB: "Archives and History of Mathematics." Presider: G. Baley Price, University of Kansas.

"Archives of American Mathematics," by Albert C. Lewis, Humanities Research Center, University of Texas at Austin. "Who Gave You the Epsilon? - or The Origins of Cauchy's Rigorous Calculus," by Judith Grabiner, California State University at Dominguez Hills. "Applications of Mathematics in Rocket Work and Computing During World War II," by J. Barkley Rosser, University of Wisconsin.

Professor Grabiner began her talk with the question, "How was calculus transformed from a collection of problem-solving techniques in the eighteenth century to the rigorously-based subject which we have inherited from the nineteenth century?" She said that the essential step in this transformation is properly credited to Augustin-Louis Cauchy (1789-1857), who gave us the first reasonably rigorous treatment of the calculus. But as revolutionary as Cauchy's achievement was, he used both ideas and techniques-for instance, approximation techniques developed by Euler and D'Alembert, algebraic techniques from Lagrange-based upon the major mathematical work of the eighteenth century.

Professor Rosser pointed to the many results from classical ballistics used in rocket work during the Second World War. He said that many new problems had been encountered when space travel began and he descibed the development in computers necessary to solve these problems.

SPECIAL SESSIONS OF THE ASSOCIATION

Films on Mathematics and Art by Michela Emmer of the University of Rome were shown in the Rackham Lecture Hall on both Monday and Tuesday afternoon--just prior to the Hedrick Lectures. The usual Film Program was held at 7:00 on Tuesday evening and the following films were shown as part of that Program: Sampling and Estimation, Inferential Statistics, Part II; Dragon Fold; Curves of Constant Width; Conics; The Definite Integral; Journey to the Center of a Triangle; Regular Homotopies in the Plane, Part I; The Seven Bridges of Konigsberg; Non-Euclidean Universe.

Professor Gerald J. Porter of the University of Pennsylvania, Governor of the Eastern Pennsylvania and Delaware Section of the Association, presented the special session, "Introduction to the Use of Computer Generated Graphics in Undergraduate Mathematics Education," on Monday evening.

The Association sponsored the mini-course "Teaching Calculus Using Infinitesimals" which was presented by Professor Frank A. Wattenberg of the University of Massachusetts at Amherst. There were 40 registrants for this mini-course and the participants heard two sample lectures and problem sessions designed to illustrate the content and pedagogy of calculus with infinitessimals. There was also a lecture on the theory of infinitessimals and a panel discussion by instructors who have utilized this approach. The mini-course met on both Monday and Tuesday evenings.

On Tuesday afternoon, the Michigan Section of the Association hosted a reception in honor of the speakers and meeting participants. All meeting participants were invited to attend this reception which featured the wines of the vineyards of Michigan.

On Tuesday evening, the Association sponsored a dinner in honor of its twenty-five year members and their spouses. One hundred and twenty five persons attended this dinner and heard MAA President Dorothy L. Bernstein analyze the statistics obtained from a study of the Association's membership records. MAA President-Elect Richard D. Anderson also spoke on influence of governmental policy on mathematics and the need for mathematicians to play an active role in the development of this policy. Professor G. Baley Price served as the official MAA host and toastmaster for the dinner.

The Committee on the Undergraduate Program in Mathematics (CUPM) held two open meetings.

ENTERTAINMENT

On Tuesday evening there was a wine-tasting. Tickets to this event were sold at the registration desk.

The traditional Summer Meeting picnic was held at Romanoff's on Pontiac Trail on Wednesday. Special bus transportation was available at several points on the campus for the 8.5 mile trip to Romanoff's.

A piano-cello concert by Joseph Gurt, Eastern Michigan University Music faculty, and Jerome Jelinek, member of the University of Michigan's Music faculty, was presented on Thursday evening. The concert was free of charge.

Two bus trips were planned for Thursday. One was a guided tour of Ford Motor Company's River Rouge Plant and the other was a tour of Greenfield Village--the Henry Ford Museum complex. Greenfield Village covers 240 acres and is an indoor-outdoor museum of nearly one hundred historic buildings and artifacts including Edison's lab and Henry Ford's early home.

The Local Arrangements Committee consisted of Paul T. Bateman (ex-officio), Morton Brown, Frederick Gehring, George E. Hay, Marshall D. Hestenes, Melvin Hochster, Fred Hoppe, Paul Howard, Phillip S. Jones, Wilfred Kaplan, Wilfred M. Kincaid, William J. LeVeque (ex-officio), Judith Q. Longyear, M. S. Ramanujan, Ethel Rathbun, Marjorie D. Reade, Maxwell O. Reade (Chairman), David P. Roselle (ex-officio), Joseph L. Ullman, and James G. Wendel.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors met on Wednesday, August 20, 1980, in the Vandenberg Room of the Michigan League with 41 members present. The meetin began with the introduction to the members of the Board of Dr. Marcia P. Sward, the Association's newly appointed Associate Director. Dr. Sward joins the Association after having served as Chairman of the Mathematics Department at Trinity College. Among the items of business approved by the Board of Governors were the following:

The Board elected Mary R. and Robert F. Wardrop of Central Michigan University as Associate Editors of the MONTHLY in charge of mathematics education. Their term will extend through December 31, 1986. The Board also elected a slate of Associate Editors of MATHEMATICS MAGAZINE nominated by Professor Doris W. Schattschneider, Editor-Elect.

The Board of Governors approved the recommendation of the Committee on Earle Raymond Hedrick Lectures that Professor Daniel Gorenstein of Rutgers University be elected the Hedrick Lecturer for the 1981 meeting at the University of Pittsburgh.

Upon recommendation of the Committee on Sections, the Board approved a revised set of Bylaws for the Eastern Pennsylvania and Delaware Section.

The Board of Governors elected J. Arthur Seebach and Lynn A. Steen as Co-Chairmen of the Editorial Committee for the newsletter now proposed for publication by the Association. It is anticipated that the newsletter will be edited by Dr. Marcia Sward, the Associate Director of the Association.

The Board of Governors approved a statement submitted by the Committee on Improving Remediation Efforts in the Colleges. This statement describes the problems facing colleges and universities in remedial instruction and describes the responsibilities and some of the efforts now underway. The document will later be circulated to the membership of the Association.

The Board of Governors approved a new mode of operation for the Association's program of visiting lecturers and consultants. Institutions desiring a speaker will thus be encouraged to directly contact the speaker. If an institution desires a speaker but does not have access to sufficient funding, the institution will be invited to apply to the Association for a grant.

The Congressional Science Fellowship Program was extended through the 1982-83 academic year. This program places mathematicians as members of a Congressional staff. Persons on sabbatical leave are eligible for appointment as a Congressional Science Fellow. Interested persons may inquire of Dr. A. B. Willcox, 1529 Eighteenth St., NW, Washington, D. C. 20036 for additional information.

At the request of the Joint Meetings Committee, the Board of Governors discussed whether to hold the usual Summer Meeting in 1982. The concern was that the meeting attendance may be low because certain members will plan to attend the International Congress in Poland. After considerable discussion, it was voted to request the Joint Meetings Committee to attempt to find a suitable location and to plan to hold the 1982 Summer Meeting.

MEETING OF SECTION OFFICERS

At 7:30 P.M. on Monday in the Vandenberg Room of the Michigan League, there was a mmeting of Section Officers. This meeting was devoted to an exchange of ideas on Section Newsletters and was attended by representatives of 25 of the 29 Sections. Persons desiring information about Section Newsletters or copy of the annual survey of Section activities are invited to request them from the Chairman of the Association's Committee on Sections, Dean Lester H. Lange, School of Science, San Jose State University, San Jose, CA 92182.

David P. Roselle
Secretary

CALENDAR OF FUTURE MEETINGS

Sixty-fourth Annual Meeting, San Francisco, California, January 9–11, 1981.
Sixty-first Summer Meeting, Pittsburgh, Pennsylvania, August 17–19, 1981.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May 15–16, 1981.
EASTERN PENNSYLVANIA AND DELAWARE, University of Delaware, Newark, November 22, 1980.
FLORIDA, Bethune Cookman College, Daytona Beach, March 6–7, 1981.
ILLINOIS, Illinois State University, Normal, May 1–2, 1981.
INDIANA
INTERMOUNTAIN
IOWA, third weekend in April. Deadline for papers February 1.
KANSAS, Benedictine College, Atchison, April 10–11, 1981.
KENTUCKY, Jefferson Community College, Louisville, April 3–4, 1981.
LOUISIANA–MISSISSIPPI, Mississippi State University, Mississippi State, February 13–14, 1981.
MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Goucher College, Towson, Maryland, November 14–15, 1980.
METROPOLITAN NEW YORK, spring. Deadline for papers two weeks before meeting.
MICHIGAN, first Friday and Saturday in May. Deadline for papers six weeks before meeting.
MISSOURI, Northwest Missouri State University, Maryville, April 10–11, 1981.
NEBRASKA, University of South Dakota, Vermillion, South Dakota, April 10–11, 1981.
NEW JERSEY, Seton Hall University, South Orange, spring 1981.
NORTH CENTRAL, end of October and April. Deadline for papers October 1 and April 1.
NORTHEASTERN, Merrimack College, North Andover, Massachusetts, November 22, 1980.
NORTHERN CALIFORNIA, first or second Saturday in February.
OHIO
OKLAHOMA–ARKANSAS, Oklahoma Christian College, Oklahoma City, March 27–28, 1981.
PACIFIC NORTHWEST, second Saturday in June. Deadline for papers six weeks before meeting.
ROCKY MOUNTAIN, Colorado College, Colorado Springs, May 1–2, 1981.
SEAWAY, Daemen College, Buffalo, New York, November 7–8, 1980.
SOUTHEASTERN, University of Alabama, Birmingham, April 10–11, 1981.
SOUTHERN CALIFORNIA, first or second Saturday in March.
SOUTHWESTERN, usually in April. Deadline for papers two weeks before meeting.
TEXAS, Friday and Saturday in early April. Deadline for papers March 1.
WISCONSIN, University of Wisconsin, La Crosse, late March–early April 1981.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Toronto, January 3–8, 1981.
AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES
AMERICAN MATHEMATICAL SOCIETY, San Francisco, California, January 7–10, 1981.
AMERICAN SOCIETY FOR ENGINEERING EDUCATION
ASSOCIATION FOR COMPUTING MACHINERY, Los Angeles, California, November 9–11, 1981.
ASSOCIATION FOR SYMBOLIC LOGIC, San Francisco, California, January 9–10, 1981.
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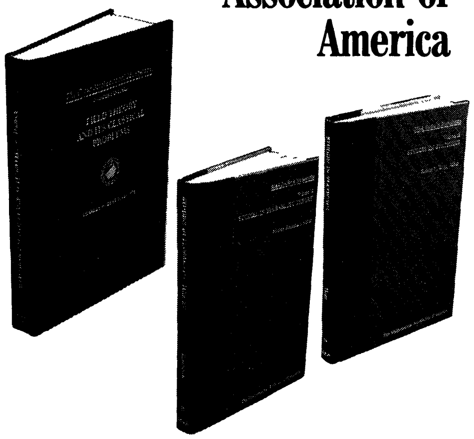

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p. 373, line 13: "10^{27}" should read "10^{24}".

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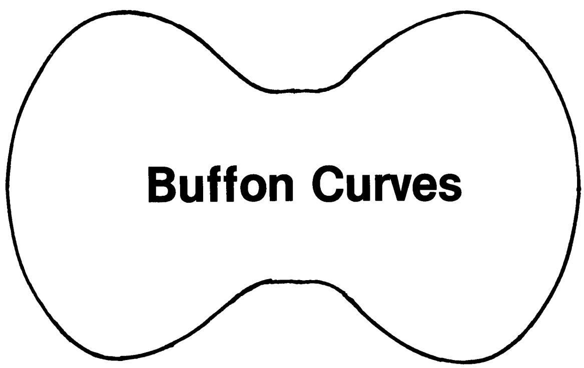

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CONSTRUCTING BUFFON CURVES FROM THEIR DISTRIBUTIONS

DUANE W. DETEMPLE AND JACK M. ROBERTSON

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1. Introduction. In 1733, Georges Louis Laclerc, Comte de Buffon (1707–1788), submitted a memoir to the *Académie des Sciences* in which he considered the now famous needle problem. In its classical formulation, a needle of length L is thrown randomly onto a plane ruled by a uniform grid of parallel lines d units apart, and one seeks the probability of at least one intersection of the needle with the grid. Buffon elegantly solved the problem using integral calculus, but as he was not yet an Academician the paper was not published. It did, however, help enable Buffon to gain admission to the Academy early the following year as a junior member. There followed numerous papers on such topics as probability, optics, forestry, and physiology, attesting to Buffon's ever broadening scientific interests. In 1749, the first volume of Buffon's *Natural History* appeared. When the thirty-sixth and last volume was published the year following his death, Buffon was clearly established as the leading naturalist of his century. In 1777, the earlier paper on the needle problem was finally published as part of an essay contained in the *Supplement to Natural History*. Here it passed almost unnoticed until discovered in 1869 by the English mathematician Morgan W. Crofton [3]. As Crofton was unaware of the memoir's 1733 origin, he was surprised, commenting in a footnote: "That one whose life was devoted to other branches of science should have had the sagacity to discern the true mathematical principles involved in a question of so entirely novel a character, and to reduce them correctly to calculation by means of the integral calculus, thereby opening up a new region of inquiry to his successors, must move us to admiration for a mind so rarely gifted." Perhaps this historical vignette explains why early researchers in geometric probability (Sylvester, Steiner, Cauchy, Barbier, et al) sëemed to mention Laplace, rather than Buffon, in connection with the needle problem. (Curiously, Crofton does refer to the *celebrated* problem of Buffon, though.)

Various extensions of the classical needle problem have been proposed since its inception. In one direction the grid is modified so as to produce improved statistics which approximate π. (See, e.g., [7], [9].) In another direction, it is the needle that is modified, say by lengthening or bending (see [5], [4], [8]). For example, the expected number of times a randomly thrown curve will cross a grid of parallel lines is dependent only on the length of the curve, not its shape; of course the probability distribution for the number of crossings does depend on shape. The intersection probabilities between more general families of geometric figures can be considered (see, e.g., [10], [6]).

In this paper, we retain the grid of parallel lines but address an inverse problem: Starting with a given probability distribution for the number of crossings, can a curve be constructed which has that distribution?

2. Preliminaries. The following notation and terminology will be used:

Curves $\mathcal{C}$ will all be assumed rectifiable with length $L(\mathcal{C})$.

The parallel lines on the plane will be assumed to be separated by one unit of distance.

With respect to a reference direction $\theta=0$ and a reference point $P\in\mathcal{C}$, the location of a curve $\mathcal{C}$ on the grid will be described by the pair $(\theta,\alpha)\in[0,2\pi)\times[0,1)$ (see Fig. 1).

For convex curves $\mathcal{C}$, the distance between supporting parallel lines at angle θ to the

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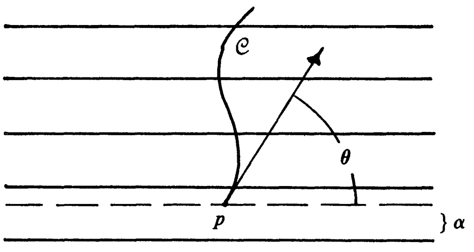

FIG. 1

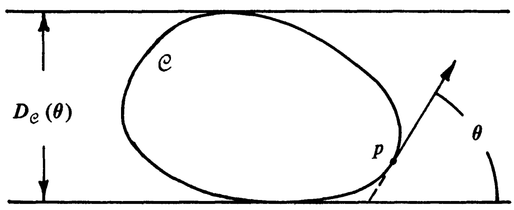

FIG. 2

reference direction will generate a width function $D_{\mathcal{C}}(\theta)$, $0 \leqslant \theta \leqslant \pi$ (see Fig. 2). We assume random tosses of $\mathcal{C}$ onto the grid; that is, $\theta \in [0,2\pi)$ and $\alpha \in [0,1)$ are independent random variables, each uniformly distributed.

For any nonnegative integer k, let $p_{\mathcal{C}}(k)$ denote the probability of exactly k intersections of $\mathcal{C}$ with the grid of parallel lines. Let $\bar{p}_{\mathcal{C}}$ denote the mean; that is, $\bar{p}_{\mathcal{C}}$ is the expected number of intersections of $\mathcal{C}$ with the grid.

THEOREM 1. *For any curve* $\mathcal{C}$, $\bar{p}_{\mathcal{C}} = 2L(\mathcal{C})/\pi$.

A complete proof of this theorem can be found in **[8]**. The result follows easily once the following two facts are established:

(i) The expected number of crossings of a line segment with the grid is proportional to the length of the segment; so for some λ, $\bar{p}_{\mathcal{C}} = \lambda L(\mathcal{C})$.
(ii) The expected number of crossings (while proportional to $L(\mathcal{C})$) is independent of the shape of $\mathcal{C}$.

COROLLARY (Barbier, 1860). *All convex curves of constant width d have the same perimeter* πd.

There is no loss of generality in assuming $d=1$. Any such curve strikes the grid exactly twice in any position; thus, by the theorem,

$$L(\mathcal{C}) = \frac{\pi \bar{p}_{\mathcal{C}}}{2} = \frac{\pi \cdot 2}{2} = \pi.$$

2. Distributions for Closed Convex Curves. Since tangencies occur with probability zero, we may assume that when a closed convex curve $\mathcal{C}$ is tossed on the grid each line of the grid is crossed twice or not at all. Furthermore, if two lines of the grid are struck twice by $\mathcal{C}$, the same is true of all lines between.

THEOREM 2. *Let* $\mathcal{C}$ *be a closed convex curve and* $E_n = \{\theta \in [0,\pi]\colon n-1 \leqslant D_{\mathcal{C}}(\theta) < n\}$. *Then the probability of* $2n$ *crossings when* $\mathcal{C}$ *is randomly placed on the grid is given by*

$$p_{\mathcal{C}}(2n) = \frac{1}{\pi}\left(\int_{E_n} (D_{\mathcal{C}}(\theta) - n + 1)\,d\theta + \int_{E_{n+1}} (n+1 - D_{\mathcal{C}}(\theta))\,d\theta\right).$$

Let $\mathcal{C}$ be located by the pair (θ,α) and fix θ. Then for $0 < \alpha < [D_{\mathcal{C}}(\theta)] + 1 - D_{\mathcal{C}}(\theta)$ ($[x]$ is the greatest integer less than or equal to x) $2[D_{\mathcal{C}}(\theta)]+2$ crossings occur. The probability α lies in this range is $D_{\mathcal{C}}(\theta) - [D_{\mathcal{C}}(\theta)]$. On the other hand, with $[D_{\mathcal{C}}(\theta)] + 1 - D_{\mathcal{C}}(\theta) < \alpha < 1$, which occurs with probability $[D_{\mathcal{C}}(\theta)] + 1 - D_{\mathcal{C}}(\theta)$, $2n$ crossings result. The lengths $[D_{\mathcal{C}}(\theta)] + 1 - D_{\mathcal{C}}(\theta)$ and $D_{\mathcal{C}}(\theta) - [D_{\mathcal{C}}(\theta)]$ are shown on the graph of $D_{\mathcal{C}}$ in Fig. 3, so it follows that the shaded area of Fig. 4 can be interpreted as that part of a rectangle $[0,\pi]\times[0,1]$ giving $2n$ crossings. Hence we see

$$p_{2n} = \frac{1}{\pi}\left[\int_{E_n} (D_{\mathcal{C}}(\theta) - (n-1))\,d\theta + \int_{E_{n+1}} ((n+1) - D_{\mathcal{C}}(\theta))\,d\theta\right].$$

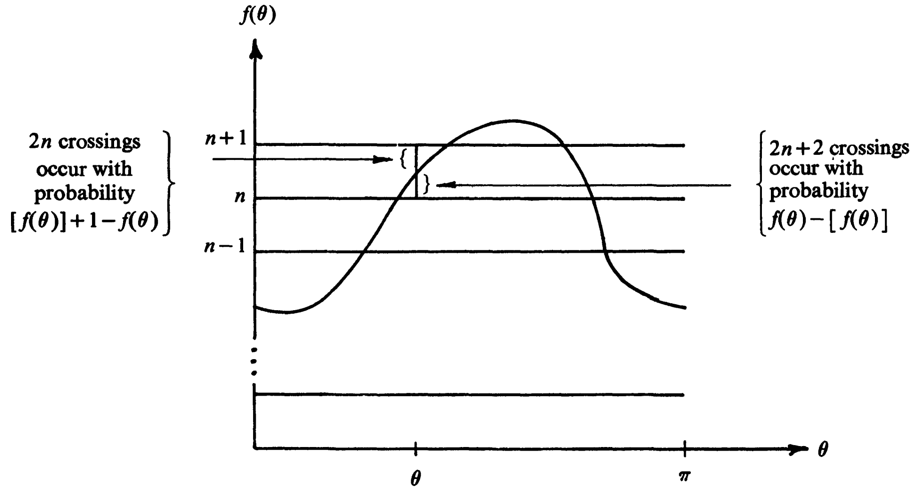

FIG. 3

COROLLARY 1. *If a closed convex curve $\mathcal{C}$ produces a degenerate distribution with $p_{\mathcal{C}}(2n)=1$ then the curve has constant integer width n.*

COROLLARY 2. *If $\mathcal{C}$ is a closed convex curve then the average width of $\mathcal{C}$ is $L(\mathcal{C})\pi$.*

From Theorem 1, $\pi\bar{p}_{\mathcal{C}}=2L(\mathcal{C})$, and by direct computation using Theorem 2

$$\pi\bar{p}_{\mathcal{C}}=\sum_{n=1}^{\infty}\int_{E_n}(2n(D_{\mathcal{C}}(\theta)-n+1)+(2n-2)(n-D_{\mathcal{C}}(\theta)))\,d\theta$$
$$=\sum_{n=1}^{\infty}\int_{E_n}2(D_{\mathcal{C}}(\theta))\,d\theta=2\int_0^{\pi}D_{\mathcal{C}}(\theta)\,d\theta.$$

So the average width is

$$\frac{1}{\pi}\int_0^{\pi}D_{\mathcal{C}}(\theta)\,d\theta=\frac{\bar{p}_{\mathcal{C}}}{2}=\frac{L(\mathcal{C})}{\pi}.$$

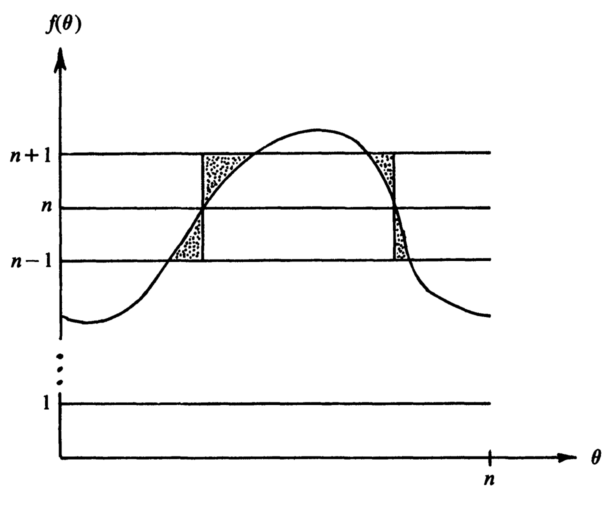

FIG. 4

This corollary shows that the average width of a closed convex curve is the same as the diameter of a circle of equal length.

If we define a relation on closed convex curves by $\mathcal{C}_1 \sim \mathcal{C}_2$ if and only if $p_{\mathcal{C}_1} = p_{\mathcal{C}_2}$ then $\sim$ is an equivalence relation. From above, the map $\mathcal{C} \to p_{\mathcal{C}}$ is many to one, $\mathcal{C}_1 \sim \mathcal{C}_2$ provided their diameter functions produce the same shaded areas from Theorem 2, and the curves of constant diameter n comprise one equivalence class by Corollary 1. Also, if one starts with a distribution and tries to reconstruct the curve from it, uniqueness is not obtained. We look at this problem next.

3. Closed Convex Curves Generated from Given Distributions. Suppose p is a given distribution with weights placed on the nonnegative integers. In order that some closed convex curve $\mathcal{C}$ generates p, i.e., $p_{\mathcal{C}} = p$, the following necessary conditions must be satisfied:

(i) $p(n) = 0$ for $n > 2([\pi\bar{p}/4] + 1)$.

(ii) $p(n) = 0$ for $n = 2k+1$, k an integer.

(iii) if $n < m$ and $p(2n) \neq 0$, $p(2m) \neq 0$, then $p(2k) \neq 0$ for $n < k < m$.

The first condition follows from knowing $L(\mathcal{C}) = \pi\bar{p}_{\mathcal{C}}/2$ from Theorem 1; thus the maximum width for the closed curve is $L(\mathcal{C})/2 = \pi\bar{p}_{\mathcal{C}}/4$. Condition (ii) has been observed earlier, since an odd number of intersections can only happen in the case of tangencies. Condition (iii) follows from Theorem 2, the continuity of D, and the intermediate value theorem. Thus not all distributions are generated by closed convex curves. However, the following theorem shows how to construct a closed convex curve generating a translate of p provided p satisfies (ii) and (iii).

THEOREM 3. *Let p be a distribution with finite support on nonnegative integers satisfying* (ii) *and* (iii) *above. Then there is a curve $\mathcal{C}$ and a positive integer m such that $p_{\mathcal{C}}(n) = p(n-m)$ for all n.*

The procedure will be to construct a $\mathcal{C}^\infty$ width function producing the appropriate areas given by Theorem 2, then translating the curve upward to ensure that a convex curve exists for that width function.

Now assume $p(2j) = p_{2j}$ for $j = n, n+1, \ldots, n+k$. We construct a function f on $[0, \pi/2]$ giving areas dictated by Theorem 2 in steps, as follows:

Step 1. Choose $\theta_1 \in (0, \pi/2)$ so that f can be defined on $[0, \theta_1]$ with all the following properties:

(i) $f \in \mathcal{C}^\infty$ on $[0, \theta_1]$.

(ii) $f(0) = n$, $f(\theta_1) = n+1$.

(iii) $f^{(i)}(0) = f^{(i)}(\theta_1) = 0$, $i = 0, 2, 3, \ldots$. ($f^{(i)}(x)$ is the ith derivative of f at x.)

(iv) f is nondecreasing.

(v) $(2/\pi)\int_0^{\theta_1}((n+1) - f(\theta))\,d\theta = p_{2n}$.

(vi) $(2/\pi)\int_0^{\theta_1}(f(\theta) - n)\,d\theta \leqslant \frac{1}{2}p_{2n+2}$.

Step 2. Choose $\theta_2 > \theta_1$ so that f can be defined on $[\theta_1, \theta_2]$ with all the following properties:

(i) $f \in \mathcal{C}^\infty$ on $[\theta_1, \theta_2]$.

(ii) $f(\theta_1) = n+1$, $f(\theta_2) = n+2$.

(iii) $f^{(i)}(\theta_1) = f^{(i)}(\theta_2) = 0$, $i = 1, 2, 3, \ldots$.

(iv) f is nondecreasing.

(v) $(2/\pi)\int_0^{\theta_1}(f(\theta) - n)\,d\theta + \int_{\theta_1}^{\theta_2}((n+2) - f(\theta))\,d\theta = p_{2n+2}$.

(vi) $(2/\pi)\int_{\theta_1}^{\theta_2}(f(\theta) - (n+1))\,d\theta \leqslant \frac{1}{2}p_{2n+4}$.

Continue inductively to define f as in Step 2 through $0 < \theta_1 < \theta_2 < \cdots < \theta_{k-1}$.

Step k. Define f on $[\theta_{k-1}, \pi/2]$ so that $f(\theta)$ has all the following properties:

(i) $f \in \mathcal{C}^\infty$ on $[\theta_{k-1}, \pi/2]$.

(ii) $f(\theta_{k-1})=n+k-1,\ f(\pi/2)=n+k$.
(iii) $f^{(i)}(\theta_{k-1})=f^{(i)}(\pi/2)=0,\ i=1,2,3,\ldots$.
(iv) f is nondecreasing.
(v) $(2/\pi)[\int_{\theta_{k-2}}^{\theta_{k-1}}(f(\theta)-(n+k-2))d\theta+\int_{\theta_{k-1}}^{\pi/2}(n+k-f(\theta))d\theta]=p_{2n+2k-2}$.

Now since

$$\frac{2}{\pi}\left[\int_0^{\theta_1}((n+1)-f(\theta))\,d\theta+\int_0^{\theta_1}(f(\theta)-n)\,d\theta+\cdots+\int_{\theta_{k-1}}^{\pi/2}((n+k)-f(\theta))\,d\theta+\int_{\theta_{k-1}}^{\pi}(f(\theta))-(n+k-1)\,d\theta\right]=1,$$

from property (v) and $\sum_{i=n}^{n+k}p_{2i}=1$, it follows that

$$1=p_{2n}+p_{2n+2}+\cdots+p_{2n+2k-2}+\frac{2}{\pi}\int_{\theta_{k-1}}^{\pi/2}(f(\theta)-(n+k-1))\,d\theta=\sum_{i=n}^{n+k}p_{2i}$$

so that

$$\frac{2}{\pi}\int_{\theta_{k-1}}^{\pi/2}(f(\theta)-(n+k-1))\,d\theta=p_{2n+2k}.$$

Thus far we have a $\mathcal{C}^{\infty}$ nondecreasing function producing the correct areas as required by Theorem 2 on $[0,\pi/2]$, which is extended to $[0,\pi]$ by setting $f(\theta+(\pi/2))=f((\pi/2)-\theta)$ for $0\leqslant\theta\leqslant\pi/2$.

We now use the following result obtained by slightly modifying the argument found on pages 115–116 of [2].

THEOREM. *If f is $\mathcal{C}^2$ with period π, if $f(\theta)+f''(\theta)>0$ for all θ, then there is a convex curve whose width function is f. The family of lines passing through $(\theta,f(\theta)/2)$ and perpendicular to the ray joining the origin and the point $(\theta,f(\theta)/2)$ are envelopes of such a curve.*

We now examine f as constructed above. If $f(\theta)+f''(\theta)>0$ for all θ, we set $D_{\mathcal{C}}=f$ and we are done. If not, for some positive integer $m, D_{\mathcal{C}}(\theta)=f(\theta)+m$ has the property that $D_{\mathcal{C}}(\theta)+D''_{\mathcal{C}}(\theta)>0$. Translating the function upward by the integral amount does not change the size of the constructed areas, only their location. Hence the weights are correct, but because the curve has been "fattened" the distribution is translated to the right m units.

An Example. By crude means of numerical integration on graph paper the "width" function shown in Fig. 5 was constructed for the distribution $p_2=.4, p_4=.2, p_6=.4$. (Condition (iii) of the inductive construction was ignored.)

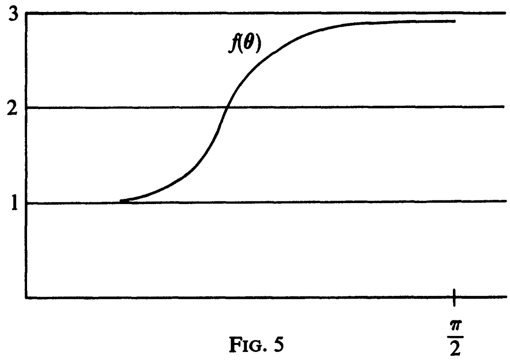

FIG. 5

This "width" function leads to the construction of a nonconvex curve (Fig. 6) which is then made convex by adding 13 uniformly to the radius for $0 \leqslant \theta \leqslant 2\pi$ (Fig. 7). A transparency was prepared for the curve and, in 100 tosses onto a grid, 28 crossings occurred 45 times, 30 crossings occurred 18 times, and 32 crossings occurred 37 times. (See Fig. 7.)

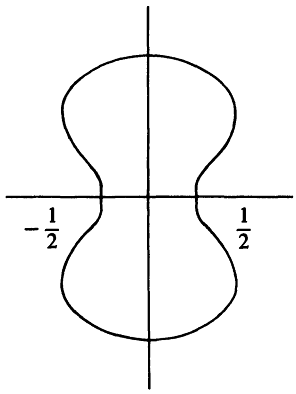

FIG. 6

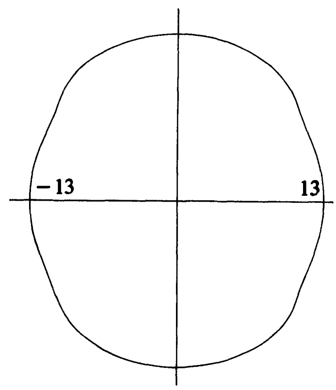

FIG. 7

Questions left open by the discussion above include:

1. What about distributions for nonconvex curves? One could hope to produce theorems like those above for the more general case.

2. What are a sufficient set of conditions on a distribution D that some closed convex curve generates that distribution without translation?

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46. Elegance . . . is a quality of universal range and strict meaning. Mathematicians say of a problem, a demonstration or a solution in their science, when it exhibits perfect lucidity and form, that it is elegant: the one remaining use of the term in its proper sense. In this austere region it reveals that the aesthetic principle is fundamental; intellect and emotion are ultimately identical.

— Isabel Paterson, Preface to *Mr. Hodge and Mr. Hazard*, in *The Novels of Elinor Wylie*, London, 1934.

INCREASING THE PARTICIPATION OF WOMEN IN FIELDS THAT USE MATHEMATICS

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Since 1974, Mills College has been striving to interest women in, and prepare them for, professional careers that traditionally have not attracted many women. We have stressed computer science, management, engineering, medicine, the physical sciences, and similar fields, all of which require strong mathematics backgrounds. We also tried to increase the mathematical and computer competence of students in fields like economics and sociology. We have therefore created a comprehensive program to increase the participation of women in regular mathematics and computer-science courses.

Although this article is about the mathematical education of women, the program's methodology and philosophy are adaptable to many educational situations.

1. Backgrounds. Mills is a liberal arts college for women, near San Francisco, with a stable undergraduate enrollment of approximately 850 students. About one-third, including foreign students, belong to ethnic minorities and about one-fifth are students returning to college after a break in their education. Traditionally, Mills has emphasized the humanities and the arts. Even with the increasing interest in professional careers during the early 1970's, Mills was certainly not thought of as a place for students inclined to mathematics or engineering.

A department of mathematics and computer science was established in 1974. (These disciplines had been part of the Physical Sciences Department.) Computing facilities have been available since the early 1960's via terminal linkages to Stanford University and to the Lawrence Hall of Science (at the University of California, Berkeley). A computer center (with a DEC PDP 11/34 computer running UNIX) was established in the spring of 1978 under a grant from the National Science Foundation.

Our program was started in 1974 by Blum, partly in response to the documented discrepancy between the number of men and the number of women who were prepared to take calculus when they entered college. In a 1973 study [17] of first-year students at the University of California, Berkeley, Sells pointed out two related facts. First, the percentage of entering students prepared to take calculus was substantially lower for women than for men. Second, calculus was a prerequisite for "majoring in every field at the university except the traditionally female, and hence lower-paying, fields." Thus, inadequate mathematics preparation served as a

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Steven Givant received his Ph.D. in 1975 at the University of California, Berkeley, under the direction of Robert Vaught. He is an Assistant Professor of Mathematics at Mills College and also holds a research position at the University of California, Berkeley, where he collaborates with Alfred Tarski on a National Science Foundation-sponsored research project in logic and the foundations of mathematics. Previously, he was the San Francisco Bay Area director of Project SEED, a program in which professional mathematicians teach advanced mathematics, on a daily basis, to elementary-school children from poverty and minority backgrounds. In 1979 he was elected to the Danforth Foundation's Associate Program.—*Editors*

"critical filter" for entry and participation of women in many fields. (This observation had also been made by the Carnegie Commission on Higher Education [5].)

Patterns of socialization (and discrimination) cause young women to avoid mathematics and fields that require mathematics [5], [6], [8], [9], [10], [11], [13], [14], [16], [17], [20]. Most people study mathematics for its "perceived usefulness" [9], [16]. However, professions requiring mathematics have long been considered part of the "male domain" and inappropriate for women [8], [9]. Therefore girls tend to stop studying mathematics early and are ill-prepared to enter scientific and technical fields. The resulting dearth of women perpetuates the masculine image of these fields.[1] Thus, as Fox [9] points out, "the problem seems circular."

2. The Mills Program. Our program is designed to break this cycle and to alter the perceptions and experiences that cause women to avoid mathematics. Regardless of whether these perceptions are the result of teacher, parent, or peer attitudes, of role stereotyping, inadequate or even misleading counseling, negative experiences, or general "math anxiety,"[2] our philosophy is that the best way of overcoming the problems is by actually doing mathematics. Rapid changes in attitudes, aspirations, and capabilities are possible. Concrete positive experiences make the difference. Therefore our program provides rapid entry into the regular mathematics curriculum and into scientific and technical fields. Our approach includes (at the introductory level) stimulation of student interest; goal-oriented beginning courses supported by peer-taught workshops; and (at higher levels) interdisciplinary courses stressing applications; active and meaningful student participation as peer teachers and as lecturers in the departmental seminar; early career experiences through internships; dual degree options in liberal-arts and engineering (in conjunction with the University of California at Berkeley and the Stanford University and Boston University schools of engineering); and a computer-literacy and computer-science program parallel to, and in conjunction with, the mathematics program (see Section 5).

During the summer, all incoming students receive a brochure describing the program and containing self-placement quizzes. The quizzes are meant more to interest students and encourage them to seek advice than to place them. An orientation meeting is held at the beginning of the academic year to discuss both the importance of mathematics for various careers and the different routes into the mathematics curriculum. Well-publicized career events are sponsored jointly by our department and the Center for Career Planning: films, panel discussions with women scientists and engineers (who serve as important role models, potential mentors, and job contacts), student discussions of their own internship experiences, and field trips to local corporations and research facilities. A list of typical events is available from the authors.

Students can enter a mathematics curriculum via pre-calculus and calculus, computer programming, finite mathematics, mathematical modeling, statistics, etc. We stress the pre-calculus/calculus route for several reasons. We have already noted the "critical filter" aspect of calculus. In addition, because of its intimate connection with modern technology, calculus forms an important part of a liberal-arts education, a part that has been missing from the college education of most women. Finally, calculus is traditionally associated with college mathematics. A student who has studied calculus rightfully feels that she has taken a real mathematics course and can go further in this direction if she wishes.

3. The Pre-calculus Program. The pre-calculus course is specifically designed to prepare students in one semester (no matter what their backgrounds) for a calculus sequence; we felt that the idea of spending a year or more just preparing for calculus would deter many students.

We have tried to avoid the shortcomings of the usual college-algebra approach, including its remedial nature and its attempt to teach large amounts of material without covering interesting and intuitive fundamental concepts; students completing college algebra are often unprepared intellectually or technically for calculus. However, we also had to come to terms with many of

the problems that college-algebra courses face: the large amount of material to be covered, the students' attitudes toward mathematics, and the diversity of their mathematical backgrounds.

The amount of mathematics usually thought necessary for calculus is far too great for a one-semester course. The usual solution is either to cover this material in several semesters or to assume that the students are familiar with basic algebra, quickly review linear and quadratic equations, and then begin a rapid study of elementary functions. Unfortunately, an important reason for the high attrition rate in lower-division mathematics courses is precisely their grueling pace. (Indeed, many well-meaning instructors teach contrary to their own pedagogical beliefs simply to cover the required material in the prescribed time.)

Many of our students have had negative experiences that have undermined their confidence in their ability to understand mathematics. It is essential to *restore* this confidence. We feel that this is best done by providing successful mathematical experiences *in a regular classroom setting and with material that the students see as challenging*. The standard remedial approach often produces poor results precisely because it conjures up the memory of past failures. Furthermore, since its aspirations are usually rather low, a remedial course does not prepare a student intellectually for more advanced work. Thus, it is important to start with mathematics that is new and fresh and that simultaneously provides a vision of what is to come.

Another reason for teaching new and fresh mathematics is the great diversity in the mathematical backgrounds of the students. Some have successfully completed more than two years of high school mathematics and need only a brief review. Others have studied little or no high school mathematics and need to spend a great deal of time working on their algebraic skills. It is extremely difficult in a large classroom setting to teach standard algebraic material without either boring the first group or losing the second. By teaching mathematics that is new and different, it is possible to capture the interest of all students; moreover, it gives those students who are not as well prepared the feeling that they can compete on an equal footing with their better-prepared peers.

Our course has two components. Component A deals with functions and emphasizes concepts that will be encountered in calculus; component B consists of basic algebraic material. Each component is taught in a different setting, component A in the regular class with a regular instructor, component B in small workshops taught by other undergraduates. This system provides students with supplementary help when they need it, in a supportive and nonthreatening setting. In addition, by stressing the ideas in the regular class and the algebraic skills in the workshop, we help the students separate the conceptual from the more computational aspects of mathematics. Detailed syllabi of the two components are available from the authors.[3]

Component A. Component A is the subject of the pre-calculus class. It meets three hours a week, and the students receive one course credit. The course consists of a streamlined presentation of the basic properties of elementary functions, with emphasis on general techniques for visualizing the graphs of these functions. Many key ideas from calculus are introduced, for example, continuity and points of discontinuity, local and absolute maxima and minima, concavity and inflection points. Ideas and techniques that are not directly applicable in calculus have, for the most part, been omitted.[4]

As much as possible, the mathematics is taught independently of the workshops, that is, using as little algebra as possible. With this approach, students with weak algebra backgrounds can still follow the course while they are working on their algebraic skills in the workshops. The pace of the pre-calculus class at Mills is *relaxed*, precisely because the workshop system frees time that would otherwise be spent reviewing algebra. Moreover, any pre-calculus topic that cannot be covered in the class can be taken up in the workshops of the calculus course during the next semester. In short, the workshop system permits a great deal of flexibility in pacing the pre-calculus class.

The lectures are given in an informal "discovery" style. This discourages passive note-taking.

It encourages students to think through and try to understand the material in class while it is being presented, not several days later when the homework is due. This increases the motivation of students appreciably. Moreover, the questions make the class much more interesting and stimulating for the students and give the instructor a continual reading of what is, and is not, being understood. Particularly stressed are the "whys," "what-fors," and "what-ifs," as well as the connections among various topics. The students are encouraged to discover mathematical facts and generalizations on their own (and the homework is often designed to guide them in this discovery).

Since we believe that the only way to learn mathematics is to do it, and since we want the students to succeed, both regular homework and regular attendance are required. Homework is assigned to *every* class and an assignment is due at the beginning of the *very next class*. It counts one-third of the total grade. Since students must review a lesson before they can do a homework assignment, this means that they will come prepared to the next class, having digested the previous lesson. Students are encouraged to discuss the assignments and work on them together; the aim here is to reduce the frustration of not being able to do a particular assignment. The idea of discussing mathematics outside of class is novel to many of our students, and they are surprised to discover that such discussions can be interesting and even exciting. (Students are also encouraged to browse in the mathematics section of bookstores and libraries, another novel activity for many of them.)

Most students who are unsure of themselves mathematically dread taking mathematics tests. The fear of not finishing on time reduces their capacity to think clearly. One successful solution to this problem is to give examinations in the evening, with no time-limit. (However, a student must complete her examination in one sitting.)

Component B. Component B consists of the basic algebraic material and is taught in pre-calculus workshops. Each workshop consists of a small group of students, usually 10–15, and is taught by a student teaching-assistant (T.A.) under faculty supervision. It meets two hours a week for the whole semester. Every student enrolled in the regular pre-calculus class is required to enroll simultaneously in one of the workshops.

The general atmosphere of the workshops is very relaxed and informal. The T.A.'s are students who have completed one year or more of calculus. (This is dictated by the small size of Mills and its mathematics department.) Generally, they have not taught before. To overcome this obstacle and to provide them with a model, the T.A.'s are required to attend twice weekly a demonstration workshop taught by the faculty instructor. They take notes on what is taught and how it is presented, specifically observing what types of questions are asked and what techniques are used to involve the whole class. After each meeting, they discuss their observations and talk about their own workshops. They then teach the lesson they have just observed at the next meeting of their own workshops. (In practice, their workshops may actually be a few lessons behind.) They are required to observe another T.A.'s workshop once a week and to write up their observations. Occasionally the faculty supervisor visits their workshops and afterwards discusses the session with them. This procedure of training T.A.'s eliminates almost entirely the frustration that both students and T.A.'s might otherwise experience. On the one hand, the students learn a great deal of basic algebra; on the other hand, the T.A.'s learn much about teaching, and their understanding of the mathematics is increased. Indeed, because of the style of teaching, the relaxed atmosphere, and the small size of the workshops, students develop a great deal of rapport with their T.A. (who is often taking some other class with them). This peer interaction between students and T.A.'s very much helps to break down the anxieties students feel about asking questions and actively participating in a mathematical discussion. They begin to see that algebra is understandable and even interesting.

To avoid the impression that the workshops are remedial, we try to begin with material that is not very familiar to students, for example, inequalities. It is important in the first few sessions to avoid boring "I know this stuff" lessons. Gradually we work in the more familiar topics, giving a

few sample questions first to determine how thorough the presentation should be. As the workshop sections are small, each T.A. can easily adjust the pace of her class, spending less time on those basic topics with which *her* students are familiar and treating more thoroughly the mathematics that causes them difficulties. Emphasis is placed on homework, but there are also occasional tests. Attendance is required.

Students receive 1/2 course credit (graded pass/fail) for the workshop component of the pre-calculus course. We feel that credit is justified because of the content and nature of the workshops and the amount of homework required. (Workshops associated with more advanced courses carry either 1/4 credit or no credit at all.) The T.A.'s receive 1 course credit (graded pass/fail), which falls under the title of Teaching Practicum. Again, their gain in mathematical maturity is substantial. Students may receive at most two such credits and may not use these to fulfill requirements in the various majors.

4. The Transition from Pre-Calculus to More Advanced Work. The pre-calculus program is meant to prepare students with weak mathematics backgrounds to enter a calculus course, not to fill all the gaps in their mathematical education. Thus, many topics that more advanced mathematics students should know (e.g., for vector calculus) have been left out. In addition, the mathematical confidence that the students have acquired in the program can easily be shaken. After all, for some, their earlier lack of confidence has deep roots, extending back even to elementary school. Nor is the program a magical remedy for all inadequacies in problem-solving and computation. Students can forget or become confused about ideas and techniques from one semester to the next. Again, these students have just begun their mathematical training and typically do not have years of solid grounding and positive experiences behind them. In short, one shot of pre-calculus is not a panacea. The students must continue to receive encouragement and careful, positive teaching.

The pre-calculus course and the subsequent calculus sequence must be closely coordinated. The calculus instructor must be aware of the student's backgrounds, what they do and don't know, in order to effect a smooth transition. Continual review in calculus workshops refreshes the pre-calculus material and simultaneously provides a nonthreatening situation in which to deepen students' understanding of, and familiarity with, this material. Topics that have been left out of pre-calculus can be covered in these workshops. In this way students get the help they need and fill the gaps in their mathematical background *while they are progressing through a regular mathematics curriculum*. (For similar reasons we would recommend the use of auxiliary workshops in other mathematics and science courses.)

5. Additional Aspects of the Program. Several aspects of our program directly concern career preparation. Here we have attempted to involve students in concrete and meaningful experiences that help them develop the self-confidence and positive self-image needed to view many options as truly feasible. For instance, the *internship program* gives students early job experiences in "real world" situations with industry or research centers, either part-time during the school year or for a few months during the summer.[5] This kind of experience is particularly important for young women who often have had little or no opportunity to be involved with technical areas of the working world. Most of the women in this program have taken calculus and introductory computer programming, but many are not mathematics or computer-science majors.

In collaboration with the University of California at Berkeley, Stanford University, and Boston University, we have developed a *dual degree engineering program*. Students in this program attend Mills College for three years and then transfer to the engineering school of one of these universities for two more years. On completing the program, they receive bachelor's degrees in both liberal arts and engineering; it is also possible for exceptional students to earn a master's degree at the same time.

Within our own department we have developed interdisciplinary courses (e.g., Mathematical Modeling for the Social Sciences); and we have introduced applications of mathematics and computer science into some standard courses, even outside our department (e.g., in sociology, economics, and chemistry).

Finally, since 1977 the department has been developing a broad-based *computer literacy program* on four levels; for the entire student body, drop-in workshops provide on-line experience in interactive use of computers; for students in introductory mathematics and science courses, emphasis is placed on the use of computers in problem-solving; upper-division science and social science students use computers to deal with large data bases and for simulation and modeling; advanced computer science students are exposed to new fields and more technical areas such as system design and computer graphics. A crucial feature is faculty training (including released time) so that faculty in different disciplines can become actively involved. This aspect of the program has been developed by our colleagues Carol Lennox and Diane McEntyre.

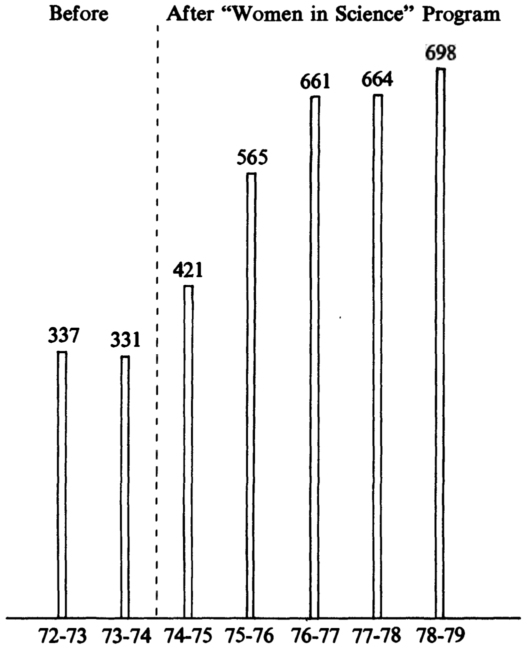

FIG. 1. Total enrollment in regular mathematics and computer-science courses at Mills College.

6. Evaluation. The Mills program has been extremely successful in increasing the participation of women in courses and fields related to mathematics (see Fig. 1). Since the program began in 1974, enrollment in regular mathematics and computer-science courses has doubled, from 331 in 1973–74 to 698 in 1978–79 (not counting enrollment in workshops). Enrollment in pre-calculus has tripled, from 27 in 1973–74 to 84 in 1978–79 (which is about 10% of the student population at Mills). Also, the number of students enrolled in beginning calculus has nearly doubled, from 57 to 101. And students are going on into more advanced courses. For example, the enrollment in Differential Equations went from 9 to 20, in Logic and the Foundations of Mathematics from 9 to 22, and in Functions of a Real Variable from 5 to 15. Given a stable *total* college-student population of 850 undergraduates, these increases, and the actual numbers themselves, are very significant and, indeed, far exceed national norms. (For example, according to a recent report of the Conference Board of the Mathematical Sciences [7], during the years

1970–1975 enrollment in pre-calculus courses in college and university mathematics departments increased 12%, while upper-division enrollment in mathematics was down 32%.) It is remarkable that total enrollments in the courses of our department now exceed those of every other department on campus (including English).

A large number of students have participated in the internship program since it was instituted in 1974. During 1975–1978, about 50 students have had technical internships, 27 at IBM in San Jose. Dr. Jean-Paul Jacob of IBM has written: "Given that Mills is a small college for women, traditionally not known for their science and engineering instruction, we initially did not expect the interns to contribute to our research projects. But we have had surprises after surprises with this program. The successes achieved by the Mills students were beyond our best possible expectations. They range from contributions to research papers [e.g., on the Volterra-Lotka equations, by Janet Walker [**19**]] to the writing of instruction manuals for the use of our special interactive systems ... By all measures we can think of, IBM considers the internship program a great success ..."

Pre-engineering is now one of the fastest-growing options on campus. Fifteen students have entered engineering schools and 35 students have been placed in technical or engineering-related jobs. This is in contrast to the situation less than ten years ago when an informal survey showed there was "little likelihood that Mills students would wish to pursue engineering." In addition, a group of about 45 students has formed a Math/Computer-Science/Engineering Club that sponsors speakers, films, tutoring projects, and science fairs; 15 students have helped install a new computer center on campus. These developments are quite unusual at a women's college.

In summary, large numbers of our students are entering and successfully completing mathematics courses and even considering careers in technical fields, despite previously weak mathematics backgrounds or long interruptions in their mathematical education.[6] (The average mathematics SAT score of entering students at Mills is about 530.)

7. Conclusion. Although our program has focused on the education of women, it has also developed strategies that can increase mathematical competence and confidence, and hence opportunities, for many people who might otherwise avoid fields that require mathematics. The important features of the Mills program are the carefully designed curriculum, the positive teaching, and the supportive and encouraging environment. We have emphasized the need for students to have positive experiences doing solid mathematics that leads to more advanced work, the need for realistic career information that points out the importance of mathematical training for many fields and helps students see that these areas offer them viable and exciting career options, and the importance of role models. Although it is particularly important to have female role models, we feel that concerned and supportive men can also play an important role.[7]

Many institutions and government agencies are now recognizing the need to develop programs and resources that encourage the full participation of women in fields based on mathematics. The Women in Science Program of the National Science Foundation has focused on career counseling by sponsoring conferences across the country that provide science-career information and role models for college-aged women. Both the Visiting Scientists Program of the NSF and the Women and Mathematics program (WAM) of the Mathematical Association of America arrange for women mathematicians, or women who use mathematics in their work, to speak to high school students, teachers, and counselors. The Women's Educational Equity Act, U. S. Office of Education, has funded the development of a number of resource materials, including videotapes and films [**1**]–[**3**].

The Carnegie Corporation is currently supporting the Math/Science Network, which coordinates the programs of many institutions and individuals and serves as a national resource and information center. Giving further recognition to this area is a bill introduced in Congress by Senator Kennedy [**12**] that would establish a ten-year program "to encourage the full participation of women in scientific, professional and technical fields through programs to improve

science education; to promote literacy in science and mathematics and educate and inform the public concerning the importance of the participation of women in science and technology."

This paper is based on an invited talk presented by the authors in MAA-NCTM Joint Sessions of the 56th Annual Meeting of the National Council of Teachers of Mathematics, San Diego, California, on April 13, 1978. This work was supported in part by grants from the San Francisco Foundation, the IBM Corporation, the National Science Foundation, the Women's Educational Equity Act Program, the Educational Foundation of America, the Xerox Fund, and the Carnegie Corporation of New York. For many valuable discussions and suggestions we especially wish to thank Nancy Kreinberg and William Johntz, as well as Ruth Cronkite, Mary Gray, Joanne Koltnow, Warren Leffler, Bonnie Miller, and Judith Roitman.

Notes

1. About 15% of the scientists and less than 3% of the engineers in the United States are women. These relatively small proportions are due in part to the limited job opportunities for women as well as to inadequate career preparation. According to the NSF report **[15]**, the proportion of women earning doctorates in science and engineering nearly doubled (from 9.7% to 18%) during the period 1970–1977. However, this increase in educational preparation was not matched by increases in job opportunities or salary levels relative to men's. For example, during 1977 the unemployment rate for women with doctorates in science and engineering was 3 to 5 times that for men; in addition, women with doctorates in these fields earned on the average of 20.5% less than their male counterparts (*up* from a gap of 17% in 1973).

2. We are pleased that the term "math anxiety" (cf. **[18]**), by attracting publicity, has helped call attention to the causes and consequences of the low participation of women, and others, in mathematics-related fields. However, we regret that the emphasis is on the individual's psychological problems—as opposed to societal factors —and that the implied solution is psychological counseling and therapy, not a carefully designed, substantial mathematics program. Indeed, many math-anxiety programs are counseling or desensitization sessions that "introduce some mathematics in small bits and never for very long." Our objections to these approaches are similar to those expressed in this paper about many remedial programs (see Section 3). In particular, they don't prepare students for more advanced work.

3. Givant is writing a pre-calculus text based on notes he has been developing for the course during the past several years. Other materials we have used include *Precalculus Mathematics*, by Thomas W. Hungerford, Richard Mercer, and Sybil R. Barrier (published by Saunders College/Holt, Rinehart and Winston); the *Series in Mathematical Modules* (published by Cummings); and *Math PS Modules*, by Gary Fitts (available from the ASUC, Berkeley, Calif.). Two texts used for a similar course at Harvard University are: *The Math Workshop*: *Algebra* and *The Math Workshop*: *Functions*, by Deborah Hughes Hallett (published by W. W. Norton).

4. Other faculty are now also becoming involved in the pre-calculus program, and we are incorporating their suggestions. In particular, two new aspects have been introduced by Richard Bassein. The first is directed at improving students' problem-solving skills, and the second is intended to help students "see the world through mathematical eyes." During the first week or so, the class is introduced to the role of examples, guessing, and hypothesis-testing in mathematics. (For instance, the students discovered the formula for the sum of the first n integers by systematically inspecting and analyzing a table of values.) The technique is reinforced by approaching much of the course material in terms of problems to be solved. In addition, students play a problem-solving computer game, Mathcave (written by Bassein). Throughout the semester, groups of students work on carefully supervised projects that interpret some aspect of the world mathematically. Projects have included charting the effect of the sun and the moon on the tides, tiling the plane, finding all regular solids, and measuring the size of the earth.

5. Dr. Jean-Paul Jacob of IBM, in a report to the National Research Council's Committee on Applied Mathematics Training, has described the Mills/IBM internship program as follows: "During the internship period each student is assigned a mentor who guides the student through a project. This project is chosen in consultation with the student, and we try to best use the student's talents and motivations. We do not expect them to do work for IBM, but to learn and enhance their skills. The internship has been a semi-academic program for which the students receive credit-units from Mills."

6. A formal evaluation of various components of the Mills program is in preparation **[4]**.

7. These features and strategies are also recommended by others. Gray **[10]** discusses the importance of special courses, role models, and career information, as well as the mathematical community's self-interest in educating college women in mathematics. Fox, Fenema, and Sherman **[9]** point out a wide range of socio-cultural factors that stereotype "mathematics as a male domain." Kirk **[13]** stresses the need for early and continuing exposure to, and

participation in, mathematics that relates to the student's life, both present and future. Casserly [6] emphasizes the need to inform parents, teachers, and counselors about the importance of mathematics for their children's future. Kreinberg [14] discusses the implications of these findings and recommendations for policy and action programs; she describes how these strategies are common to a variety of successful programs spanning all age levels and different educational contexts offered by the Math/Science Network, based in the San Francisco Bay Area.

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MISCELLANEA

47. At the core of science are the logical truisms, beginning, of course, with mathematics, and all of these truisms are obvious once they have been pointed out. Anything that is not ultimately obvious cannot be mathematics.

— Kenneth E. Boulding, in "Science: Our Common Heritage," *Science*, 207 (22 February 1980) 833–834.

TEACHING PROBLEM-SOLVING SKILLS

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1. Introduction and Overview. This paper deals with two questions:

A. Can we accurately describe the strategies used by "expert" mathematicians to solve problems? and

B. Can we teach students to use those strategies?

I make two basic assumptions. First: as a result of their problem-solving experience, mathematicians develop consistent and useful problem-solving strategies. Second: most students are not aware of, or do not use, these strategies. For example, consider the following problems.

Problem 1: Let a, b, c, and d be given numbers between 0 *and* 1. *Prove that*

$$(1-a)(1-b)(1-c)(1-d) > 1-a-b-c-d.$$

Problem 2: Determine the sum $\dfrac{1}{2!}+\dfrac{2}{3!}+\cdots+\dfrac{n}{(n+1)!}$.

Problem 3: Prove that if 2^n-1 is a prime, then n is a prime.

Ostensibly, all three of these problems are accessible to high school students. None of them require mathematical knowledge beyond algebra, and all of them have straightforward solutions. Yet college students and professional mathematicians attack these problems in dramatically different ways.

On Problem 1 most students will laboriously multiply the four factors on the left, subtract the terms on the right, and then try to prove that $(ab+ac+ad+bc+bd+cd-abc-abd-acd-bcd+abcd)>0$—usually without success. Virtually all of the mathematicians I've watched solving it begin by proving the inequality $(1-a)(1-b)>1-a-b$. Then they multiply this inequality, in turn, by $(1-c)$ and $(1-d)$ to prove the three- and four-variable versions of it.

Likewise in Problem 2, most students begin by doing the addition and placing all the terms over a common denominator. A typical expert, on the other hand, begins with the observation: "That looks messy. Let me calculate a few cases." The inductive pattern is clear and easy to prove.

The colleague who read Problem 3 and said, "That's got to be done by contradiction," was typical; given the structure of the problem, one really has no alternative. Yet this almost automatic expert observation is alien to students: a large number of those to whom I have given the problems either respond with comments like "I have no idea where to begin" or try a few calculations to see whether the result is plausible and then reach a dead end.

Of course, these are special problems for which expert and novice performance is remarkably consistent. While the experts did not consciously follow any strategies, their behavior was at least consistent with these "heuristic" suggestions:

a. For complex problems with many variables, consider solving an analogous problem with fewer variables. Then try to exploit either the method or the result of that solution.

b. Given a problem with an integer parameter n, calculate special cases for small n and look for a pattern.

Alan H. Schoenfeld received his Ph.D. from Stanford in 1973, working under Karel deLeeuw in topology and measure theory. After two years at Davis he joined the SESAME group at U.C. Berkeley, where he began his current "research and development in 'human AI,' where the research goal is to create detailed models of human problem-solving, the development goal to use the models to teach students to solve problems the way experts do." He would like to hear from colleagues who are either studying or trying to teach problem solving.

The author dedicates this paper to the memory of Karel deLeeuw, in appreciation of his spirit of inquiry and his devotion to his students.

c. Consider argument by contradiction, especially when extra "artillery" for solving the problem is gained by negating the desired conclusion.

Many of the novices were unaware of the strategies, and many others "knew of them" (that is, upon seeing the solution they acknowledged having seen similar solutions) but hadn't thought to use them. Expert and novice problem-solving are clearly different. The critical question is: Can we train novices to solve the problems as experts do?

My answer is a provisional "yes." I think it is possible to give a course in which we can teach students to solve a wide variety of problems—including problems unlike any solved in the course—better and more efficiently than they could otherwise. But there are many questions to be answered. How much sophistication and background do students need before such instruction can be effective? What does it take to understand a strategy like "establish subgoals" and how to use it? What do you need in addition to the mastery of individual strategies? Briefly, my thesis is this.

First, the strategies are more complex than their simple descriptions would seem to indicate. If we want students to use them, we must describe them in detail and teach them with the same seriousness that we would teach any other mathematics. Second, there is clear evidence that the strategies do make a difference—when there are only a small number of them and they are taught under closely controlled circumstances. Third, being able to use individual strategies is not enough: you have to know which ones to use, and when. We can provide students with a reasonable structure for efficient problem-solving and can actually demonstrate improvement.

2. The Complexity of Heuristic Strategies. The first person to describe problem-solving strategies in such a way that they could be *taught* (although he does not claim that they can be and makes no promises about the results) was Pólya. In *How to Solve It* (1945) and the two volumes of *Mathematical Discovery* (1962 and 1965) Pólya laid the foundation for explorations in heuristics.

Let us define a *heuristic strategy* as a general suggestion or technique which helps problem-solvers to understand or to solve a problem. Heuristic strategies include the "fewer variables," "calculating special cases," and "argument by contradiction" strategies described in section 1. Fig. 3 gives many more. Many investigators have attempted to show that these strategies can help students to solve problems. However, the results are generally inconclusive, in part because these apparently simple strategies can turn out to be very complex. Consider the following strategy and a few problems.

"To solve a complicated problem, it often helps to examine and solve a simpler analogous problem. Then exploit your solution."

Problem 4: Two points on the surface of the unit sphere (in 3-space) are connected by an arc A which passes through the interior of the sphere. Prove that if the length of A is less than 2, then there is a hemisphere H which does not intersect A.

Problem 5: Let a, b, and c be positive real numbers. Show that not all three of the terms $a(1-b)$, $b(1-c)$, and $c(1-a)$ can exceed $1/4$.

Problem 6: Find the volume of the unit sphere in 4-space.

Problem 7: Prove that if $a^2+b^2+c^2+d^2=ab+bc+cd+da$, then $a=b=c=d$.

These four problems, like Problem 1, can be solved by the "analogous problem" strategy. Yet it is unlikely that a student untrained in using the strategy would be able to apply it successfully to many of these. Part of the reason is that the strategy needs to be used *differently* in the solution of each problem.

In solving Problem 1, we built up an inductive solution from the two-variable case, using the result of the analogous problem as a stepping stone in the solution of the original.

In contrast, analogy is used in Problem 4 to furnish the idea for an argument. The problem is hard to visualize in 3-space but easy to see in the plane: we want to construct a diameter of a unit circle which does not intersect an arc of length 2 whose endpoints are on the circle. Observing that the diameter parallel to the straight line between the endpoints has this property enables us to return to 3-space and to construct the analogous plane.

Problem 5 is curious. It looks as though the two-variable analogy should be useful, but I haven't found an easy way to solve it. At first the one-variable version looks irrelevant, but it's not. If you solve it, and think to take the product of the three given terms, you can solve the given problem. So again we exploit a result, but this time a different result in a different way.

Problem 6 exploits both the methods and results of the lower-dimensional problems. We integrate cross-sections using the same method; the measures of the cross-sections are the results we exploit.

In Problem 7 it would seem apparent that the two-variable problem is the appropriate one to consider. However, *which* two-variable problem is not at all clear to students. A large number of those I have watched tried to solve:

Problem 7′: Prove that $a^2+b^2=ab$ *implies that* $a=b$, *instead of*

Problem 7″: Prove that $a^2+b^2=ab+ba$ *implies* $a=b$.

We conclude that the description "exploiting simpler analogous problems" is really a convenient *label* for a collection of similar, but not identical, strategies. To solve a problem using this strategy, one must (a) think to use the strategy (this is nontrivial!), (b) be able to generate analogous problems which are appropriate to look at, (c) select among the analogies the appropriate one, (d) solve the analogous problem, and (e) be able to exploit either the method or the result of the analogous problem appropriately.

This strategy isn't especially complex. "Look for an inductive solution when you see an integer parameter" is easier, but "establish and exploit subgoals" is far more difficult. The moral of this section is that it's easy to underestimate the amount of work that would go into teaching students to use even a single strategy. We should single it out as a useful strategy; we should give sample problems (like the ones above) showing how it works; we should remind the students of its use in other problems when we use it; and we should chide them when they fail to use it.

3. Teaching Strategies to Students Makes a Difference (Sometimes). There is some convincing evidence that "experts" do use the kind of heuristic strategies we have been discussing. But these strategies are rarely taught explicitly: they might be called "good habits learned through experience in problem solving." Thus some might claim (indeed, have claimed) that any gain in a problem-solving environment is not due to the strategies taught but to the practice in solving problems. For that reason I conducted an experiment in which two groups of students received essentially the same problem-solving training, except that the strategies were mentioned explicitly to only one of the groups.

Each of the seven upper-division science majors I worked with was taught and tested individually. "Instruction" was provided on tape recordings, so that it was replicable and could be checked by colleagues. The students were trained to solve problems "out loud" and the tests were tape-recorded. The experiment was designed to test the students' use of five particular heuristic strategies:

1. Draw a diagram if at all possible.
2. If there is an an integer parameter, look for an inductive argument.
3. Consider a logical alternative: arguing by contradiction or contrapositive.
4. Consider a similar problem with fewer variables.
5. Try to establish subgoals.

During the instruction, each of the students worked twenty problems and then saw solutions

to each of them. Each had the same amount of time for problem solving and for seeing the solutions. However, four of the students received a heuristic "extra." They were given a list and explanation of the five strategies used in the experiment and an "overlay" to each solution explaining how the strategy had been used. Fig. 1 gives the solution to a problem we have already considered. The right-hand side is the solution seen by all students. The left-hand side was seen only by the "heuristics" students. All the students were reminded periodically (during practice sessions and tests) to review carefully what they were doing, and the reminders to the heuristics group included the phrase "Look over the list of strategies."

The four-variable problem is too complicated.	You are given the real numbers a, b, c, and d, each of which lies between 0 and 1. Prove the inequality $(1-a)(1-b)(1-c)(1-d)>1-a-b-c-d.$
	Solution
Can we learn something from a similar one-variable problem? No.	Suppose we start by proving the equation
How about the comparable two-variable problem? It's easy to solve.	$(1-a)(1-b)>1-a-b.$ (*) If we multiply out the left, (*) is true if and only if $1-a-b+ab>1-a-b,$ which is true if and only if $ab>0$. But $ab>0$, since we were given that a and b are both positive. This proves (*). Now let's build on this. The number c is between 0 and 1, so $(1-c)$ is positive. Multiplying both sides of (*) by $(1-c)$, we get
Can we use the result? Yes . . . build up to 3 variables . . . Then build up to 4.	$(1-a)(1-b)(1-c)>(1-a-b)(1-c)$, or $(1-a)(1-b)(1-c)>1-a-b-c+ac+bc.$ Since ac and bc are both positive, we obtain $(1-a)(1-b)(1-c)>1-a-b-c.$ (**)
Remember, when a problem is complicated...consider a similar problem with fewer variables. Then try to use either the method or the result to solve the original problem.	Continuing in the same vein, we notice that $(1-d)$ is positive; multiplying both sides of (**) by $(1-d)$, we obtain $(1-a)(1-b)(1-c)(1-d)>(1-a-b-c)(1-d)$, or $(1-a)(1-b)(1-c)(1-d)>1-a-b-c-d+ad+bd+cd.$
	As before, we see that ad, bd, and cd are all positive. Thus $(1-a)(1-b)(1-c)(1-d)>1-a-b-c-d,$ which is what we wanted to prove.

FIG. 1

I give a detailed description of the results in "Explicit heuristic training..." Even this small sample yielded a statistically significant difference in pretest to post-test gains: each of the four "heuristics" students outscored the non-heuristics students, a one-in-35 chance. But, more important, the transcripts of the solutions show that explicit use of the strategies accounted for the differences between the two groups.

The "fewer variables" practice included Problems 1, 5, and 7, as given above, and the following.

Problem 8: Show that it is impossible to find real numbers a,b,c,d,e,A,B,C,D,E *such that*

$$x^2+y^2+z^2+r^2+s^2=(ax+by+cz+dr+es)(Ax+By+Cz+Dr+Es)$$

for all values of x, y, z, r, *and* s.

For each of these, all of the students saw how the one- or two-variable analog was used to solve the original problem. If "practice" is what counts, all of them should have solved this post-test problem:

Problem 9: Suppose p, q, r, *and* s *are positive real numbers. Prove the inequality*

$$\frac{(p^2+1)(q^2+1)(r^2+1)(s^2+1)}{pqrs} \geqslant 16.$$

All four of the "heuristics" students solved it, but only one of the others did. The other two non-heuristics students multiplied through by $pqrs$ and tried (unsuccessfully) to deal with the resulting inequality.

This shows that we cannot rely on students' abilities to grasp useful problem-solving strategies when the students are not given explicit instructions on their use and that the instruction "made a difference." More precisely, we can say that heuristics made a difference under experimental conditions in which (1) there were a limited number of strategies to "worry" about, (2) there were periodic reminders to consider using the strategies, and (3) the test problems were clearly amenable to one of the suggested "heuristic" approaches. Taking heuristics instruction out of the laboratory and into the real world will be no easy task.

4. The Need for Global Strategies. If heuristics can make a difference, why are the results in the literature so equivocal? The major studies (see those by Goldberg, Kantowski, Lucas, Smith, Webb, Wilson) are generally encouraging, but that's about all. I can see two possible reasons for this, one of them easily remedied.

We have seen that it is easy to underestimate the amount of work required to teach a particular strategy. For example, Kantowski reported at the 1978 NCTM meetings that students in a problem-solving experiment failed to "look back" over their solutions, in spite of the fact that 40 per cent of instruction time was spent "looking back." Videotapes of the class sessions showed that after each problem the teacher had stepped aside and said, "Now let's look at what we've done," and proceeded to do so. But the value of the strategy was not stressed. Students were not shown why it is useful to "look back"—so they didn't. If we want students to take a strategy seriously, we have to convince them of its usefulness.

But even if we succeed in teaching students to use a series of important heuristic strategies, I see no guarantee that there will be clear signs of improvement in their general problem solving. Knowing how to use a strategy isn't enough: the student must think to use it when it's appropriate. To justify this claim I argue first by analogy, then briefly describe a supportive experiment.

We can think of a heuristic strategy as a "key" to unlock a problem. There are a large number of such "keys," and a given problem may be "openable" by only a few of them. Imagine facing a locked door with a key ring on which there are thirty keys, two of which will open the door. If you only have time to try three or four keys in the lock, the fact that the "right" key is somewhere on the chain may not help you very much. On the other hand, a strategy for selecting the right key might. If you could narrow down the collection of "candidate" keys to ten, the opportunity to try three or four of these gives you a much better chance of success.

Consider techniques of integration in elementary calculus. There are fewer than a dozen important techniques, all of them algorithmic and relatively easy to learn. Most students can

learn integration by parts, substitution, and partial fractions as individual techniques and use them reasonably well, as long as they know which techniques they are supposed to use. (Imagine a test on which the appropriate technique is suggested for each problem. The students would probably do very well.) When they have to select their own techniques, however, things often go awry. For example,

$$\int \frac{x\,dx}{x^2-9},$$

a "gift" first problem on a test, caused numerous students trouble when they tried to solve it by partial fractions or, even worse, by a trigonometric substitution!

In "Presenting a Strategy for Indefinite Integration" (this MONTHLY, Oct. 1978) I discuss an experiment in which half the students in a calculus class (not mine) were given a strategy for selecting techniques of integration, based on a model of "expert" performance. The other students were told to study as usual—using the miscellaneous exercises in the text to develop their own approaches to problem solving. Average study time for members of the "strategy" group was 7.1 hours, while for the others it was 8.8 hours; yet the "strategy" group significantly out-performed the rest on a test of integration skills—in spite of the fact that they were not given training in integration, just in selecting the techniques of integration.

The "moral" to the experiment is that students who cannot choose the "right" approach to a problem—even in an area where there are only a few useful straightforward techniques—do not perform nearly as well as they "should." If we leap from techniques of integration to general mathematical problem-solving, the number of potentially useful techniques increases substantially, as does the difficulty and subtlety in applying the techniques. An efficient means for selecting approaches to problems, for avoiding "blind alleys," and for allocating problem-solving resources in general thus becomes much more critical. Without it, the benefits of training in individual heuristics may be lost.

5. The Model. The model of "expert" performance described below serves as the framework for my courses in problem solving. Of course, any attempt to characterize mathematical problem solving on just a few sheets of paper must leave out much more than it includes.

The global outline of the strategy is given in Fig. 2. I use a flow chart to indicate the generally dynamic but structured nature of the process; it is meant as a guide to profitable behaviors, not as a straitjacket that orders and restricts them. Various individual heuristics often come into play most appropriately at certain phases of the process. These are listed in Fig. 3.

The problem-solving process begins with an *analysis* of what the problem entails. This includes getting a "feel" for the problem by looking for what is given, what is asked for, why the "givens" are given, whether what is asked for seems plausible, what major mechanisms seem to apply, what mathematical context the problem fits into, and so on. Which heuristics (if any) are brought to bear during analysis may depend on both the problem and who is solving it (how much of a "problem" or routine exercise is this "task" to the individual?). But examples of the appropriate use of some heuristic strategies at this stage of problem-solving are:

(1) to draw a diagram even when the problem appears amenable to a different kind of argument, such as in the following:

Problem 10: Find those values of t for which the equations

$$\left\{\begin{matrix} x^2-y^2=0 \\ \textit{and} \\ (x-t)^2+y^2=1 \end{matrix}\right\} \textit{have } 0, 1, 2, 3, \textit{or } 4 \textit{ solutions}.$$

(2) to examine special cases and try to solve them or to determine patterns. For example, in:

Problem 11: Given $a, b>0$, determine $\lim_{n\to\infty}(a^n+b^n)^{1/n}$.

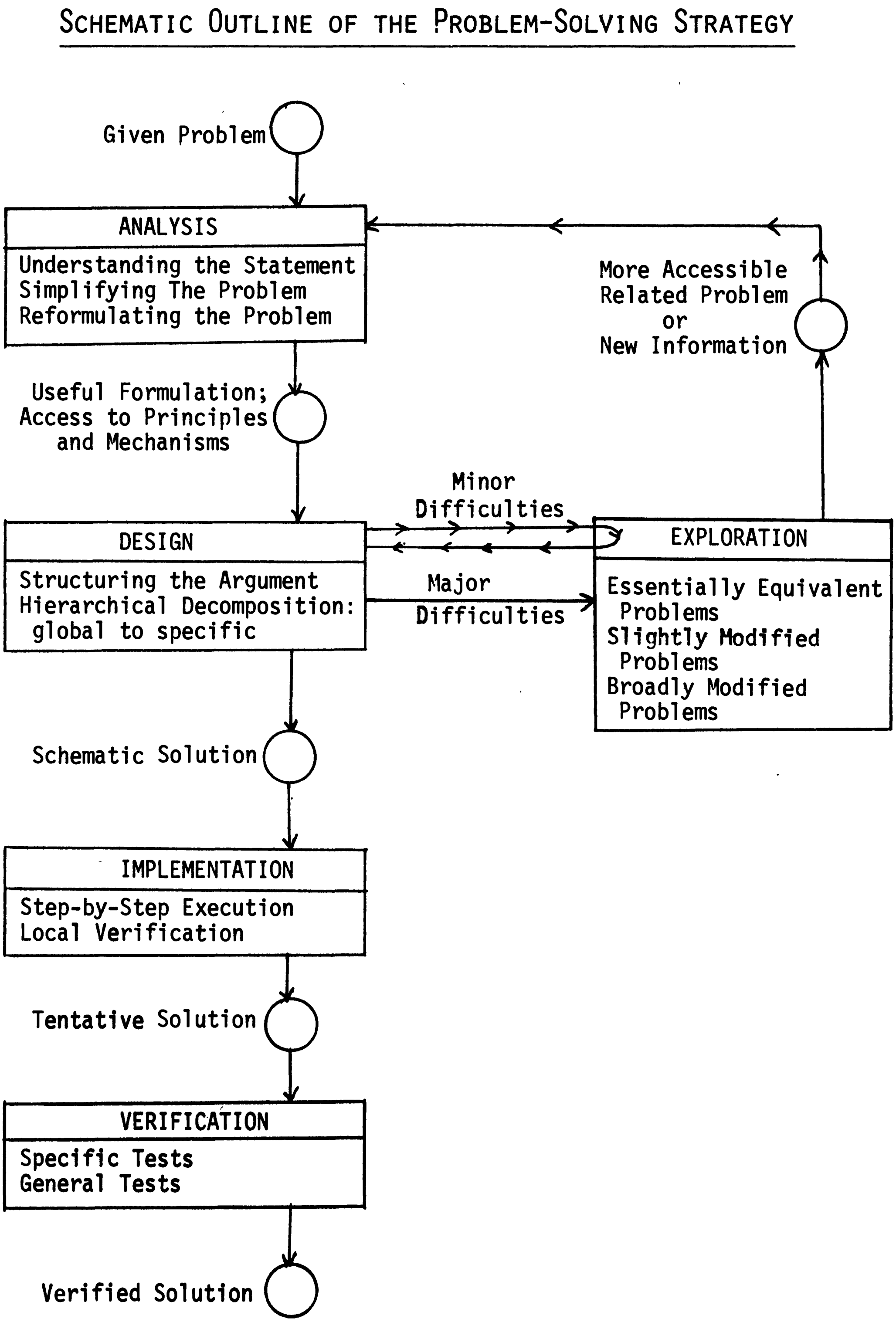
SCHEMATIC OUTLINE OF THE PROBLEM-SOLVING STRATEGY
Given Problem
ANALYSIS
Understanding the Statement
Simplifying The Problem
Reformulating the Problem
More Accessible
Related Problem
or
New Information
Useful Formulation;
Access to Principles
and Mechanisms
Minor
Difficulties
DESIGN
Structuring the Argument
Hierarchical Decomposition:
global to specific
Major
Difficulties
EXPLORATION
Essentially Equivalent
Problems
Slightly Modified
Problems
Broadly Modified
Problems
Schematic Solution
IMPLEMENTATION
Step-by-Step Execution
Local Verification
Tentative Solution
VERIFICATION
Specific Tests
General Tests
Verified Solution

FIG. 2

Frequently Used Heuristics

Analysis

1. Draw a diagram if at all possible.
2. Examine special cases:
   a. Choose special values to exemplify the problem and get a "feel" for it.
   b. Examine limiting cases to explore the range of possibilities.
   c. Set any integer parameters equal to $1, 2, 3, \ldots$, in sequence, and look for an inductive pattern.
3. Try to simplify the problem by
   a. exploiting symmetry, or
   b. "without loss of generality" arguments (including scaling).

Exploration

1. Consider essentially equivalent problems:
   a. Replacing conditions by equivalent ones.
   b. Re-combining the elements of the problem in different ways.
   c. Introduce auxiliary elements.
   d. Re-formulate the problem by
      (i) change of perspective or notation
      (ii) considering argument by contradiction or contrapositive
      (iii) assuming you have a solution and determining its properties.
2. Consider slightly modified problems:
   a. Choose subgoals (obtain partial fulfillment of the conditions)
   b. Relax a condition and they try to re-impose it
   c. Decompose the domain of the problem and work on it case by case.
3. Consider broadly modified problems:
   a. Construct an analogous problem with fewer variables.
   b. Hold all but one variable fixed to determine that variable's impact.
   c. Try to exploit any related problems that have similar
      (i) form
      (ii) "givens"
      (iii) conclusions.

   Remember: when dealing with easier related problems, you should try to exploit both the *result* and the *method of solution* on the given problem.

Verifying Your Solution

1. Does your solution pass these specific tests?
   a. Does it use all the pertinent data?
   b. Does it conform to reasonable estimates or predictions?
   c. Does it withstand tests of symmetry, dimension analysis, and scaling?
2. Does it pass these general tests?
   a. Can it be obtained differently?
   b. Can it be substantiated by special cases?
   c. Can it be reduced to known results?
   d. Can it be used to generate something you know?

FIG. 3

one might want to set $a = 1$; in:

Problem 12: Find $\sum_{n=1}^{\infty} 1/n(n+1)$.

one might want to compute the sums for 1, 2, 3, 4, and 5 terms to see the (surprisingly obvious) answer.

(3) to look for preliminary simplifications. In:

Problem 13: Find the largest area of any triangle which can be inscribed in a circle of radius R.

one might (i) consider first the unit circle, (ii) note that, without loss of generality, one can assume that the base of the triangle is horizontal, and (iii) examine several sketches and try to guess an answer before jumping into an analytic solution.

Design is in a sense a "master control." It is not really a separate box on the flow chart but rather pervades the entire solution process. Its function is to ensure that the problem solver is engaged in activities most likely to be profitable. It entails keeping a global perspective on the problem and proceeding hierarchically. An outline of the solution should be developed at a rough qualitative level and then elaborated in detail as the solution process proceeds. For example, detailed calculations or complex operations should not be performed until (i) alternatives have been explored, (ii) there is clear justification for them, and (iii) other stages of the problem solution have proceeded to the point where the results of the calculations either are necessary or will clearly prove useful. (How painful it is to expend time and energy solving a differential equation only to discover that the solution is of no real help in the "next" global phase of the problem!)

Exploration is the heuristic "heart" of the strategy; it is in the exploratory phase that most of the problem-solving heuristics come into play. Fig. 3 shows that exploration is divided into three stages. Generally, the suggestions in the first stage are either easier to employ or more likely to provide direct access to a solution of the original problem than those in the second stage; likewise for the relation between stages 2 and 3. All other factors being equal, the problem solver in the exploration phase would briefly consider those suggestions in stage 1 for plausibility, select one or more, and try to exploit it. If the strategies in stage 1 prove insufficient, one proceeds to stage 2; if need be, when stage 2 has been exhausted one tries the strategies in stage 3. If substantial progress is made at any stage, the problem solver may either return to *design* a plan for the balance of the solution, or may decide to re-enter *analysis*, with the belief that the insights gained in *exploration* can help re-cast the problem in a way not previously seen.

Implementation needs little comment, save that it should be the last step in the actual problem solution. *Verification*, on the other hand, deserves attention if only because it is so often slighted. At a local level, one can catch silly mistakes. At a global level, a review of the solution can yield alternative methods, show connections to other seemingly unrelated subject matter, and, on occasion, clarify a useful technique that then can be incorporated into one's global problem-solving approach.

6. The Instruction and Some Results. The model described in the previous section has served as the foundation for two courses in problem solving—one given to eight upper-division mathematics majors at Berkeley in 1976, and one given to nineteen (mostly) lower-division liberal arts students at Hamilton College in 1979. In each course the students were given the model as a guide to the problem-solving process. Each class session was devoted to a series of problems solvable by one (or more) of the strategies listed in Fig. 3. We would go over the solution, stressing both the use of the particular strategy and (relative to the model) how to approach the whole problem with some efficiency.

There were substantial differences between the two courses, largely because of the difference in mathematical sophistication between the two groups of students. For example, the suggestion to consider the argument "by contradiction or contrapositive" meant very different things to the two groups. For the upper-division mathematics majors, a few examples and a discussion of when it might be appropriate to consider the strategy were enough; I was essentially "pulling together" in coherent form what they had seen used as a tool in a variety of places. It was an entirely different story with the freshmen. Many of them were unconvinced that there is a need to prove things mathematically *at all*, and many had never seen an argument by contradiction! I had assigned the following as part of a take-home midterm at Berkeley.

Problem 14: Let a be a digit from 1 to 9. Which numbers of the form aaaa...a, where a is repeated n times, are perfect squares?

All but one student solved it. In contrast, it occupied us (in bits and pieces) in class for a number of days at Hamilton.

There were, however, many similarities between the two groups. Few of the students had had conscious access to any of the heuristics strategies we have been discussing. During the first class session at Berkeley, only two of the eight students succeeded in finding $\sum_{n=1}^{\infty} 1/n(n+1)$, using the telescoping series. None had thought to try values of 1,2,3,4 for n. Similarly, they did not draw diagrams where appropriate, etc. At the end of the course there was clear evidence that the students were consciously using heuristics effectively and recognizing the appropriateness of particular heuristics to particular types of problems. For example, this was on the final examination.

Problem 15: Let S be any nonempty finite set. We define $E(S)$ to be the number of subsets of S which have an even *number of elements, including the null set and possibly S. Determine $E(S)$ in closed form for any finite set S, and prove your answer.*

Seven of the eight students approached the problem by looking for an inductive pattern, a far cry from their entering behavior. (The eighth student, the only one who claimed to have seen the problem before, outlined a combinatoric argument.)

There were similar results for this strategy in the course at Hamilton. On a test at the beginning of the course, 4 students of 19 thought to calculate sums in this:

Problem 16: What is the sum of the first 89 *odd numbers?*

and some others used Gauss's pairing of terms (which they had seen before) to get the answer. On the final examination, 18 of 19 solved the following.

Problem 17: What is the sum of the coefficients of $(x+1)^{31}$?

Nothing resembling Problem 17 had been discussed in the course.

Likewise in both courses there were clear differences on the "fewer variables" strategy and on other strategies that are equally well defined. Student performance on Problem 1, which I used at the beginning of both courses, has been discussed. Problem 9 was on both final exams, and more than three-fourths of the students in each course solved it.

I should balance these "success stories" somewhat. As we saw in section 2, heuristics are subtle and students can easily go astray when trying to use them. There we saw that choosing the "right" analogous problem was not easy. Also, we "experts" have the ability to "see through" certain forms which even the more advanced undergraduates are unable to recognize. The last line in our proof of Problem 7 at Berkeley was: "Since $(a-b)^2+(b-c)^2+(c-d)^2+(d-a)^2=0$, then $a=b$, $b=c$, $c=d$, and $d=a$." Leaving this line on the board, I gave the class the following problem.

Problem 18: Let the numbers a_i and b_i be given for $i=1,2,\ldots,n$. Determine necessary and sufficient conditions on the a_i and b_i such that there exist real numbers A and B such that $(a_1x+b_1)^2+\cdots+(a_nx+b_n)^2=(Ax+B)^2$ for all x.

The morass of symbols in the second problem, including the variable x, the subscripts, and so on, obscured the similarity between the two problems. The students failed to see the essentially analogous structure that "a sum of squares equals something that is or can be made equal to zero." They were thus unable to solve the second problem.

We cannot expect students to use any heuristic in ways that go significantly beyond the way they have been shown to use it. Asked to find the number of positive integer divisors $D(N)$ of the integer N, my students had no trouble in seeing that they should calculate $D(N)$ for different values of N. When the results did not look suggestive, however, they did not think to ask, "What values of N give particular values of $D(N)$?"—which unlocks the problem. Likewise problems that call for a clever synthesis of two heuristics they have studied (like Pick's theorem, which calls for first fixing one variable and then doing an induction on the free one) will often prove

beyond the students' reach. These are not grounds for despair, but merely a call for realistic expectations.

Our reasonable expectations can actually be rather high. At least in the short term, testing before and after the course indicates some substantial progress on the part of the students. Of course, the more important question is the long-term impact of the instruction and the effect, if any, that it has on the students' performance outside the class. It's still too early to tell, but preliminary reports from students who have taken the courses have been enthusiastic and favorable.

To be perfectly honest, I should mention that a course in problem solving requires a substantial commitment from all concerned. The teacher has to be especially flexible, because it's the *process* of problem solving that counts and the teacher is essentially serving as a "coach" to the students. The students are being asked to *think*, and to create, rather than to "recite" subject matter. That's not an easy task, but it is a critically important one—and ultimately a very rewarding one, well worth the effort on the part of the students. It is also, of course, a source of tremendous gratification for the successful instructor.

Note. While this article was in press, I taught the problem-solving course again—this time with an extensive barrage of before-and-after testing and a "control group" for comparison. The results were quite dramatic and will be written up in later reports.

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THE FORMULA OF FAA DI BRUNO

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1. Introduction. Almost every calculus student is familiar with the formula of Leibniz for the nth derivative of the product of two functions

$$D^n f(t)g(t)=\sum_{k=0}^{n}\binom{n}{k}D^k f(t)D^{n-k}g(t).$$

A much less well known formula is that of Faà di Bruno for the nth derivative of the composition $f(g(t))$ (see Theorem 2). It is the purpose of this paper to give a new proof of this formula.

Several proofs of this formula have appeared in the literature. For example, in [1] there is a

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brief sketch of a proof using Taylor series. However, the omitted details are quite cumbersome. In [2] there is a proof involving the Bell polynomials. In [3] there is a proof which relies on the old-style umbral calculus developed in the mid 19th century [3], [4]. However, this technique is sometimes not mathematically rigorous and must resort to justification by other means. The umbral calculus has taken great strides in the past decade [5]–[15] and is now a completely rigorous theory. We shall use this theory to prove the formula of Faà di Bruno.

2. The Umbral Calculus. For our purposes the basic ideas of the umbral calculus may be summarized as follows. Let P be the algebra of polynomials in a single variable x over a field C, usually the real or complex numbers. Let P^* be the dual vector space of linear functionals on P. We use the notation $\langle L|p(x)\rangle$, borrowed from Physics, for the action of the linear functional L on the polynomial $p(x)$. For each nonnegative integer k we define the linear functional A^k by

$$\langle A^k|x^n\rangle = n!\,\delta_{n,k}$$

for all $n \geqslant 0$, where $\delta_{n,k}$ is the Kronecker delta function (that is, $\delta_{n,k}=1$ if $n=k$ and $\delta_{n,k}=0$ if $n\neq k$). Then A^k is extended to any polynomial by linearity. Now any linear functional on P can be expressed as a formal series in A^k. By a formal series in A^k we mean an expression of the form

$$\sum_{k=0}^{\infty} a_k A^k$$

where $a_k \in C$. A series of this form represents a well-defined linear functional if we set

$$\langle \sum_{k=0}^{\infty} a_k A^k|p(x)\rangle = \sum_{k=0}^{\infty} a_k\langle A^k|p(x)\rangle.$$

This follows because $\langle A^k|p(x)\rangle = 0$ for all but a finite number of integers k and so the sum on the right is a finite one. Now we can prove:

THEOREM 1. *If L is a linear functional on P then L can be written as*

$$L = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} A^k. \tag{1}$$

Proof. We have seen that the sum above is a well-defined linear functional. Moreover,

$$\begin{aligned}\langle \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} A^k|x^n\rangle &= \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!}\langle A^k|x^n\rangle \\ &= \langle L|x^n\rangle\end{aligned}$$

for all $n \geqslant 0$. Thus (1) holds and the proof is complete.

Theorem 1 implies that the vector space P^* is isomorphic to the vector space F of all formal power series in the variable A. But F is also an algebra (in fact an integral domain). Therefore, so is P^*. To be explicit we make P^* into an algebra by setting

$$A^k A^j = A^{k+j}.$$

Then if L and M are given in the form of equation (1) we set

$$\begin{aligned}LM &= \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} A^k \sum_{j=0}^{\infty} \frac{\langle M|x^j\rangle}{j!} A^j \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left[\sum_{k=0}^{n} \binom{n}{k}\langle L|x^k\rangle\langle M|x^{n-k}\rangle\right] A^n,\end{aligned}$$

and so we have the formula

$$\langle LM|x^n\rangle = \sum_{k=0}^{n} \binom{n}{k}\langle L|x^k\rangle\langle M|x^{n-k}\rangle. \tag{2}$$

From this formula an induction argument easily establishes the formula for a multiple product

$$\langle L_1 \cdots L_j | x^n \rangle = \sum_{k_1 + \cdots + k_j = n} \binom{n}{k_1, \ldots, k_j} \langle L_1 | x^{k_1} \rangle \cdots \langle L_j | x^{k_j} \rangle.$$

We call the algebra P^* the *umbral algebra*. Whenever we write $L = f(A)$ we shall mean that $f(A)$ is the series given in equation (1).

We need to establish only one simple fact about the umbral algebra before turning to Faà di Bruno's formula. If $L = f(A)$, then by L' or $f'(A)$ we mean the linear functional obtained by taking the formal derivative of the series $f(A)$ with respect to the variable A. Thus for example $(A^k)' = kA^{k-1}$.

LEMMA 1. *If $L = f(A)$ is a linear functional on P, then*

$$\langle f'(A) | p(x) \rangle = \langle f(A) | xp(x) \rangle$$

for all polynomials $p(x)$.

Proof. By linearity we need only establish this for $f(A) = A^k$ and $p(x) = x^n$. But then we have

$$\begin{aligned} \langle (A^k)' | x^n \rangle &= \langle kA^{k-1} | x^n \rangle \\ &= kn!\,\delta_{n,k-1} \\ &= (n+1)!\,\delta_{n+1,k} \\ &= \langle A^k | x^{n+1} \rangle \end{aligned}$$

and the proof is complete.

3. The Formula of Faà di Bruno. We are now ready for the main result.

THEOREM 2. *If $f(t)$ and $g(t)$ are functions for which all the necessary derivatives are defined, then*

$$D^n f(g(t)) = \sum \frac{n!}{k_1! \cdots k_n!} (D^k f)(g(t)) \left(\frac{Dg(t)}{1!} \right)^{k_1} \cdots \left(\frac{D^n g(t)}{n!} \right)^{k_n},$$

where $k = k_1 + \cdots + k_n$ and the sum is over all $k_1, \ldots, k_n$ for which $k_1 + 2k_2 + \cdots + nk_n = n$.

Proof. We shall follow the general lines of the proof given in [3]. Let us write

$$\begin{aligned} h(t) &= f(g(t)) \\ h_n &= D_t^n h(t) \\ g_n &= D_t^n g(t) \\ f_n &= D_u^n f(u)|_{u=g(t)}. \end{aligned}$$

Then

$$h_1 = D_t h(t) = D_u f(u)|_{u=g(t)} D_t g(t) = f_1 g_1,$$

and similarly

$$\begin{aligned} h_2 &= f_1 g_2 + f_1 g_1^2 \\ h_3 &= f_1 g_3 + f_2 3 g_1 g_2 + f_1 g_1^3. \end{aligned}$$

It is easily established by induction that h_n has the form

$$h_n = \sum_{k=1}^{n} f_k l_{n,k}(g_1, \ldots, g_n) \tag{3}$$

where $l_{n,k}(g_1, \ldots, g_n)$ does not depend on any of the functions f_j. Now, since we wish only to determine $l_{n,k}(g_1, \ldots, g_n)$, we are free to choose $f(t)$ arbitrarily. Let us take $f(t) = e^{at}$ where a is an arbitrary constant. Then

$$f_k = D_u^k f(u)|_{u=g(t)} = a^k e^{ag(t)} \tag{4}$$

and

$$h_n = D_t^n e^{ag(t)}. \tag{5}$$

Substituting (4) and (5) into (3) and multiplying by $e^{-ag(t)}$ gives

$$e^{-ag(t)}D_t^n e^{ag(t)} = \sum_{k=1}^{n} a^k l_{n,k}(g_1,\ldots,g_n).$$

If we set $B_n(t)=e^{-ag(t)}D_t^n e^{ag(t)}$, then for $n \geqslant 1$ we have

$$\begin{aligned} B_n(t) &= e^{-ag(t)}D_t^{n-1}ag_1(t)e^{ag(t)} \\ &= ae^{-ag(t)}\sum_{k=0}^{n-1}\binom{n-1}{k}g_{k+1}(t)D_t^{n-k-1}e^{ag(t)} \\ &= a\sum_{k=0}^{n-1}\binom{n-1}{k}g_{k+1}(t)B_{n-k-1}(t) \end{aligned} \tag{6}$$

where we have used Leibniz's formula for the second equality. Now we may think of t as being fixed; write $B_n(t)=B_n$ and $g_n(t)=g_n$ and define two linear functionals L and M on P by

$$\langle L|x^n\rangle = B_n$$
$$\langle M|x^n\rangle = g_n.$$

Notice that $\langle L|1\rangle = B_0 = 1$, $\langle M|1\rangle = g_0 = g$ and

$$L = \sum_{k=0}^{\infty}\frac{B_k}{k!}A^k$$
$$M = \sum_{k=0}^{\infty}\frac{g_k}{k!}A^k.$$

Equation (6) now becomes, by virtue of equation (2),

$$\begin{aligned} \langle L|x^n\rangle &= a\sum_{k=0}^{n-1}\binom{n-1}{k}\langle M|x^{k+1}\rangle\langle L|x^{n-1-k}\rangle \\ &= a\sum_{k=0}^{n-1}\binom{n-1}{k}\langle M'|x^k\rangle\langle L|x^{n-1-k}\rangle \\ &= a\langle M'L|x^{n-1}\rangle \end{aligned}$$

and so

$$\langle L'|x^{n-1}\rangle = a\langle M'L|x^{n-1}\rangle.$$

Since this holds for all $n \geqslant 1$, we conclude that

$$L' = aM'L. \tag{7}$$

This formal differential equation is easily solved. The linear functional $F(A)=e^{a(M-g_0)}$ clearly satisfies (7) and if $G(A)$ also satisfies (7) then $F(A)/G(A)$ has derivative equal to zero and is therefore a constant. Hence all solutions are of the form

$$L = ce^{a(M-g_0)}$$

where c is a constant. In order to determine c we consider the initial condition

$$1 = B_0 = \langle L|1\rangle = \langle ce^{a(M-g_0)}|1\rangle = c$$

and so

$$L = e^{a(M-g_0)}.$$

Thus

$$B_n = \langle L|x^n\rangle$$

$$
\begin{aligned}
&=\langle e^{a(M-g_0)}|x^n\rangle\\
&=\sum_{k=0}^{\infty}\frac{a_k}{k!}\langle (M-g_0)^k|x^n\rangle\\
&=\sum_{k=0}^{\infty}\frac{a_k}{k!}\sum_{j_1+\cdots+j_k=n}\binom{n}{j_1,\ldots,j_k}\langle M-g_0|x^{j_1}\rangle\cdots\langle M-g_0|x^{j_k}\rangle\\
&=\sum_{k=0}^{n}\frac{a_k}{k!}\sum_{\substack{j_1+\cdots+j_k=n\\ j_i\geq 1}}\binom{n}{j_1,\ldots,j_k}g_{j_1}\cdots g_{j_k}
\end{aligned}
$$

and so equating coefficients of a^k in the two expressions for B_n gives

$$
\begin{aligned}
l_{n,k}(g_1,\ldots,g_n)&=\frac{n!}{k!}\sum_{\substack{j_1+\cdots+j_k=n\\ j_i\geq 1}}\left(\frac{g_{j_1}}{j_1!}\right)\cdots\left(\frac{g_{j_k}}{j_k!}\right)\\
&=\frac{n!}{k!}\sum\binom{k}{k_1,\ldots,k_n}\left(\frac{g_1}{1!}\right)^{k_1}\cdots\left(\frac{g_n}{n!}\right)^{k_n}
\end{aligned}
$$

where the last sum is over all $k_1,\ldots,k_n$ for which $k_1+\cdots+k_n=k$ and $k_1+2k_2+\cdots+nk_n=n$. Finally,

$$
\begin{aligned}
h_n(t)&=\sum_{k=1}^{n}f_k l_{n,k}(g_1,\ldots,g_n)\\
&=\sum_{k=1}^{n}f_k\sum\frac{n!}{k_1!\cdots k_n!}\left(\frac{g_1}{1!}\right)^{k_1}\cdots\left(\frac{g_n}{n!}\right)^{k_n}\\
&=\sum\frac{n!}{k_1!\cdots k_n!}f_k\left(\frac{g_1}{1!}\right)^{k_1}\cdots\left(\frac{g_n}{n!}\right)^{k_n}
\end{aligned}
$$

where $k=k_1+\cdots+k_n$ and the last sum is over all $k_1,\ldots,k_n$ for which $k_1+2k_2+\cdots+nk_n=n$. This is the desired formula and the proof is complete.

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MATHEMATICAL NOTES

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BURNSIDE'S THEOREM ON ALGEBRAS OF MATRICES

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BURNSIDE'S THEOREM. *Let $\mathcal{V}$ be a finite-dimensional vector space of dimension larger than* 1 *over an algebraically closed field. If $\mathcal{A}$ is an algebra of linear transformations on $\mathcal{V}$ such that the only subspaces invariant under all the members of $\mathcal{A}$ are $\{0\}$ and $\mathcal{V}$, then $\mathcal{A}$ contains every linear transformation on $\mathcal{V}$.*

The following proof of Burnsides's Theorem, suggested by a reading of [**2**, Lemma 1], appears to be shorter and simpler than the standard proofs (cf. [**1**, p. 276]) (although nothing in it is very new).

Note that $x \neq 0$ implies $\{Ax: A \in \mathcal{A}\} = \mathcal{V}$.

We begin by showing that $\mathcal{A}$ contains some T_0 of rank 1. To see this, choose $T_0 \in \mathcal{A}$ with minimal nonzero rank, say d. If $d > 1$, choose x_1 and x_2 so that $\{T_0x_1, T_0x_2\}$ is linearly independent, and then choose $A \in \mathcal{A}$ so that $AT_0x_1 = x_2$. Then $T_0AT_0x_1$ and T_0x_1 are linearly independent, and $T_0AT_0 - \lambda T_0$ is different from 0 for all scalars λ. Since the field of scalars is algebraically closed, there is a λ_0 such that the restriction of $T_0A - \lambda_0$ to $T_0\mathcal{V}$ is not invertible. Then $(T_0A - \lambda_0)T_0$ has rank less than d and greater than 0, contradicting the minimality of d. Hence $d = 1$.

Every linear transformation on $\mathcal{V}$ is a sum of linear transformations of rank 1; so to finish the proof of Burnside's Theorem it suffices to show that $\mathcal{A}$ contains all such, that is, that $\mathcal{A}$ contains each nonzero linear transformation T of the form $Tx = \phi(x)y$ for $y \in \mathcal{V}$ and ϕ a linear functional on $\mathcal{V}$. We know that $\mathcal{A}$ contains at least one such T, say $T_0x = \phi_0(x)y_0$. Now the set of all ϕ such that the operator T defined by $Tx = \phi(x)y_0$ is in $\mathcal{A}$ includes $\phi_0(Ax)$ for each $A \in \mathcal{A}$ (since $T_0A \in \mathcal{A}$); so there is no nonzero vector x that is annihilated by all such ϕ. Thus every T of the form $Tx = \phi(x)y_0$ for some ϕ is in $\mathcal{A}$. Now, given any y and any ϕ, if $Tx = \phi(x)y_0$ and if $A \in \mathcal{A}$ is chosen so that $Ay_0 = y$, then $ATx = \phi(x)y$ is shown to be in $\mathcal{A}$. Thus $\mathcal{A}$ contains every transformation of rank 1 and the proof is complete.

Some infinite-dimensional analogues of Burnside's Theorem can be found in [**3**, Chapter 8].

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THE COUNTABILITY OF THE RATIONAL POLYNOMIALS: A DIRECT METHOD

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Traditionally, the countability of the set of polynomials with rational coefficients has been demonstrated by using Cantor's first diagonal method repeatedly to show the countability of

zeroth order polynomials, then first order ones, and so on. A proof will be given in this note in which the rational polynomials are placed in a one-to-one correspondence with a subset of the integers, using the technique of Gödel numbering [1].

Consider the general rational polynomial

$$r_0x^0+r_1x^1+\cdots+r_nx^n, r_n\neq 0.$$

We first observe that this polynomial may be expressed in the form

$$\frac{1}{d}((-1)^{s_0}a_0x^0+(-1)^{s_1}a_1x^1+\cdots+(-1)^{s_n}a_nx^n).$$

In this expression, n, the order of the polynomial, and d, the least common denominator of the r_i's, are both positive integers, each s_i is zero or one, and each a_i is a nonnegative integer. Now let s be the number

$$s_02^0+s_12^1+\cdots+s_n2^n.$$

Note that s is also a nonnegative integer. Also, let p_i denote the ith prime number, beginning with $p_1=2$.

From these definitions, we can produce the number

$$p_1^s\cdot p_2^d\cdot p_3^{a_0}\cdot p_4^{a_1}\cdot\cdots\cdot p_{n+3}^{a_n}.$$

Since s, d, and the a_i uniquely specify the polynomial, no other polynomial produces this set of factors. Furthermore, the unique decomposition theorem assures us that this is the only set of factors for this integer. Thus, a unique integer is associated with each rational polynomial and the set of such polynomials is countable.

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COMMENT ON THE NOTE: "THE CONGRUENCE $a^{r+s}\equiv a^r \pmod m$" BY A. E. LIVINGSTON AND M. L. LIVINGSTON

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1. Introduction. In "The Congruence $a^{r+s}\equiv a^r \pmod m$" [1] the authors wish to obtain "the ideal generalization of the Euler-Fermat theorem." By means of [6] it is possible to get all lemmas and theorems of [1] in a straightforward, simple manner. In the next section we give the ideal generalization of the Euler-Fermat theorem for finite semigroups. This will be applied in the third part of this paper to get a short proof of the main results in [1]. We conclude our note with some hints concerning a further generalization of the concepts developed in Sections 2 and 3.

2. The Euler-Fermat Theorem in Finite Semigroups. *Let S be a finite semigroup and let a be an element of S.*

It is well known (see [6, p. 48] or [2, p. 106]) that the cyclic semigroup S_a generated by a has the following structure:

$$a,a^2,\ldots,a^h,\ldots,a^{h+d-1},a^{h+d},\ldots$$

$$a^h=a^{h+d}(h>0,\ d>0) \tag{1}$$

where h, d are the least positive integers such that (1) holds. The ordered pair (h,d) of positive integers is called the *type* of S_a. We can easily derive the following lemma.

LEMMA 1. *Let r and s be positive integers and let S_a be of type (h,d); then*

$$a^{r+s}=a^r$$

if and only if $r \geqslant h$ *and* $d|s$.

Definition 1. If S is a finite semigroup, we define the integers H, D in the following way:

$$H=\max\{h_a|a\in S\},$$
$$D=\text{l.c.m.}\{d_a|a\in S\}.$$

Theorem 1. *Let S be a finite semigroup and H, D as in Definition* 1. *Then*

$$a^{r+s}=a^r$$

for all $a\in S$ *if and only if* $r\geqslant H$ *and* $D|s$.

Proof. Take any $a\in S$; then $r\geqslant H$ and $D|s$ imply $r\geqslant h_a$ and $d_a|s$. Therefore, by Lemma 1, $a^{r+s}=a^r$. If $a^{r+s}=a^r$ is true for all $a\in S$, Lemma 1 gives $r\geqslant h_a$ and $d_a|s$ for all $a\in S$. But this means $r\geqslant H$ and $D|s$.

Thus, for finite semigroups, the ideal generalization of the Euler-Fermat theorem is given by the following corollary.

Corollary 1.1. *Let S be a finite semigroup and let H and D be as above. Then*

$$a^{H+D}=a^H$$

for all a in S.

A fairly simple example is $S=G$, where G is a finite group. We know G_a is of type $(1,d_a)$ for all $a\in G$. From group theory we get $d_a\|G|$, where $|G|$ denotes the order of G. That means $D\|G|$, and Theorem 1 gives $a=a^{|G|+1}$ for all $a\in G$. If, in particular, G is a finite abelian group, we define $\exp G$, the exponent of G, to be the maximum of the orders for all $a\in G$. Then it is known that $d_a|\exp G$ for all $a\in G$, and it follows that $D=\exp G$. The Euler-Fermat theorem in a finite abelian group becomes

$$a=a^{\exp(G+1)}$$

for all $a\in G$.

3. The Multiplicative Semigroup Z_m. The problem to be solved is: compute H and D if $S=Z_m$ is the multiplicative semigroup of integers modulo m $(m>0)$. Following [6] we define:

Definition 2. Let $a\in Z$, and let $m=m_1\cdot m_2$, with $(m_1,m_2)=1$, and where each prime divisor of m that divides one of the pair a, m_1 divides the other also. In consequence of this, $(a,m_2)=1$. *The nildegree* n_a *of a modulo* m_1 is the least positive integer n such that $a^n\equiv 0 \pmod{m_1}$. *The order* t_a *of a modulo* m_2 is the least positive integer t with $a^t\equiv 1 \pmod{m_2}$. If $(a,m)=1$, then $n_a=1$ and t_a is the order of a modulo m. If $m_1=m$, then $t_a=1$ and n_a is the nildegree of a modulo m. If $a\equiv 0 \pmod m$, this means $n_a=t_a=n_0=t_0=1$.

Such a decomposition of m certainly always exists. Let $\bar a$ denote the residue class of $a\in Z$ modulo m.

Theorem 2. *Let* $a\in Z$ *and* $m=m_1\cdot m_2$ *as in Definition* 1; *then* $S_{\bar a}$ *is of type* $(h_{\bar a},d_{\bar a})=(n_a,t_a)$.

Proof. If $(a,m)>1$, this is just Theorem 8.2.3 in [6, p. 49]. But $(a,m)=1$ means $\bar a$ generates in Z_m a cyclic group of order t_a; that is, a semigroup $S_{\bar a}$ of type $(h_{\bar a},d_{\bar a})=(1,d_{\bar a})=(1,t_a)=(n_a,t_a)$. This completes the proof.

Lemma 2. *If* $S=Z_m$ $(m>0)$, *then* $H=N$ *and* $D=T$ *where*

$$N=\max\{n_i|i=0,1,\ldots,m-1\},$$
$$T=\text{l.c.m.}\{t_i|i=0,1,\ldots,m-1\}=\lambda(m),$$

where H, D are defined as in Definition 1, n_i, t_i $(i=0,1,\ldots,m-1)$ *are as in Definition* 2, *and* $\lambda(m)$ *denotes the "universal exponent" of* m (*also called the Carmichael-function*).

Proof. Of course $Z_m=\{\bar{0},\bar{1},\bar{2},\ldots,\overline{m-1}\}$; but then from Definition 1 and Theorem 2 we get $H=N$ and $D=T=\lambda(m)$ since $\lambda(m)$ is exp U_m where $U_m=\{\bar{a}|\bar{a}\in Z_m$ and $(a,m)=1\}$.

THEOREM 3. *If Z_m is the finite multiplicative semigroup of integers modulo m $(m>0)$, then*

$$\bar{a}^{r+s}=\bar{a}^r$$

for all $\bar{a}\in Z_m$, if and only if $r\geqslant N$ and $\lambda(m)|s$.

Proof. This is only a special case of Theorem 1 if we take $S=Z_m$.

COROLLARY 3.1 (Generalized Euler-Fermat theorem).

$$\bar{a}^{N+\lambda(m)}=\bar{a}^N$$

for all $\bar{a}\in Z_m$. Alternatively, in terms of congruences,

$$a^{N+\lambda(m)}\equiv a^N(\mathrm{mod}\ m)$$

for all $a\in Z$.

COROLLARY 3.2. *If r, s, and m are positive integers, then*

$$a^{r+s}\equiv a^r\ (\mathrm{mod}\ m)\qquad (m>0) \tag{2}$$

for all $a\ \in Z$ if and only if $r\geqslant N$, $\lambda(m)|s$.

COROLLARY 3.3. *If m is a positive integer and r is a nonnegative integer, then*

$$a^{r+\phi(m)}\equiv a^r(\mathrm{mod}\ m)\qquad (m>0) \tag{3}$$

for all integers a if and only if m is $(r+1)$th power free.

Proof. At first we define:

$$\epsilon(m)=\max\{1\}\cup\{r|p^r\|m\}$$

(see [1, p. 100]). It is easy to prove $N=\epsilon(m)$ and (2) may be rewritten: $a^{r+s}\equiv a^r(\mathrm{mod}\ m)$ for all $a\in Z$ if and only if $r\geqslant\epsilon(m)$, $\lambda(m)|s$. But $\lambda(m)|\phi(m)$ is always fulfilled and $r\geqslant\epsilon(m)$ means the same as saying m is $(r+1)$th power free. For reasons of symmetry "l.c.m." in the definition of T (but not of D!) could be replaced by "max":

$$T=\max\{n_i|i=0,1,\ldots,m-1\}.$$

4. Generalizations and Remarks. Given a finite semigroup it is tedious to compute H and D. For example, the set of all binary relations B_Ω on a finite set Ω becomes a semigroup when the usual multiplication of binary relations is introduced. (If $|\Omega|=n$ this is the semigroup of all $n\times n$ Boolean matrices.) This semigroup was studied by S. Schwarz in a series of papers (see [4]). In a short communication given at the ICM 78 at Helsinki [5] he announces H and D to be computed for that semigroup and gives results concerning other types of semigroups.

The approach taken in Section 3 of this paper can be generalized to a class of rings containing Z_m. Let R be a principal ideal domain; if then (m) $(m\in R)$ is an ideal of R with $\bar{R}=R/(m)$ a finite ring, we can get an Euler-Fermat theorem that looks similar to (2). $\lambda(m)$ must be replaced by T and the definitions of N, T have to be modified.

A ring of special interest is the polynomial ring $K[x]$, where K is a finite field. Instead of a, m we take $a(x)$, $m(x)\in K[x]$, and if ϕ denotes the generalized ϕ-function of Dedekind (see [3, p. 33 Theorem II.19]), then (3) holds. What about an Euler-Fermat theorem in infinite semigroups? First of all, this must be a kind of semigroup that has all its cyclic subsemigroups finite. Such semigroups are called "periodic" and are known (see [2, Chap. 3]; [4, p. 113]). Second, there must be a fixed upper bound u such that $h_a,d_a\leqslant u$ for all $a\in S$, where S is a periodic semigroup.

If $S=G$ is an abelian group, we have defined abelian groups of bounded order. Their structure is known; they are direct sums of finite cyclic groups. We may now introduce (periodic) semigroups of bounded order. I don't know whether there are any results on such semigroups.

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UNSOLVED PROBLEMS

EDITED BY RICHARD GUY

In this department the MONTHLY *presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada T2N 1N4.*

OVERLAPPING CONGRUENT CONVEX BODIES

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Let R and R' be congruent closed rectangular regions in the plane whose interiors intersect. Must the ratio

$$\frac{\text{length}(\partial R\cap R')}{\text{length}(\partial R'\cap R)} \tag{1}$$

always lie between $1/3$ and 3? (Here ∂ is the function which associates with any region its boundary.) To put the problem another way, must at least a fourth of the length of the polygon $\partial(R\cap R')$ come from ∂R, and at least a fourth from $\partial R'$?

Reversing the roles of R and R' inverts the fraction (1), so it is only necessary to consider, say, upper bounds. No number smaller than 3 will always bound (1) from above, as is shown in Fig. 1.

Rather than requiring R and R' to be congruent, it is probably only necessary to require them to have the same width. Generalizing to $\mathbb{R}^n$, we make the following conjecture.

CONJECTURE. *Let R and R' be closed rectangular parallelepipeds in $\mathbb{R}^n$ whose interiors intersect. Suppose that the shortest edge of R and the shortest edge of R' have the same length. Then*

$$\frac{\lambda_{n-1}(\partial R\cap R')}{\lambda_{n-1}(\partial R'\cap R)}<2n-1 \tag{2}$$

(λ_{n-1} is $(n-1)$-dimensional measure; for example λ_2 is surface area.)

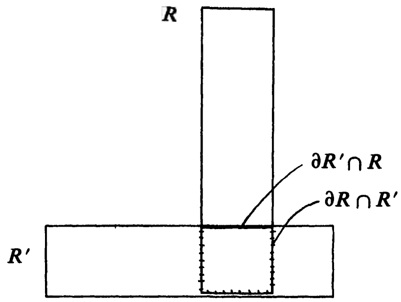

FIG. 1

The example of Fig. 1 generalizes to show that no number smaller than $2n-1$ will work in (2).

We have one positive result in $\mathbb{R}^2$: It is easy to show that the ratio (1) is always less than 4. Consider all possibilities for the number of sides of the polygon $\partial(R\cap R')$ and how they are divided between ∂R and $\partial R'$ (Fig. 2). In each case elementary geometry shows that the length of each segment of $\partial R\cap R'$ is less than the total length of $\partial R'\cap R$. Since $\partial R\cap R'$ has at most four segments, the conclusion follows.

We can consider sets more general than parallelepipeds. For an arbitrary compact convex subset R of $\mathbb{R}^n$ we define $p(R)$, the *pointedness* of R, by the formula

$$p(R)=\sup_{R'}\frac{\lambda_{n-1}(\partial R\cap R')}{\lambda_{n-1}(\partial R'\cap R)},$$

where the sup is taken over all R' congruent to R, with the interiors of R and R' intersecting.

The pointedness of a ball is 1. It would be interesting to have some other cases worked out. If R is a triangular region in the plane, we guess that $p(R)=1/\sin(\theta/2)$, where θ is the smallest angle of R.

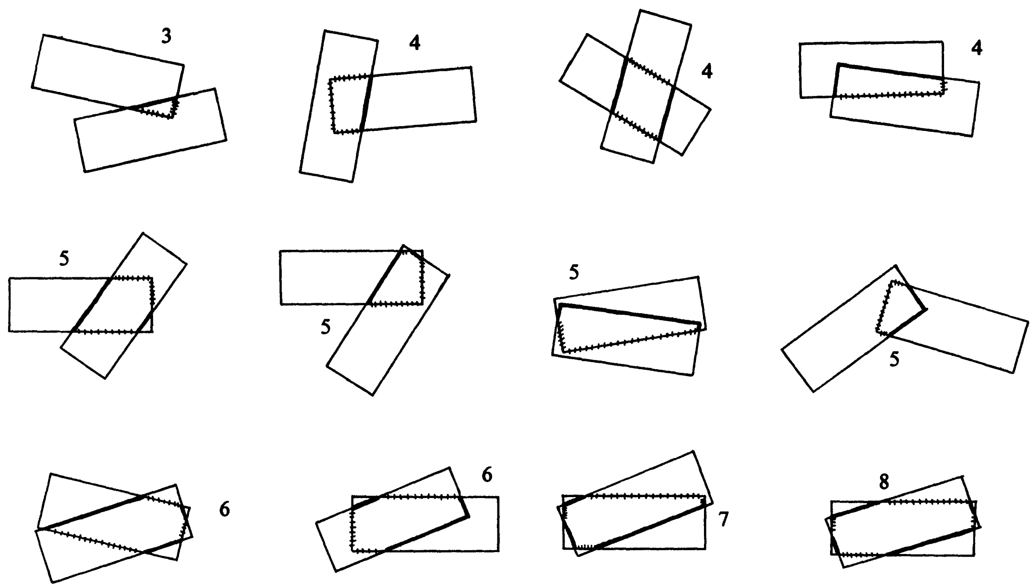

FIG. 2

EDITED BY DEBORAH TEPPER HAIMO AND FRANKLIN TEPPER HAIMO

Material for this department should be sent to Professor Deborah Tepper Haimo, Department of Mathematical Sciences, University of Missouri, St. Louis, MO 63121.

FUNCTIONS WITH ARBITRARILY SMALL PERIODS

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Recently R. Cignoli and J. Hounie [2] gave a new proof, together with applications, of Burstin's Theorem: *A Lebesgue-measurable function $f:R\to R$ having arbitrarily small periods is constant a.e.* The following is a more direct, self-contained proof.

Let I be any closed interval, and let $D=f^{-1}(I)$. Then the measure of D intersected with any interval depends linearly on the length of the interval. To see this, let $\alpha=m(D\cap[0,1])$ and suppose we are given $\varepsilon>0$ and $a<b$. Choose a period p of f so that $p<\varepsilon$ and $|m/n-(b-a)|<\varepsilon$, where $n=[1/p]$ and $m=[(b-a)/p]$. Since p is a period of f, the measure of D intersected with any interval of length p is the same. Thus if $d=m(D\cap[0,p])$, then $\alpha=m(D\cap[0,1])=nd+\varepsilon_1$ and $m(D\cap(a,b))=md+\varepsilon_2$, with $\varepsilon_1,\varepsilon_2<\varepsilon$. We then have:

$$\begin{aligned}|m(D\cap(a,b))-\alpha(b-a)|&=\left|nd\left(\frac{m}{n}\right)+\varepsilon_2-(nd+\varepsilon_1)(b-a)\right|\\&=\left|nd\left(\frac{m}{n}-(b-a)\right)+\varepsilon_2-\varepsilon_1(b-a)\right|<\alpha\varepsilon+\varepsilon_2+\varepsilon_1(b-a);\end{aligned}$$

hence $m(D\cap(a,b))=\alpha(b-a)$.

The theorem results from the following lemma.

LEMMA. *If the measure of a set D intersected with any interval depends linearly on its length, then either $m(D)=0$ or $m(D^c)=0$.*

Using this, for each $n>0$, let k_n, $I_n=[k_n/n,(k_n+1)/n]$ be such that $f^{-1}(I_n)$ is not of measure 0. By the lemma, $m(f^{-1}(I_n)^c)=0$. Also, $\cap_{n<\omega}I_n$ is not empty, since

$$m\left(f^{-1}\left(\left(\bigcap_{n<\omega}I_n\right)^c\right)\right)=m\left(f^{-1}\left(\bigcup_{n<\omega}I_n^c\right)\right)=m\left(\bigcup_{n<\omega}f^{-1}(I_n)^c\right)=0.$$

Since there can be no more than one point q in $\cap_{n<\omega}I_n$, $0=m(f^{-1}(\{q\})^c)$ implies $f(x)=q$ a.e.

The lemma is proved as follows: let $\alpha=m(D\cap[0,1])$. Given $\varepsilon>0$, cover $D\cap[0,1]$ with open intervals O_n so that $\Sigma_n m(O_n)<\alpha+\varepsilon$. Since $m(O_n\cap D)=\alpha m(O_n)$, we have

$$\alpha=m(D\cap[0,1])\leqslant\sum_n m(D\cap O_n)\leqslant\alpha\sum_n m(O_n)<\alpha^2+\alpha\varepsilon.$$

As ε is arbitrary, this shows $\alpha\leqslant\alpha^2$, and so $\alpha=0$ or 1. If $\alpha=0$, $m(D)=0$; and if $\alpha=1$, $m(D^c)=0$.

Another proof depends on the well-known principle that a set that covers at most a fixed fraction of every interval covers almost none of every interval.

I am informed that A. B. Novikoff has found that Burstin's original proof is incorrect.

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We have seen that the sums of rearrangements of the alternating harmonic series depend only on the asymptotic density α. This behavior is in some sense specific to series like the harmonic series, as Theorem 2 indicates. Readers are invited to construct proofs for themselves or to consult Pringsheim's paper [3].

THEOREM 2 [3]. *Suppose* $\{a_n\}_{n=1}^{\infty}$ *is a sequence of real numbers such that* $|a_1| \geqslant |a_2| \geqslant |a_3| \geqslant \cdots$, $\lim_{n\to\infty} a_n = 0$, *and* $a_{2k-1} > 0 > a_{2k}$ *for* $k = 1,2,3,\ldots$.

(i) *If* $\lim_{n\to\infty} n|a_n| = \infty$, *and if* S *is a real number, there is a simple rearrangement of the series* $\sum_{k=1}^{\infty} a_k$ *with asymptotic density* $\frac{1}{2}$ *whose sum is* S.

(ii) *If* $\lim_{n\to\infty} na_n = 0$, *if* $\sum_{k=1}^{\infty} b_k$ *is a simple rearrangement of the series* $\sum_{k=1}^{\infty} a_k$ *for which the asymptotic density* α *exists, and if* $0 < \alpha < 1$, *then* $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} a_k$.

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A STRONG CONVERSE TO GAUSS'S MEAN-VALUE THEOREM

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The theorem of Gauss in the title affirms that

$$h(a) = \frac{1}{2\pi}\int_0^{2\pi} h(a + re^{i\theta})\,d\theta \tag{1}$$

holds for all a in a region Ω, all $r > 0$ such that the closure of the disc $D(a,r) = \{z \in \mathbb{C}: |z-a| < r\}$ lies in Ω, and all functions h that are harmonic throughout Ω. Most books on function theory or potential theory prove this elementary result as well as the following converse due to Koebe [4]: *If* h *is continuous in the region* Ω *and* (1) *holds for all* a *and* r *such that* $\overline{D}(a,r) \subset \Omega$, *then* h *is harmonic in* Ω. In fact, the somewhat stronger version in which the equality is required to hold only at each a for some sequence $r_n(a) \to 0$ is often proved. What does not seem to be well known is that, when h is continuous on $\overline{\Omega}$, one radius suffices. This strong converse of Gauss's theorem is due to Kellogg [3] and is not trivial. However, for Dirichlet regions this strong converse is as easy to prove as Koebe's theorem and should be presented in elementary texts. The theorem for Dirichlet regions is due to Volterra [7] (with a supplemental hypothesis) and to Vitali [6] (where the supplemental hypothesis is removed). Here is their proof in modern dress, presented in dimension two, though the reader will see that it is valid in any dimension.

LEMMA. *Let* U *be a bounded open subset of the complex plane* $\mathbb{C}$ *and let* $f: \overline{U} \to \mathbb{R}$ *be continuous and for each* $a \in U$ *have the following* restricted mean-value property:

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{i\theta})\,d\theta \text{ for some } r = r(a) > 0 \text{ such that } \overline{D}(a,r) \subset U. \tag{2}$$

Then $\max f(\overline{U}) = \max f(\partial U)$.

Proof. (Cf. Cimmino [1]) Let $M = \max f(\overline{U})$. It suffices to see that the closed subset $f^{-1}(M)$ of

$\overline{U}$ meets ∂U. If this is not the case, then by compactness there is a point $a \in f^{-1}(M) \subset U$ that is nearest to ∂U. From

$$M = f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + r(a)e^{i\theta})\,d\theta$$

and the maximality of M we see that $f(a + r(a)e^{i\theta}) = M$ for each $\theta \in [0, 2\pi]$. That is, the circle $\{z \in \mathbb{C}: |z - a| = r(a)\}$ lies wholly in $f^{-1}(M)$. Since, obviously, some point of this circle is closer to ∂U than a, we have a contradiction to the choice of a.

THEOREM. *Let U be a bounded open subset of $\mathbb{C}$ for which the Dirichlet problem is solvable. Then any continuous real-valued function on $\overline{U}$ that has the restricted mean-value property is harmonic in U.*

Proof. Let g be such a function. Since the Dirichlet problem is solvable for U, there exists a continuous function h on $\overline{U}$ that is harmonic in U and coincides with g on ∂U. Since h has the (unrestricted) mean-value property (by Gauss's theorem), the functions $f_1 = g - h$, $f_2 = h - g$ each satisfy the hypotheses of the lemma. Since $f_1 = f_2 = 0$ on ∂U, we infer from the lemma that $f_1 \leqslant 0$, $f_2 \leqslant 0$ throughout U. That is, $g \equiv h$.

FINAL REMARKS. A version of this proof occurs in Courant and Hilbert [**2**, pp. 279–281]. There the reader will also find elementary examples showing that both the boundedness of U and the continuity of h on $\overline{U}$ are essential to the validity of the theorem. There are, however, some fascinating versions of the theorem for the cases $U = \mathbb{C}$ or h only continuous on U. For discussions of these results and extensive references to the literature see Netuka [**5**] and Zalcman [**8**], [**9**]. The latter paper is especially readable.

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3. O. D. Kellogg, Converses of Gauss' theorem on the arithmetic mean, Trans. Amer. Math. Soc., 36 (1934) 227–242; Jbuch 60, p. 430; Zbl 9, p. 112.

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9. ______, Offbeat integral geometry, this MONTHLY, 87 (1980) 161–175.

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MATHEMATICAL EDUCATION

EDITED BY W. E. MASTROCOLA

Material for this department should be sent to Robert F. Wardrop, Department of Mathematics, Central Michigan University, Mount Pleasant, MI 48859.

MATHEMATICS BEYOND THE CLASSROOM: THE NEWSLETTER APPROACH

SABRA S. ANDERSON

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A common weakness in the mathematics education of undergraduates on large, predominantly commuter campuses is the lack of mathematical involvement outside the classroom. Although opportunities abound, they are notoriously underutilized. How do you communicate with students who disappear into the corridor crush at the end of each class? To increase the awareness and involvement of our students in mathematical activities and also to disseminate information, we have established a weekly newsletter. While the format is not unique, the other examples we know of are in small, private, residential colleges. This note describes our experience in publishing the newsletter and suggests that other departments might profit from a similar undertaking.

Our newsletter (*Hyperbole of a Single Sheet*) is published weekly during the academic year. It contains announcements, a calendar of the week's mathematical events, a "Problem of the Week," and a potpourri of other mathematical items—news, book reviews, history, humor, employment information.

The announcements are the primary reason for the newsletter's existence. In the past we have relied on posters, bulletin boards, and announcements read in classes to distribute routine items, such as club meetings, lectures, changes in the published course schedule, common examination locations and times, and job openings. The newsletter is a more effective way to furnish students with this information. At the same time, we give added publicity and visibility to the activities of the Math Club and the Computer Club. Each quarter we encourage new mathematics students to take advantage of our (free) tutoring service. We advertise mathematics-related employment opportunities for students on campus, such as assistance on research projects and jobs in the Computer Center, and announce the visits of recruiters from prospective employers. We promote the weekly colloquium by including a short description of the topic and its mathematical prerequisites.

The informational items seldom fill up the available space. We use the remainder for sharing mathematical tidbits that rarely find their way into the classroom. Birthdays of famous mathematicians are noted, accompanied by short biographies. We report on mathematical news, such as the selection of the Fields medal and the Cole prize winners and the discovery by undergraduates (Nickel and Noll) of a new prime. We include reviews of books an undergraduate might read for pleasure, such as Constance Reid's biographies and S. Ulam's autobiography. We have news items on the employment and salary outlook for graduates in the mathematical sciences.

We highlight special activities of students and faculty members, such as a high standing in the Putnam competition or the publication of a paper. Our departmental requirement that every major complete an individual project has always been a great mystery to students who are (understandably) baffled by what is expected and are inclined to delay the project until the last minute. We have begun to publish reports of completed projects in an attempt to illustrate the nature of acceptable projects.

There have been some unexpected benefits. Each year we poll our readership to monitor the

newsletter's effectiveness. The first year we discovered that we had an unanticipated audience among extension students who take courses only in the evenings. We previously had virtually no communication with these students. They now feel less isolated. The second year we had a "Name in the News" contest, resulting in a multitude of clever titles. We have discovered that printing the names of the people who solve the problem of the week provides an incentive for others to work on these problems. Requests for copies of the newsletter have expanded our circulation to other segments of the campus.

There are three other newsletters in the North Central Section of the Mathematical Association of America, one annual and two weekly. Sharing newsletters within the Section has not only kept us in touch with our colleagues but has also led to joint ventures, such as the sponsoring of outside speakers and the interchange of local colloquium speakers.

Publishing a weekly newsletter does represent an expense both in time and money. Is the pay-off worth the price? We have tried to keep the costs low. The newsletter is brief (two sides of one page), the style is informal, and department members are generous in their contributions of announcements, items, and suggestions. Editing takes an hour or two per week. Secretarial work requires another one or two hours. The benefits certainly justify a cost of this magnitude.

PROBLEMS AND SOLUTIONS

EDITED BY VLADIMIR DROBOT

*Send all **proposed** problems, in duplicate if possible, to Professor Vladimir Drobot, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053. Please include solutions, relevant references, etc.*

An asterisk (*) indicates that neither the proposer nor the editors supplied a solution.

***Solutions** should be sent to the addresses given at the head of each problem set.*

A publishable solution must, above all, be correct. Given correctness, elegance and conciseness are preferred. The answer to the problem should appear right at the beginning. If your method yields a more general result, so much the better. If you discover that a MONTHLY *problem has already been solved in the literature, you should of course tell the editors; include a copy of the solution if you can.*

SOLUTIONS OF PROBLEMS DEDICATED TO EMORY P. STARKE

Switching the Stairway Light Switches

S 17 [1979, 591]. *Proposed by Leonard Gillman, University of Texas, Austin.*

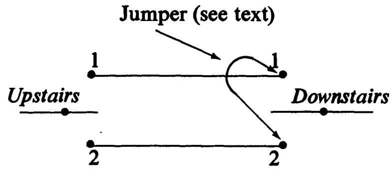
When the upstairs switch is in one position, the downstairs switch turns the stairway light on or off as it should, but when the upstairs switch is in the other position the stairway light remains off irrespective of the position of the downstairs switch. Which is the defective switch?

I. *Solution by Edward T. Ordman, New England College.* Set the switches so the light is on. The condition may be rephrased as: When the switches are in this position, the light is on, but if either or both switches are changed the light will be off. Stated this way, the condition is

symmetrical with regard to the two switches. Thus, no conclusion as to which switch is at fault can be drawn.

II. *Solution by Clayton W. Dodge, University of Maine at Orono.* In the usual wiring of 3-way switches to control a light from two locations the line feeds into the switch and is connected to terminal 1 when the switch is in one position and to terminal 2 when the switch is in the other position. The number 1 terminals of the two switches are connected by a wire, as also are the number 2 terminals. The line then feeds to the light from the second switch, which is a duplicate of the first switch. If, say, contact 2 of either switch fails to carry current, then each switch works normally when the other is in position 1; current will flow when both switches are in position 1, and only then. So if either switch is in position 2, the other switch cannot turn on the light. Hence either switch could be defective. Furthermore, any break in the wire joining the number 2 contacts would produce the same results; so it could be that neither switch is defective.

Assuming exactly one switch is defective, remove the cover plate from either switch and connect a jumper wire between the number 1 and number 2 terminals of that switch without removing the existing wiring. Then the switch that turns the light on and off is the defective one because it is the only one that can break the circuit. The good switch will not change the condition of the light; the bad one will turn it on and off.



Also solved by Ron M. Adin (Haifa), John D. Baildon, Ken Brown, Michael Brozinsky, Douglas E. Cameron, Randall J. Covill, Dan J. Eustice, James E. Falk, Joyce Killen Gendler, Michael Goldberg, Sylvan H. Greene, P. R. Halmos, Dale T. Hoffman, H. Kestelman (England), Uri Leron (Israel), O. P. Lossers (Netherlands), Maurice Nadler, G. W. Peck, Alan Shuchat, Robert Singleton, David Singmaster, Arthur J. Waldo, Howard J. Wilcox, Harald Ziehms (Federal Republic of Germany), Gene Zirkel, and the proposer.

ELEMENTARY PROBLEMS

Solutions of these Elementary Problems should be mailed in duplicate to Dr. J. L. Brenner, 10 Phillips Road, Palo Alto, CA 94303 (USA), by April 30, 1981. Please place the solver's name and mailing address on each (double-spaced) sheet. Include a self-addressed card or label (for acknowledgment).

E 2859. *Proposed by Ulrich Faigle, Technische Hochschule, Darmstadt, Germany.*

Let the sequence $\{a_n\}$ of real numbers be defined by $a_n=a_{n-1}(a_{n-1}-1)$ for $n\geqslant 2$. For what (initial) values of a_1 will this sequence converge?

E 2860. *Proposed by J. Martin Borden, University of Illinois.*

Let $\{a_n\}$ $(n=1,2,\ldots)$ be a nondecreasing sequence, $0\leqslant a_n\leqslant a_{n+1}$. Assume $a_{mn}\geqslant ma_n$ for all m, n, and also $\sup(a_n/n)=c<\infty$. Must a_n/n have a limit?

E 2861. *Proposed by George Shulman, Teaneck, N.J.*

Let $p>3$ be a prime; A_l is the lth elementary symmetric function of the set $\{1,2,\ldots,p-1\}$. If l is odd, $1<l<p$, prove $A_l\equiv 0 \bmod p^2$. (Wolstenholme's theorem is the case $l=p-2$.) Can the relation $A_l\equiv 0 \bmod p^2$ hold if l is even?

E 2862. *Proposed by T. Keller, Honolulu, Hawaii.*

For $n\geqslant 3$, show that $n-1$ straight lines are sufficient to go through the interior of every square of an $n\times n$ chessboard. *Are $n-1$ lines necessary?

SOLUTIONS OF ELEMENTARY PROBLEMS

Chebyshev Interpolation

E 2796 [1979, 703]. *Proposed by P. Henrici, Eidgenossische Technische Hochschule, Zürich, Switzerland.*

Prove that the polynomial p with degree less than or equal to n that agrees with a given function $f(x)$ at the Chebyshev points

$$x_k = \cos\phi_k, \text{ where } \phi_k = (2k+1)\pi/(2n+2) \qquad (k=0,1,\dots,n),$$

is, for x not in $\{x_0,\dots,x_n\}$, given by $p(x) = N(x)/D(x)$ with

$$N(x) = \sum_{k=0}^{n} \frac{(-1)^k f(x_k)\sin\phi_k}{x - x_k}, \qquad D(x) = \sum_{k=0}^{n} \frac{(-1)^k \sin\phi_k}{x - x_k}.$$

Solution by Charles A. DeCarlucci, A.P.L., Johns Hopkins University; Jeffrey Mitchell Cohen, University of Pittsburgh; Michael J. Dixon, C.S.U., Chico, Calif.; S. J. Goodenough & T. M. Mills, Bendigo College of Advanced Education, Victoria, Australia; A. A. Jagers, Technische Hogeschool Twente, Enschede, Netherlands; L. Kuipers, Mollens (Vs), Switzerland; L. Van Hamme, Vrije Univ., Brussels; and Pei Yuan Wu, Indiana University. Since 1 is its own Lagrange interpolation polynomial (on the Chebyshev points), the relation

$$1 \equiv (n+1)^{-1} \sum_{0}^{n} (-1)^k \cos(n+1)\phi \sin\,\phi_k/(x - x_k)$$

is clear, since $T_{n+1}(x) = T_{n+1}(\cos\phi) = \cos(n+1)\phi$. In fact for any function f, the Lagrange interpolation polynomial that agrees with f at the $n+1$ zeros x_k of $T_{n+1}(x)$ is given by

$$p(x) = \sum_{0}^{n} [f(x_k) T_{n+1}(x)]/[(x - x_k) T'_{n+1}(x_k)].$$

The assertion of the problem follows. Note $T'_{n+1}(x_k) = (-1)^k (n+1)/\sin\phi_k$.

$k = (q-1)/p$, and 2 (or b) Is a kth Power mod q

E 2798 [1979, 785]. *Proposed by Doug Hensley, Texas A & M University, College Station, Texas.*

Prove that there are infinitely many pairs (p,q) of primes such that $(q-1)/p$ is an integer k and 2 is a kth power modulo q.

Solution by Lorraine L. Foster, California State University, Northridge. "2" may be replaced by any integer b ($b \neq 0, \pm 1$). Let p be an arbitrary odd prime such that $p \nmid b-1$. Let q be a prime divisor of $(b^p - 1)/(b-1)$. All congruences are taken modulo q. Clearly $b \not\equiv 1$, since otherwise, $b^{p-1} + b^{p-2} + \cdots + 1 \equiv p \equiv 0$, so that $p = q$, $p | b-1$. Thus b has order p (mod$q$) so that $p | q-1$. Set $q - 1 = kp$. Let g be a primitive root, and suppose $b = g^s$. Then $g^{sp} \equiv 1$, so that $kp | sp$, $k | s$, and b is a kth power.

Foster's generalization was also mentioned by J. M. Cohen and by L. Somer. L. E. Mattics referred to *On integral divisors of $a^n - b^n$*, by G. D. Birkhoff and H. S. Vandiver, Ann. of Math (2) 5 (1904) 173–180. E. Trost showed that infinitely many *mutually disjoint* pairs (p, q) exist. Some solvers considered the numbers $2^\alpha - 1$, with α composite.

Also solved by P. T. Bateman, E. Bator & B. Spearman, R. Breusch, D. Broline, J. M. Cohen, L. Jones, J. Leech, R. E. Shafer, E. Triesch (student, West Germany), and the proposer.

Superfactorials and Catalan Numbers

E 2799 [1979, 785]. *Proposed by Marlow Sholander and E. B. Leach, Case Western Reserve University.*

For n a positive integer, let $n!!$ denote the *superfactorial* $\prod_{i=1}^{n} i!$ and let $0!!=1$. Set $A_n=(2n-1)!!/[(n-1)!!]^4$. Prove that A_n is an integral multiple of $(2n-1)!$. (A_n is the reciprocal of the determinant of the n by n Hilbert matrix $H_n=(h_{ij})$ with $h_{ij}=(i+j-1)^{-1}$. See Pólya-Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Dover, 1945, vol. 2, Chap. 7, Prob. 3.)

Solution by H. L. Abbott, University of Alberta, and Nick Franceschine, Sebastopol, Calif. (independently). In fact, if $B_n=A_n/(2n-1)!$, then B_n is an integral multiple of n. To see this, note the recurrence relation

$$B_{n+1}/(n+1)=(B_n/n)C_n\binom{2n+1}{n},$$

where $C_n=\binom{2n}{n}/(n+1)$ is the Catalan number (the number of ways to parenthesize the product $a_1\cdots a_n$).

R. E. Schafer showed that B_n is divisible also by $1\cdot 3\cdot 5\cdot 7\cdots(2n-3)$ if $n>2$; O. G. Ruehr noted that B_n is divisible by $n!$; T. Hermann (Hungary) found $B_n=\prod_{i=1}^{n-1}\binom{n+1}{n}\binom{n}{i}$.

Also solved by M. Ascher, W. Boucher (Canada), R. Breusch, D. Broline, F. Buckley, C. T. Giel, K. W. Heuer & G. A. Heuer, C. Hurd, J. Leech, L. E. Mattics, M. Vowe (Switzerland), W. V. Webb & W. A. Johnson, and the proposer.

A Test for Composite Numbers

E 2800 [1979, 785]. *Proposed by B. de la Rosa, University of the Orange Free State, Bloemfontein, South Africa.*

Show that an odd·positive integer c is composite if and only if there exists a positive integer $a\leqslant(c-3)/3$ such that $(2a-1)^2+8c$ is a square

Solution by H. L. Abbott, University of Alberta; R. Breusch, Amherst, Mass.; B. Brindza, student, University of Debrecen, Hungary; D. Broline, University of Evansville; J. M. Cohen, student, University of Pittsburgh; L. L. Foster, California State University, Northridge; N. Franceschine, Sebastopol, Calif.; B. J. Gaitley, Pacific-Sierra Research Corp.; J. Oppenheim, University of Peshawar, Pakistan; S. Singh, Clarion State College; E. Trost, Zürich, Switzerland. If $c=(2k-1)Q$, $k\geqslant 2$, $Q\geqslant 2k-1$, let $a=Q-k+1\leqslant c/(2k-1)-1\leqslant(c-3)/3$. Then $(2a-1)^2+8c=(2Q+2k-1)^2$. On the other hand, if (*) $(2a-1)^2+8c=(2t+1)^2$, then $c=(t+a)(t-a+1)/2$; hence c is composite; indeed, $t-a+1>2$ follows from $c\geqslant 3a+3>a+1$ and (*). Q can be even.

Also solved by A. Adelberg, W. Boucher, M. W. Ecker, C. A. Ellard & J. R. Hays (students), G. Gagola, R. Heller, C. Hurd, L. Jones, L. Kuipers (Switzerland), M. Laidecker, J. Leech, N. A. Martin (Canada), J. McCleary, I. A. Sakmar (France), G. Shulman, L. Somer, C. Wells, and the proposer.

ADVANCED PROBLEMS

Solutions of these Advanced Problems should be mailed in duplicate to Prof. R. C. Lyndon, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109 (USA), by April 30, 1981. The solver's full post-office address should be on each sheet.

6322*. *Proposed by M. J. Pelling, University of Malaya, and P. Erdős, Hungarian Academy of Sciences.*

Find the largest constant k such that there exists a set S of planar measure k, no three points of which form the vertices of a triangle of area 1. In particular, is $k=4\pi 3^{-3/2}$?

6323. *Proposed by P. Erdős, Hungarian Academy of Sciences.*

Let k, n be integers.

(i) Let $n-1\leqslant k\leqslant \frac{1}{2}n(n-1)$. Prove that there exist n distinct points $x_1,x_2,\ldots,x_n$ on the line that determine exactly k distinct distances $|x_i-x_j|$.

(ii) Let $[\frac{1}{2}n]\leqslant k\leqslant \frac{1}{2}n(n-1)$. Prove there are n distinct points in the plane that determine exactly k distinct distances.

(iii) *Is it true that, for every $\epsilon>0$, there is an $n_0=n_0(\epsilon)$ such that, if $n>n_0$ and if $\epsilon n<k\leqslant \frac{1}{2}n(n-1)$, there are n points in the plane that determine exactly k distances?

6324. *Proposed by I. N. Herstein, University of Chicago.*

Let R be a ring (not necessarily with 1) having no nilpotent ideals. Suppose that $L\neq 0$ is a left ideal in R such that Ra is a minimal left ideal of R for every $a\neq 0$ in L. Prove that L itself must be a minimal left ideal of R.

6325*. *Proposed by Barry J. Powell, Kirkland Washington.*

Let $E_0=1$, $E_2=-1$, $E_4=5$, $E_6=-61$, $E_8=1385$, $E_{10}=-50521,\ldots,$ be the set of Euler numbers defined by $\sec x=\Sigma_0^\infty E_n x^n/n!$ $(|x|<\pi/2)$. Prove or disprove that, for any prime $p\equiv 1 \bmod 4$, $E_{(p-1)/2}\not\equiv 0 \bmod p$. (It is true for $p\equiv 5 \bmod 8$. See E. Lehmer, *On congruences involving Bernoulli numbers and the quotients of Fermat and Wilson*, Annals of Math., 39 (1938) 350–360.)

SOLUTIONS OF ADVANCED PROBLEMS

Cubes with Integral Vertices

6179 [1977, 744]. *Proposed by E. Ehrhart, University of Strasbourg, France.*

Find all cubes in a cubic lattice whose vertices are lattice points.

Solution by Georges Glaeser, Université de Strasbourg. We use the following proposition, which can be found, for example, in the *Traité de Mécanique* of P. Appell, vol 1.

THEOREM OF OLINDE RODRIGUES. *The set of* 3-*by*-3 *orthogonal matrices can be given by four real parameters a,b,c,d, not simultaneously zero, as follows*:

$$M=\frac{\pm 1}{a^2+b^2+c^2+d^2}\begin{bmatrix} a^2+b^2-c^2-d^2 & 2(bc+da) & 2(bd-ca) \\ 2(bc-da) & a^2-b^2+c^2-d^2 & 2(cd+ba) \\ 2(bd+ca) & 2(cd-ba) & a^2-b^2-c^2+d^2 \end{bmatrix}.$$

REMARK. In fact, the unit cube defined by M depends upon only three parameters, for example, the ratios of a,b,c,d.

LEMMA 1. *The following four assertions are equivalent*:

(1) *all entries of M are rational*;

(2) *the ratio of every pair of entries is rational*;

(3) *the ratios of a,b,c,d are rational*;

(4) *the matrix can be given by rational a,b,c,d.*

Proof. That (3) implies (4) follows from the remark; and it is trivial that (4) implies (1) and (1) implies (2). To show that (2) implies (3) we note that if $d\neq 0$, then (writing A_j^i for the entry in the ith column and jth row)

$$\frac{c}{d}=\frac{A_2^1+A_1^2}{A_3^1+A_1^3}=\left(\frac{A_2^1}{A_3^2}+\frac{A_1^2}{A_3^2}\right):\left(\frac{A_3^1}{A_3^2}+\frac{A_1^3}{A_3^2}\right).$$

(This proof works only if there are enough nonzero $A_j^i + A_i^j$. This is the case if all $|A_i^i| \neq 1$. If some $|A_i^i| = 1$, the formula simplifies and the result can be obtained directly.)

LEMMA 2. *The length of the side of a cube with integral vertices is an integer.*

Proof (E. Ehrhart). Let the cube be spanned by vectors $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ of length l. The vector formulas for the volume V and the area S of a face show that these are integers. Thus $l = V/S$ is rational. Since $l^2 = \overrightarrow{OA} \wedge \overrightarrow{OB}$ is an integer, l is an integer.

THEOREM. *Every cube with integral vertices can be obtained as follows.* (1) *Substitute integers for* a,b,c,d. (2) *Reduce the entries* A_j^i *to lowest terms.* (3) *Multiply the* A_j^i *by the least common multiple* k *of their denominators.* (4) *Multiply all entries by any integer* $k' \neq 0$.

REMARKS. (1) The length of the side is the integer $l = k \cdot |k'|$. (2) The cube depends upon five parameters a,b,c,d,k', where a,b,c,d can be taken pairwise relatively prime without loss of generality.

Note. The method resembles that by which Pythagorean triangles $a^2 + b^2 = c^2$ are obtained starting from the parametric representation

$$\frac{a}{c} = \frac{1-t^2}{1+t^2}, \qquad \frac{b}{c} = \frac{2t}{1+t^2}.$$

Reference

1. E. Ehrhart, Sur les polygones et les polyèdres réguliers entiers, l'Enseign. Math., 5 (1959) 81–85.

Also solved by A. Viricel (France).

Bases for Piecewise Continuous Functions

6184 [1977, 829]. *Proposed by Ole Jørsboe, Technical University of Denmark.*

Let $(\phi_n)_{n=1}^{\infty}$ be an orthonormal system of real-valued piecewise continuous functions on the interval $[0,1]$ with the property that if f is a real-valued piecewise continuous function on $[0,1]$ fulfilling $(f,\phi_n) = \int_0^1 f(x)\phi_n(x)\,dx = 0$ for all $n \in N$, then f is 0 at all points of continuity.

Does this imply that (ϕ_n) spans the space of all real-valued piecewise continuous functions on $[0,1]$, i.e., can every piecewise continuous function f be written in the form

$$f = \lim_{N\to\infty} \sum_{n=1}^{N} a_n \phi_n?$$

Solution by the proposer. The answer is no, as can be seen from the following example. Let K be a "Cantor-like" set in $[0,1]$ of Lebesgue measure $\frac{1}{2}$. The complement of K is a countable union of open intervals I_n, and in each of these intervals I_n we take a complete orthonormal system $(\phi_{n,k})_{k=1}^{\infty}$, e.g., the trigonometric system. For every $n \in N$ we extend the functions $\phi_{n,k}$ to $[0,1]$ by putting them equal to 0 outside I_n.

The system $(\phi_{n,k})$ is an orthonormal system in $[0,1]$ and it has the stated property that, if $(f,\phi_{n,k}) = 0$ for all n and k, and, if f is piecewise continuous, then $f = 0$ at all continuity points (here we use the fact the K is nowhere dense). But the system does not span the space of all piecewise continuous functions on $[0,1]$, e.g., Parseval's equation is *not* fulfilled for the function f which is equal to 1 in $[0,1]$.

Comment. The problem has for some years been included in some lecture notes (in Danish) in measure theory used at the University of Aarhus, Denmark, but I have not seen the problem in any English text.

Determining Heavy and Light Balls by Weighings

6224 [1978, 600]. *Proposed by David P. Robbins, Hamilton College.*

Suppose we are given N balls which are indistinguishable except that some are heavy and some are light (the heavy balls are alike in weight, as are the light balls). Using a pan balance what is the minimum number of weighings in which it is always possible

(a) to identify one heavy and one light ball?

(b) to determine the number of heavy and light balls?

Partial solution by Zane C. Motteler, Michigan Technological University. At most $N-1$ weighings are necessary in either case, in order to weigh an arbitrarily chosen ball against each of the others. The question is then whether it is possible to do better.

(a) One can identify one heavy and one light ball in at most the number of weighings indicated below:

$$\begin{array}{ll} k & \text{if } N=2^k \quad \text{or} \quad 2^{k-1}<N\leqslant 3\cdot 2^{k-2}; \\ k+1 & \text{if } 3\cdot 2^{k-2}<N\leqslant 4\cdot 2^{k-2}-1. \end{array}$$

(i) *Proof if* $N=2^k$ (by induction on k). If $k=1$, then one weighing suffices. Assume this is true for $k=l$. Then for $k=l+1$, divide the 2^{l+1} balls into two equal piles of 2^l balls and weigh them.

Case I. The piles balance. Then each pile contains both light and heavy balls; choose one pile arbitrarily, and by hypothesis it will take l more weighings, for a total of $l+1$.

Case II. The two piles do not balance. Then take half the balls from one pile and half from the other and compare these two piles of 2^{l-1} balls. If they do not balance, discard the other two piles of 2^{l-1} balls and repeat the process. If they do balance, discard them and repeat the process on the other two piles. In any event, at each step we are comparing half as many balls as at the previous step.

(ii) *Proof for* $2^{k-1}<N\leqslant 3\cdot 2^{k-2}$. We shall show that in at most two steps one can identify two piles, each containing 2^{k-2} balls, to which the process described in (i) can be applied. Choose arbitrarily two piles each containing 2^{k-2} balls and weigh them. If they do not balance, proceed as in Case II above. (This case will take at most $k-2$ further weighings, for a total of $k-1$.) If they do balance, replace all or part of one pile with the 2^{k-2} or fewer balls remaining. Weigh again. If they balance, proceed as in Case I; if not, as in Case II. In any event we have at most $k-2$ further weighings, for a total of k.

(iii) For $3\cdot 2^{k-2}<N\leqslant 4\cdot 2^{k-2}-1$ we proceed as in (ii), noting that it may take as many as three weighings to reduce the problem to two piles of 2^{k-2} to which we may apply in the above.

It is clear that k is the least number of weighings required for 2^k balls; we cannot do better than to halve the number we must examine at each step. It is also fairly obvious that 2^k+1 will, in general, take one more weighing than 2^k (if the piles of 2^{k-1} balance, one must test the extra ball in the event that all the others are equal). It is not clear to this author, however, why 2^k-1 should necessarily require one more weighing than 2^k.

(b) One can determine the number of heavy and light balls in at most the number of weighings indicated below:

$$w=k+\left[\frac{N-2}{2}\right] \quad \text{if} \quad N=2^k;$$

$$w=k+\left[\frac{N-2}{2}\right]+N-2\left[\frac{N}{2}\right] \quad \text{if} \quad 2^{k-1}<N\leqslant 3\cdot 2^{k-2};$$

$$w=k+1+\left[\frac{N-2}{2}\right]+N-2\left[\frac{N}{2}\right] \quad \text{if} \quad 3\cdot 2^{k-2}<N\leqslant 4\cdot 2^{k-2}-1;$$

where "[·]" denotes the greatest integer function. The reasoning behind this is that we isolate a heavy-light pair in k or $k+1$ steps as in part (a); we then weigh it against the $[(N-2)/2]$ remaining pairs, and, if N is odd, perform one more weighing to determine the status of the leftover ball. In general, this takes fewer than $N-1$ weighings. For some smaller values of N it is more economical to weigh one ball against each of the $N-1$ others: e.g., $w(4)=3$, $w(5)=5$, $w(6)=5$, $w(7)=7$; but $w(32)=20$, and as $N\to\infty$, $w(N)\to N/2$.

Additional partial solution to part (b) by A. Nijenhuis, University of Pennsylvania. We use merge-sort. A merge-sort of m objects arranges them in increasing order by comparison of pairs in $[\log_2 m]$ phases, each phase requiring at most m comparisons. In phase i, sorted sublists of length 2^{i-1} (plus a shorter sorted sublist) are combined into lists of length 2^i (plus remainder) by pairing up and merging the sublists; repeatedly, the smaller of the minimal elements of the two lists is moved to the new sublist.

We begin by forming m groups of $k=[N/m]$ balls each, plus a left-over group of k' balls, where $0\leqslant k'<k$. The value of m will be chosen at the end, to optimize the answer. Use merge-sort to place the m groups into an order $G_1,\ldots,G_m$ so the numbers $h_1,\ldots,h_m$ of heavy balls in these groups form an ascending sequence; the sorting finds repetitions among $h_1,\ldots,h_m$ but does not find their actual values. The number of weighings in the merge-sort does not exceed $m[\log_2 m]$.

Next, use $k-1$ weighings to determine h_1 (if all balls have the same weights, subsequent comparisons will determine which is which), and if $h_m>h_1$ (else skip the rest of this paragraph) do the same for h_m, another $k-1$ weighings. The $2k$ balls thus identified as heavy or light are used to form a pool from which to form test groups $G(h)$ of k balls, h of which are heavy $(h_1<h<h_m)$. In fact, test $G(h)$ against the first G_i of unknown weight, starting with $h=h_1+1$, and incrementing h until equality is achieved. Then continue with the next unknown G_i and the next value of h, etc. In no more than h_m-h_1-1 weighings all the h_i will have been determined.

Another k' weighings determine the number h' of heavy balls in the left-over group. The total number of heavy balls is $h'+h_1+\cdots+h_m$. The total number T of weighings satisfies

$$T\leqslant m[\log_2 m]+2(k-1)+h_m-h_1-1+k'\leqslant m[\log_2 m]+4k-4.$$

This bound on T can be minimized by making the two terms equal, that is, by taking $m=[(8N\log_2 N)^{\frac{1}{2}}]$, which yields $T<(8N\log_2 N)^{\frac{1}{2}}$.

The proposer notes that this problem was inspired by Problem 5 in the ***Russian Olympiad Problem Book***, by Shlarsky, Chentzov, and Yaglom, W. H. Freeman, 1962, p. 7. See also Problem 6.

Ratio of Derangement Number to Ménage Number

6234* [1978, 770]. *Proposed by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Canada.*

Let D_n and M_n denote the derangement number and the ménage number respectively. Prove or disprove that the sequence $\{M_n/D_n\}, n=4,5,6,\ldots$ is monotonically increasing and $\lim_{n\to\infty}(M_n/D_n)=1/e$. (For definitions see, e.g., H. J. Ryser, *Combinatorial Mathematics*, Chap. 2, Sec. 3 and Chap. 3, Sec. 2.)

I. *Solution by Thomas H. Foregger, Bell Telephone Laboratories, Murray Hill, N.J.* The sequence $\{M_n/D_n\}, n=4,5,6$ has first 3 terms .22,.29,.30; so assume $n\geqslant 6$. The sequence $\{M_n\}$ satisfies the recurrence

$$(n-2)M_n=n(n-2)M_{n-1}+nM_{n-2}+4(-1)^{n+1}.$$

(*An Introduction to Combinatorial Analysis*, J. Riordan, Wiley, 1958, p. 201, Eq. (19)) and the sequence $\{D_n\}$ satisfies

$$D_n=nD_{n-1}+(-1)^n$$

(Riordan, p. 59, Eq. (21)).

Suppose first that n is odd. Then

$$\frac{M_n}{D_n}=\frac{nD_{n-1}}{D_n}\left(\frac{M_{n-1}}{D_{n-1}}\right)+\frac{n}{n-2}\,\frac{M_{n-2}}{D_n}+\frac{4}{(n-2)D_n}$$

$$>\left(\frac{D_n+1}{D_n}\right)\frac{M_{n-1}}{D_{n-1}}>\frac{M_{n-1}}{D_{n-1}}.$$

Next, suppose that n is even. Then $M_{n-2}\geqslant 2$ and

$$M_n=nM_{n-1}+\left(1+\frac{2}{n-2}\right)M_{n-2}-\frac{4}{n-2},$$

so

$$\frac{M_n}{D_n}=\frac{nD_{n-1}}{D_n}\left(\frac{M_{n-1}}{D_{n-1}}\right)+\frac{M_{n-2}}{D_n}+\frac{2M_{n-2}-4}{(n-2)D_n}$$

$$\geqslant\left(\frac{D_n-1}{D_n}\right)\frac{M_{n-1}}{D_{n-1}}+\frac{M_{n-2}}{D_n}$$

$$=\frac{M_{n-1}}{D_{n-1}}+\frac{1}{D_n}\left(M_{n-2}-\frac{M_{n-1}}{D_{n-1}}\right)$$

$$\geqslant\frac{M_{n-1}}{D_{n-1}},$$

since $\dfrac{M_{n-1}}{D_{n-1}}\leqslant 1\leqslant M_{n-2}$.

This establishes the monotonicity of the sequence $\{M_n/D_n\}$ for $n\geqslant 4$.

The limit is $1/e$ since $M_n\sim n!e^{-2}f(n)$, where $f(n)\to 1$ as $n\to\infty$ (Riordan, Problem 7(b), p. 225) and $D_n/n!\to e^{-1}$.

II. *Solution by C. C. Rousseau, Memphis State University*. We change M_n to U_n to conform to standard usage. The sequence $\{U_n\}$ is the increasing sequence of integers generated by the Lucas recursion formula

$$(n-1)U_{n+1}=(n^2-1)U_n+(n+1)U_{n-1}+4(-1)^n \tag{1}$$

together with the initial values $U_2=0, U_3=1$. Also

$$D_{n+1}=(n+1)D_n+(-n)^{n+1} \tag{2}$$

for $n=1,2,3,\ldots$, with $D_1=0$. From (1) and (2) it follows that $U_{n+1}/D_{n+1}>U_n/D_n$ is equivalent to

$$\frac{n+1}{n-1}U_{n-1}>(-1)^{n+1}\left(\frac{U_n}{D_n}+\frac{4}{n-1}\right). \tag{3}$$

From their combinatorial definitions [3, p. 22,31] we know that $0<U_n/D_n<1$ for $n=3,4,5,\ldots$. It is thus clear that (3) holds for all $n\geqslant 4$. It is well known that $D_n\sim n!/e$ and $U_n\sim n!/e^2 (n\to\infty)$. See [1], [2], and [3, p. 23]. Thus $\lim_{n\to\infty}U_n/D_n=1/e$.

References

1. I. Kaplansky, Solution of the "problème des ménages," Bull. Amer. Math. Soc., 49 (1943) 784–785.
2. I. Kaplansky and J. Riordan, The problème des ménages, Scripta Math., 12 (1946) 113–124.
3. H. J. Ryser, Combinatorial Mathematics, Carus Mathematical Monograph No. 14, M.A.A., 1963.

Also solved by L. E. Clarke (England), C. R. Pranesachar, Lajos Takács, and (partially) Ken Yocum.

Norm of a Matrix

6249 [1979, 59]. *Proposed by H. Kestelman, University College, London, England.*

The norm $\|A\|$ of a real 2×2 matrix A is by definition the maximum of $\|A\hat{x}\|$ when $\|\hat{x}\|=1$; if $\|x\|$ is the euclidean norm $(x^Tx)^{\frac{1}{2}}$, then $\|A\|\leqslant \||A|\|$ where $|A|$ is the matrix whose elements are the absolute magnitudes of those of A. Find necessary and sufficient conditions on an invertible 2×2 matrix N in order that $\|A\|\leqslant\||A|\|$ for all A when $\|x\|$ is defined as the euclidean norm of Nx.

(One tends to use the inequality $\|A\|\leqslant\||A|\|$ automatically in matrix analysis and might "naturally" assume it, when $\|x\|$ is the euclidean norm of Nx, for all N.)

Solution by Emeric Deutsch, Polytechnic Institute of New York. Let K be the field of real or complex numbers, let ϕ be a norm on K^n $(n=2,3,\ldots)$, and let lub_ϕ denote the operator norm induced by ϕ on the algebra $M_n(K)$ of $n\times n$ matrices over K. For a vector $x\in K^n$ (matrix $A\in M_n(K)$) we denote by $|x|(|A|)$ the vector (matrix) whose elements are the absolute values of the elements of $x(A)$. The norm ϕ is said to be *absolute* if $\phi(|x|)=\phi(x)$ for all $x\in K^n$. Let $\|x\|$ denote the Euclidean norm $(x^*x)^{\frac{1}{2}}$ and for an invertible $N\in M_n(K)$ let $\|x\|_N=\|Nx\|$ $(x\in K^n)$.

The solution of the problem follows at once from the following two propositions which have an independent interest:

(1) $\operatorname{lub}_\phi A\leqslant\operatorname{lub}_\phi|A|$ *for all* $A\in M_n(K)$ *if and only if* ϕ *is absolute.*

(2) *The norm* $x\to\|x\|_N$ *is absolute if and only if* N^*N *is a diagonal matrix* (*i.e., the columns of* N *are orthogonal*).

Proof of (1). ($\Rightarrow$): Let $x=(\xi_1,\ldots,\xi_n)^T\in K^n$ and let D be the diagonal matrix such that $|x|=Dx$. Then $|D|=I$ (I is the $n\times n$ identity matrix) and we have

$$\phi(|x|)=\phi(Dx)\leqslant(\operatorname{lub}_\phi D)\phi(x)\leqslant(\operatorname{lub}_\phi|D|)\phi(x)=\phi(x).$$

Similarly, since $x=D^{-1}|x|$ and $|D^{-1}|=I$, we have $\phi(x)\leqslant\phi(|x|)$. Thus ϕ is an absolute norm.

($\Leftarrow$): For $x\in K^n$ and $A\in M_n(K)$ we have $|Ax|\leqslant|A||x|$, where the ordering of vectors is meant componentwise. Taking into account that an absolute norm ϕ is also monotone (i.e., $\phi(y)\leqslant\phi(z)$ whenever $|y|\leqslant|z|$; see, for example, P. Lancaster, Theory of Matrices, Academic Press, New York, 1969, p. 214), we have

$$\phi(Ax)=\phi(|Ax|)\leqslant\phi(|A||x|)\leqslant(\operatorname{lub}_\phi(|A|))\phi(x),$$

whence $\operatorname{lub}_\phi A\leqslant\operatorname{lub}_\phi|A|$.

Proof of (2). Taking $x=(\xi_1,\ldots,\xi_n)^T\in K^n$ and denoting $N^*N=(p_{ij})$, we have

$$\|x\|_N^2=x^*N^*Nx=\sum_{i,j=1}^n p_{ij}\bar{\xi}_i\xi_j.$$

From this relation one can easily deduce that $\||x|\|_N=\|x\|_N$ for all $x\in K^n$ if and only if $p_{ij}=0$ whenever $i\neq j$.

(For the less obvious "only if" part, fixing i,j, $i\neq j$ and taking $\xi_k=0$ for $k\neq i,j$, the relation $\|x\|_N=\||x|\|_N$ yields $\operatorname{Re}p_{ij}\bar{\xi}_i\xi_j=|\xi_i\xi_j|\operatorname{Re}p_{ij}$. Taking $\bar{\xi}_i\xi_j=-1$, we obtain $\operatorname{Re}p_{ij}=0$. If K is the field of complex numbers, then, taking $\bar{\xi}_i\xi_j=-i$, we obtain $\operatorname{Im}p_{ij}=\operatorname{Re}(-ip_{ij})=\operatorname{Re}p_{ij}=0$ and so $p_{ij}=0$.)

REMARK. The "if" part of Proposition (1) is known; see: F. L. Bauer, J. Stoer and C. Witzgall, *Absolute and monotonic norms*, Numer. Math., 3, 1961, pp. 257–264. The "only if" part of the Proposition (1) is also known; it was proved by J. Menkoski, *Some notes on absolute norms*, Reprint A10, Dept. of Mathematics, University of Tampere, Finland.

Also solved by Alan Suchat and the proposer.

Limit of a Combinatorial Sum

6252 [1979, 131]. *Proposed by Ioan Tomescu, University of Bucharest, Romania.*

Let $f(n)=\Sigma_{i=1}^{n}\Sigma_{j=1}^{n}\binom{n}{i}\binom{n}{j}\, i^{n-j}j^{n-1}$ and show that

$$\lim_{n\to\infty}\frac{[f(n)]^{1/2n}\ln n}{n}=\frac{1}{e}.$$

Solution by Robert Breusch, Amherst College. "lim" and "$\to$" will always mean: as $n\to\infty$.

Let $r=r(n)$ and $s=s(n)$ be such that $a_{rs}=\binom{n}{r}\binom{n}{s}r^{n-s}s^{n-r}$ is maximal among the n^2 terms of $f(n)$. Then $a_{rs}<f(n)<n^2a_{rs}$ and therefore

$$\lim\left(\frac{f(n)}{a_{rs}}\right)^{1/(2n)}=1. \tag{1}$$

For $1\leqslant m\leqslant n$, $m^me^{-m}<m!<m^me^{-m}(me)$ and thus

$$\lim\left(\frac{m!e^m}{m^m}\right)^{1/(2n)}=1. \tag{2}$$

This also holds for $m=0$ if 0^0 is defined as $\lim_{m\to 0^+}(m^m)=1$.

$$a_{rs}=\frac{n!n!r^{n-s}s^{n-r}}{r!(n-r)!s!(n-s)!}. \tag{3}$$

By (1), (2), and (3)

$$\begin{aligned}\lim\frac{(f(n))^{1/(2n)}(\log n)}{n}&=\lim\frac{(a_{rs})^{1/(2n)}(\log n)}{n}\\ &=\lim\left(\frac{n^{2n}r^{n-s}s^{n-r}}{r^r(n-r)^{n-r}\cdot s^s(n-s)^{n-s}}\right)^{1/(2n)}\frac{\log n}{n}\\ &=\lim\left(\frac{(rs)^{n-r-s}}{(n-r)^{n-r}\cdot(n-s)^{n-s}}\right)^{1/(2n)}(\log n),\end{aligned}$$

The function

$$g_n(x,y)=(n-x-y)\cdot\log(xy)-(n-x)\log(n-x)-(n-y)\log(n-y)$$

is defined on the square $R_n=\{(x,y)|0<x,y\leqslant n\}$ and differentiable in its interior.

If M_n is the absolute maximum of g_n on $S_n=\{(x,y)|1\leqslant x,y\leqslant n\}$ occurring at (x_n,y_n), then we are to show that $\exp(M_n/(2n))(\log n)\to 1/e$ and that the same is true for lattice points $(\underline{x}_n,\underline{y}_n)$ sufficiently close to (x_n,y_n).

On $x=n$,

$$g_n(n,y)=-y\log(ny)-(n-y)\log(n-y)<-n\log(n/2).$$

Thus

$$\exp(g_n(n,y)/(2n))(\log n)<(n/2)^{-1/2}(\log n)=o(1).$$

The same is true on $y=n$. Every other point of S_n is an interior point of R_n. An absolute maximum at such a point is also a relative maximum characterized by

(i) $-\log(xy)=\dfrac{n-x-y}{x}+1+\log(n-x)=0$

(ii) $-\log(xy)+\dfrac{n-x-y}{y}+1+\log(n-y)=0.$

If (iii) $n-x-y\leqslant 0$, then by (i) and (ii) $n-x\geqslant xy/e$, $n-y\geqslant xy/e$, and thus by (iii) $y\geqslant xy/e$,

$x \geqslant xy/e$; therefore $x \leqslant e, y \leqslant e$, and $n \leqslant 2e < 6$. From now on we assume that $n \geqslant 6$. Then (i) and (ii) imply that $n-x-y>0$. Now subtract (ii) from (i):

$$(n-x-y)(1/x-1/y)+\log\left(\frac{n-x}{n-y}\right)=0.$$

This is impossible if $x<y$ because in this case $1/x-1/y>0$ and $\log(n-x)/(n-y)>0$. The inequality $x>y$ is equally impossible. Therefore, for a relative maximum at (x,y), necessarily $x=y$. Then

$$g_n(x,y)=h_n(x)=2(n-2x)(\log x)-2(n-x)\log(n-x) \qquad (1 \leqslant x<n),$$

$$h_n'(x)=-4(\log x)+2\frac{n}{x}-4+2\log(n-x)+2$$

$$h_n'(1)=2n+2\log(n-1)-2>0, \qquad h_n'(n-\epsilon)\sim -4(\log n)+2\log\epsilon<0,$$

and $h''(x)=-4/x-2n/x^2-2/(n-x)<0$ throughout.

Therefore h_n has exactly one relative maximum on $(1,n)$. For $1/2<\alpha<2$,

$$\begin{aligned} h_n'(\alpha n/(\log n)) &= -4(\log\alpha)-4(\log n)+4\log\log n \\ &\quad +\frac{2}{\alpha}(\log n)-2+2\log n+2\log\left(1-\frac{\alpha}{\log n}\right) \\ &= 2\left(\frac{1}{\alpha}-1\right)(\log n)+o(\log n). \end{aligned}$$

Thus for sufficiently large n

$$h_n'(\alpha n/(\log n))\begin{cases} >0 & \text{if } \alpha<1 \\ <0 & \text{if } \alpha>1 \end{cases}.$$

Therefore the maximum occurs at some $x_n=(n/\log n)/(1+o(1))$. It is found that

$$h_n(x_n)=2n(-\log\log n-1+o(1)),$$

$$\exp(h_n(x_n)/(2n))=(e\log n)^{-1}(1+o(1)),$$

and thus

$$\lim\frac{(f(n))^{1/(2n)}(\log n)}{n}=\lim\exp(h_n(x_n)/(2n))(\log n)=1/e.$$

The argument shows that the same is true for integers $\underline{x}_n=x_n(1+o(1))$, for instance for $\underline{x}_n=[x_n]$. This completes the proof.

Also solved by the proposer.

MISCELLANEA

48. Computer scientists have become so immersed in the study of their tools that they cannot provide "poets" with an appreciation of programming without subjecting them to a staggering amount of absolutely irrelevant detail. Declarations, formats, tedious syntax, and "18th century" etiquette descend on unsuspecting beginners and conspire to hide from them the beauty and pleasures of programming. Admittedly, the "poet" must establish communication with the computer, and to do so requires observance of the computer's rules and behavior. But both PL/1 and Fortran carry this to ridiculous extremes.

—Adapted from Alan Perlis, *American Scientist*, 67 (1979) 731.

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook
P = professional reading
S = supplementary reading
L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, S**, P**, L***. *Mathematics: The Loss of Certainty.* Morris Kline. Oxford U Pr, 1980, 366 pp, $19.95. [ISBN: 0-19-502754X] A popular historical account of "the rise and decline of the majesty of mathematics." "We know today that mathematics does not possess the qualities that in the past earned for it universal respect and admiration...the acme of exact reasoning, a body of truths in itself, and the truth about the design of nature." Kline's literate and lucid tale, devoid of the blatant hyperbole characteristic of his recent books on pedagogy, moves beyond the "loss of truth" to suggest two issues for a new foundation of mathematics: effectiveness as the criterion of correctness ("Of course, such a criterion is provisional. What is considered correct today may prove wrong in the next application.") and the mystery of why, despite its contradictions and unsatisfactory foundations, it has been so incredibly effective. ("Are we performing miracles with imperfect tools?") LAS

GENERAL, P, L*. *Table of Integrals, Series, and Products, Corrected and Enlarged Edition.* I.S. Gradshteyn, I.M. Ryzhik. Transl: Alan Jeffrey. Acad Pr, 1980, xlv + 1160 pp, $19.50. [ISBN: 0-12-294760-6] Enlarged by 73 pages of short sections on vector field theory, algebraic inequalities, integral inequalities, matrices and quadratic forms, determinants, vector norms, ordinary differential equations and integral transforms. In today's market this book is a real bargain for those in need of a table of integrals. JK

GENERAL, T(13: 2). *Modern Mathematics with Applications to Business and the Social Sciences, Third Edition.* Ruric E. Wheeler, W.D. Peeples, Jr. Brooks/Cole, 1980, xvi + 613 pp, $17.95. [ISBN: 0-8185-0366-1] Substantial revision, with less emphasis on the structure of mathematics and new material on applications. (*Second Edition,* TR, April 1976.) LCL

BASIC, T(13: 1). *Essentials of Mathematics, Fourth Edition.* Russell V. Person. Wiley, 1979, 865 pp, $18.95. [ISBN: 0-471-05184-5] Only minor changes and additions from earlier editions (*Second Edition,* TR, February 1969; *Third Edition,* TR, March 1974). Arithmetic, algebra, geometry, logarithms, trigonometry and the elements of calculus for use in technical institutes, junior colleges, and in first-year college general mathematics courses. Uninspiring but honest. Suitable for home study. Separate study guide available. JK

BASIC, T?(13). *Operational Mathematics for Business.* R.C. Pierce, Jr., W.J. Tebeaux. Wadsworth, 1980, xii + 495 pp, $16.95. [ISBN: 0-534-00789-9] In spite of the title, this is a very elementary treatment of "Business mathematics." Begins with fractions and decimals and proceeds to debt amortization, how to figure sales tax, etc. Very precalculus! TAV

PRECALCULUS, T(13: 1). *Algebra and Trigonometry, A Skills Approach.* J. Louis Nanney, John L. Cable. Allyn, 1980, xi + 513 pp, $16.95. [ISBN: 0-205-06917-7] Combined edition of 1980 Lecture Versions of *College Algebra: A Skills Approach* (TR, June-July 1980; TR, March 1978) and *Trigonometry: A Skills Approach* (TR, June-July 1980). LCL

PRECALCULUS, T(13). *Algebra and Trigonometry: A Straightforward Approach.* Martin M. Zuckerman. W.W. Norton, 1980, xii + 594 pp, $15.95. [ISBN: 0-393-95020-4] A no-nonsense approach reminiscent of older textbooks; no sets, no winding functions, plenty of examples. Logarithms, progressions, topics from theory of equations, determinants, induction; it's all here. Should appeal to those wanting a text with substance. One criticism: the existence of calculators is barely acknowledged. AWR

PRECALCULUS, T(13: 1), L. *College Algebra.* Vivian Shaw Groza. Saunders Coll, 1980, ix + 438 pp, $13.95. [ISBN: 0-03-040376-6] After a first chapter on an axiomatic introduction to the real numbers the treatment and content are traditional. Exposition is somewhat terse, but examples and exercises are numerous. Answers, index. JS

PRECALCULUS, T(13: 1). *College Algebra.* Steven Bryant, Daniel Saltz. Goodyear, 1980, viii + 343 pp, $17.95. [ISBN: 0-87620-198-2] Algebra needed for calculus, plus logarithms, matrices, polynomials, sequences, and probability. Very direct presentation with emphasis on skills and techniques. LCL

PRECALCULUS, T(13: 1, 2). *College Algebra, Second Edition.* Edward D. Gaughan. Brooks/Cole, 1980, xiii + 416 pp, $16.95. [ISBN: 0-8185-0351-3] The reals, equations, inequalities, polynomials, exponents, functions, graphs, polynomial equations, systems of equations, matrices, logarithms, series, and probability. Nice section on translating into mathematical language. Optional exercises for calculators. (First Edition, TR, June-July 1974.) FLW

PRECALCULUS, T(13: 1). *Algebra and Trigonometry, Second Edition.* Margaret L. Lial, Charles D. Miller. Scott F, 1980, xiii + 560 pp, $15.95. [ISBN: 0-673-15272-3] An extensive reorganization and rewriting of the First Edition (TR, October 1978), this edition incorporates an introduction to calculators and calculator problems, trigonometry presented from a unit-circle approach, and Pólya's techniques for solving word problems. JNC

EDUCATION, P. *Activities for the Maintenance of Computational Skills and the Discovery of Patterns.* Bonnie H. Litwiller, David R. Duncan. NCTM, 1980, iv + 92 pp, $4.50 (P). Nearly 100 activities using

addition, subtraction, and multiplication tables, and the hundred square. Preservice and inservice elementary through middle school teachers will find non-routine skill building and enrichment ideas. Permission to copy granted. MW

EDUCATION, P, *Teaching the Gifted and Talented in the Mathematics Classroom.* Kevin G. Bartkovich, William C. George. NEA, 1980, 48 pp, $3.50 (P). [ISBN: 0-8106-0738-7] Specific suggestions for identification and education of mathematically gifted secondary students. Based on experience at Johns Hopkins University. Includes references and lists of screening tests. MW

EDUCATION, P, *An Agenda for Action, Recommendations for School Mathematics of the 1980s.* NCTM, 1980, ii + 29 pp, $1 (P). [ISBN: 0-87353-166-3] A detailed set of recommendations, with supporting rationale, in response to three major crises in school mathematics: school mathematics is not keeping pace with changing needs; most students do not take as much mathematics in high school as they will need in their careers; and the present shortage of teachers is increasing dramatically. Recommendations include these: that school mathematics, at all levels, should emphasize problem solving and make full use of calculators and computers; that three years of mathematics be required of all high school graduates; and that colleges stop awarding credit for courses covering mathematics ordinarily taught in high school. LAS

EDUCATION, T(13-14: 1, 2), *Mathematics for Elementary Teachers, A Content Approach.* Ruth E. Heintz. A-W, 1980, xi + 492 pp, $16.95. [ISBN: 0-201-03227-9] Author's experience shows. Thoughtful non-condescending treatment of the usual material, with special attention to problem-solving techniques, a plug for using estimation to detect gross errors and exercises for hand-hand calculators. Interesting marginal notes. Suggestions for further reading. Aimed at prospective teachers, but appropriate for in-service courses and liberal arts students. JK

EDUCATION, *Teaching Elementary School Mathematics: Methods and Content for Grades K-8.* Frederick H. Bell. Wm C Brown Pub, 1980, xvi + 582 pp, $14.95 (P). [ISBN: 0-697-06018-7] Emphasis on mathematical content as well as methods of teaching mathematics. Excellent list of references and teaching resources. Will remain useful as a reference when students become practicing teachers. MW

HISTORY, P, L*, *The Tragicomical History of Thermodynamics 1822-1854.* C. Truesdell. Stud. in History of Math. and Phy. Sci., No. 4. Springer-Verlag, 1980, xii + 372 pp, $48. [ISBN: 0-387-90403-4] A detailed exegesis of the published work of Fourier, Carnot, Joule, Kelvin, Clausius, Rankine and others. "In no other discipline have the same equations been published over and over again so many times by different authors in ill-defined notations and therefore claimed as his own by each; in no other has a single author seen fit to publish essentially the same ideas over and over again within a period of twenty years; and nowhere else is the ratio of talk and excuse to reason and result so high." LAS

HISTORY, S, P, L, *Coded Character Sets, History and Development.* Charles E. Mackenzie. A-W, 1980, xxi + 513 pp, $24.95. [ISBN: 0-201-14460-3] A description of many of the different binary codes used for alphanumeric communication together with a discussion of the factors that shaped these codes. Although the topic is inherently technical, the author's glossary and indices are more than adequate to make the book accessible to a general readership. In fact the glossary is useful on its own. JAS

HISTORY, S, L*, *Quest--An Autobiography.* Leopold Infeld. Chelsea, 1980, 361 pp, $17.50. [ISBN: 0-8284-0309-0] A re-issue of an important book first published in 1941. From reviews of that edition: "An enthralling human document." "Infeld is Boswell for the mature Einstein." "The personality of Einstein is pertinently analysed and reconstructed with a wealth of picturesque detail." LAS

FOUNDATIONS, T(18), P, L, *Descriptive Set Theory.* Yiannis N. Moschovakis. Stud. of Logic and Found. of Math., V. 100. North-Holland, 1980, xii + 637 pp, $72.25. [ISBN: 0-444-85305-7] The central problem of descriptive set theory is "to find and study the characteristic properties of definable objects" (primarily sets of integers and reals). Analysts initiated the subject at the turn of the century in reaction to the (dubious) notion of function as an *arbitrary* correspondence between objects without regard for how the function was defined. Logicians dominate the modern theory because of the deep connections with recursion theory and set theory (independence results, large cardinals, etc.). The present volume does a fine job of systematically presenting the main results and methods of the theory, including necessary background material. Central results are stressed, with ramifications and extensions (and hints) in the exercises. I suspect it may become *the* standard reference for many years. GHM

FOUNDATIONS, T(18: 1), S, P, *Cardinal Functions in Topology--Ten Years Later.* I. Juhász. Math. Centre Tracts, No. 123. Math Centrum, 1980, iv + 160 pp, Dfl. 20 (P). [ISBN: 90-6196-196-3] A completely rewritten and expanded edition (First Edition, TR, May 1972). Fundamental results that can be obtained in ZFC are included; independence results are not discussed in detail. Original appendix on combinatorial set theory, deleted; the references, expanded. PDH

FOUNDATIONS, P, *Mathematical Logic in Latin America.* Ed: A.I. Arruda, R. Chuaqui, N.C.A. Da Costa. Stud. in Logic and Found. of Math., V. 99. North-Holland, 1980, xii + 392 pp, $44. [ISBN: 0-444-85402-9] Proceedings of the IV Latin American Symposium on Mathematical Logic, Santiago, Chile, 1978. Papers dealing with topics in model theory, foundations of probability, nonclassical logics, axioms for arithmetic and other areas. Also includes two survey articles: one on "Variable binding term operators," the other on "Paraconsistent logic." A witness to the strength of research in logic in Latin America. GHM

COMBINATORICS, T(15: 1), S, P, L*, *An Introduction to Combinatorial Analysis.* John Riordan. Princeton U Pr, 1978, xii + 244 pp, $6.95 (P). [ISBN: 0-691-02365-4] A new printing of this 1968 classic. Still a valuable resource. CEC

COMBINATORICS, T(16-18), S, P, L. *Graphs and Questionnaires.* Claude François Picard. Math. Stud., V. 32. North-Holland, 1980, xiii + 431 pp, $39 (P). [ISBN: 0-444-85239-5] Questionnaires are mathematical models designed to aid in making choices and decisions. Here is a self-contained development of the theory: general properties, construction, analysis. Much of this research is easily accessible, even to undergraduates. Problems, with solutions; bibliography. LCL

NUMBER THEORY, S(13), L? *Criteria for Divisibility.* N.N. Vorob'ev. Trans: Daniel A. Levine, Timothy McLarnan. U of Chicago Pr, 1980, ix + 69 pp, $6 (P). [ISBN: 0-226-86516-9] A brief yet self-contained introduction to divisibility. Includes divisibility criteria and Euler's Theorem, but does not introduce congruence notation. Includes exercises. Expensive. CEC

NUMBER THEORY, P. *Lecture Notes in Mathematics-776: Arithmetic on Elliptic Curves with Complex Multiplication.* Benedict H. Gross. Springer-Verlag, 1980, 95 pp, $9.80 (P). [ISBN: 0-387-09743-0] A study of Q-curves, elliptic curves arithmetically "like" curves defined over Q. The author presents a review of the general theory of elliptic curves with complex multiplication, local and global properties of Q-curves, a close study of a particular curve, and some open questions. SG

NUMBER THEORY, T**(14: 1), S*, P, L*. *An Introduction to the Theory of Numbers, Fourth Edition.* Ivan Niven, Herbert S. Zuckerman. Wiley, 1980, xii + 335 pp, $19.95. [ISBN: 0-471-02851-7] Extensive refinements have been made in the *Fourth Edition.* The approach is less formal; historical observations, more easy problems and an introductory section on algorithms and computing have been added. (*Third Edition,* TR, June-July 1972.) CEC

NUMBER THEORY, T*(13-14: 1), S, L*. *Elementary Number Theory, Revised Printing.* David M. Burton. Allyn, 1980, ix + 390 pp, $19.95. [ISBN: 0-205-06965-7] A revised printing of this highly readable 1976 text. Errors have been corrected. (First Edition, TR, March 1976.) CEC

LINEAR ALGEBRA, P. *Algorithmes et Pratique de Programmation Linéaire.* Philippe Chrétienne, Yvon Pesqueux, Jean-Claude Grandjean. Éditions Technip, 1980, xviii + 332 pp, 250 FF (P). [ISBN: 2-7108-0364-X] Presentation of the simplex method, the revised simplex method and duality. A number of completely worked examples from business and industry. Few exercises. Note price. JG

LINEAR ALGEBRA, T(14: 1). *Applied Linear Algebra.* Alden F. Pixley. U Pr of America, 1980, vi + 258 pp, $10.50 (P). [ISBN: 0-8191-1169-4] Misnamed--this should be entitled Rigorous Linear Algebra. The only applications are to differential equations and a discussion of the Monte Carlo method. LLK

ALGEBRA, P, L**. *Handbook of Applicable Mathematics, Volume I: Algebra.* Ed: Walter Ledermann, Steven Vajda. Wiley, 1980, xix + 524 pp, $89.50. [ISBN: 0-471-27704-5] First of six core volumes of an innovative series aimed at professional adult users of mathematics. Contains 17 independent but carefully coordinated and cross-referenced chapters (by different authors) on such topics as sets, eigenvalues, groups, integer programming, financial mathematics, and units of measurement. Each chapter contains basic definitions, examples and theorems (but no proofs), designed to enable a user of mathematics to find just what he or she needs to know. "The Handbook has been written as a contribution to the *practice* of mathematics, not to the *theory.*" The editors' intention is "to ensure that no branch of mathematics is omitted which is useful, and conversely that none is included which is not." (The six core volumes will be supplemented by a series of *Guidebooks,* cross-referenced to the core volumes, that discuss the application of mathematics to particular problems in the various sciences.) LAS

ALGEBRA, P. *Affine Representations of Grothendieck Groups and Applications to Rickart C*-Algebras and $\aleph_0$-Continuous Regular Rings.* K.R. Goodearl, D.E. Handelman, J.W. Lawrence. Memoirs No. 234. AMS, 1980, vii + 163 pp, $6 (P). [ISBN: 0-8218-2234-9]

ALGEBRA, S(18), P, L. *Combinatorial, Algebraic and Topological Representations of Groups, Semigroups and Categories.* Ales Pultr, Vera Trnková. Math. Lib., V. 22. North-Holland, 1980, x + 372 pp, $48.75. [ISBN: 0-444-85083-X] An approach by way of category theory to representations of groups as automorphism groups for some structured set. No background in category theory is assumed, though some previous algebra is desirable. Three introductory chapters are followed by applications to combinatorics, algebra, and topology. Some new results and unsolved problems are mentioned. Exercises, bibliography, index. JS

ALGEBRA, P. *Lecture Notes in Mathematics-795: Séminaire d'Algèbre Paul Dubreil et Marie-Paule Malliavin.* Ed: M.P. Malliavin. Springer-Verlag, 1980, v + 433 pp, $25.70 (P). [ISBN: 0-387-09980-8] Proceedings of the 1979 session. JAS

ALGEBRA, S(18), P. *Cyclotomic Fields II.* Serge Lang. Grad. Texts in Math., V. 69. Springer-Verlag, 1980, xi + 164 pp, $19.80. [ISBN: 0-387-90447-6] A natural continuation of *Volume I* (it begins with Chapter 10), "the main concern is with class number formulas, Gauss sums and the like." Includes Ferrero-Washington theorems leading to proof of an Iwasawa conjecture, applications to Iwasawa invariants, Gross-Koblitz formula for Gauss sums, and Ferrero-Greenberg theorems. Bibliography, index. JS

ALGEBRA, P. *Richard Brauer: Collected Papers.* Ed: Paul Fong, Warren J. Wong. MIT Pr, 1980. *Volume I: Theory of Algebras, and Finite Groups,* liv + 615 pp; *Volume II, Finite Groups,* viii + 586 pp; *Volume III, Finite Groups, Lie Groups, Number Theory, Polynomials and Equations; Geometry, and Biography,* x + 689 pp, $165 set. [ISBN: 0-262-02157-9] 120 of Brauer's 129 papers in three major parts: 20 papers on the theory of algebras in *Volume I;* 74 papers on finite groups in *Volumes I-III;* and, concluding *Volume III,* 26 papers on miscellaneous topics. *Volume I* contains, in addition, a brief recounting by Brauer of his mathematical education, a thorough professional autobiography by J.A. Green (reprinted from *Bull. London Math. Soc.* 10 (1978) 317-342), and a complete list of Brauer's publications. LAS

ALGEBRA, P, *Generators and Relations for Discrete Groups, Fourth Edition.* H.S.M. Coxeter, W.O.J. Moser. Ergebnisse der Math., B. 14. Springer-Verlag, 1980, ix + 169 pp, $37.40. [ISBN: 0-387-09212-9] Two changes from the *Third Edition* (TR, March 1973), only one of which can be attributed to the authors: (1) A new exposition of the Todd-Coxeter coset enumeration procedure, (2) A price increase to $37.40 from $13.40. SG

ALGEBRA, T(14-16: 1), L, *Abstract Algebra: A First Course.* Dan Saracino. A-W, 1980, v + 233 pp, $16.95. [ISBN: 0-201-07391-9] Somewhat modest in content and size, this is a carefully written introduction to abstract algebra suitable for a one-semester course. Fairly extensive treatment of groups, including the Fundamental Theorem for finite abelian groups and the Sylow theorems. Much more limited treatment of rings and practically nothing on fields, but does discuss unique factorization domains and Fermat theorem. Many examples, exercises, index, bibliography. JS

ALGEBRA, T(17-18: 1), P, *Lectures on Character Theory.* David M. Goldschmidt. Publish or Perish, 1980, vi + 243 pp, $12. [ISBN: 0-914098-17-9] Study of arithmetic properties of group characters. Substantial review of character theory. Inspired by Brauer, follows his treatment. Notation index. Few exercises. JG

CALCULUS, T?(14: 1), S, *Introduction to Analysis.* W. Vance Underhill. U Pr of America, 1980, viii + 424 pp, $12.75 (P). [ISBN: 0-8191-1205-4] Intended as a text for a transition course from intuitive freshman calculus to more rigorous mathematics, this photocopy edition consists of rigorous proofs of standard first year calculus theorems without introducing new concepts. Probably better suited as a supplementary reference. JNC

CALCULUS, T**(13-14: 2), S, L*, *Calculus, Second Edition.* Michael Spivak. Publish or Perish, 1980, xii + 647 pp, $19.95. [ISBN: 0-914098-77-2] A great book returns! The *Second Edition* contains appendices on polar coordinates, uniform continuity, parameterized curves, Riemann sums, and applications of integration--topics which were previously slighted. There are also 160 new problems and corrections and refinements throughout the text. JAS

CALCULUS, T(13: 2), *Introductory Mathematical Analysis for Students of Business and Economics, Third Edition.* Ernest F. Haeussler, Jr., Richard S. Paul. Reston Pub, 1980, xv + 910 pp, $18.95. [ISBN: 0-8359-3282-6] New chapter on mathematics of finance; expanded discussion and coverage of the simplex method; additional examples, exercises and problems scattered throughout. (*First Edition*, TR, May 1974; *Second Edition,* TR, December 1976.) LCL

CALCULUS, T(14: 1), *Advanced Calculus and Its Applications to the Engineering and Physical Sciences.* John C. Amazigo, Lester A. Rubenfeld. Wiley, 1980, viii + 407 pp, $20.95. [ISBN: 0-471-04934-4] Good text. Chapters 1-5 in the same league with *Vector Calculus* by Marsden-Tromba and *Intermediate Calculus* by Hurley. Chapters 6-8 could be covered with a more advanced class. LLK

CALCULUS, T(13: 2), *Calculus and Its Applications, Second Edition.* Larry J. Goldstein, David C. Lay, David I. Schneider. P-H, 1980, xx + 610 pp, $19.95. [ISBN: 0-13-111963-X] Intuitive approach written especially for students of biology, social sciences, and management science. This edition adds an intuitive treatment of limits and additional chapters on Taylor polynomials, infinite series, and continuous probability. (First Edition, TR, February 1977.) LCL

CALCULUS, T(13: 2), *Mathematics and Calculus with Applications.* Margaret L. Lial, Charles D. Miller. Scott F, 1980, 754 pp, $19.95. [ISBN: 0-673-15352-5] This text is an expanded version of *Mathematics with Applications, Second Edition* (TR, April 1979). This volume extends the calculus coverage to include further applications of the derivative and the integral, and functions of several variables. LCL

COMPLEX ANALYSIS, T(16-17: 2), L, *Complex Function Theory.* Anthony S.B. Holland. North-Holland, 1980, xii + 304 pp, $26.95. [ISBN: 0-444-00342-8] A careful, complete treatment from the algebra of complex numbers through the Weierstrass factorization theorem for entire functions. For a one-year course that covers every topic along that route, this text is a good choice. TAV

COMPLEX ANALYSIS, P, *Complex Approximation.* Ed: Bernard Aupetit. Progress in Math., No. 4. Birkhäuser, 1980, ix + 118 pp, $8 (P). [ISBN: 3-7643-3004-X] Texts of the invited lectures and some supplementary papers given during a conference held at Quebec on July 3-8, 1978. JAS

COMPLEX ANALYSIS, S(18), P, *Lecture Notes in Mathematics-783: Wertverteilung meromorpher Funktionen in ein- und mehrfach zusammenhängenden Gebieten.* Alexander Dinghas. Springer-Verlag, 1980, xiii + 145 pp, $12.70 (P). [ISBN: 0-387-09759-7] A unified treatment of results due to the Nevanlinnas and others on the distribution of values of functions meromorphic in simply- and multiply-connected regions. JD-B

DIFFERENTIAL EQUATIONS, T(16-18: 1, 2), S, P, L, *Differential Equations, Introduction and Qualitative Theory.* Jane Cronin. Pure and Appl. Math., V. 54. Dekker, 1980, viii + 372 pp, $37.50. [ISBN: 0-8247-6819-1] Basic theory and some recently developed qualitative theory of ordinary differential equations. Emphasis on stability theory. No treatment of eigenvalue problems. Examples in biological, physiological and chemical systems. Advanced calculus and linear algebra assumed. Exercises, many not routine. Over seventy selected references to books and papers. Very brief but very interesting collection of historical comments in Chapter 0. JK

DIFFERENTIAL EQUATIONS, P, *Lecture Notes in Mathematics-810: Geometrical Approaches to Differential Equations.* Ed: R. Martini. Springer-Verlag, 1980, vii + 339 pp, $19.50 (P). [ISBN: 0-387-10018-0] Proceedings of the Fourth Scheveningen (The Netherlands) conference on differential equations held on August 26-31, 1979. JAS

NUMERICAL ANALYSIS, T(15-16: 1), L. *Elementary Numerical Analysis: An Algorithmic Approach, Third Edition.* S.D. Conte, Carl de Boor. McGraw-Hill, 1980, xii + 432 pp, $21.95. [ISBN: 0-07-012447-7] Revised edition of successful textbook. Incorporates discussion of several topics not in previous editions. Programs in Fortran 77 are presented for many of the algorithms. (*First Edition,* TR, March 1967; *Second Edition,* TR, February 1973.) AO

NUMERICAL ANALYSIS, P. *Computational Methods in Nonlinear Mechanics.* Ed: J.T. Oden. North-Holland, 1980, viii + 539 pp, $68.25. [ISBN: 0-444-85382-0] 26 papers selected from a conference held at the University of Texas in March 1979 for the purpose of exploring new ideas concerning the "nonlinear barrier" in mechanics emerging from recent research in mechanics, mathematics and computer science. LAS

NUMERICAL ANALYSIS, S(17-18), P*, L. *Sparse Matrices and Substructures with a Novel Implementation of Finite Element Algorithms.* F.J. Peters. Math. Centre Tracts, No. 119. Math Centrum, 1980, ii + 98 pp, Dfl. 12 (P). [ISBN: 90-6196-192-0] Newly developed, highly efficient recursive algorithm for solving sparse sets of linear equations. These important results promise broad application, for example, wherever the finite element method of calculation is encountered. LCL

FUNCTIONAL ANALYSIS, T(18), *Introduction to Functional Analysis, Second Edition.* Angus E. Taylor, David C. Lay. Wiley, 1980, xi + 467 pp, $23.95. [ISBN: 0-471-84646-5] A clearly written, thorough text, ideal for a second-year graduate course. Some features include an emphasis on topological linear spaces (rather than merely normed linear spaces) and a de-emphasis on completeness in the study of linear operators. Includes chapters on spectral theory in Hilbert space and Banach algebras. Contains almost 500 exercises. SG

FUNCTIONAL ANALYSIS, P. *Holomorphic Maps and Invariant Distances.* Tullio Franzoni, Edoardo Vesentini. Math. Stud., V. 40. North-Holland, 1980, viii + 226 pp, $31.75 (P). [ISBN: 0-444-85436-3] A self-contained treatment of the Carathéodory-Kobayashi distances and differential matrices on domains in complex Banach spaces. Various maximum principles and pseudometrics are developed, with a complete discussion of the semigroup of holomorphic isometries of the unit ball in a complex Hilbert space. TAV

FUNCTIONAL ANALYSIS, T(18: 2), S, P. *Applications of Functional Analysis and Operator Theory.* V. Hutson, J.S. Pym. Math. in Sci. and Eng., V. 146. Acad Pr, 1980, xi + 389 pp, $39.50. [ISBN: 0-12-363260-9] Chapters 1-5 review introductory functional analysis. Authors "have chosen to concentrate on a basic area which might be described roughly as 'the solution of equations.'" Non-linear equations emphasized throughout. Includes chapters on degree theory and bifurcation theory. More exposition than usual; many examples. SES

OPTIMIZATION, P. *Dynamic Optimization and Mathematical Economics.* Ed: Pan-Tai Liu. Math. Concepts and Methods in Sci. and Eng., V. 19. Plenum Pr, 1980, x + 269 pp, $29.50. [ISBN: 0-306-40245-9] In the hope that the book will stimulate interaction between control theorists and economists, fifteen papers are presented, each with its own perspective on future developments in dynamic economic theory, including one by Nobel laureate Kenneth Arrow. TAV

OPTIMIZATION, P. *Integer Programming: Facets, Subadditivity, and Duality for Group and Semi-group Problems.* Ellis L. Johnson. CBMS Reg. Conf. in Appl. Math., No. 32. SIAM, 1980, vii + 68 pp, $9 (P). [ISBN: 0-89871-162-2] This monograph contains the lectures from an NSF sponsored regional conference; intended as a contribution to the research literature on integer programming. Printed (not duplicated from typewritten notes) with attractive drawings. AWR

OPTIMIZATION, T(17: 1, 2), S, P, L. *Conjugate Direction Methods in Optimization.* Magnus R. Hestenes. Appl. of Math., No. 12. Springer-Verlag, 1980, x + 325 pp, $29.80. [ISBN: 0-387-90455-7] A relatively full account of conjugate direction methods in optimization. Well written and replete with examples and exercises. PDH

OPTIMIZATION, P. *Advances in Geometric Programming.* Ed: Mordecai Avriel. Math. Concepts and Methods in Sci. and Eng., V. 21. Plenum Pr, 1980, x + 460 pp, $39.50. [ISBN: 0-306-40381-1] Of the 19 chapters, 14 are reprinted from two 1978 issues of the *Journal of Optimization* that were devoted to geometric programming. Three other chapters are reprints of earlier papers included to make this book as self-contained as possible. AWR

OPTIMIZATION, T**(15-16), S*, L**. *Methods of Mathematical Economics: Linear and Nonlinear Programming, Fixed-Point Theorems.* Joel Franklin. Springer-Verlag, 1980, x + 297 pp, $24. [ISBN: 0-387-90481-6] A delightfully written text, accessible to advanced undergraduates. Emphasizes linear programming models as well as the techniques for solving them--simplex, duality. Contains an introduction to quadratic and geometric programming and three proofs of Brouwer's fixed point theorem. Hard to put down as it is so well written! TAV

ANALYSIS, S(18), P. *Approximation by Polynomials with Integral Coefficients.* LeBaron O. Ferguson. Math. Surveys, No. 17. AMS, 1980, xi + 160 pp, $25.60. [ISBN: 0-8218-1517-2] A fairly complete and readable treatment of the theory of approximation of functions by polynomials with integral coefficients. Many examples, illustrative remarks, and historical notes. PDH

ANALYSIS, T(16-17: 1), S, P, L. *Measure and Category: A Survey of the Analogies between Topological and Measure Spaces, Second Edition.* John C. Oxtoby. Grad. Texts in Math., V. 2. Springer-Verlag, 1980, x + 106 pp, $19.80. [ISBN: 0-387-90508-1] A "Supplementary Notes and Remarks" section has been appended to the text to describe recent developments. The reference list has been expanded. Continues to be a lovely little book. (First Edition, TR, February 1972.) PDH

ANALYSIS, T(18: 2), P. *Einführung in die harmonische Analyse.* Walter Schempp, Bernd Dreseler. B.G. Teubner, 1980, 300 pp, (P). [ISBN: 3-519-02220-6] A sophisticated and difficult introduction to harmonic analysis, assuming some knowledge of general topology, integration theory and functional analysis. JD-B

ALGEBRAIC GEOMETRY, P. *Lecture Notes in Mathematics-777: Séminaire sur les Singularités des Surfaces.* Ed: M. Demazure, H. Pinkham, B. Teissier. Springer-Verlag, 1980, ix + 339 pp, $20.40 (P). [ISBN: 0-387-09746-5] Most of the papers presented at the 1976-1977 seminar at the Centre de Mathématique de l'Ecole Polytechnique. JAS

ALGEBRAIC GEOMETRY, T(18: 1), S, P, L. *Introduction to Algebraic Geometry and Algebraic Groups.* Michael Demazure, Peter Gabriel. Math. Stud., V. 39. North-Holland, 1980, xiv + 357 pp, $36.50 (P). [ISBN: 0-444-85443-6] In order to make effective use of category theory the authors adopt a functorial (as opposed to geometric) point of view in their approach to algebraic schemes. There are two chapters. Chapter I is an introduction to algebraic geometry, schemes, and morphisms. Chapter II is concerned with algebraic groups, working through cohomology and calculus on group schemes up to Lie algebras, including some discussion for characteristic p = 0. Includes functorial dictionary and indexes for notation and terminology. JS

ALGEBRAIC GEOMETRY, T(18: 1), S, P, L. *Affine Sets and Affine Groups.* D.G. Northcott. London Math. Soc. Lect. Note Ser., No. 39. Cambridge U Pr, 1980, x + 285 pp, $22.50 (P). [ISBN: 0-521-22909-X] Assuming a background in commutative algebra and field theory, Part I develops a theory of affine sets up to derivatives and tangent spaces. Part II deals with affine groups up to the development of the associated Lie algebra with a last chapter on power series and exponentials. Bibliography, index. JS

DIFFERENTIAL GEOMETRY, P. *Lecture Notes in Mathematics-805: Generalized Symmetric Spaces.* Oldrich Kowalski. Springer-Verlag, 1980, xii + 187 pp, $11.80 (P). [ISBN: 0-387-100024-2] The author defines formally and studies the notion of a generalized symmetric space. Topics include reductive spaces, s-manifolds, and classification of generalized symmetric Riemannian spaces in low dimensions. SG

DIFFERENTIAL GEOMETRY, P. *All Compact Orientable Three Dimensional Manifolds Admit Total Foliations.* Detlef Hardorp. AMS, 1980, vi + 74 pp, $4 (P). [ISBN: 0-8218-2233-0] A slight revision of the author's Ph.D. thesis which proved the assertion made in the title. JAS

TOPOLOGY, T(16-18: 1, 2), S, L. *Topological Methods in Euclidean Spaces.* Gregory L. Naber. Cambridge U Pr, 1980, x + 230 pp, $19.95. [ISBN: 0-521-22746-1] Brief but broad introduction to concepts and methods of basic point-set topology, combinatorial techniques, homotopy theory, simplicial homology theory and differential techniques. For the mathematically mature. In particular, readers should know differential equations, advanced calculus, linear and abstract algebra. Emphasis on motivation. Exercises play vital role in text development. Guide to further study keyed to bibliography. JK

PROBABILITY, T(16-17: 1), S, P, L. *Lattice Path Counting and Applications.* Sri Gopal Mohanty. Prob. and Math. Stat. Acad Pr, 1979, xi + 185 pp, $21. [ISBN: 0-12-504050-4] The author has synthesized and summarized results of the past two decades related to lattice path combinatorics. Includes discussions of counting methods, invariance properties of paths and relations to fluctuation theory, applications, and convolution identities. Includes exercises and substantial lists of references. CEC

PROBABILITY, S? *Craps, A Smart Shooter's Guide.* Thomas Midgley. GBC Pr, 1980, 219 pp, $6.95 (P). [ISBN: 0-89650-625-8] Written by someone who believes dice have a memory! Contains 64 pages of actual data accumulated over 40 hours of playing craps at five Las Vegas casinos. RSK

PROBABILITY, P. *Lecture Notes in Mathematics-784: Séminaire de Probabilités XIV, 1978/79.* Ed: J. Azéma, M. Yor. Springer-Verlag, 1980, vii + 546 pp, $33.70 (P). [ISBN: 0-387-09760-0] The "continuation" of the Strasbourg seminar, now at Paris. JAS

STATISTICS, P. *Lecture Notes in Statistics-2: Mathematical Statistics and Probability Theory.* Ed: W. Klonecki, A. Kozek, J. Rosiński. Springer-Verlag, 1980, xxii + 373 pp, $20 (P). [ISBN: 0-387-90493-X] Thirty papers from a conference on mathematical statistics held in Wisła, Poland, during December 1978. Dedicated to Jerzy Neyman, this volume also contains three very short papers concerning his contributions. RSK

STATISTICS, T(13-14: 1, 2). *Statistics for Management.* Lincoln L. Chao. Brooks/Cole, 1980, xiv + 738 pp, $17.95. [ISBN: 0-8185-0367-X] Presupposes only intermediate algebra. Non-mathematical, intuitive treatments of the usual topics along with some nonparametric tests, an introduction to decision theory, multiple regression, time series, and quality control. FLW

STATISTICS, T(13: 1). *Elementary Statistics.* Donald R. Burleson. Winthrop Pub, 1980, xiv + 368 pp, $17.95. [ISBN: 0-87626-213-2] Presupposes no college level mathematics. Contains the usual topics and several nonparametric tests, but no Bayesian methods. FLW

STATISTICS, T(13: 1, 2). *Introduction to Statistics.* Lincoln L. Chao. Brooks/Cole, 1980, xiii + 512 pp, $18.95. [ISBN: 0-8185-0321-1] Presupposes only intermediate algebra. The usual topics plus some nonparametric tests and an introduction to decision theory. FLW

COMPUTER PROGRAMMING, S, P. *BCPL--The Language and its Compiler.* Martin Richards, Colin Whitby-Strevens. Cambridge U Pr, 1979, x + 173 pp, $21.95. [ISBN: 0-521-21965-5] An introduction to BCPL paying particular attention to programming style. BCPL is a simple systems programming language with a portable compiler. Includes a number of extended examples. Suitable as a handbook for established users. CEC

COMPUTER PROGRAMMING, S(15), P, L. *Software Manual for the Elementary Functions.* William J. Cody, Jr., William Waite. P-H, 1980, x + 269 pp, $16.95. [ISBN: 0-13-822064-6] Written for a broad audience but especially for systems programmers not familiar with numerical methods, this manual provides specific recipes for the preparation and testing of elementary function subroutines for non-vector oriented digital computers. Will be of interest to anyone concerned or curious about how the elementary functions might be computed. TRS

COMPUTER PROGRAMMING, T(13: 1), S. *BASIC for Business for the PDP-11.* Alan J. Parker, Val Silbey. Reston Pub, 1980, viii + 264 pp, $10.95 (P). [ISBN: 0-8359-0349-4] Very good pedagogically. Objectives clearly stated and summaries excellent. Carefully written, easy to follow, good examples and problems. LLK

COMPUTER SCIENCE, P*, L. *TEX and METAFONT, New Directions in Typesetting.* Donald E. Knuth. Digital Pr, 1979, xi + 105 pp, $12 (P). [ISBN: 0-932376-02-9] Three documents in a single volume: Knuth's 1978 Gibbs Lecture, reprinted from the *Bull. Amer. Math. Soc.* 1 (1979) 337-372; the manual for TEX, a computer system for a technical text first published by the Amer. Math. Soc. in 1979 (TR, December 1979); and a manual for METAFONT, a general system for designing alphabets. Intended not as a finished document, but as an interim report so that a large community of users can experiment with the system. LAS

COMPUTER SCIENCE, T(14-15: 1). *Microcomputers for Engineers and Scientists.* Glenn Gibson, Yu-cheng Liu. P-H, 1980, xv + 479 pp, $24.95. [ISBN: 0-13-580886-3] Text uses Intel 8080 and associated devices to present basic concepts of microprocessor based hardware and software design. Also includes chapters on 16-bit microprocessors and bit-sliced architectures. AO

COMPUTER SCIENCE, P. *Lectures on the Logic of Computer Programming.* Zohar Manna. CBMS Reg. Conf. in Appl. Math., No. 31. SIAM, 1980, iv + 49 pp, $7.50 (P). [ISBN: 0-89871-164-9] Lecture Notes which deal with aspects of computer programming that involve techniques derived from mathematical logic. Includes a proof that a given program produces the intended result whenever it halts, that a given program eventually halts, that a given program is partially correct and terminates, and that a system of rewriting rules always halts. CEC

COMPUTER SCIENCE, P. *Lecture Notes in Computer Science-84: New Theory and Applications.* Ed: Wilfried Brauer. Springer-Verlag, 1980, xiii + 537 pp, $29.50 (P). [ISBN: 0-387-10001-6] Nets, in particular Petri nets, offer a way of describing complex situations that cannot be described by classical sequential system models. This volume consists of the Proceedings of the Advanced Course on General Net Theory of Processes and Systems held at Hamburg, October 8-19, 1979. Includes introductory material. JAS

COMPUTER SCIENCE, P. *Automata, Languages and Programming.* Ed: J.W. de Bakker, J. van Leeuwen. Springer-Verlag, 1980, viii + 671 pp, $31.90 (P). [ISBN: 0-387-10003-2] Proceedings of the Seventh Annual Colloquium of the European Association for Theoretical Computer Science held at Noordwijkerhout, The Netherlands on July 14-18, 1980. JAS

COMPUTER SCIENCE, T(13: 1). *Computers in Action with FORTRAN: Data Processing.* Perry Edwards, Bruce Broadwell. Wadsworth, 1980, 472 pp, $16.95. [ISBN: 0-534-00805-4]; *Study Guide,* William L. Harrison, vi + 244 pp, $6.95 (P). [ISBN: 0-534-00879-8] Introductory computing text in hardware/software. Combines general study of computers with a specific language (Fortran IV). Divided into four modules: Hardware, Language, Management, and Supplements. Includes a good selection of problems. LLK

COMPUTER SCIENCE, P. *Lecture Notes in Computer Science-83: International Symposium on Programming.* Ed: B. Robinet. Springer-Verlag, 1980, vii + 341 pp, $19.50 (P). [ISBN: 0-387-09981-6] 23 of the 24 papers presented at a conference in Paris on April 22-24, 1980. LAS

COMPUTER SCIENCE, S*, L. *Microcomputer Interfacing.* Bruce A. Artwick. P-H, 1980, x + 341 pp, $18.95. [ISBN: 0-13-580902-9] An informal and readable presentation for the serious hobbyist, intermediate student, or mathematician faced with new gadgets. The book contains a general discussion of many technical problems which are considered dull by manufacturers and programmers, but which "become very interesting when you try to use a microprocessor in an actual application." Although jargon and technical vocabulary are kept under control (in spite of what must have been great temptation), the index and glossary don't cover the language gap as well as they might; the book contains much valuable information that's hard to find quickly. It's still a potentially very useful volume. JAS

COMPUTER SCIENCE, P, L. *Computers and Privacy in the Next Decade.* Ed: Lance J. Hoffman. Acad Pr, 1980, xv + 215 pp, $16.50. [ISBN: 0-12-352060-6] Eight papers from two meetings sponsored by NSF and AFIPS designed to highlight the most pressing issues in the field. LAS

COMPUTER SCIENCE, P. *Lecture Notes in Computer Science-88: Mathematical Foundations of Computer Science 1980.* Ed: P. Dembiński. Springer-Verlag, 1980, viii + 723 pp, $37.20 (P). [ISBN: 0-387-10027-X] Texts of 7 invited lectures (e.g., discrete optimization, abstract data types, electronic category theory) and 46 contributed papers from the Ninth Annual Poland-Czechoslovakia Symposium held this year at Rydzyna, Poland, September 1-5, 1980. These proceedings were in print within a month of the Symposium! LAS

SYSTEMS THEORY, P. *Lecture Notes in Control and Information Sciences-9: Learning Systems: Decision, Simulation, and Control.* Yousri M. El-Fattah, Claude Foulard. Springer-Verlag, 1978, vii + 119 pp, $9 (P). [ISBN: 0-387-09003-7] A learning system is a stochastic mechanical, electronic or economic system controlled by a feedback mechanism that is able to learn by comparing the system's performance to a predetermined goal. This short monograph introduces the theory (especially Bayesian inference and Markov chains) and applications (e.g., to resource allocation and price regulation) of such systems. LAS

SYSTEMS THEORY, T(16-18: 1, 2), S, P, L. *Linear Systems.* Thomas Kailath. P-H, 1980, xxi + 682 pp, $27.95. [ISBN: 0-13-536961-4] An integrated development of linear systems theory offering a synthesis of the state-space (first-order differential equation) approach and the transfer-function (high order differential equation) approach. Emphasis on discussion and motivation. LCL

SYSTEMS THEORY, P. *Control and Dynamic Systems: Advances in Theory and Application, V. 16.* Ed: C.T. Leondes. Acad Pr, 1980, xvii + 371 pp, $27.50. [ISBN: 0-12-012716-4] Eight research surveys on optimization of dynamic systems, model methods, Kalman filtering, time delays, noise statistics, and related topics. LAS

APPLICATIONS, S(18), P. *Number Theory in Digital Signal Processing.* James H. McClellan, Charles M. Rader. P-H, 1979, xii + 276 pp, $18.95. [ISBN: 0-13-627349-1] Primarily a collection of papers representative of the applications of number theoretic ideas to the development of algorithms for the two most basic signal processing computations: convolution and the discrete Fourier transform. Includes an introduction to relevant elementary number theory. CEC

APPLICATIONS, T(17-18: 1, 2), S, P, L*. *The Geometry of Biological Time.* Arthur T. Winfree. Biomathematics, V. 8. Springer-Verlag, 1980, xiv + 530 pp, $32. [ISBN: 0-387-09373-7] A rich, kaleidoscopic survey of theory (first half) and examples (second half, called the Bestiary) of cycles in nature (e.g., fireflies, circadian clocks, slime mold, oscillating reactions, female cycles), supported by large doses of elementary topology. The author, a biologist, eschews equations ("only in part because I seldom manage to get equations right"), but believes that the life sciences have much to gain from (and something to contribute to) topology. His reference list covers over 1000 items, 80% published since 1970, ranging over the literature of biology, chemistry, engineering, mathematics, medicine and physics. An extraordinary invitation to a timely and obviously growing interdisciplinary research area that uses mathematical models. LAS

APPLICATIONS, T(15-17: 1), S, L*. *An Introduction to Catastrophe Theory.* P.T. Saunders. Cambridge U Pr, 1980, xii + 144 pp, $22.95; $7.95 (P). [ISBN: 0-521-23042-X; 0-521-29782-6] A brief elementary exposition designed for students and scientists who have had no more mathematical training than university calculus. Includes complete descriptions of the elementary catastrophes, and applications to physics, social science, and biology. LAS

APPLICATIONS (BIOLOGY), P. *Nonlinear Oscillations in Biology.* Ed: Frank C. Hoppensteadt. Lect. in Appl. Math., V. 17. AMS, 1979, x + 253 pp, $29.20. [ISBN: 0-8218-1117-7] The proceedings of the Tenth Summer Meeting on Applied Mathematics, sponsored jointly by the AMS and SIAM at the University of Utah, June, 1978. TRS

APPLICATIONS (CODING THEORY), S(16-18), P. *Algebraic Coding Theory and Applications.* Ed: G. Longo. Springer-Verlag, 1979, x + 527 pp, $52.60 (P). [ISBN: 0-387-81544-9] Ten rather extensive survey papers presented by leading scholars at the two week session on coding theory held in Udine, Italy in July, 1978. LCL

APPLICATIONS (CONTROL THEORY), T(16-18: 1, 2), S, L. *Lectures on the Calculus of Variations and Optimal Control Theory.* L.C. Young. Chelsea, 1980, xiii + 337 pp, $14.95. [ISBN: 0-8284-0304-X] A corrected reprint in one volume of Young's engaging, personal, two-volume survey first published in 1969. The central elements in this work are generalized curves (called sliding regimes or chattering controls), analogous to distributions, that Young introduced by completing a certain topology on the space of ordinary curves. LAS

APPLICATIONS (ECONOMICS), P. *Demand Functions and the Slutsky Matrix.* S.N. Afriat. Princeton U Pr, 1980, xii + 269 pp, $16.50. A book for economists who understand mathematics. An in-depth investigation of the Slutsky theory in which it is argued that the theory has mathematical interest but does not serve for the purposes Slutsky had in mind. AWR

APPLICATIONS (ENGINEERING), P. *Fluid Mechanics in Energy Conversion.* Ed: John David Buckmaster. SIAM, 1980, ix + 315 pp, $28.50 (P). [ISBN: 0-89871-165-7] Proceedings of a SIMS conference on fluid mechanics and energy producing processes held at Alta, Utah, June 25-29, 1979. Conference focused on energy from the oceans, energy from the ground, nuclear reactor cooling, and combustion. TRS

APPLICATIONS (GEOGRAPHY), T(13-14: 2), S. *Mathematics for Geographers and Planners, Second Edition.* A.G. Wilson, M.J. Kirkby. Clarendon Pr, 1980, xvi + 408 pp, $36; $19.95 (P). [ISBN: 0-19-874114-6; 0-19-874115-4] The number of mathematics topics covered in these 400 pages is staggering. It is suggested for those with the mathematical education of 15 or 16 year olds. One would hope it would be used mainly by those who have a higher mathematics background and wish to relate it to such subjects as town and regional planning, resource management, and environmental planning. Topics include algebra, coordinate geometry, calculus, differential and difference equations, probability and statistics, stochastic processes, catastrophe theory and bifurcation. LLK

APPLICATIONS (SIMULATION), P. *Simulation of Distributed-Parameter and Large-Scale Systems.* Ed: S.G. Tzafestas. North-Holland, 1980, ix + 380 pp, $58.50. [ISBN: 0-444-85447-9] Proceedings of the meeting held at the University of Patras, Greece, October 2-4, 1979. JAS

APPLICATIONS (SOCIAL SCIENCE), S(14-16), L. *Mathematical Models in the Social, Management and Life Sciences.* D.N. Burghes, A.D. Wood. Ellis Horwood (Distr: Wiley), 1980, 287 pp, $38.95. [ISBN: 0-85312-097-8] Built around fictitious case studies ranging from the serious (drug concentration in the blood stream) to the frivolous (profits at the Yummy Jam Company). Required mathematics ranges over calculus, differential equations, linear programming, game theory. AWR

Reviewers Whose Initials Appear Above

Judith N. Cederberg, St. Olaf; Clifton E. Corzatt, St. Olaf; John Dyer-Bennet, Carleton; Jennifer Galovich, St. Olaf; Steven Galovich, Carleton; Paul D. Humke, St. Olaf; Lorraine L. Keller, St. Olaf; Richard S. Kleber, St. Olaf; Joseph Konhauser, Macalester; Loren C. Larson, St. Olaf; George H. Mills, Carleton; Arnold Ostebee, St. Olaf; A. Wayne Roberts, Macalester; Thomas R. Savage, St. Olaf; John Schue, Macalester; J. Arthur Seebach, Jr., St. Olaf; Stanley E. Seltzer, Carleton; Lynn Arthur Steen, St. Olaf; T.A. Vessey, St. Olaf; Martha Wallace, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY FRANK KOCHER. The Pennsylvania State University

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036

PERSONAL ITEMS

B.A. Fusaro, Chairman at Salisbury State College, Md., is spending 1980-81 as Adjunct Professor at the Department of Environmental Engineering Sciences, University of Florida, Gainesville.

E.L. May, Associate Professor at Salisbury State College, Md., will spend 1980-81 as Visiting Associate Professor at Wake Forest University, North Carolina.

Bernard Greenspan has retired at Drew University as of June 1 1980, with the title of Professor Emeritus.

Associate Professor *Robert A. Melter* of Southampton College Long Island University, has been promoted to Professor.

Jeanne Tamaki, formerly of Tulane University, has been appointed Assistant Professor of Mathematics at the University of Santa Clara.

Assistant Professor *Edward T. Ordman,* New England College, Henniker, New Hampshire, has been promoted to Associate Professor.

Thomas W. Hungerford of the University of Washington has been appointed Professor and Chairman of the Department of Mathematics at Cleveland State University.

Rasul A. Khan of Case Western Reserve University has been appointed Visiting Associate Professor of Mathematics at Cleveland State University.

Jerzy Szulga of Wroclaw University, Poland, has been appointed Research Fellow in Mathematics at Cleveland State University.

Walter Walker, formerly of Clemson University, has been appointed Assistant Professor of Mathematics at Eckerd College, St. Petersburg, FL.

At the University of Oklahoma, Associate Professor *David Kay* has been promoted to Professor and *William Ray,* formerly of Iowa State University, has been appointed Assistant Professor.

Assistant Professor *Jerrold W. Grossman* of Oakland University, Rochester, Michigan, has been promoted to Associate Professor.

Associate Professor *Donald Z. Spicer,* Vassar College, has been appointed Associate Dean of the College.

Associate Professor *David E. Dobbs* of the University of Tennessee, Knoxville, has been promoted to Professor.

Associate Professor *Don Hill* of Florida A&M University has been promoted to Professor.

Simon W. Straus of Camp Springs, Md., recently received a gold medal and the Decoration for Exceptional Civilian Service of the United States Air Force. This is the highest recognition which the Secretary of the Air Force can confer upon a civilian.

At the University of California, Berkeley, the following Lecturers have been appointed: *Timothy Carlson* of the University of Minnesota, *Andrejs Treibergs* of Stanford University, and *Phillip Yasskin* of the University of Maryland. *Adrian Wadsworth,* Associate Professor at U.C., San Diego, will be a Visiting Associate Professor during 1980-81. *Shmuel Winograd* of the IBM Research Center was Hitchcock Professor during November, 1980. *Stephen Paneitz* of M.I.T. has been awarded a Fellowship by the Miller Institute for Basic Research in Science for 1980-82. Professor *David Gale* was a co-recipient of the Von Neumann Prize for work in the theory of operations research. He also received a Lester R. Ford prize for an expository paper. *Paul R. Chernoff* was promoted to Professor effective July 1, 1980. Professor *Edwin Spanier* is on leave for 1980-81. He is spending the year at U.C., San Diego. Professor *John L. Kelley* began phased retirement on July 1, 1980.

Associate Professor *M. Ray Perryman* of Baylor University has been named Herman Brown Professor of Economics and Director of the Center for the Advancement of Economic Analysis.

At Fort Lewis College, Durango, Colorado, *Richard Gibbs* has been promoted to Professor and has been appointed Chairman. Associate Professor *Gary Grefsrud* has been promoted to Professor. Dr. *John Sopka* of Framingham, Massachusetts, has joined the faculty as Professor.

Dr. *Buchanan Cargal* died August 11, 1980, at Fort Walton Beach, Florida, at the age of 57. He was a member of the Association for 27 years.

SUGGESTION BOX

Members of the MAA are encouraged to send in suggestions, questions, etc. about the operations of the Association. Communications will be referred to the appropriate officer of the Association for answering; from time to time, those of general interest may also be answered in one or both of the official journals. Communications should be addressed to: Suggestion Box, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036.

CONFERENCE ON COMBINATORICS, GRAPH THEORY, AND COMPUTING

The twelfth Southeastern Conference on Combinatorics, Graph Theory, and Computing will be held March 2-5, 1981, at Pleasant Hall, Louisiana State University, Baton Rouge, Louisiana. There will be daily sessions for contributed papers together with twice daily instructional lectures by well known lecturers. The deadline for abstracts of contributed papers is February 16, 1981. For program and housing information write

Prof. K.B. Reid
Department of Mathematics
Louisiana State University
Baton Rouge, Louisiana 70803

THE FACULTY EXCHANGE CENTER

John Joseph, Program Coordinator for the Faculty Exchange Center, 952 Virginia Avenue, Lancaster, PA 17603, has announced a new function of the Center which may interest readers of the MONTHLY. For faculty members who may not wish to exchange teaching positions but who are willing to exchange houses for purposes of travel and study, the Center will publish directory supplements in the Spring and Fall of 1981.

To obtain a copy of the directory or the housing supplements or to be listed in either one, write to Dr. Joseph at the address shown above.

SHORT COURSE IN NUMERICAL LINEAR ALGEBRA

The Ohio Section of the MAA is sponsoring a short course in Numerical Linear Algebra to be held June 16-19, 1981, at Ohio State University in Columbus. The lectures, to be given by *Bostwick Wyman*, Associate Professor of Mathematics at Ohio State University, will treat some of the basic "theoretical" ideas of linear algebra in a practical, computer-oriented context.

For further information contact:

Barbara Miller, Assistant Professor
Division of Science and Mathematics
Lorain County Community College
1005 North Abbe Road
Elyria, Ohio 44035

COMING SOON IN THIS MONTHLY

The following articles will appear in the AMERICAN MATHEMATICAL MONTHLY for January 1981:

Bernard Grofman, Fair Apportionment and the Banzhaf Index
Ben G. Roth, Rigid and Flexible Frameworks
Bruce Rose and *Robert Stafford*, An Elementary Course in Mathematical Symmetry
J.R. Quine, Plücker Equations for Curves
James G. Kennedy, Arithmetic with Roman Numerals
Kenneth R. Meyer, An Application of Poincaré's Recurrence Theorem to Academic Administration
Paul Schaefer, Sum-Preserving Rearrangements of Infinite Series

The following articles are among those which have been accepted by the editors for later issues of the MONTHLY. The order of listing does not indicate the order in which they will appear.

William Abikoff, The Uniformization Theorem
E. Ray Bobo, Foursomes, Fivesomes and Orgies
Joel Brenner and *Roger Lyndon*, Proof of the Fundamental Theorem of Algebra
Ezra A. Brown, The First Proof of the Quadratic Reciprocity Law Revisited
Robert B. Burckel, Iterating Analytic Self-Maps of Discs
Philip J. Davis, Are There Coincidences in Mathematics?
Michael C. Gemignani, What Is a Computer Program?
Solomon W. Golomb, Irrational Sums and Twin Primes
Samuel L. Greitzer, The Ninth U.S.A. Mathematical Olympiad
H.B. Griffiths, Cayley's Version of the Resultant of Two Polynomials
Yakar Kannai, An Elementary Proof of the No-Retraction Theorem
R. Arthur Knoebel, Exponentials Reiterated
Saunders MacLane, Mathematical Models, A Sketch for the Philosophy of Mathematics
George M. Phillips, Archimedes, the Numerical Analyst
Louis J. Ratliff, Jr., A Brief History and Survey of the Catenary Chain Conjecture
Michael Rosen, Abel's Theorem on the Lemniscate

SAN FRANCISCO SHORT COURSE
January 5-6, 1981

The American Mathematical Society, in conjunction with its eighty-seventh annual meeting, will present a short course entitled *Cryptology in Revolution: Mathematics and Models* on Monday and Tuesday, January 5 and 6, 1981, in the San Francisco Hilton.

Cryptology is rapidly changing. Ever since the invention of asymmetric cryptosystems, public cryptology has changed in fundamental ways. There are now "unbreakable" cryptosystems. Applications range from data bases to legal contracts; electronics fund transfers to the U.S. census.

Cryptology depends for its success on several areas of mathematics. It draws mostly on classic number theory and computational complexity. However, other branches such as aspects of ergodic theory, information theory, and combinatorics also play fundamental roles.

This short course aims to survey the nature and scope of the research in public cryptology. In addition to the basic technical ideas the role of public research in this area will also be discussed.

The program is under the direction of Richard J. Lipton of the Department of Electrical Engineering and Computer Science at Princeton University. The timetable, including the names of the speakers and the titles of their lectures, appears below. Synopses of the talks may be found on pages 516, 524 of the October 1980 issue of the NOTICES of the American Mathematical Society.

A basic knowledge of elementary number theory including congruences, Euler's theorem, primitive roots, factorization, greatest common divisors, etc., will be presumed. For general information about the subject of the course, participants may consult *A new kind of cipher that would take millions of years to break*, by Martin Gardner in the August 1977 issue of *Scientific American*, pages 120-124, and *Cryptology in transition* by Abraham Lempel (Department of Electrical Engineering, Technion-Israel Institute of Technology, Haifa Israel) in *ACM Computing Surveys* (special issue on cryptology), December 1979, pages 285-303.

The registration fee will be \$30, with a special fee of \$10 for students and unemployed individuals. Further information about registration and accommodations can be found in the October and November issues of the NOTICES, or may be obtained by writing or calling the American Mathematical Society's Meeting Arrangements Department, P.O. Box 6887, Providence, Rhode Island 02940; Telephone (401) 272-9500, Ext. 239.

MONDAY, January 5

9:00 a.m. - 4:00 p.m.	REGISTRATION
2:00 p.m. - 3:15 p.m.	*A short history of public cryptology* George Davida, Department of Electrical Engineering, University of Wisconsin, Milwaukee, and Department of Information and Computer Sciences, Georgia Institute of Technology
3:30 p.m. - 4:45 p.m.	*Asymmetric cryptosystems* David P. Dobkin, Department of Computer Science, University of Arizona

TUESDAY, January 6

9:00 a.m. - 10:15 a.m.	*How safe are cryptosystems?* Richard J. Lipton
10:30 a.m. - 11:45 a.m.	*How secure are data bases?* David P. Dobkin
1:30 p.m. - 2:45 p.m.	*Cryptographic protocol* Richard A. DeMillo, Department of Information and Computer Sciences, Georgia Institute of Technology
3:30 p.m. - 4:15 p.m.	*Access control structures* Michael A. Harrison, Department of Electrical Engineering and Computer Science, University of California, Berkeley
4:15 p.m. - 5:00 p.m.	*General discussion: What is the future of public work in crytography?*

CONFERENCE ON UNDERGRADUATE MATHEMATICS

The sixth annual CONFERENCE ON UNDERGRADUATE MATHEMATICS, sponsored by the *Journal* of UNDERGRADUATE MATHEMATICS, will be held at Hendrix College, Conway, Arkansas, on April 10 and 11, 1981.

The program of the meeting will include presentation of papers written by students during their undergraduate careers, and talks by *R.H. Bing*, University of Texas; *Paul R. Halmos*, Indiana University; *Burton W. Jones*, University of Colorado; *M.Z. Nashed*, University of Delaware; *John W. Neuberger*, North Texas State University.

All student papers submitted before February 15, 1981, for publication in JUM will be considered for presentation at the CONFERENCE. Notification of acceptance will be made before March 15, 1981.

The CONFERENCE will be partially supported by a grant from the EXXON EDUCATIONAL FOUNDATION to Hendrix College. For those participants presenting papers, board and lodging will be paid and limited travel support will be available.

For information regarding the CONFERENCE, you may contact:

J.R. Boyd Department of Mathematics Guilford College Greensboro, North Carolina 27410 (919-292-5511, Ext. 276)	or	Robert C. Eslinger Department of Mathematics Hendrix College Conway, Arkansas 72032 (501-450-1254)

CALENDAR OF FUTURE MEETINGS

Sixty-fourth Annual Meeting, San Francisco, California, January 9–11, 1981.

Sixty-first Summer Meeting, Pittsburgh, Pennsylvania, August 17–19, 1981.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May 15–16, 1981.

EASTERN PENNSYLVANIA AND DELAWARE, Saturday before Thanksgiving.

FLORIDA, Bethune Cookman College, Daytona Beach, March 6–7, 1981.

ILLINOIS, Illinois State University, Normal, May 1–2, 1981.

INDIANA

INTERMOUNTAIN, Brigham Young University, Provo, Utah, April 10–11, 1981.

IOWA, Coe College, Cedar Rapids, April 10–11, 1981 (tentative date).

KANSAS, Benedictine College, Atchison, April 11–12, 1981.

KENTUCKY, Jefferson Community College, Louisville, April 3–4, 1981.

LOUISIANA–MISSISSIPPI, Mississippi State University, Mississippi State, February 13–14, 1981.

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, William and Mary College, Williamsburg, Virginia, April 11, 1981 (tentative date).

METROPOLITAN NEW YORK, spring. Deadline for papers two weeks before meeting.

MICHIGAN, Oakland University, Rochester, May 1–2, 1981.

MISSOURI, Northwest Missouri State University, Maryville, April 10–11, 1981.

NEBRASKA, University of South Dakota, Vermillion, South Dakota, April 10–11, 1981.

NEW JERSEY, Seton Hall University, South Orange, spring 1981.

NORTH CENTRAL, end of October and April. Deadline for papers October 1 and April 1.

NORTHEASTERN, Saturday before Thanksgiving and third week in June.

NORTHERN CALIFORNIA, California State University, Hayward, March 1981 (tentative date).

OHIO

OKLAHOMA–ARKANSAS, Oklahoma Christian College, Oklahoma City, March 27–28, 1981.

PACIFIC NORTHWEST, second Saturday in June. Deadline for papers six weeks before meeting.

ROCKY MOUNTAIN, Colorado College, Colorado Springs, May 1–2, 1981.

SEAWAY, first Saturday in November and Saturday in late April. Deadline for papers six weeks before meeting.

SOUTHEASTERN, University of Alabama, Birmingham, April 10–11, 1981.

SOUTHERN CALIFORNIA, first or second Saturday in March.

SOUTHWESTERN, New Mexico State University, Las Cruces, April 1981.

TEXAS, San Antonio College, San Antonio, April 10–11, 1981.

WISCONSIN, University of Wisconsin, La Crosse, late March–early April 1981.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Toronto, January 3–8, 1981.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES

AMERICAN MATHEMATICAL SOCIETY, San Francisco, California, January 7–10, 1981.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION

ASSOCIATION FOR COMPUTING MACHINERY, Los Angeles, California, November 9–11, 1981.

ASSOCIATION FOR SYMBOLIC LOGIC, San Francisco, California, January 9–10, 1981.

ASSOCIATION FOR WOMEN IN MATHEMATICS, San Francisco, California, January 7–11, 1981.

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHÉMATIQUES

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, St. Louis, Missouri, April 22–25, 1981.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Four Seasons Sheraton, Toronto, Canada, May 4–6, 1981.

PI MU EPSILON

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Troy, New York, June 8–10, 1981.

REPORTS AND ANNOUNCEMENTS OF THE ASSOCIATION AND ITS SECTIONS

Meetings and Announcements of the Association

Meetings of Its Sections

ERRATA

Vol. 85

In the 1978 index of Problems Solved, the following solutions and their solvers were omitted.

5608 (p. 500), solved by M. L. Glasser.
6080 (p. 502), solved by Paul G. Chauveheid (partial solution) and by R. N. Hevener, Jr., the proposer (partial solution).
6102 (p. 503), solved by William C. Waterhouse.
6102 (p. 504), solved by Scot Adams.
6116 (p. 505), solved by Leonard Scott and Douglas Costa.
6117 (p. 505), solved by Lee A. Rubel.
6118 (p. 506), solved by Chen-Han Sung.

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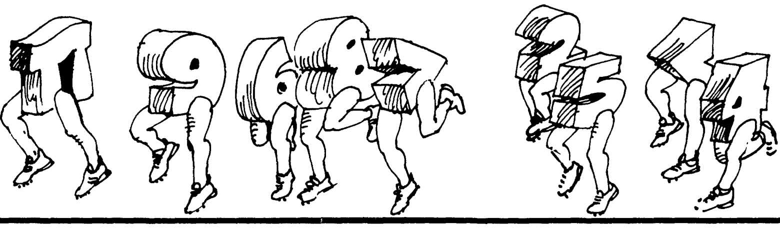
p. 373, line 13: "10^{27}" should read "10^{24}".



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